E-solid locally inverse semigroups as extensions

Outline of Ph.D. Thesis

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1 Introduction

Group extensions play a fundamental role both in the structure theory and in the theory of varieties of groups. In 1950, Kaloujnine and Krasner proved that any extension of a group N by a group H is embeddable in the wreath product of N by H, see [6]. Note that the wreath product of N by H is a special semidirect product of a direct power of N by H.

Semigroups are natural generalisations of groups. One of the important classes of semigroups where the influence of the Kaloujnine–Krasner Theorem is fundamental is the class of regular semigroups.

Inverse semigroups are one of the most natural generalisations of groups. By Cayley's Theorem we can think of groups (up to isomorphism) as sets of permutations on a given set which are closed under composition and taking inverse. A similar result, the Wagner–Preston Theorem, shows that inverse semigroups are, (also up to isomorphism) sets of partial permutations on a set X (i.e., bijection between subsets of X) which are closed under composition of partial maps and taking inverse.

A regular semigroup is completely simple if it is a union of its maximal subgroups and it contains only one $\mathcal{D}$-class. Note that in a completely simple semigroup all maximal subgroups are isomorphic to each other. Completely simple semigroups are also natural generalisations of groups.

Let K and T be semigroups. The semidirect product of K by T and the wreath product of K by T are defined analogously to the group case. They are denoted by $K \rtimes T$ and $K \wr T$, respectively. If K is a semigroup and T is a group then $K \rtimes T$ and $K \wr T$ are regular [inverse, completely simple] if and only if K is. However, in general, a semidirect product $K \rtimes T$ is not regular even if both K and T are inverse. This led Billhardt [2] to adapt these constructions to the inverse case. This construction is called the λ-semidirect product of K by T.

A congruence on an inverse semigroup S is said to be idempotent separating if every congruence class contains at most one idempotent and so, every idempotent class is a subgroup of S. Billhardt and Szittyai [3] proved that if S is an inverse semigroup and ϱ is an idempotent separating congruence such that every idempotent ϱ-class is from a group variety $\mathcal{V}$ then S is embeddable in a λ-semidirect product of a group from $\mathcal{V}$ by S/ϱ.

The thesis concentrates on E-solid locally inverse semigroups which are extensions by inverse semigroups and the idempotent classes are completely simple. The main problem we will give an answer to is whether such extensions are embeddable in a λ-semidirect product of a completely simple semigroup by an inverse semigroup. This result is a generalisation of Billhardt and Szittyai's result.

2 Preliminaries

A congruence ϱ is said to be a *group [semilattice, ...] congruence*, if S/ϱ is a group [semilattice, ...]. The kernel of a group congruence $\operatorname{Ker}\varrho$ is the inverse image of the identity element of S/ϱ. If ϱ is a semilattice congruence, and $\varphi\colon S \to Y$ is a surjective homomorphism inducing the congruence ϱ on S (and so $Y \cong S/\varrho$), then S is said to be the *semilattice Y of the subsemigroups* S_α $(\alpha \in Y)$ of S where S_α is the inverse image of α. If there are certain kinds of homomorphisms between these classes, called structure homomorphisms, and we can express the multiplication of S with the help of the multiplication of the S_α's and the structure homomorphisms, then S is said to be a *strong semilattice Y of the subsemigroups S_α* $(\alpha \in Y)$.

A semigroup S is *completely regular* if it is the union of its maximal subgroups. Recall that S is *completely simple* if it is completely regular and it contains only one $\mathcal{D}$-class. Every completely regular semigroup is known to be a semilattice of completely simple semigroups.

By a *Rees matrix semigroup* we mean a semigroup $S = \mathcal{M}[G; I, \Lambda; P]$ where G is a group, I, Λ are non-empty sets and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with elements from G, called a *sandwich matrix*. The underlying set of S is $I \times G \times \Lambda$, and the multiplication is defined by

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu).$$

Every Rees matrix semigroup is completely simple and conversely, by the Rees–Suschkewitsch Theorem, every completely simple semigroup is isomorphic to a Rees matrix semigroup. We say that P is *normalised* if there exists $i \in I$ and $\lambda \in \Lambda$ such that $p_{\mu i} = p_{\lambda j} = 1_G$ for every $j \in I$ and $\mu \in \Lambda$. Every Rees matrix semigroup is isomorphic to one with normalised sandwich matrix.

A completely simple semigroup is called *central* if the product of any two of its idempotents lies in the centre of the containing maximal subgroup. It is well known that a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ with P normalized is central if and only if each entry of P belongs to the centre of G.

The group congruences of a Rees matrix semigroup with a normalized sandwich matrix are characterized as follows.

