TEACHING GEOMETRY USING COMPUTER VISUALIZATION

Thesis of Ph.D dissertation

Elvira Ripco Sipos

Supervisor:

József Kosztolányi

Doctoral School in Mathematics and Computer Science
Bolyai Institute
University of Szeged, 2011.

Acknowledgment

I would like to express my gratitude to everybody who helped me in some way in the elaboration of my dissertation.

Special thanks to my supervisor, Dr. József Kosztolányi, professor at the Bolyai Institute, University of Szeged whose useful suggestions and ideas helped me in completing this work. I thank him ever wholeheartedly for his help and for the great moral support.

I would like to thank professor Dr. János Karsai for leading me to wonders of Mathematica software and for making the IPA-Project stylesheet available to me (HU-SRB/0901/221/088 TEAMATHMODSCI).

I am also thankful to professors Dr. Djurdjica Takači and Dr. Árpád Takácsi from the University of Novi Sad for their help and useful suggestions regarding my work.

I also owe special thanks to my family for encouraging and supporting me during the completion of this work.

It is by logic that we prove, but by intuition that we discover.

Jules Henri Poincaré

1. Preliminaries

In the Bolyai Secondary Grammar School and Dormitory for Gifted Students in Senta I have worked as a geometry teacher since the beginning of the school's existence in 2003. My method, teaching geometry using visualizations on the computer has been applied since 2004. We use "Euklides" DGS (Dynamic Geometry System) to help pupils in understanding the difficult problems and constructing the figure with given properties. The DGS on the computer draws figures with special accuracy and one can move basic elements on the figure and follow changes of the figure. This way one can find new connections and proof of a given problem. This increases the students' motivation. They like lessons much more in the computer laboratory classroom; they ask for and wait for it. I expect growth of achievement on the graduation exam. During the first six years of teaching geometry using computers my experience shows that our pupils at the end of their secondary education receive the kind of tools and knowledge which help them solve problems in their own way using special skills, and find new solutions to technical problems as engineering experts. They learn to apply computers and visualizations to their own problems far from the field of geometry and mathematics in every special case of life and their mind is open to new discoveries in the world.

Introduction

Teaching mathematics in a secondary school is a very difficult task.Its most interesting area is axiomatically built geometry. [21]

Helping students to understand and learn new theorems in geometry is an exciting task from a teachers point of view. Some psychological and didactical surveys show that the acquisition and learning of geometry in secondary school is a very difficult task. This could be due to a lack of problem-solving skills or to the decreasing creativity of the students. Consequently, few schoolchildren get to know the amazing world of science of the Ancient Greek geometers'. The incompleteness and imperfectness of representations on paper and limited time available hinder the learning of axiomatically-based theoretical geometry. A great number of geometrical figures, their symmetries, perfectness and variety remain unknown for some students who have finished secondary school.

One of the most important tools for children in the 21^{st} Century is their companion, the computer. Children born after 2000 are called Digital Natives by Marc Prensky, while we, the others born in the 20^{th} Century are Digital Immigrants with methodical thinking. Applying the computer in the teaching and learning processes plays an important role. It is important how we introduce the computer during the course of teaching and learning from the aspect of the achievement of the cognitive processes. The appearance and rapid development of dynamic geometry systems(DGS) has brought rapid progress in the teaching of geometry.

The criteria to measure creative mathematical potential based on the Balka's formulation in my translation are the following:

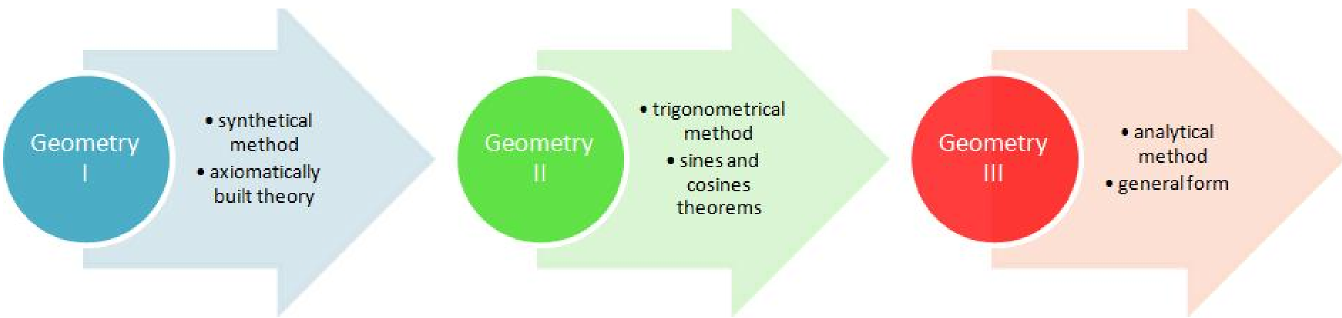
- Learning the use of computer visualization in GeoGebra or Euklides helps to formulate mathematical hypothesis about cause and effect in mathematical situation.
- The use of computers is an effective method to help students determine patterns and to break from established mind sets to obtain solutions in mathematical (and geometrical) situations.
- Introducing the different systems of geometry axioms-theorems helps to consider and evaluate unusual mathematical (and geometrical) ideas, to think through the possible consequences for mathematical situation.
- Verification of errors aids students to sense what is missing from a given mathematical situation and to ask questions that will enable one to fill in the missing mathematical information.
- The software Mathematica develops the ability to split general mathematical problems into specific subproblems.

