Estimation of the offspring means for critical 2-type Galton–Watson processes with immigration

Outline of Ph.D. Thesis

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Introduction

This thesis is devoted to developing a toolkit for asymptotic study of estimators in 2-type critical Galton–Watson processes. We introduce the basic notations for our model. Define a criticality parameter, namely the spectral radius of the offspring mean matrix, and describe the classification of 2-type Galton–Watson processes into subcritical, critical and supercritical cases based on its value. Then we state a functional limit theorem for the process by Ispány and Pap [6]. This limit is curious, because it is degenerate in the sense that it is concentrated on a single line whose direction is determined by the right Perron vector of the offspring mean matrix.

Then we develop the toolkit. We define a decomposition of the process based on the phenomena observed at the end of the previous section. We want to use these random variables as building blocks of any estimator whose asymptotic properties we want to investigate. In order to do that we need a firm understanding of their behaviour, so we estimate their growth as the number of generations in the underlying process tends to infinity. Our first upper bounds are too big, so in few select cases we refine them. Then we state a joint limit theorem for our building blocks.

Finally we demonstrate the applicability of this method. We state results that can be proven using our toolkit. The results for the joint estimator of both the offspring mean matrix and the immigration mean is new. The other theorems have been published in the following papers.

Ispány, M., Körmendi, K. and Pap, G. (2014).
Asymptotic behavior of CLS estimators for 2-type doubly symmetric critical Galton–Watson processes with immigration.
Bernoulli **20(4)** 2247–2277.

Körmendi, K. and Pap, G. (2018). Statistical inference of 2-type critical Galton–Watson processes with immigration.
Statistical Inference for Stochastic Processes **21(1)** 169–190.

1 Preliminaries

For each $k, j \in \mathbb{Z}_+$ and $i, \ell \in \{1, 2\}$, the number of individuals of type i in the k^{th} generation will be denoted by $X_{k,i}$, the number of type ℓ offsprings produced by the j^{th} individual who is of type i belonging to the $(k-1)^{\text{th}}$ generation will be denoted by $\xi_{k,j,i,\ell}$, and the number of type i immigrants in the k^{th} generation will be denoted by $\varepsilon_{k,i}$. Then we have

$$\begin{bmatrix} X_{k,1} \\ X_{k,2} \end{bmatrix} = \sum_{j=1}^{X_{k-1,1}} \begin{bmatrix} \xi_{k,j,1,1} \\ \xi_{k,j,1,2} \end{bmatrix} + \sum_{j=1}^{X_{k-1,2}} \begin{bmatrix} \xi_{k,j,2,1} \\ \xi_{k,j,2,2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{k,1} \\ \varepsilon_{k,2} \end{bmatrix}, \qquad k \in \mathbb{N}. \tag{1}$$

Here $\big\{\boldsymbol{X}_0,\, \boldsymbol{\xi}_{k,j,i},\, \boldsymbol{\varepsilon}_k : k, j \in \mathbb{N},\, i \in \{1, 2\}\big\}$ are supposed to be independent, where

$$\boldsymbol{X}_k := \begin{bmatrix} X_{k,1} \\ X_{k,2} \end{bmatrix}, \qquad \boldsymbol{\xi}_{k,j,i} := \begin{bmatrix} \xi_{k,j,i,1} \\ \xi_{k,j,i,2} \end{bmatrix}, \qquad \boldsymbol{\varepsilon}_k := \begin{bmatrix} \varepsilon_{k,1} \\ \varepsilon_{k,2} \end{bmatrix}.$$

Moreover, $\{\boldsymbol{\xi}_{k,j,1} : k, j \in \mathbb{N}\}$, $\{\boldsymbol{\xi}_{k,j,2} : k, j \in \mathbb{N}\}$ and $\{\boldsymbol{\varepsilon}_k : k \in \mathbb{N}\}$ are supposed to consist of identically distributed random vectors.

We suppose $\mathbb{E}(\|\boldsymbol{\xi}_{1,1,1}\|^2) < \infty$, $\mathbb{E}(\|\boldsymbol{\xi}_{1,1,2}\|^2) < \infty$ and $\mathbb{E}(\|\boldsymbol{\varepsilon}_1\|^2) < \infty$. Introduce the notations

$$\begin{aligned} &\boldsymbol{m}_{\boldsymbol{\xi}_i} := \mathbb{E}\big(\boldsymbol{\xi}_{1,1,i}\big) \in \mathbb{R}_+^2, \qquad \boldsymbol{m}_{\boldsymbol{\xi}} := \begin{bmatrix} \boldsymbol{m}_{\boldsymbol{\xi}_1} & \boldsymbol{m}_{\boldsymbol{\xi}_2} \end{bmatrix} \in \mathbb{R}_+^{2\times 2}, \\ &\boldsymbol{m}_{\boldsymbol{\varepsilon}} := \mathbb{E}\big(\boldsymbol{\varepsilon}_1\big) \in \mathbb{R}_+^2, \end{aligned}$$