Proposition 2.1. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup where P is normalized. Assume that N is a normal subgroup of G such that every entry of P belongs to N. Define a relation ϱ on S such that, for every (i, g, λ), $(j, h, \mu) \in S$, let*

$$(i, g, \lambda)\, \varrho\, (j, h, \mu) \quad \textit{if and only if} \quad gh^{-1} \in N.$$

Then ϱ is a group congruence on S such that S/ϱ is isomorphic to G/N and $\operatorname{Ker} \varrho = \mathcal{M}[N; I, \Lambda; P]$*.*

Conversely, every group congruence on S is of this form for some normal subgroup N of G where all entries of P belong to N.

A semigroup S is called an *inverse* semigroup if every element a of S has a unique inverse element denoted by a^{-1}. Equivalently, a semigroup is inverse if it is regular and the set of idempotents is a subsemilattice. An inverse semigroup S is a group if and only if $|E_S| = 1$.

Let S be a semigroup, and $\mathcal{K}$ a class of semigroups. If ϱ is an inverse semigroup congruence on S (i.e., S/ϱ is an inverse semigroup) then ϱ is said to be a *congruence over* $\mathcal{K}$ if each idempotent ϱ-class, as a subsemigroup of S, belongs to $\mathcal{K}$. In this case, the union of the idempotent ϱ-classes, called the *kernel of* ϱ and denoted $\operatorname{Ker} \varrho$, is a semilattice of subsemigroups belonging to $\mathcal{K}$. If S is regular then each idempotent ϱ-class and also $\operatorname{Ker} \varrho$ are regular subsemigroups.

A regular semigroup S is called *locally inverse* if each local submonoid eSe ($e \in E_S$) is an inverse subsemigroup. Note that each inverse semigroup and each completely simple semigroup is locally inverse.

We can introduce another binary operation $\wedge$ on a locally inverse semigroup S, called the *sandwich operation* on S. It has the properties that $s \wedge t$ is an idempotent and $s \wedge t = ss^* \wedge t^*t$ for every $s, t \in S$ and for any $s^* \in V(s)$ and $t^* \in V(t)$.

A regular semigroup S is called *E-solid* if the core of S, that is, the subsemigroup generated by the idempotents of S is completely regular. In particular, completely regular semigroups and inverse semigroups are E-solid. It is also known, that a regular semigroup is E-solid if and only if the least inverse semigroup congruence is over the class of all completely simple semigroups, see Yamada (and Hall) [8]. Thus the kernel of the least inverse semigroup congruence of an E-solid locally inverse semigroup is a locally inverse completely regular semigroup, that is, a strong semilattice of completely simple semigroups.

Let K be a semigroup and T an inverse semigroup. If S is a semigroup and ϱ is a congruence on S such that S/ϱ is isomorphic to T and $\operatorname{Ker}\varrho$ is isomorphic to K then the pair (S,ϱ) is called an *extension of K by T*.

Let K, T be semigroups. We denote the endomorphism monoid of K by $\operatorname{End} K$. We say that T acts on K by endomorphisms on the left, in short, T acts on K if an antihomomorphism $\varepsilon\colon T \to \operatorname{End} K$, $t \mapsto \varepsilon_t$ is given, that is a map, where $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ for any $u, t \in T$. For brevity, we will use the usual notation ${}^t a$ to denote $a\varepsilon_t$ $(a \in K, t \in T)$. The *semidirect product* $K \rtimes T$ is defined on the set $K \times T$ by multiplication

$$(a,t)(b,u) = (a \cdot {}^t b, tu).$$

A related construction is the following. For any semigroups K,T, an action of T on the direct power K^T can be defined in the following natural way: for any $f \in K^T$ and $t \in T$, let ${}^t f$ be the element of K^T where $u({}^t f) = (ut)f$ for any $u \in T$. The semidirect product $K^T \rtimes T$ defined by this action is called the *wreath product of K by T*, and is denoted by $K \wr T$. In case K and T are groups, these are the usual definitions of a semidirect product $K \rtimes T$ and of the wreath product $K \wr T$ of K and T.

If K is a semigroup and T is a group then $K \rtimes T$ and $K \wr T$ are regular [inverse, completely simple] if and only if K is. However, in general, a semidirect product $K \rtimes T$ is not regular even if both K and T are inverse. This led Billhardt [2] to adapt these constructions to the inverse case in the following way. Let K be a semigroup and T an inverse semigroup acting on K. The *λ-semidirect product* $K \rtimes_\lambda T$ is defined on the underlying set

$$\{(a,t) \in K \times T \colon {}^{tt^{-1}}a = a\}$$

by multiplication

$$(a,t)(b,u) = ({}^{(tu)(tu)^{-1}}a \cdot {}^t b, tu),$$

for all $a, b \in K$, $t, u \in T$.

