

# **Periodic Orbits and Global Dynamics for Delay Differential Equations**

## **Ph.D. Thesis**

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# 1 Introduction

Since Newton and Leibniz invented differential and integral calculus in the seventeenth century, numerous problems in physics, biology and economics have been analyzed using ordinary differential equations. Ordinary differential equations are adequate models for systems that satisfy the principle of causality, meaning that the rate of change of the state of the system depends solely on the present state and not on the past one. However, in many processes time delays are not negligible: a signal needs time to travel to the controlled object, a driver needs time to react or animals need time to mature before reproducing. In these examples, the effect of any change is not necessarily instantaneous, hence the future of the system depends on past states as well. Such systems are modeled by functional differential equations or delay differential equations.

In the eighteenth century, Euler, Lagrange and Laplace already studied delay differential equations in relation to various geometrical problems. At the 1908 International Congress of Mathematicians, Picard highlighted the importance of hereditary effects in physical systems. In the late 1920s and early 1930s, Volterra, during his research on predator-prey models and viscoelasticity, proposed some general differential equations with delay, and he was the first one to study such equations systematically. Approximately ten years later, Minorsky, who investigated ship stabilization and automatic steering, showed the importance of delays in feedback mechanism. The lack of sufficient theoretical tools, however, limited the study of functional differential equations until the 1950's. Since then, the theoretical background of this field has been vigorously developing.

There are many similarities between the theory of ordinary differential equations and functional differential equations. The analytical tools developed for ordinary differential equations have been extended to the latter class of equations when possible. There are important differences as well: while the phase space for an ordinary differential equation is always finite dimensional, a functional differential equation generates an infinite dimensional dynamical system. This feature results from the fact that instead of an initial value, an initial function has to be given to determine a solution.

Delay differential equations, in particular equations of the form

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \quad (1.1)$$

play an essential role in the study of artificial neural networks. Wu gives a general

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overview of this field in [50]. The present thesis focuses on Eq. (1.1) with parameter  $\mu > 0$  and monotone continuous nonlinearity  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and is motivated by the following examples:

- Eq. (1.1) with  $f(x) = \alpha \tanh(\beta x)$  or  $f(x) = \alpha \tan^{-1}(\beta x)$ ,  $\alpha \neq 0, \beta > 0$ , models the voltage of a single, self-excitatory neuron [18, 39]. A complete picture is available for such nonlinearities (see the results of Krisztin, Walther and Wu in [22, 25, 26, 27]).
- A model of artificial neural networks introduced by Hopfield [19] in 1984 assumes that voltage amplifiers (or neurons) communicate and respond instantaneously. If such a network is connected symmetrically and consists of analogous neurons, then there is no oscillation in the system. Marcus and Westervelt [35] improved the Hopfield model in 1989 by adding time delays due to the finite switching speeds of the amplifiers. They found that delays can induce sustained oscillation for certain connection topologies. The general form of their model of  $N$  identical saturating voltage amplifiers is

$$C\dot{x}_i = -\frac{1}{R}x_i(t) + \sum_{j=1}^N T_{ij}f(x_j(t-\tau)), \quad i = 1, \dots, N,$$

where  $x_j(t)$  represents the voltage on the input of the  $j$ th neuron at time  $t$ ,  $C$  is the input capacitance of the neuron,  $R$  is the resistance of the neuron, and  $\tau$  is the delay. Transfer function  $f$  is sigmoidal, saturating at  $\pm 1$  with maximum slope at  $x = 0$ . If for the connection matrix  $T = (T_{ij})$ ,

$$\sum_{j=1}^N T_{ij} = \sum_{j=1}^N T_{kj}$$

holds for all  $i, k$  in  $\{1, \dots, N\}$ , then there exist synchronized solutions, that is solutions with  $x_1(t) = x_2(t) = \dots = x_N(t)$ . It is easy to see that synchronized solutions are characterized by Eq. (1.1) with suitable  $\mu$  and monotone  $f$ .

- The system

$$\begin{aligned} \dot{x}^0(t) &= -\mu x^0(t) + f(x^1(t)) \\ &\vdots \\ \dot{x}^{N-1}(t) &= -\mu x^{N-1}(t) + f(x^N(t)) \\ \dot{x}^N(t) &= -\mu x^N(t) + f(x^0(t-1)) \end{aligned}$$

with  $N \geq 1$ ,  $\mu > 0$  and feedback function  $f$  models a unidirectional ring of interacting neurons (see [31] and the references therein). It is verified in [6, 9] that

the periodic solutions of the above system and the periodic solutions of Eq. (1.1) correspond to each other in case  $f$  is strictly increasing, odd, continuously differentiable and satisfies some convexity property.