and

$$\boldsymbol{V}_{\boldsymbol{\xi}_i} := \operatorname{Var}\big(\boldsymbol{\xi}_{1,1,i}\big) \in \mathbb{R}^{2\times 2}, \qquad \boldsymbol{V}_{\boldsymbol{\varepsilon}} := \operatorname{Var}\big(\boldsymbol{\varepsilon}_1\big) \in \mathbb{R}^{2\times 2}, \qquad i \in \{1, 2\}.$$

Note that many authors define the offspring mean matrix as $\boldsymbol{m}_{\boldsymbol{\xi}}^\top$. For $k \in \mathbb{Z}_+$, let $\mathcal{F}_k := \sigma\big(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k\big)$. By (1),

$$\mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1}) = \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}.$$

Consequently,

$$\mathbb{E}(\boldsymbol{X}_k) = \boldsymbol{m}_{\boldsymbol{\xi}}^k \, \mathbb{E}(\boldsymbol{X}_0) + \sum_{j=0}^{k-1} \boldsymbol{m}_{\boldsymbol{\xi}}^j \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N}.$$

Hence, the asymptotic behaviour of the sequence $(\mathbb{E}(\boldsymbol{X}_k))_{k\in\mathbb{Z}_+}$ depends on the asymptotic behaviour of the powers $(\boldsymbol{m}_{\boldsymbol{\xi}}^k)_{k\in\mathbb{N}}$ of the offspring mean matrix, which is related to the spectral radius $r(\boldsymbol{m}_{\boldsymbol{\xi}}) =: \varrho \in \mathbb{R}_+$ of $\boldsymbol{m}_{\boldsymbol{\xi}}$. A 2-type Galton–Watson process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ with immigration is referred to respectively as *subcritical*, *critical* or *supercritical* if $\varrho < 1$, $\varrho = 1$ or $\varrho > 1$. We will write the offspring mean matrix of a 2-type Galton–Watson process with immigration in the form

$$\boldsymbol{m}_{\boldsymbol{\xi}} := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

We will focus only on *positively regular* 2-type Galton–Watson processes with immigration, i.e., when there is a positive integer $k \in \mathbb{N}$ such that the entries of $\boldsymbol{m}_{\boldsymbol{\xi}}^k$ are positive (see Kesten and Stigum [7]), which is equivalent to $\beta, \gamma \in (0, \infty)$, $\alpha, \delta \in \mathbb{R}_+$ with $\alpha + \delta > 0$. Then the matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ has eigenvalues

$$\lambda_+ := \frac{\alpha + \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2},$$
$$\lambda_- := \frac{\alpha + \delta - \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2},$$

satisfying $\lambda_+ > 0$ and $-\lambda_+ < \lambda_- < \lambda_+$, hence the spectral radius of $\boldsymbol{m}_{\boldsymbol{\xi}}$ is $\varrho = r(\boldsymbol{m}_{\boldsymbol{\xi}}) = \lambda_+$. By the Perron theorem,

$$\lambda_+^{-k} \boldsymbol{m}_{\boldsymbol{\xi}}^k \to \boldsymbol{u}_{\mathrm{right}} \boldsymbol{u}_{\mathrm{left}}^\top \qquad \text{as } k \to \infty,$$

where $\boldsymbol{u}_{\mathrm{right}}$ is the unique right eigenvector of $\boldsymbol{m}_{\boldsymbol{\xi}}$ (called the right Perron vector of $\boldsymbol{m}_{\boldsymbol{\xi}}$) corresponding to the eigenvalue λ_+ such that the sum of its coordinates is 1, and $\boldsymbol{u}_{\mathrm{left}}$ is the unique left eigenvector of $\boldsymbol{m}_{\boldsymbol{\xi}}$ (called the left Perron vector of $\boldsymbol{m}_{\boldsymbol{\xi}}$) corresponding to the eigenvalue λ_+ such that $\langle \boldsymbol{u}_{\mathrm{right}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 1$, hence we have

$$\boldsymbol{u}_{\mathrm{right}} = \frac{1}{\beta + \lambda_+ - \alpha} \begin{bmatrix} \beta \\ \lambda_+ - \alpha \end{bmatrix},$$
$$\boldsymbol{u}_{\mathrm{left}} = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} \gamma + \lambda_+ - \delta \\ \beta + \lambda_+ - \alpha \end{bmatrix}.$$