2. Research aim and methods

My research aim is to map out in which proportions it is necessary and sufficient to apply DGS, since visualization on the computer is not the only teaching-device we have at our disposal to use during geometry lessons. Besides understanding the theorems, our students should find good proofs and it is necessary to practice the elementary geometrical constructions using compass and rulers. It is important to find an optimal qualitative and quantitative balance during lessons of time spent on visualizations and theoretical proofing of constructions. My task is to follow the development of these students for four years and to test their learning of Euler's line in the fourth-grade, because I think that concepts learned from different methods will give the expected results.

Using a computer in complicated calculations helps to reach the main purpose, aim: theorems had been proven synthetically by elementary methods of axiomatically built Euclidian geometry, then they were proven in trigonometry and after in general form by analytical geometry. The principle of spirality to deepen the students' geometrical knowledge is observable here.

The three steps of spirality are:



3. Main results

Geometry I, Planimetry

In Geometry I there is a problem, which we can use to follow the students' cognitive development in geometrical thinking using computer visualizations and elementary proofs. The given problem is Euler's line. The students' preliminary knowledge contained the important points of the triangles such as the center of the circumcircle, the orthocenter, the center of the inscribe circle and the centroid, which were studied during two lessons. First, the definition was given, and we constructed the Euler's line on the acute triangle. The next step was its proof, to show correctly the related theorem and the properties. The third step was the construction of these important points in special cases of the isosceles, the equilateral and the right-angle triangle. The homework was classical construction on paper in the case of the obtuse triangle. The students were asked to construct the triangle in adequate accuracy. I checked the homework and pointed out incorrect details and imperfect constructions.

The examination contained three parts (phases):

1. Visualization on the computer;
2. Comprehensive tasks for assessment;
3. Revision after the winter vacation.

Until we got to the proof of the property of the Euler-line OTH, my students knew the basics of vector algebra such as their addition and multiplication with real numbers, and we proved the Hamilton-theorem:

$$\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OA} = \overrightarrow{OH}$$

as well as vector equality:

$$\overrightarrow{CH} = 2 \cdot \overrightarrow{OC_1}$$

if C_1 is the midpoint of the segment AB in the triangle ABC. In these phases the students' works are introduced, with their evaluation. In the first phase, visualization helped in the exploration of new concepts, their properties, and to find and understand new theorems.

In the second phase, students recalled to the accurate construction made on the computers by the dynamic geometry system. Their drawings were as accurate as possible. They made an effort to prepare as exact construction as possible using graphite pencils on paper.

The third phase took place one month later, after the assimilation period. There was an emphasis on the theoretical proof. It was enough to remind them of Euler's line, and they knew what their task was: from the given triangle, across the important points of the triangle, how one can get to the line OHT. But not all the solutions were satisfactory (to my disappointment). The most accurate solutions varied in their use of geometrical tools, one used isometric transformation, and another used perspective transformation in the proof, the third student recalled the proof applying vectors.

Every three months there is a 90-minute-long written exam in the school, which contains tasks that have been taught in the previous three months, problems to solve and to prove.

The maximum number of points students can get on the second exam is 20. Points and marks are based on the following classification:

18-20 points excellent (5)

15-17 points very good (4)

12-14 points good (3)

9-11 points sufficient (2)

0-8 points insufficient (1)

Description of researched groups:

Old = the control group who did not do visualizations on computers, in 2003; there were 20 students.
New = the experimental group who had 25 percent of all lessons using computer techniques and visualization, in 2004; there were 14 students.

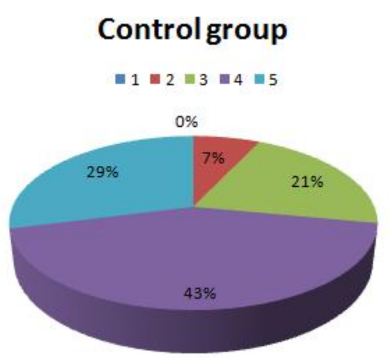
The results, the dispersion of the points in the tasks of the second exam in the control and experimental groups are shown in the table below, where tasks are noted vertically in green color, and the points of each task are noted horizontally in red color:

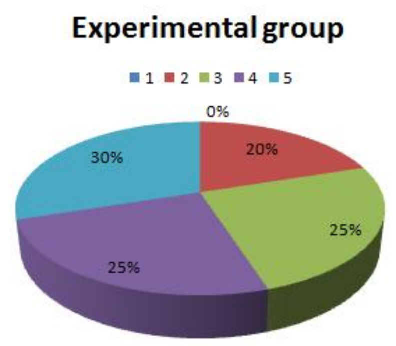
	0	1	2	3	4
1	0	0	4	6	9
2/a	0	4	6	9	
2/b	0	6	13		
3	0	2	8	4	5
4	1	8	3	7	
5/a	1	10	8		
5/b	4	10	5		

	0	1	2	3	4
1	0	0	1	4	9
2/a	0	1	9	4	
2/b	0	0	14		
3	0	3	2	5	4
4	0	5	1	8	
5/a	0	7	7		
5/b	1	9	4		

These results show and prove the advantage of the experimental group using visualization on computer.