A class of regular semigroups is termed an *existence variety*, or, for short, an *e-variety* if it is closed under taking direct products, homomorphic images and regular subsemigroups. For example, $\mathcal{LI}$, $\mathcal{ES}$ and $\mathcal{CS}$ form e-varieties.

Note also that a class of inverse semigroups or a class of completely simple semigroups constitutes an e-variety if and only if it forms a variety of semigroups with an additional operation ${}^{-1}$.

If S is a regular semigroup then an *inverse unary operation* is defined to be a mapping ${}^{\dagger}\colon S \to S$ with the property that $s^{\dagger} \in V(s)$ for every $s \in S$.

By a *binary semigroup* we mean a semigroup having an additional binary operation denoted by $\wedge$. A homomorphism or a congruence of a binary semigroup is always supposed to respect both the multiplication and the $\wedge$ operation. As noticed above, each locally inverse semigroup is also a binary semigroup with respect to the sandwich operation, and the homomorphisms and congruences of locally inverse semigroups, considered as usual semigroups and binary semigroups, respectively, coincide.

Let X be a non-empty set. The free semigroup on X is denoted by X^{+}. We 'double' X by forming $\overline{X} = X \cup X'$ where x' is a 'formal' inverse of x $(x \in X)$, and extend the definition of ${}'$ by putting $(x')' = x$ $(x \in X)$.

Let S be a regular semigroup. A mapping $\nu\colon \overline{X} \to S$ is called *matched* if $x'\nu$ is an inverse of $x\nu$ in S for each $x \in X$. Now let $\mathcal{K}$ be a class of regular semigroups. We say that a semigroup $B \in \mathcal{K}$ together with a matched mapping $\xi\colon \overline{X} \to B$ is a *bifree object in* $\mathcal{K}$ *on* X if, for any $S \in \mathcal{K}$ and any matched mapping $\nu\colon \overline{X} \to S$, there is a unique homomorphism $\varphi\colon B \to S$ extending ν, that is, for which $\xi\varphi = \nu$ holds. It was essentially proved by Yeh [9] that an e-variety admits a bifree object on any alphabet (or, equivalently, on an alphabet of at least two elements) if and only if it is contained either in $\mathcal{LI}$ or in $\mathcal{ES}$.

The *free binary semigroup* $F_{\langle 2,2\rangle}(Y)$ *on the alphabet* Y can be interpreted as follows. Its underlying set is the smallest one among the sets W which fulfill the following conditions:

(i) $Y \subseteq W \subseteq (Y \cup \{(,\ \wedge,\ )\})^{+}$,

(ii) $u, v \in W$ implies $uv \in W$,

(iii) $u, v \in W$ implies $(u \wedge v) \in W$.

The operations $\cdot$ and $\wedge$ are the concatenation and the operation

$$F_{\langle 2,2\rangle}(Y) \times F_{\langle 2,2\rangle}(Y) \to F_{\langle 2,2\rangle}(Y)\,,\ (u, v) \mapsto (u \wedge v)\,,$$

respectively.

A *bi-identity* in $\mathcal{LI}$ is a formal equality $u \mathrel{\hat{=}} v$ among terms $u, v \in F_{\langle 2,2\rangle}\left(\overline{X}\right)$. We say that a semigroup $S \in \mathcal{LI}$ *satisfies the bi-identity* $u \mathrel{\hat{=}} v$ if $u\overline{\nu} = v\overline{\nu}$ for each matched mapping $\nu \colon \overline{X} \to S$. The *bi-identity* $u \mathrel{\hat{=}} v$ *holds in the class* $\mathcal{K}$ *of locally inverse semigroups* if it holds in every member of $\mathcal{K}$. For an e-variety $\mathcal{V}$ of locally inverse semigroups, define

$$\begin{aligned}\Theta(\mathcal{V}, X) = \{(u,v) \in F_{\langle 2,2\rangle}\left(\overline{X}\right) \times F_{\langle 2,2\rangle}\left(\overline{X}\right) :& \\ \text{the bi-identity } u \mathrel{\hat{=}} v \text{ holds in } \mathcal{V}\}.&\end{aligned}$$

Then $\Theta(\mathcal{V}, X)$ is a congruence, and $F_{\langle 2,2\rangle}\left(\overline{X}\right)/\Theta(\mathcal{V}, X)$ is the bifree object in $\mathcal{V}$ on X.

In the sequel we need the description, published in [1], of the bi-identities satisfied in the e-variety $\mathcal{CS}$.

For any term $w \in F_{\langle 2,2\rangle}\left(\overline{X}\right)$, denote by ιw $[w\tau]$ the first [last] letter (i.e., element of $\overline{X}$) appearing in w (reading w from the left to the right as a word in the alphabet $\overline{X} \cup \{(, \wedge,)\}$). Similarly to the well-known model of free groups, reductions of terms in $F_{\langle 2,2\rangle}\left(\overline{X}\right)$ are introduced. It is proved that every word $w \in F_{\langle 2,2\rangle}\left(\overline{X}\right)$ has a uniquely determined reduced form which is denoted by $\mathbf{s}\,(w)$.