Using the so-called Putzer's spectral formula [8], the powers of $\boldsymbol{m}_{\boldsymbol{\xi}}$ can be written in the form

$$\boldsymbol{m}_{\boldsymbol{\xi}}^k = \lambda_+^k \boldsymbol{u}_{\mathrm{right}} \boldsymbol{u}_{\mathrm{left}}^\top + \lambda_-^k \boldsymbol{v}_{\mathrm{right}} \boldsymbol{v}_{\mathrm{left}}^\top, \qquad k \in \mathbb{N},$$

where $\boldsymbol{v}_{\mathrm{right}}$ and $\boldsymbol{v}_{\mathrm{left}}$ are appropriate right and left eigenvectors of $\boldsymbol{m}_{\boldsymbol{\xi}}$, respectively, belonging to the eigenvalue λ_-, for instance,

$$\boldsymbol{v}_{\mathrm{right}} = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} -\beta - \lambda_+ + \alpha \\ \gamma + \lambda_+ - \delta \end{bmatrix},$$
$$\boldsymbol{v}_{\mathrm{left}} = \frac{1}{\beta + \lambda_+ - \alpha} \begin{bmatrix} -\lambda_+ + \alpha \\ \beta \end{bmatrix}.$$

The process $(\boldsymbol{X}_k)_{k \in \mathbb{Z}_+}$ is critical and positively regular if and only if $\alpha, \delta \in [0, 1)$ and $\beta, \gamma \in (0, \infty)$ with $\alpha + \delta > 0$ and $\beta\gamma = (1-\alpha)(1-\delta)$, and then the matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ has eigenvalues $\lambda_+ = 1$ and

$$\lambda_- = \alpha + \delta - 1 \in (-1, 1).$$

Next we will recall a convergence result for critical and positively regular 2-type Galton–Watson processes with immigration. For each $n \in \mathbb{N}$, consider the random step process

$$\boldsymbol{\mathcal{X}}_t^{(n)} := n^{-1} \boldsymbol{X}_{\lfloor nt \rfloor}, \qquad t \in \mathbb{R}_+.$$

The following theorem is a special case of the main result in Ispány and Pap [6, Theorem 3.1].

Theorem. 1.1. *Let $(\boldsymbol{X}_k)_{k \in \mathbb{Z}_+}$ be a 2-type Galton–Watson process with immigration such that $\alpha, \delta \in [0, 1)$ and $\beta, \gamma \in (0, \infty)$ with $\alpha + \delta > 0$ and $\beta\gamma = (1-\alpha)(1-\delta)$ (hence it is critical and positively regular), $\boldsymbol{X}_0 = \boldsymbol{0}$, $\mathbb{E}(\|\boldsymbol{\xi}_{1,1,1}\|^2) < \infty$, $\mathbb{E}(\|\boldsymbol{\xi}_{1,1,2}\|^2) < \infty$ and $\mathbb{E}(\|\boldsymbol{\varepsilon}_1\|^2) < \infty$. Then*

$$\left(\boldsymbol{\mathcal{X}}_t^{(n)}\right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{X}}_t)_{t \in \mathbb{R}_+} := (\mathcal{Y}_t \boldsymbol{u}_{\mathrm{right}})_{t \in \mathbb{R}_+}$$

as $n \to \infty$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, where $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\begin{aligned} \mathrm{d}\mathcal{Y}_t &= \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle \,\mathrm{d}t + \sqrt{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \mathcal{Y}_t^+} \,\mathrm{d}\mathcal{W}_t, \quad t \in \mathbb{R}_+, \\ \mathcal{Y}_0 &= 0, \end{aligned} \tag{2}$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion and*

$$\overline{\boldsymbol{V}}_{\boldsymbol{\xi}} := \sum_{i=1}^{2} \langle \boldsymbol{e}_i, \boldsymbol{u}_{\text{right}} \rangle \boldsymbol{V}_{\boldsymbol{\xi}_i} = \frac{\beta \boldsymbol{V}_{\boldsymbol{\xi}_1} + (1-\alpha)\boldsymbol{V}_{\boldsymbol{\xi}_2}}{\beta + 1 - \alpha}$$

is a mixed offspring variance matrix.

In fact, in Ispány and Pap [6, Theorem 3.1], the above result has been proved under the higher moment assumptions

$$\mathbb{E}(\|\boldsymbol{\xi}_{1,1,1}\|^4) < \infty, \qquad \mathbb{E}(\|\boldsymbol{\xi}_{1,1,2}\|^4) < \infty, \qquad \mathbb{E}(\|\boldsymbol{\varepsilon}_1\|^4) < \infty,$$

which have been relaxed in Danka and Pap [3, Theorem 3.1].