The average grade of the control group is 3.65 and the average grade of the experimental group is 3.93 on the end of the school year, which is higher than the previous result by 10 percent.





Geometry II, Trigonometry

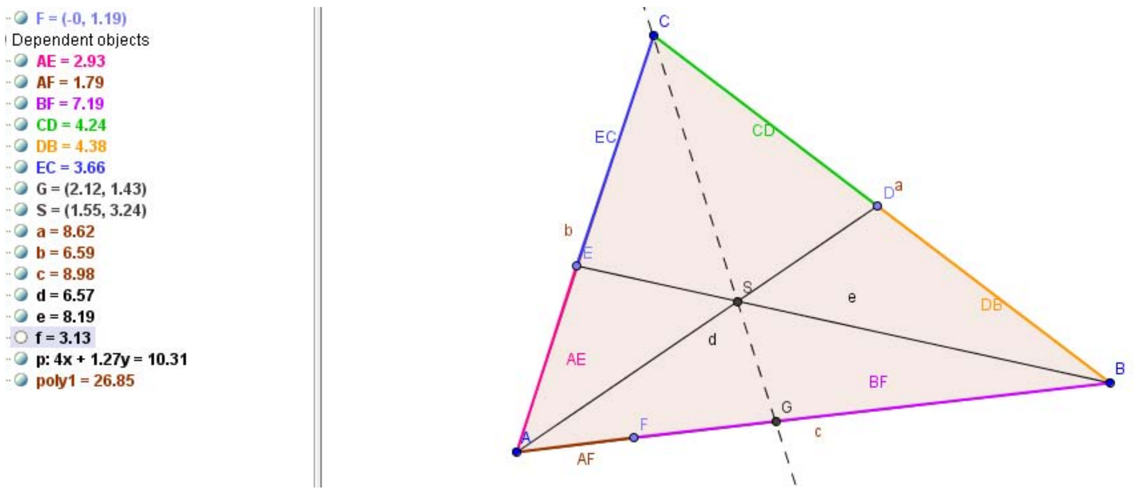
In Geometry II Trigonometry is applied and used while proving important theorems of basic geometry . Theorems of Menelaus', Ceva's and Euler's line are proven. We verify these statements with the sine and cosine laws and then we apply them in our work the emphasis here is on experimenting. The aim is to develop the students' intuition skill using heuristic methods aided by visualization on the computer.

Experimenting in DGS GeoGebra calculating the value of formula

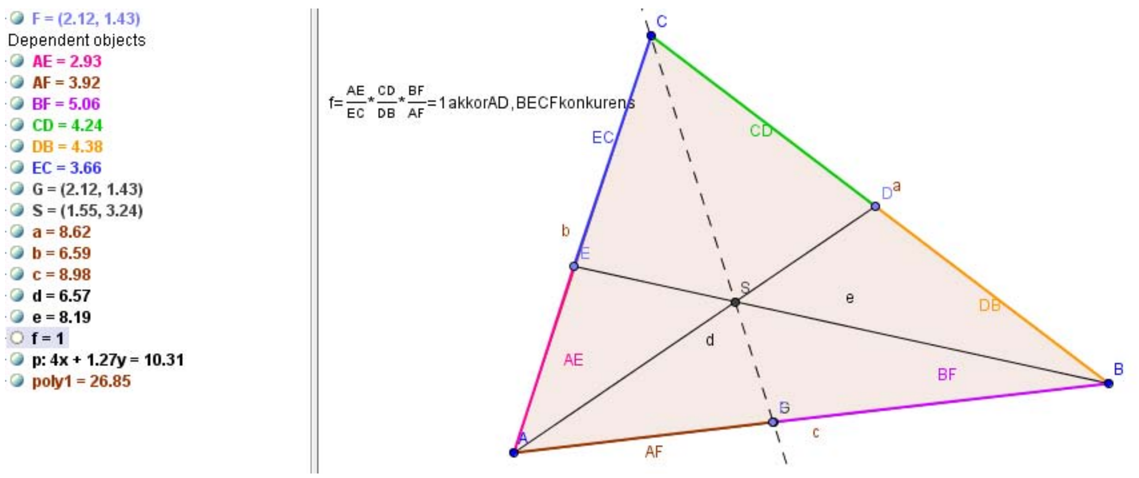
$$f = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$$

in Ceva's theorem students can follow how it changes while point F (or point E or point D) moves on the sides of triangle ABC. The conclusion is:

- if $F \neq G$, but point F is on the line AB, then the value of formula $f \neq 1$.