One of these reductions is $(u \wedge v) \rightsquigarrow (\iota u \wedge v\tau)$ for any $u, v \in F_{\langle 2,2\rangle}\left(\overline{X}\right)$. Notice that, applying this reduction, we obtain for any term in $F_{\langle 2,2\rangle}\left(\overline{X}\right)$ an element of the free semigroup $\widetilde{X}^+$ on the alphabet $\widetilde{X} = \overline{X} \cup (\overline{X} \wedge \overline{X})$ where $(\overline{X} \wedge \overline{X})$ stands for the set $\{(x \wedge y) : x, y \in \overline{X}\}$. This means that it is enough to consider words of $\widetilde{X}^+$ instead of $F_{\langle 2,2\rangle}\left(\overline{X}\right)$.

Proposition 2.2. *For any non-empty set X, we have*

$$\Theta(\mathcal{CS}, X) = \{(u,v) \in \widetilde{X}^+ \times \widetilde{X}^+ : \mathbf{s}\,(u) = \mathbf{s}\,(v)\}.$$

By considering all reductions, we have the following:

Lemma 2.3. *The congruence $\Theta(\mathcal{CS}, X)$ is generated on $\widetilde{X}^+$, as a semigroup congruence, by the relation $\mathrm{I} \cup \Upsilon$ where*

$$\mathrm{I} = \{(xx'x, x) : x \in \overline{X}\},$$

and Υ is the union of the following three relations:

$$\begin{aligned}\Upsilon_3 &= \{\big((x \wedge y)(x \wedge z), (x \wedge z)\big) : x, y, z \in \overline{X}\},\\ \Upsilon_4 &= \{\big((z \wedge x)(y \wedge x), (z \wedge x)\big) : x, y, z \in \overline{X}\},\\ \Upsilon_5 &= \{\big(x'x, (x' \wedge x)\big) : x \in \overline{X}\}.\end{aligned}$$

A *graph* $\mathcal{X}$ consists of a set of *objects* denoted by $\operatorname{Obj}\mathcal{X}$ and, for every pair $g, h \in \operatorname{Obj}\mathcal{X}$, a set of *arrows* from g to h which is denoted by $\mathcal{X}(g,h)$. The sets of arrows corresponding to different pairs of objects are supposed to be disjoint, and the set of all arrows is denoted by $\operatorname{Arr}\mathcal{X}$. If $a \in \mathcal{X}(g,h)$ then we write that $\alpha(a) = g$ and $\omega(a) = h$.

A *semigroupoid* is a graph $\mathcal{X}$ equipped with a composition which assigns to every pair of consecutive arrows $a \in \mathcal{X}(g,h),\ b \in \mathcal{X}(h,i)$ an arrow in $\mathcal{X}(g,i)$, usually denoted by $a \circ b$, such that the composition is associative, that is, for any arrows $a \in \mathcal{X}(g,h),\ b \in \mathcal{X}(h,i)$ and $c \in \mathcal{X}(i,j)$, we have $(a \circ b) \circ c = a \circ (b \circ c)$.

Let $\mathcal{X}$ be a semigroupoid and S a semigroup. If $\ell \colon \mathcal{X} \to S$ is a morphism of semigroupoids, i.e., $\ell(a \circ b) = \ell(a) \cdot \ell(b)$ for any pair of consecutive arrows a, b in $\mathcal{X}$ then ℓ is said to be a *labelling of* $\mathcal{X}$ *by* S. For an arrow $a \in \operatorname{Arr}\mathcal{X}$, the element $\ell(a)$ of S is called the *label of* a. Note that if both $\mathcal{X}$ and S are locally inverse then ℓ is also a binary morphism.

3 Extensions of completely simple semigroups by groups

In this section we present results of Chapter 3 of the thesis, which is based on [4].

We present an isomorphic copy of the wreath product $T \wr H$ of a Rees matrix semigroup $T = \mathcal{M}[G; I, \Lambda; P]$ by a group H which allows us to make the calculation in this section easier. First, it is routine to see that the direct power T^H is isomorphic to $\mathcal{M}[G^H; I^H, \Lambda^H; P^H]$ where $P^H = (p^H_{\xi\eta})$ is the following sandwich matrix: for any $\xi \in \Lambda^H$ and $\eta \in I^H$ we have $Ap^H_{\xi\eta} = p_{A\xi, A\eta}$ $(A \in H)$. Moreover, the action in the definition of the wreath product determines the following action when replacing T^H by $\mathcal{M}[G^H; I^H, \Lambda^H; P^H]$: for any $A \in H$ and $(\eta, f, \xi) \in \mathcal{M}[G^H; I^H, \Lambda^H; P^H]$ we have ${}^A(\eta, f, \xi) = ({}^A\eta, {}^Af, {}^A\xi)$, where ${}^A\eta \in I^H$, ${}^Af \in G^H$ and ${}^A\xi \in \Lambda^H$ are the maps defined by $B({}^A\eta) = (BA)\eta$, $B({}^Af) = (BA)f$ and $B({}^A\xi) = (BA)\xi$, respectively, for every $B \in H$.