In this section we have introduced a number of assumptions on the process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ For the sake of easier reference we collect those assumptions here. First a condition that guarantees that our process is critical and positively regular. The process satisfies the *criticality condition* if

$$\alpha, \delta \in [0,1), \ \beta, \gamma \in (0,\infty), \ \alpha + \delta > 0, \ \beta\gamma = (1-\alpha)(1-\delta). \quad \text{(CPR)}$$

Then we have condition that we start from an empty initial population, that is $\boldsymbol{X}_0 = \boldsymbol{0}$. If we don't want to be stuck in $\boldsymbol{0}$ we have to assume that the immigration distribution isn't degenerate $\boldsymbol{0}$, it is sufficient to assume $\boldsymbol{m}_{\boldsymbol{\varepsilon}} \neq \boldsymbol{0}$ for this. The process satisfies the *zero start condition* if

$$\boldsymbol{X}_0 = 0, \qquad \boldsymbol{m}_{\boldsymbol{\varepsilon}} \neq \boldsymbol{0}. \quad \text{(ZS)}$$

Next we have a condition on the moments of the process, where we assume the finiteness of ℓ^{th} moments of the offspring and immigration distributions. This in terms implies the finiteness of the ℓ^{th} moment of the process itself. The process satisfies the *moment condition* for some $\ell \in \mathbb{N}$ if

$$\mathbb{E}\left(\|\boldsymbol{\xi}_{1,1,1}\|^\ell\right) < \infty, \qquad \mathbb{E}\left(\|\boldsymbol{\xi}_{1,1,2}\|^\ell\right) < \infty, \qquad \mathbb{E}\left(\|\boldsymbol{\varepsilon}_1\|^\ell\right) < \infty. \quad \text{(M)}$$

Finally we have a condition that doesn't appear in this section however it will be necessary later. The process satisfies the *nondegeneracy condition* if

$$\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{v}_{\text{left}}, \boldsymbol{v}_{\text{left}} \rangle \neq 0. \quad \text{(ND)}$$

The reason for this condition can be understood if one looks at Lemma 2.5, as that describes a relation between the two parts of the upcoming decomposition.

2 A toolkit for asymptotic study of estimates

2.1 A decomposition of 2-type Galton–Watson processes

In the previous section we saw that the eigenvectors of the matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ play an important role in the asymptotic behaviour of the process itself. It is curious in Theorem 1.1 that the limit of a 2-dimensional process is degenerate in the sense that it is concentrated on a single line whose direction is determined by $\boldsymbol{u}_{\mathrm{right}}$. In this section we define a decomposition of the process based on the eigenvectors of $\boldsymbol{m}_{\boldsymbol{\xi}}$.

Applying (1), let us introduce the sequence

$$\boldsymbol{M}_k := \boldsymbol{X}_k - \mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1}) = \boldsymbol{X}_k - \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1} - \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N}, \tag{3}$$

of martingale differences with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}_+}$. By (3), the process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ satisfies the recursion

$$\boldsymbol{X}_k = \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1} + \boldsymbol{m}_{\boldsymbol{\varepsilon}} + \boldsymbol{M}_k, \qquad k \in \mathbb{N}. \tag{4}$$

We derive a useful decomposition for X_k, $k \in \mathbb{N}$. Let us introduce the sequence

$$U_k := \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_k \rangle = \frac{(\gamma + 1 - \delta) X_{k,1} + (\beta + 1 - \alpha) X_{k,2}}{1 - \lambda_-}, \qquad k \in \mathbb{Z}_+.$$

One can observe that $U_k \geq 0$ for all $k \in \mathbb{Z}_+$, and

$$U_k = U_{k-1} + \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle + \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle, \qquad k \in \mathbb{N}.$$

Hence $(U_k)_{k\in\mathbb{Z}_+}$ is a nonnegative unstable AR(1) process with positive drift $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle$ and with heteroscedastic innovation $(\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle)_{k\in\mathbb{N}}$. Note that the solution of the recursion is

$$U_k = \sum_{j=1}^{k} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_j + \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle, \qquad k \in \mathbb{N},$$

and applying the continuous mapping theorem to Theorem 1.1 yields

$$(n^{-1} U_{\lfloor nt \rfloor})_{t\in\mathbb{R}_+} = (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t^{(n)} \rangle)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle)_{t\in\mathbb{R}_+} = (\mathcal{Y}_t)_{t\in\mathbb{R}_+}$$

as $n \to \infty$, where $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE (2). We could think of the variables $(U_k)_{k\in\mathbb{Z}_+}$ as the *well* *behaved* part of our decomposition, because they allow us to get the underlying 1-dimensional stochastic process in Theorem 1.1 . Moreover, let

$$V_k := \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_k \rangle = \frac{-(1-\alpha)X_{k,1} + \beta X_{k,2}}{\beta + 1 - \alpha}, \qquad k \in \mathbb{Z}_+.$$