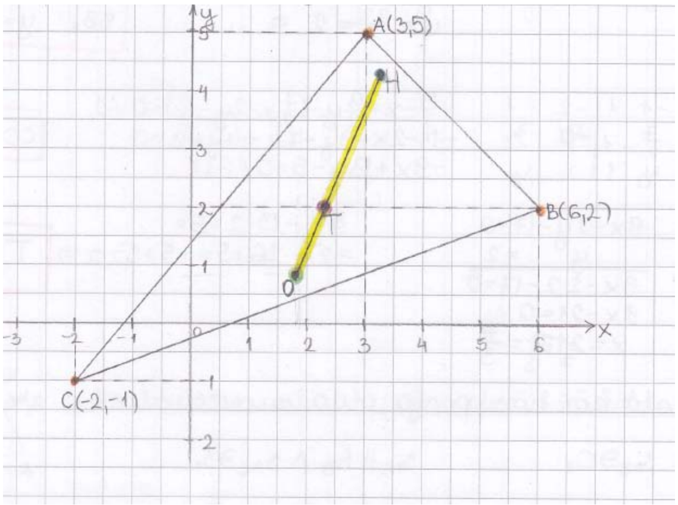


- if $F = G$, point F was moved to the point of intersection of lines CS and AB, then $f = 1$.

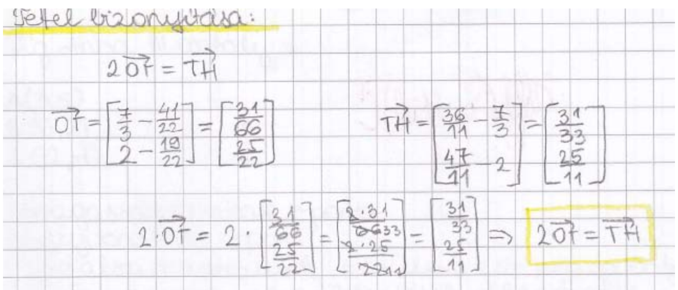


Geometry III, Linear algebra and analytical geometry

In Geometry III there is analytical geometry using coordinates of the point in Descartes-coordinates, and equations of lines, circles, ellipses, hyperbolas and parabolas. Their visualization on the computer helps student understand the theory and make his/her own models on graph paper. The useful Dynamic Geometry System is ***"GeoGebra"***. It is not unknown to the students of the third grade because they had met it in the second grade, but without the algebraic forms. Students are familiar with basic geometry constructions, important theorems of axiomatically built Euclidian and non-Euclidian geometry. However to prove important theorems in analytical form generally, there is a possibility to use software ***Mathematica*** 6.2. This software helps calculate very long and complicated formulas and to simplify them more easier than human brain ever could. The computer helps in calculating, but the operator has to know what to calculate. Leadership is in the programmer's hand. The student has to know the algorithm of the process, the aim of the work, to lead the process to the end algorithmically. Using the computer made the work easier, however the main problem was solved in the student's mind.
After a few practice lessons with and without computers in the computer laboratory the test for assessment was done without using computers. Students had to draw visualization on graph paper $A4$. The coordinates of the given triangle ΔABC were $A(3,5)$, $B(6,2)$ and $C(-2,-1)$. One of the best works is below, a very nice image, precisely constructed. Euler's line is emphasized with a different color, yellow.



At the end they had to prove the vector-theorem of the Euler's line:



The results show that 40 percent of the class in 2008 of 20 students; and 56 percent of the class in 2009 of 18 students executed the task perfectly or very well.

Apollonius' problems

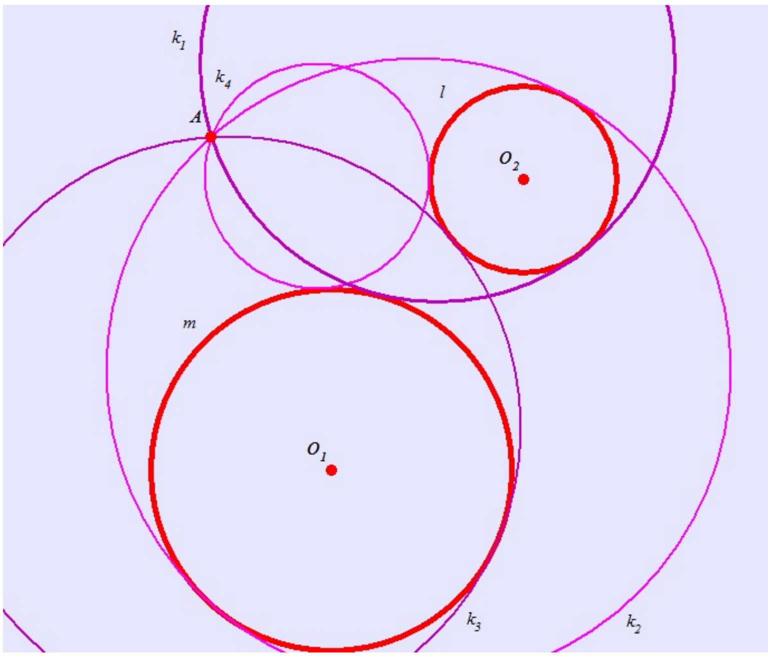
The Geometry I program was built on the introduction of basic geometrical knowledge. There are:

1. *Isometric transformations*: rotation, reflection, symmetry, translation;
2. *Dilatation*, symbol $H_{O,k}$, where O is the center of the dilatation, k is the ratio;
3. *Inversion*, symbol $\Psi_i(O, r)$, where O is the center and r is the radius of the inversion circle i.