Notice that, for any $A \in H$, we have

$$A({}^Bp^H_{\xi\eta}) = (AB)p^H_{\xi\eta} = p_{(AB)\xi,(AB)\eta} = p_{A({}^B\xi),A({}^B\eta)} = Ap^H_{{}^B\xi,{}^B\eta},$$

and so

$${}^Bp^H_{\xi\eta} = p^H_{{}^B\xi,{}^B\eta}$$

for any $B \in H$.

Let $S = \mathcal{M}[G; I, \Lambda; P]$ be an extension of a completely simple semigroup U by a group H where P is chosen to be normalized. By Proposition 2.1, we can assume that there is a normal subgroup N of the group G such that all entries of the sandwich matrix P belong to N, and we have $H = G/N$ and $U = \mathcal{M}[N; I, \Lambda; P] \subseteq S$.

First suppose that S is central, i.e., each entry of P belongs to the centre of the group G. Note that, in this case, U is necessarily also central. In this case, we can mimic the proof of the Kaloujnine–Krasner Theorem. For, it is routine to check that the map

$$\nu \colon S \to U \wr H = U^H \rtimes H,\ (i, g, \lambda) \mapsto (f_g^{i\lambda}, gN)$$

where

$$f_g^{i\lambda} \colon H \to U,\ A \mapsto (i, Af_g, \lambda)$$

is an embedding. This verifies the following statement.

Proposition 3.1. *Each central completely simple semigroup which is an extension of a (necessarily also central) completely simple semigroup U by a group H is embeddable in the wreath product of U by H.*

Now we turn to investigating the general case where S is an arbitrary completely simple semigroup. Suppose that there exists an embedding $S \to U \wr H$, i.e., an embedding

$$\varphi \colon S \to \mathcal{M}[N^H; I^H, \Lambda^H; P^H] \rtimes H$$

where $\mathcal{M}[N^H; I^H, \Lambda^H; P^H] \rtimes H$ is the isomorphic copy of $U \wr H$ introduced above. In this case φ is necessarily of the form

$$(i, g, \lambda)\varphi = [(\eta_i, f_g^{i\lambda}, \xi_{gN,\lambda}), gN],$$

where indexes on the right hand side show dependencies of elements. We proved several important properties of such embeddings. Two of the more important ones are

$$f_{p_{\lambda j}}^{i\mu} = f_1^{i\lambda} p_{\xi_\lambda \eta_j}^H f_1^{j\mu}, \quad \text{for every } i, j \in I \text{ and } \lambda, \mu \in \Lambda$$

and

$$f_{p_{\lambda i}^{-1}}^{i\lambda} = (p_{\xi_\lambda \eta_i}^H)^{-1}, \quad \text{for every } i \in I \text{ and } \lambda \in \Lambda.$$

We give a suitable group G, a normal subgroup N of G and a Rees matrix semigroup $S = \mathcal{M}[G; I, \Lambda; P]$ for which no such injective morphism φ exists.

Let G be the non-commutative group of order 21. To ease our calculations, we present G in the form $G = \mathbb{Z}_7 \rtimes [\overline{2}]$ where $\mathbb{Z}_7$ is the additive group of the ring of residues modulo 7, $[\overline{2}] = \{\overline{1}, \overline{2}, \overline{4}\}$ is the subgroup of the (multiplicative) group of units of the same ring generated by $\overline{2}$, and $[\overline{2}]$ acts on $\mathbb{Z}_7$ by multiplication. Clearly, G is an extension of $N = \{(a, \overline{1}) : a \in \mathbb{Z}_7\}$ by $[\overline{2}]$, Define S as follows. Let $I = \Lambda = \{1, 2\}$, and denote by P the normalized sandwich matrix of type $\Lambda \times I$ over G consisting of the elements $p_{11} = p_{12} = p_{21} = (\overline{0}, \overline{1})$, the identity element of N, and $p_{22} = (\overline{1}, \overline{1}) \in N$, an element of order 7.

Applying the above mentioned properties, we expressed a suitable element of S by means of sandwich elements, and so we showed a contradiction to injectivity of φ. This gives us the following result.

Theorem 3.2. *There exists a completely simple semigroup which is an extension of a completely simple semigroup U by a group H and which is not embeddable in the wreath product of U by H.*

Next we present a modified version of the Kaloujnine–Krasner Theorem which holds for all extensions of completely simple semigroups by groups.