Note that we have

$$V_k = \lambda_- V_{k-1} + \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle + \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle, \qquad k \in \mathbb{N}.$$

Thus $(V_k)_{k\in\mathbb{N}}$ is a stable AR(1) process with drift $\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle$ and with heteroscedastic innovation $(\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle)_{k\in\mathbb{N}}$. Note that the solution of the recursion is

$$V_k = \sum_{j=1}^{k} \lambda_-^{k-j} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_j + \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle, \qquad k \in \mathbb{N},$$

and applying the continuous mapping theorem to Theorem 1.1 yields

$$(n^{-1} V_{\lfloor nt \rfloor})_{t\in\mathbb{R}_+} = (\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t^{(n)} \rangle)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle)_{t\in\mathbb{R}_+} = 0.$$

We could think of the variables $(V_k)_{k\in\mathbb{Z}_+}$ as the *problematic* part of our decomposition, because the continuous mapping theorem does not find the nonzero limit of them. The recursion (4) has the solution

$$\boldsymbol{X}_k = \sum_{j=1}^{k} \boldsymbol{m}_{\boldsymbol{\xi}}^{k-j} (\boldsymbol{m}_{\boldsymbol{\varepsilon}} + \boldsymbol{M}_j), \qquad k \in \mathbb{N}.$$

Consequently, using (1) yields

$$\boldsymbol{X}_k = U_k \boldsymbol{u}_{\mathrm{right}} + V_k \boldsymbol{v}_{\mathrm{right}} = \begin{bmatrix} \frac{\beta}{\beta+1-\alpha} U_k - \frac{\beta+1-\alpha}{1-\lambda_-} V_k \\ \frac{1-\alpha}{\beta+1-\alpha} U_k + \frac{\gamma+1-\delta}{1-\lambda_-} V_k \end{bmatrix},$$

for all $k \in \mathbb{Z}_+$.

We want to use this decomposition as a tool to investigate asymptotic properties of various estimators of the matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$. Any estimator based on the sample $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ can be rewritten in terms of the variables $U_1, \ldots, U_n, V_1, \ldots, V_n$, thus a good understanding of their behaviour can gain insight into the behaviour of the estimator itself. We note that this reformulation of an estimator is strictly a theoretical tool to prove theorems about it, as without knowing $\boldsymbol{m}_{\boldsymbol{\xi}}$ we also don't know $\boldsymbol{u}_{\mathrm{left}}$ and $\boldsymbol{v}_{\mathrm{left}}$ therefore we can't calculate U_k and V_k.

2.2 An estimation of moments

We want to bound the growth of $(\boldsymbol{M}_k)_{k\in\mathbb{Z}_+}$, $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$, $(U_k)_{k\in\mathbb{Z}_+}$ and $(V_k)_{k\in\mathbb{Z}_+}$ and some related expressions as $k\to\infty$. The reader will find statements in this section that allows us to identify negligible terms in an expression, that is terms that with the right scaling disappear in the limit. We will establish nonzero limits for some of these expression in the next section.

First note that, for all $k\in\mathbb{N}$, $\mathbb{E}(\boldsymbol{M}_k \mid \mathcal{F}_{k-1}) = \boldsymbol{0}$ and thus $\mathbb{E}(\boldsymbol{M}_k) = \boldsymbol{0}$, since $\boldsymbol{M}_k = \boldsymbol{X}_k - \mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1})$.

Lemma 2.1. *Let $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (M) *with some $\ell\in\mathbb{N}$ and $\boldsymbol{X}_0=\boldsymbol{0}$. Then $\mathbb{E}(\|\boldsymbol{X}_k\|^i) = \mathrm{O}(k^i)$ and further*

$$\mathbb{E}(\boldsymbol{M}_k^{\otimes i}) = \mathrm{O}(k^{\lfloor i/2\rfloor}), \qquad \mathbb{E}(U_k^i) = \mathrm{O}(k^i), \qquad \mathbb{E}(V_k^{2j}) = \mathrm{O}(k^j)$$

for $i,j\in\mathbb{Z}_+$ with $i\le\ell$ and $2j\le\ell$.

The next corollary can be derived exactly as Corollary 9.2 of Barczy et al. [2].