As the example of using inversion on a circle there are Apollonius' problems, their constructions, discussion, analysis and proof. Those who have some experiences in construction using chalk on the blackboard with wooden compass and rulers have an idea of how to show and prove these interesting but difficult problems to their own students. The emphasis is on convincing instead of proving. A great help in visualization is some DGS. The most general and difficult case is obviously the case of three circles, which was covered in Book II of Apollonius' "On Tangencies".

The basic problem is:
Given three objects, each of which may be a point, line, or circle, draw a circle that is tangent to each.

There are ten cases introduced.
For example the solution of the sixth problem, (The circle across one point A touches two given circles l, m) has the final image below.



At the end of the school year my students are able to construct these geometry problems using compass and rulers on paper using technical pencil. Two are from five lessons in computer laboratories to help students understand, experiment and learn the basics of the problems. The remaining three lessons are classical constructions to develop the students' psychomotorical skills. Psychomotorical aims of teaching geometry are in [1]:

- grasping the clear solutions of the tasks;
- smart and clever use of compass and rulers;
- free sketching;
- making mathematical and geometric models of the problem;
- using computers and calculators.

These targets are satisfied during the 144 Geometry I lessons of the year. Students have to construct the fifth and/or the sixth Apollomius' problem as homework. Marks are based on the following classification:

precise construction, nice drawing excellent (5)

one little error very good (4)

two or three inaccuracies good (3)

incorrect drawing sufficient (2)

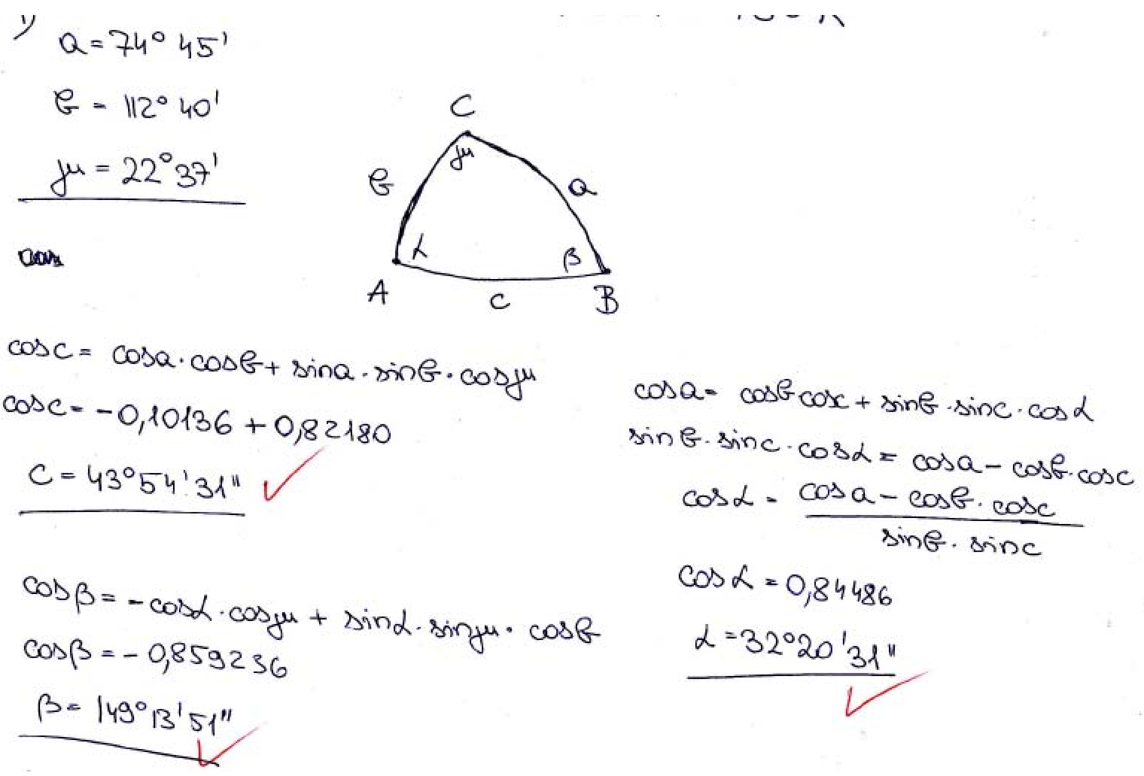
not worked insufficient (1)

Is there geometry after Euclid?