Let S be an extension of a completely simple semigroup U by a group H. Our goal is to give an embedding of S into a semidirect product $V \rtimes H$ of a completely simple semigroup V by H such that, in the special case where S is a group (i.e., I and Λ are singletons), it is just the embedding constructed to prove the Kaloujnine–Krasner Theorem. Unlike in the wreath product $U \wr H$, in this semidirect product $V \rtimes H$ the $\mathcal{R}$- and $\mathcal{L}$-classes of V, its sandwich matrix and the action of H on V can be chosen appropriately.

Theorem 3.3. *Any extension of a completely simple semigroup U by a group H is embeddable in a semidirect product of a completely simple semigroup V by the group H, where the maximal subgroups of V are direct powers of the maximal subgroups of U.*

More preciously, let S be an extension of U by H. As above, we can assume that $S = \mathcal{M}[G; I, \Lambda; P]$ where the sandwich matrix P is normalized, and by Proposition 2.1, there is a normal subgroup N of G such that every entry of P belongs to N, and $H = G/N$, $U = \mathcal{M}[N; I, \Lambda; P] \subseteq S$. Consider the action of H on N^H defining the wreath product $N \wr H$, and, for any

$g \in G$, the map $f_g \in N^H$ defined in the proof of the Kaloujnine–Krasner Theorem.

By means of S, we define a suitable semigroup V, an action of H on V, and an embedding of S into the semidirect product of V by H. Let $V = \mathcal{M}[N^H; I, H \times \Lambda; Q]$, where the entries of Q belong to the direct power N^H: for any $(B, \lambda) \in H \times \Lambda$ and $j \in I$, let $q_{(B,\lambda),j} = {}^B f_{p_{\lambda j}}$.

Define an action of H on $H \times \Lambda$ by the rule ${}^A(B, \lambda) = (AB, \lambda)$ $((B, \lambda) \in H \times \Lambda,\ A \in H)$. Now we give an action of H on V as follows: for any $A \in H$ and $(i, f, (B, \lambda)) \in V$, let ${}^A(i, f, (B, \lambda)) = (i, {}^A f, {}^A(B, \lambda))$.

We proved that the mapping

$$\psi \colon S \to \mathcal{M}[N^H; I, H \times \Lambda; Q] \rtimes H,$$

where

$$(i, g, \lambda)\psi = ((i, f_g, (gN, \lambda)), gN)$$

is an embedding.

4 Extensions of completely simple semigroups by inverse semigroups

In this section we present results of Chapter 4 of the thesis, which is based on [5].

The main result of the thesis is the following:

Theorem 4.1. *Let S be an E-solid locally inverse semigroup and ϱ an inverse semigroup congruence on S such that the idempotent classes of ϱ are completely simple subsemigroups in S. Then the extension (S, ϱ) can be embedded into a λ-semidirect product extension of a completely simple semigroup by S/ϱ.*

Recall that, in an E-solid semigroup, the idempotent congruence classes of the least inverse semigroup congruence are completely simple subsemigroups. Taking into account a result from [7] and that both classes of E-solid and of locally inverse semigroups are closed under taking regular subsemigroups, we deduced the following characterization of E-solid locally inverse semigroups.

Corollary 4.2. *A regular semigroup is E-solid and locally inverse if and only if it is embeddable in a λ-semidirect product of a completely simple semigroup by an inverse semigroup.*

In particular, this statement provides a structure theorem that constructs E-solid locally inverse semigroups from completely simple and inverse semigroups by means of two fairly simple constructions: forming λ-semidirect product and taking regular subsemigroup.

Next we will summarize the construction and the main idea of this proof.

Let (S,ϱ) be an extension by an inverse semigroup where S is an E-solid locally inverse semigroup and ϱ is over the class $\mathcal{CS}$ of all completely simple semigroups. For brevity, denote the factor semigroup S/ϱ by T and its elements by lower case Greek letters.

First we define the *derived semigroupoid* $\mathcal{C}$ corresponding to the extension (S,ϱ) as follows. Let $\operatorname{Obj}\mathcal{C}=T$ and, for any $\alpha,\beta\in T$, let

$$\mathcal{C}(\alpha,\beta)=\big\{(\alpha,s,\beta)\in T\times S\times T:\alpha\cdot s\varrho=\beta \text{ and } \beta\cdot(s\varrho)^{-1}=\alpha\big\}.$$

Furthermore, by putting $\ell(a)=s$ for every arrow $a=(\alpha,s,\beta)\in\operatorname{Arr}\mathcal{C}$, we define a labelling of $\mathcal{C}$ by S.