Corollary 2.2. *Let $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (M) *with some $\ell\in\mathbb{N}$ and $\boldsymbol{X}_0=\boldsymbol{0}$. Then*

(i) *for all $i,j\in\mathbb{Z}_+$ with $\max\{i,j\}\le\lfloor\ell/2\rfloor$, and for all $\kappa > i+\frac{j}{2}+1$, we have*

$$n^{-\kappa}\sum_{k=1}^{n}\left|U_k^i V_k^j\right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty,$$

(ii) *for all $i,j\in\mathbb{Z}_+$ with $\max\{i,j\}\le\ell$, for all $T>0$, and for all $\kappa > i+\frac{j}{2}+\frac{i+j}{\ell}$, we have*

$$n^{-\kappa}\sup_{t\in[0,T]}\left|U_{\lfloor nt\rfloor}^i V_{\lfloor nt\rfloor}^j\right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty,$$

(iii) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \le \lfloor \ell/4 \rfloor$, *for all* $T > 0$, *and for all* $\kappa > i + \frac{j}{2} + \frac{1}{2}$, *we have*

$$n^{-\kappa} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} [U_k^i V_k^j - \mathbb{E}(U_k^i V_k^j \mid \mathcal{F}_{k-1})] \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Unfortunately the above corollary doesn't always give good enough bounds. In a few a select cases we provide sharper bounds on the growth of these variables.

Remark 2.3. *In the special case* $(\ell, i, j) = (2, 1, 0)$, *one can show*

$$n^{-\kappa} \sup_{t\in[0,T]} U_{\lfloor nt \rfloor} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty \text{ for } \kappa > 1,$$

see Barczy et al. [2].

Lemma 2.4. *Let* $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ *be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (ZS) *and* (M) *with* $\ell = 4$. *Then for each* $T > 0$,

$$n^{-3/2} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} V_{k-1} \right| \xrightarrow{\mathbb{P}} 0, \quad n^{-5/2} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1} V_{k-1} \right| \xrightarrow{\mathbb{P}} 0$$

as $n \to \infty$. *If further the process satisfies the higher moment condition,* (M) *with* $\ell = 8$, *then for each* $T > 0$,

$$n^{-7/2} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1}^2 V_{k-1} \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty.$$

2.3 Limit theorems for building blocks

Up to this point we have defined a decomposition of the process and proven some zero limit theorems about a few expression related to it. We will use these results to find nonzero limits.

First we relate the sums of squares of the variables V_k to the well-behaved part of our decomposition, the variables U_k. If the process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ satisfies the condition (ND), then this can be used to find the nonzero limit of the aforementioned sum.

Lemma 2.5. *Let* $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ *be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (ZS) *and* (M) *with* $\ell = 8$. *Then for each* $T > 0$, *we have*

$$n^{-2} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt\rfloor} V_k^2 - \frac{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}}\, \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\lambda_-^2} \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as} \quad n\to\infty.$$

The following corollary is the essential piece of our toolkit. We will make heavy use of this statement in the following section.

Corollary 2.6. *Let* $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ *be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (ZS) *and* (M) *with* $\ell = 8$. *Then we have*

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-3}U_{k-1}^2 \\ n^{-2}V_{k-1}^2 \\ n^{-1}\boldsymbol{M}_k \\ n^{-2}\boldsymbol{M}_k U_{k-1} \\ n^{-3/2}\boldsymbol{M}_k V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t^2\,\mathrm{d}t \\ \frac{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\lambda_-^2} \int_0^1 \mathcal{Y}_t\,\mathrm{d}t \\ \boldsymbol{\mathcal{M}}_1 \\ \int_0^1 \mathcal{Y}_t\,\mathrm{d}\boldsymbol{\mathcal{M}}_t \\ \frac{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle^{1/2}}{(1-\lambda_-^2)^{1/2}} \int_0^1 \mathcal{Y}_t \overline{\boldsymbol{V}}_{\boldsymbol{\xi}}^{1/2}\,\mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t \end{bmatrix}$$

as $n\to\infty$.

3 Estimates for the offspring mean matrix

Here is a showcase of the power of the toolkit developed in the previous section. We derive a limit theorem for the estimation of the offspring mean matrix, $\boldsymbol{m}_{\boldsymbol{\xi}}$ in three different settings. The notations introduced in each subsection are unique to that subsection, for example the matrix $\boldsymbol{A}_n$ has a different meaning in each of the following subsections.

3.1 The doubly symmetric process

The aim of this section is to reproduce the results of [4, Theorem 3.1.]. We call a 2-type Galton–Watson process *doubly symmetric* if its offspring mean matrix has the form

$$\boldsymbol{m}_{\boldsymbol{\xi}} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

In this case $\gamma = \beta$, $\delta = \alpha$ and condition (CPR) takes the form

$$\alpha \in (0,1), \qquad \beta = 1 - \alpha \in (0,1) \tag{CPR*}$$

We have $\lambda_+ = 1$, $\lambda_- = 1 - 2\beta$, and

$$\boldsymbol{u}_{\text{right}} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}, \quad \boldsymbol{u}_{\text{left}} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \boldsymbol{v}_{\text{right}} = \begin{bmatrix}-1\\1\end{bmatrix}, \quad \boldsymbol{v}_{\text{left}} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}.$$