During the introduction and study of Hilbert's axioms, we become acquainted with the fact that there are 21 assumptions which underline the geometry published in Hilbert's classic text "Grundlagen der Geometrie" [11]: the incidence axioms, ordering axioms, congruence axioms, continuity axioms and the single parallel axiom equivalent to Euclid's fifth (parallel) postulate, as we can read in Euclid's "Elements" [9]. But it is our mission - the teacher's duty - to show the different possibilities of this axiom of parallels. At this point, the three basic axiom-systems are presented by Playfair's, Lobachevsky-Bolyai's and Riemann's axiom of parallel lines, but projective geometry is introduced by Desargues' and Pappus' theorems.[7].

It is very important to develop the cognitive attitude of the students in Geometry I, to help their development of thinking skills, spatial perception and awareness, and the approach to the modern sciences. Fifteen year-old students have undeveloped mathematical thinking; they do not possess problem solving skills because they do not have experiences in solving such tasks. Geometry gives many opportunities for an effective development of such skills.

The spherical laws of the sine and cosine laws are applied on the Lénárd-sphere.
The nice example is with precise calculations, and a good picture of a spherical triangle is below. He used cosines to the third angle and he could not make mistakes because the sign of the cosines of the angle showed that the angle was acute or obtuse.



Summary the assessed/evaluated test ended with good results, students created beautiful constructions, very impressive works and interesting, amazing mistakes. Their fantasy produced wonderful ideas.

4. Conclusions

This study examined several factors of the process of teaching and learning geometry using computers. The following research questions (with subcomponents) formulated the basis of the research:

1. How did computer-use help each student achieve good results and develop?
2. Which software was used for which exercise?

In the initial exercises in planar geometry it is useful to use the DGS "Euklides" because, as professor Szilassi Lajos put it,

"Euklides is a great software for its simplicity"

Later, the DGS "GeoGebra" is more useful with its geometry window at first and its algebraic window used later. For long proofs, for accuracy software "Mathematica" is very practical.

Teaching geometry using visualizations on the computer prepares the scientist of the future to try every case of new problems that may arise to find and discover our environment and life.

5. Publications and Conferences

List of publications:

1. **Ripco Sipos Elvira**: ***Apollóniosz problémái - tanítás, szerkesztés, vizualizáció***, ÚJ KÉP, XI évfolyam, 4-5 szám, 2007 április-május, 63-74, ISSN 1450-5010

2. **Ripco Sipos Elvira**: ***A geometria tanátása számítógép segétségével***, ÚJ KÉP, XI évfolyam, 1-2 szám, 2007 január-február, 51-60, ISSN 1450-5010

3. **Ripco Sipos Elvira**: ***Euklides - a geometriai szerkesztőprogram***, ÚJ KÉP, IX évfolyam, 1-2 szám, 2005 január-február, 30-33, ISSN1450-5010

4. **Ripco Sipos Elvira**: ***Hogyan alkalmazom a szmítógépet a matematikaórán (geometriaórán)?***, ÚJ KÉP, XI. évfolyam, 2007 október-november, 19-35, ISSN 1450-5010

5. **Ripco Sipos Elvira**: ***Geometria tanítása a zentai Bolyaiban***, ÚJ KÉP, XII. évfolyam, 2008 október-november, 32-36, ISSN 1450-5010

6. **Ripco Sipos Elvira**: ***Apollóniosz problémái***, TAVASZI SZÉL Konferenciakiadvány, 2007, 284-289, ISBN 978-963-87569-1-6

7. **Ripco Sipos Elvira**: ***Problémamegoldás és tehetségfejlesztés a számítógép segítségével a geometriaórán***, A TEHETSÉGEK SZOLGÁLATÁBAN I. NEMZETKÖZI TEHETSÉGGONDOZÓ KONFERENCIA, Magyarkanizsa, 2009. március 23, Konferenciakiadvány, 83-88, ISBN 978-86-84699-42-0

8. **Ripco Sipos Elvira**: ***Teaching geometry using computer visualizations***, TEACHING MATHEMATICS AND COMPUTER SCIENCE, (2009), 1-19, ISSN 1589-7389

9. **Ripco Sipos Elvira**: ***Apollonius' problems in grammar school***, TEACHING MATHEMATICS AND COMPUTER SCIENCE, (2009), 7/1, 69-85, ISSN 1589-7389

10. **Ripco Sipos Elvira**: ***A geometria tanításának útjai***, MAGYAR SZÓ, ÜVEGGOLYÓ, 2010. március 1-8

11. **Ripco Sipos Elvira**: ***Alkalmazzuk a számítógépet a geometriaórán!***, A MATEMATIKA TANÍTÁSA, MÓDSZERTANI FOLYÓIRAT, XVIII évfolyam, 2 szám, 2010, MOZAIK Kiadó Kft, Szeged, ISSN 1216-6650

12. **Ripco Sipos Elvira**: ***Matematika érettségi vizsgák a zentai "Bolyai" gimnáziumban***, 169-177, KUTATÓ TANÁROK TUDOMÁNYOS KÖZLEMÉNYEI, 2007-2008, ISBN 963 87225 1 7