Let us 'double' the graph $\mathcal{C}$ by forming $\overline{\mathcal{C}}=\mathcal{C}\cup\mathcal{C}'$ where $\mathcal{C}'$ consists of the 'formal inverses' a' of the arrows $a\in\operatorname{Arr}\mathcal{C}$. Here $\alpha(a')=\omega(a)$ and $\omega(a')=\alpha(a)$ for any $a\in\operatorname{Arr}\mathcal{C}$ and define $(a')'=a$ ($a\in\operatorname{Arr}\mathcal{C}$). Put $A=\operatorname{Arr}\mathcal{C}$, $A'=\operatorname{Arr}\mathcal{C}'$. Then we have $\overline{A}=A\cup A'=\operatorname{Arr}\overline{\mathcal{C}}$. Moreover, denote by $\widetilde{\mathcal{C}}$ the graph obtained from $\overline{\mathcal{C}}$ by adding an arrow $(a\wedge b)$ with $\alpha(a\wedge b)=\omega(a\wedge b)=t$ whenever $a,b\in\operatorname{Arr}\overline{\mathcal{C}}$ with $\alpha(a)=t=\omega(b)$. Finally, consider the free category $\widetilde{\mathcal{C}}^+$ where the arrows are paths in $\widetilde{\mathcal{C}}$. These paths are also called 'binary paths' in $\widetilde{\mathcal{C}}$.

Let us choose and fix an inverse unary operation † on S. This determines an inverse unary operation, also denoted by †, on $\mathcal{C}$ by letting $(\alpha,s,\beta)^\dagger=(\beta,s^\dagger,\alpha)$ for every $(\alpha,s,\beta)\in\operatorname{Arr}\mathcal{C}$. Consider the congruence θ on the semigroup $\widetilde{A}^+$ generated by

$$\Theta(\mathcal{CS},A)\cup\Xi_1\cup\Xi_2$$

(see Proposition 2.2), where

$$\begin{aligned}\Xi_1 &= \big\{(a',a^\dagger):a\in A\big\}\,,\\ \Xi_2 &= \big\{(ab,c):a,b,c\in A \text{ and } a\circ b=c \text{ in } \mathcal{C}\big\}\,.\end{aligned}$$

The following important property is implied by the main result of [7]:

Result 4.3. *Let S be an E-solid locally inverse semigroup, and let ϱ be an inverse semigroup congruence on S over $\mathcal{CS}$. Then the extension (S,ϱ)*

is embeddable in a λ-semidirect product extension of a completely simple semigroup by an inverse semigroup if and only the relations $s\,\varrho\,t$ in S and $(s\varrho(s\varrho)^{-1}, s, s\varrho)\,\theta\,(t\varrho(t\varrho)^{-1}, t, t\varrho)$ in $\widetilde{A}^+$ imply $s = t$ for every $s, t \in S$.

We show that there are some special arrows in the semigroupoid, called *stable arrows*, which played in important role in the proof. The set of stable arrows is denoted by $\operatorname{Arr}\widehat{\mathcal{C}}$. Moreover a stable arrow $\widehat{a}$ can be assigned to each arrow $a \in \operatorname{Arr}\mathcal{C}$.

Essentially, to prove Theorem 4.1, we had to show that each θ-class contains at most one word of the form $(s\varrho(s\varrho)^{-1}, s, s\varrho)$ $(s \in S)$. We had to examine the combinatorial properties of the words in the congruence class of these words. Compared to the paths of the semigroupoid, these words can contain 'breaks'. We defined two types of brackets to mark these breaks in the words. We used these bracketed words to prove properties satisfied by words in these congruence classes.

Consider the free monoid $(\widetilde{A} \cup \{\llbracket, \rfloor, \lceil, \rrbracket\})^*$ where the empty word is denoted ε, and let $\widetilde{W}$ be its smallest subset which has the following four properties:

(i) $\varepsilon \in \widetilde{W}$;

(ii) $a \in \widetilde{W}$ for all $a \in \widetilde{A}$;

(iii) $w_1 w_2 \in \widetilde{W}$ for all $w_1, w_2 \in \widetilde{W}$;

(iv) $\llbracket w \rfloor, \lceil w \rrbracket \in \widetilde{W}$ for all $w \in \widetilde{W}$, where $w \neq \varepsilon$.

Notice that $\widetilde{A}^+ \subseteq \widetilde{W}$. In order to distinguish the elements of $\widetilde{A}^+$, called words, from those of $\widetilde{W}$, the latter will be called *bracketed words*.

Now we define three subsets W_n, W_n^{right} and W_n^{left} of $\widetilde{W}$ for every $n \in \mathbb{N}_0$. Simultaneously, we attach a 'binary path' $\wp(w) \in \operatorname{Arr}\widetilde{\mathcal{C}}^+$ to each element w of these subsets. If $\wp(w)$ is defined then we use $\widehat{\wp}(w)$ to denote $\widehat{\wp(w)}$.