The decomposition then simplifies to

$$U_k = X_{k,1} + X_{k,2}, \quad V_k = \frac{1}{2}\left(X_{k,2} - X_{k,1}\right)$$

Lemma 3.1. *The joint CLS estimator for α and β has the form*

$$\begin{bmatrix}\widehat{\alpha}_n\\ \widehat{\beta}_n\end{bmatrix} = \boldsymbol{A}_n^{-1}\boldsymbol{B}_n,$$

on the set $\Omega_n := \{\omega \in \Omega \colon \det(\boldsymbol{A}_n) > 0\}$, where

$$\boldsymbol{A}_n(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n) = \sum_{k=1}^{n} \begin{bmatrix}X_{k-1,1} & X_{k-1,2}\\ X_{k-1,2} & X_{k-1,1}\end{bmatrix}^2$$

$$\boldsymbol{B}_n(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n) = \sum_{k=1}^{n} \begin{bmatrix}X_{k-1,1} & X_{k-1,2}\\ X_{k-1,2} & X_{k-1,1}\end{bmatrix}(\boldsymbol{X}_k - \boldsymbol{m}_{\boldsymbol{\varepsilon}}).$$

In the critical, doubly symmetric case the spectral radius of $\boldsymbol{m}_{\boldsymbol{\xi}}$ is

$$\varrho = \lambda_+ = \alpha + \beta,$$

so we can define a natural estimator for ϱ by $\widehat{\varrho}_n := \widehat{\alpha}_n + \widehat{\beta}_n$.

Theorem. 3.2. *Let $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ be a 2-type doubly symmetric Galton–Watson process with immigration satisfying conditions* (CPR*), (ZS) *and* (M) *with $\ell = 8$. If the process satisfies* (ND) *as well, then the probability of the existence of the estimators $\widehat{\alpha}_n$, $\widehat{\beta}_n$ and $\widehat{\varrho}_n$ tends to 1 as $n \to \infty$, and further*

$$n^{1/2}\begin{bmatrix}\widehat{\alpha}_n - \alpha\\ \widehat{\beta}_n - \beta\end{bmatrix} \xrightarrow{\mathcal{D}} \sqrt{\alpha\beta}\,\frac{\int_0^1 \mathcal{Y}_t \,\mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t \,\mathrm{d}t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$n\left(\widehat{\varrho}_n - 1\right) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{Y}_t \,\mathrm{d}\left(\mathcal{Y}_t - \langle \boldsymbol{u}_{\text{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}}\rangle t\right)}{\int_0^1 \mathcal{Y}_t^2 \,\mathrm{d}t}$$

as $n \to \infty$, *where* $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ *is defined in* (2).

3.2 The general process with known immigration mean

Lemma 3.3. *The CLS estimator of* $\boldsymbol{m}_{\boldsymbol{\xi}}$ *has the form* $\widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)} = \boldsymbol{B}_n \boldsymbol{A}_n^{-1}$ *on the set* $\Omega_n := \{\omega \in \Omega\colon \det(\boldsymbol{A}_n) > 0\}$, *where*

$$\boldsymbol{A}_n = \sum_{k=1}^{n} \boldsymbol{X}_{k-1} \boldsymbol{X}_{k-1}^\top, \qquad \boldsymbol{B}_n = \sum_{k=1}^{n} (\boldsymbol{X}_k - \boldsymbol{m}_{\boldsymbol{\varepsilon}}) \boldsymbol{X}_{k-1}^\top.$$

Theorem. 3.4. *Let* $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ *be a 2-type Galton–Watson process with immigration satisfying conditions* (CPR), (ZS) *and* (M) *with* $\ell = 8$. *If the process satisfies* (ND) *as well, then the probability of the existence of the estimators* $\widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)}$ *and* $\widehat{\varrho}_n$ *tends to 1 as* $n \to \infty$, *and further*

$$n^{1/2}(\widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)} - \boldsymbol{m}_{\boldsymbol{\xi}}) \stackrel{\mathcal{D}}{\longrightarrow} \frac{(1-\lambda_-^2)^{1/2}}{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle^{1/2}} \frac{\overline{\boldsymbol{V}}_{\boldsymbol{\xi}}^{1/2} \int_0^1 \mathcal{Y}_t \, \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t}{\int_0^1 \mathcal{Y}_t \, \mathrm{d}t} \boldsymbol{v}_{\mathrm{left}}^\top$$

$$n(\widehat{\varrho}_n - 1) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}(\mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle t)}{\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t}$$

as $n \to \infty$, *with* $\mathcal{Y}_t := \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle$, $t \in \mathbb{R}_+$, *where* $(\boldsymbol{\mathcal{M}}_t)_{t\in\mathbb{R}_+}$ *is the unique strong solution of the SDE*

$$\begin{aligned} \mathrm{d}\boldsymbol{\mathcal{M}}_t &= (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle^+)^{1/2} \, \overline{\boldsymbol{V}}_{\boldsymbol{\xi}}^{1/2} \, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+, \\ \boldsymbol{\mathcal{M}}_0 &= \boldsymbol{0}, \end{aligned}$$

where $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *and* $(\widetilde{\boldsymbol{\mathcal{W}}}_t)_{t\in\mathbb{R}_+}$ *are independent 2-dimensional standard Wiener processes.*