13. **Ripco Sipos Elvira**: ***Tehetséggondozás a zentai Bolyaiban***, 340-347, KUTATÓ TANÁROK TUDOMÁNYOS KÖZLEMÉNYEI, 2007-2008, ISBN 963 87225 1 7

14. **Ripco Sipos Elvira**: ***Tehetségfejlesztés a számítógép segítségével***, A MATEMATIKA TANÍTÁSA, MÓDSZERTANI FOLYÓIRAT, XVIII évfolyam, 3 szám, 2010, MOZAIK Kiadó Kft, Szeged, ISSN 1216-6650

15. **Ripco Sipos Elvira**: ***Hyperbolic geometry and tiling***, ISIS-Symmetry congress-festival, 342-345, Gmuend, ISSN 1447-607X

Konferenciákon tartott előadások

1. Miskolc: History of Mathematics and Teaching of Mathematics, 2006 május 19-21, "Teaching Geometry using Computers"

2. Karcag: Kutató Tanárok I konferenciája, 2006 október 6-7, "A geometria tanítása számítógép segítségével"

3. Budapest: DOSz konferencia, 2007 május 18-21, "Apollóniosz problémái- tanítás, szerkesztés, vizualizáció"

4. Szabadkai Nyári Akadémia, 2007 augusztus 6-10, " Hogyan alkalmazom a számítógépet a matematikaórákon?"

5. Novi Sad: Međunarodna Konferencija za Nastavu Matematike, 2007 augusztus 22-23, "Problemsko rešavanje zadataka iz geometrije pomoću računara"

6. Győr: Kutató Tanárok II konferenciája, 2007. október 12-13, " Matematikai érettségi vizsgák a zentai Bolyaiban"

7. Budapest: Varga Tamás Módszertani napok, 2007. november 9-10 , "Hogyan alkalmazom a számítógépet a geometriaórákon?"

8. Zenta: T-day, 2008. április 4-5, "Nastava geometrije u gimnaziji Bolyai"

9. Novi Sad: XII Kongres Matematičara Srbije, 2008. augusztus 28-szeptember 2, "Apolonijevi problemi/ nastava u srednjoj školi Boljai"

10. Budapest: Vraga Tamás Módszertani Napok, 2008. november 7-8, "Geometrical Transformations Using Computer"

11. Győr: Kutató Tanárok II konferenciája, 2008. október 10-11, "Tehetséggondozás a zentai Bolyaiban"

12. Magyarkanizsa: III Tehetség Nap, 2009. március 23, "Problémamegoldás és tehetségfejlesztés a számítógép segítségével a geometriaórán"

13. Szeged: History of Mathematics and Teaching of Mathematics, 2010. május 20-23, "Important Theorems in geometry"

14. Szeged: Szakmódszertani kutatások a természettudományos, illetve a matematika és az informatika tantárgyhoz kapcsolódóan, 2010. május 20-21, "A geometriák alapjai"

15. Gmuend: ISIS-Symmetry:Art and Science 8th Interdisciplinary Study of Symmetry congress-festival, 2010. augusztus 23-28, "Hyperbolic geometry and tiling, learn and teach using computer"
16. Zenta: I. Vajdasági Tehetségpont Konferencia, 2010. október 1-2, "Nevezetes tételek a GeoGebrában"
17. Budapest: Varga Tamás Módszertani napok, 2010. november 6
18. Novi Sad: GeoGebra Conference for Southeast Europe 2011, január 15,16, "Izometrijske transformacije u GeoGebra DGS"

References

[1] Ambrus András: ***Bevezetés a matematikadidaktikába***, ELTE Eötvös kiadó, Budapest, 1995

[2] Balka, D. S. (1974): ***Creative ability in mathematics.*** Arithmetic Teacher, 21, 633-636.

[3] Anton Bilimović: ***Apolonijevi problemi***, www.matf.bg.ac.yu, (2007-03-21)

[4] Johanne Bolyai: ***Scientiam Spatii***, Polygon, Szeged, 2002

[5] Cofman, Judita: ***What to Solve?*** –Problems and Suggestions for Young Mathematicians, Clarenden Press, Oxford, 1994

[6] Cofman, Judita: ***Numbers and Shapes Revisited***–More problems and Suggestions for Young Mathematicians, Clarenden Press, Oxford, 1995

[7] H.S.M. Coxeter: ***A geometriák alapjai***, Müszaki Könyvkiadó, Budapest, 1987, ISBN 963 10 6843 9

[8] H.S.M. Coxeter: ***Projektív geometria***, Gondolat, Budapest, 1986, ISBN 963 281 678 1

[9] Euclid, ***Elements***, Gondolat, Budapest, 1983

[10] Hajós György: ***Bevezetés a geometriába***, Tankönyvkiadó, Budapest, 1984

[11] David Hilbert: ***Grundlagen der Geometrie***, ***Osnovi geometrije***, Matematicki Institut, Beograd, 1957