Let $W_0 = \operatorname{Arr}\widetilde{\mathcal{C}}^+$, $W_0^\varepsilon = W_0 \cup \{\varepsilon\}$, and for any $w \in W_0$, define $\wp(w) = w$. Moreover, define

$$W_0^{\text{right}} = \{p(y \wedge x) : p \in W_0^\varepsilon,\ \alpha(y) \neq \omega(x),\ \text{and}\ \omega(p) = \alpha(y)\ \text{if}\ p \neq \varepsilon\},$$

and for any $w = p(y \wedge x) \in W_0^{\text{right}}$, let $\wp(w) = p(y \wedge y')$. By assumptions, this, indeed, belongs to $\operatorname{Arr}\widetilde{\mathcal{C}}^+$. Similarly, let

$$W_0^{\text{left}} = \{(x \wedge y)p : p \in W_0^\varepsilon,\ \alpha(x) \neq \omega(y),\ \text{and}\ \omega(y) = \alpha(p)\ \text{if}\ p \neq \varepsilon\},$$

and for any $w = (x \wedge y)p \in W_0^{\mathrm{left}}$, let $\wp(w) = (y' \wedge y)p$. Notice that $W_0 \cup W_0^{\mathrm{right}} \cup W_0^{\mathrm{left}} \subseteq \widetilde{A}^+$.

Assume that W_n [W_n^{right}, W_n^{left}] is defined for some $n \in \mathbb{N}_0$, and a path $\wp(w) \in \operatorname{Arr}\widetilde{\mathcal{C}}^+$ is assigned to each of its elements w. For brevity, denote the set of all idempotent arrows of $\mathcal{C}$ by E. Define the set W_{n+1} [W_{n+1}^{right}, W_{n+1}^{left}] to consist of the bracketed words in W_n [W_n^{right}, W_n^{left}] and, additionally, of all bracketed words $w \in \widetilde{W}$ of the form

$$w = p_0 B_1 C_1 p_1 B_2 C_2 \cdots B_k C_k p_k \quad (k \in \mathbb{N}), \tag{4.1}$$

where the following conditions are satisfied:

(E0)

- (E0a) $p_1, \ldots, p_{k-1} \in W_0$, $p_0 \in W_0^{\varepsilon}$ [W_0^{ε}, W_0^{left}], $p_k \in W_0^{\varepsilon}$ [W_0^{right}, W_0^{ε}], and $\omega(p_{i-1}) = \alpha(p_i)$ for every i ($1 \le i \le k$),
- (E0b) $B_1C_1, \ldots, B_kC_k \neq \varepsilon$;

(E1) for any i ($1 \le i \le k$), we have

- (E1a) $B_i = \llbracket w_1 \rfloor \llbracket w_2 \rfloor \cdots \llbracket w_s \rfloor$, where $s \in \mathbb{N}_0$ and $w_j \in W_n^{\mathrm{right}}$ ($1 \le j \le s$), and
- (E1b) for any j ($1 \le j \le s$), if $w_j\mathrm{T} = (y_j \wedge x_j)$ then
  - (E1bi) $\widehat{\wp}(w_j) \in E$ and $\widehat{y_j}\ \mathcal{R}\ \widehat{\wp}(w_j)$, and
  - (E1bii) $\widehat{x_j}\ \mathcal{L}\ \widehat{\wp}(p_{i-1})$ (in particular, $p_0 \neq \varepsilon$ if $B_1 \neq \varepsilon$);

(E2) for any i ($1 \le i \le k$), we have

- (E2a) $C_i = \lceil w_1 \rrbracket \lceil w_2 \rrbracket \cdots \lceil w_s \rrbracket$, where $s \in \mathbb{N}_0$ and $w_j \in W_n^{\mathrm{left}}$ ($1 \le j \le s$), and
- (E2b) for any j ($1 \le j \le s$), if $\mathrm{I}w_j = (x_j \wedge y_j)$ then
  - (E2bi) $\widehat{\wp}(w_j) \in E$ and $\widehat{y_j}\ \mathcal{L}\ \widehat{\wp}(w_j)$, and
  - (E2bii) $\widehat{x_j}\ \mathcal{R}\ \widehat{\wp}(p_i)$ (in particular, $p_k \neq \varepsilon$ if $C_k \neq \varepsilon$).

We prove that this condition is satisfied when changing parts of a path by rules defined by the congruence θ. More preciously, if $w \in \widetilde{A}^+$ is a word that can be 'bracketed', that is we can add brackets to w such that the result is in $\bigcup_{n \in \mathbb{N}_0} W_n$, then all words in the θ-class of w can be 'bracketed', and $\widehat{\wp}$ is constant in the θ-class. This is the above mentioned condition we used to prove the implication in Result 4.3.

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