3.3 The general process with unknown immigration mean

Lemma 3.5. *The joint CLS estimator of* $\boldsymbol{m}_{\boldsymbol{\xi}}$ *and* $\boldsymbol{m}_{\boldsymbol{\varepsilon}}$ *has the form*

$$\begin{aligned} \widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)} &= \boldsymbol{B}_n \boldsymbol{A}_n^{-1} \\ \widehat{\boldsymbol{m}_{\boldsymbol{\varepsilon}}}^{(n)} &= \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{X}_k - \widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)} \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{X}_{k-1}, \end{aligned}$$

on the set $\Omega_n := \{\omega \in \Omega\colon\ \det(\boldsymbol{A}_n) > 0\}$, *where*

$$\begin{aligned}
\boldsymbol{A}_n(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n) &= \sum_{k=1}^{n} \boldsymbol{X}_{k-1} \boldsymbol{X}_{k-1}^\top - \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{X}_{k-1} \sum_{k=1}^{n} \boldsymbol{X}_{k-1}^\top, \\
\boldsymbol{B}_n(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n) &= \sum_{k=1}^{n} \boldsymbol{X}_k \boldsymbol{X}_{k-1}^\top - \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{X}_k \sum_{k=1}^{n} \boldsymbol{X}_{k-1}^\top.
\end{aligned}$$

Theorem. 3.6. *Let* $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ *be a 2-type Galton–Watson process with immigration that satisfies conditions* (CPR), (ZS) *and* (M) *with* $\ell = 8$. *If the process also satisfies* (ND), *then the probability of the existence of the estimators* $\widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)}$, $\widehat{\boldsymbol{m}_{\boldsymbol{\varepsilon}}}^{(n)}$ *and* $\widehat{\varrho}_n$ *tends to 1 as* $n \to \infty$, *and further*

$$\begin{aligned}
&n^{1/2}(\widehat{\boldsymbol{m}_{\boldsymbol{\xi}}}^{(n)} - \boldsymbol{m}_{\boldsymbol{\xi}}) \stackrel{\mathcal{D}}{\longrightarrow} \frac{(1-\lambda_-^2)^{1/2}}{\langle \overline{\boldsymbol{V}}_{\boldsymbol{\xi}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle^{1/2}} \frac{\overline{\boldsymbol{V}}_{\boldsymbol{\xi}}^{1/2} \int_0^1 \mathcal{Y}_t \,\mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t}{\int_0^1 \mathcal{Y}_t \,\mathrm{d}t} \boldsymbol{v}_{\mathrm{left}}^\top, \\
&\qquad \widehat{\boldsymbol{m}_{\boldsymbol{\varepsilon}}}^{(n)} - \boldsymbol{m}_{\boldsymbol{\varepsilon}} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{M}}_1,
\end{aligned}$$

also

$$n(\widehat{\varrho}_n - 1) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 \mathcal{Y}_t \,\mathrm{d}(\mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle t) - (\mathcal{Y}_1 - \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle) \int_0^1 \mathcal{Y}_t \,\mathrm{d}t}{\int_0^1 \mathcal{Y}_t^2 \,\mathrm{d}t - \left(\int_0^1 \mathcal{Y}_t \,\mathrm{d}t\right)^2}$$

as $n \to \infty$, *with* $\mathcal{Y}_t := \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle$, $t \in \mathbb{R}_+$, *where* $(\boldsymbol{\mathcal{M}}_t)_{t\in\mathbb{R}_+}$ *is the unique strong solution of the SDE*

$$\begin{aligned}
\mathrm{d}\boldsymbol{\mathcal{M}}_t &= (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}} \rangle^+)^{1/2}\, \overline{\boldsymbol{V}}_{\boldsymbol{\xi}}^{1/2} \,\mathrm{d}\boldsymbol{\mathcal{W}}_t, \quad t \in \mathbb{R}_+, \\
\boldsymbol{\mathcal{M}}_0 &= \boldsymbol{0},
\end{aligned}$$

where $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *and* $(\widetilde{\boldsymbol{\mathcal{W}}}_t)_{t\in\mathbb{R}_+}$ *are independent 2-dimensional standard Wiener processes.*

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