[12] Jovan D. Kečkić: ***Matematika za treći razred gimnazije***, Naučna knjiga, Beograd, 1990

[13] Lénárt István, ***Nem-euklideszi kalandok***, Key Curriculum Press , 2009

[14] Rosa Massa, Ftima Romero, Iolanda Guevara: ***Teaching mathematics through history: Some trigonometric concepts***, The Global and the Local: The History of Science and the Cultural Integration of Europe. Proceedings of the 2nd ICESHS (Cracow, Poland, September 69, 2006)

[15] Eric Louis Mann: ***Mathematical Creativity and School Mathematics: Indicators of Mathematical Creativity in Middle School Students***, Ph.D. dissertation, University of Connecticut, 2005

[16] Milan Mitrović, Srdjan Ognjanović, Mihailo Veljković, Ljubinka Petković, Nenad Lazarević: ***Geometrija I za matematičke gimnazije***, Krug Beograd, 2003

[17] Munkácsy Katalin: ***A matematikai bizonyításfogalom változásának hatása a tanításra***, http://tandem.lauder.hu/ujsag/cikkek/20000209.html

[18] Presnky Marc: ***Digital Natives, Digital Immigrants,*** On the Horizon , MCB University Press, Vol. 9 No. 5, October 2001

[19] Pólya György: ***A gondolkodás iskolája***, Bibliotheca Kiadó, Budapest, 1957

[20] Pólya György: ***Mathematical Discovery on understanding, learning and teaching problem solving***, John Wiley and Sons Inc., New York, 1962

[21] Ripco Sipos Elvira: ***Teaching geometry using computer visualization***, Teaching Mathematics and Computer Science, TMCS 7 (2009)2, ISSN 1589-7389

[22] Ripco Sipos Elvira: ***Apollonius' problems in grammar school***, Teaching Mathematics and Computer Science, TMCS 7 (2009)1, ISSN 1589-7389

[23] Ripco Sipos Elvira: ***Matematika érettségi vizsgák a zentai "Bolyai" gimnáziumban***, 169-177, Kutató Tanárok Tudományos Közleményei, 2007-2008, ISBN 963 87225 1 7

[24] Alan H. Schoenfeld: ***Mathematical Problem Solving***, Academic Press, INC., New York, 1985

[25] Alan H. Schoenfeld: ***Mathematical Thinking and Problem Solving***, Lawernce Erlbaum Associates, Publishers, Hillsdale, New Jersey, 1994

[26] Richard R. Skemp: ***The Psychology of Learning Mathematics***, Lawrence Erlbaum Associates, New Jersey, U.S.A, Hungarian Translation ©1975 Klein Sándor, pages: 42-44, 138, 152

[27] Szilasi Lajos: http://www.jgytf.u-szeged.hu/tanszek/matematika/Bolyai/index.html, 05.06.2009

[28] Patrick Suppes: ***The Aims of Education***, from Suppes, P. (1968). Can there be a normative philosophy of education? In G.L. Newsome, Jr. (Ed.), Philosophy of Education. Studies in Philosophy and Education series, Proceedings of the 24th annual meeting of the Philosophy of Education Society–Santa Monica, April 7-l0, 1968, Edwardsville, IL: Southern Illinois University, 1–12. Reprinted in J.P. Strain (Ed.), Modern Philosophies of Education. New York: Random House, 1971, 277–288.

[29] David Tall: ***The Cognitive Development of Proof: Is Mathematical proof For All or For Some?***, Conference of the University of Chicago Scholl Mathematics Project, August 1998

[30] David Tall: ***Computers and the Link Between Intuition and Formalism***, Published in Proceedings of the Annual International Conference on technoogy in Collegiate Mathematics, Addson-Wesley Longman., pp.417-421

[31] David Tall: ***Advanced Mathematical Thinking and The Computer***,Published in Proceedings of the 20th University Mathematics Teaching Conference, Shell Centre, Nottingham, pp. 1-8 (1996)

[32] David Tall: ***The Psychology of Advanced Mathematical Thinking: Biological Brain and Mathematical Mind***, Conference of the International group for the Psychology of Mathematics Education, Lisbon, July 1994

[33] David Tall: ***Interrelationships between mind and computer: pro- cesses, images symbols***,Advanced Technologies in the Teaching of Math- ematics and Science, New York: Springer-Verlag, 385-413, 1993

[34] David Tall: ***Enviroments for enactive and visual manipulation***, The Psychology of Advanced Mathematical Thinking: Biological Brain and Math- ematical Mind, Conference of the International Group for the Psychology of Mathematical Education, Lisbon, July, 1994

Internetes oldalak

[35] www.en.wikipedia.org, History of General Relativity, (07.10.2008)

[36] Internet:***MathWorld***,Wolfram Research, http://www.mathworld.wolfram.com, (2007-03-21)

[37] Wolfram Mathematica homepage, http://demonstrations.wolfram.com/-PtolemysTheorem/, (25.03.2010)

[38] http://www-history.mcs.st-andrews.ac.uk/Biographies, (02.03.2011)