- The scalar equation,

$$C\dot{x}(t) = -\frac{x(t)}{R} + \alpha f(x(t)) + \beta f(x(t - \tau)) + \tilde{I}$$

introduced in [17], models a single neuron or the averaged potential of a population of neurons coupled by mutual inhibitory synapses. Here  $C > 0$ ,  $R > 0$  and  $\tilde{I}$  are the capacitance, resistance and external current input constants;  $x(t)$  denotes the voltage of the neuron, and  $f$  is the Hopfield activation function  $f : \mathbb{R} \ni x \mapsto 0.5(|x + 1| - |x - 1|) \in [-1, 1]$ . Time delay appears due to finite conduction velocities or synaptic transmission.

- Equations of the form (1.1) with unimodal feedback functions ( $f$  has exactly one extremum and changes the monotonicity only at one point) appear in biological applications. Two examples are the Mackey–Glass equation with  $f(x) = a \cdot x / (1 + x^n)$  modeling the production of red blood cells and the Nicholson’s blowflies equation with  $f(x) = axe^{-bx}$ ,  $a > 0$ ,  $b > 0$ . Liz, Röst and Wu showed in [29, 41] that certain choices of parameters imply that all solutions enter the domain where  $f'$  is negative, so the results for Eq. (1.1) with monotone nonlinearity can be applied to describe the long-term behavior of solutions.

The aim of this work is to describe the global attractor as thoroughly as possible for special feedback functions, as this is the subset of the phase space  $C = C([-1, 0], \mathbb{R})$  that determines the asymptotic behavior of all bounded solutions. The investigation of the global attractor includes the study of equilibria, identification of the exact number and the stability properties of periodic orbits, and, if possible, characterization of the so-called connecting orbits. Efficient analytical methods are available to explore the stability properties of equilibrium points, but the problem of detecting periodic orbits, their hyperbolicity and stability features is far from trivial.

The present thesis considers a wide variety of monotone nonlinear maps: step functions, the piecewise linear Hopfield activation function and continuously differentiable functions as well. Step feedback functions are easy to handle as Eq. (1.1) with a step function  $f$  is reduced to ordinary differential equations, hence specific infinite dimensional problems related to the equation (e.g. the construction of periodic orbits) can be simplified to finite dimensional ones. It is expected that many dynamical properties found for step nonlinearities can be carried over to smooth nonlinearities close to the step functions. A goal of the present thesis is to show that the existence of periodic orbits for equations with smooth nonlinear maps can be proved by considering step feedback functions first, and then by using perturbation theorems. This is a highly

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nontrivial task as several important technical tools (the theory of invariant manifolds, discrete Lyapunov functionals, Floquet multipliers, etc.) are not available for equations with step nonlinearities because of the lack of smoothness. A key technical property in the carry-over procedure is the hyperbolicity of periodic orbits in question. This feature can be verified in a straightforward way for equations with step functions, and the perturbation techniques preserve the hyperbolicity for smooth nonlinearities. We remark that confirming hyperbolicity of periodic orbits of delay differential equations is still an infinite dimensional problem, which is solved only in some particular cases like our one.

The thesis discusses the following new results in detail.

Firstly, Eq. (1.1) may have several equilibria determined by the fixed points of  $\xi \mapsto f(\xi)/\mu$ . In case  $f$  strictly increases and is continuously differentiable, Krisztin, Walther and Wu have described spindle-like structures between consecutive stable equilibria in terms of pointwise ordering. Chapter 3 shows that the structure of the global attractor can be more complex than the union of spindles. For a special class of strictly increasing and continuously differentiable feedback functions, exactly two large-amplitude periodic orbits are given in the sense that the orbits are not between consecutive stable equilibria. Verifying the existence of such large-amplitude periodic orbits poses a challenge as they cannot arise via local bifurcation. In the course of the proof, step nonlinearities are considered first. The problem of finding periodic solutions for step functions is reduced to the finite dimensional problem of solving systems of algebraic equations. Thereby two periodic solutions can be determined explicitly. As a second step, the implicit function theorem is applied in order to extend the result for smoothened step functions. Finally, perturbations of Poincaré maps guarantee the existence of periodic solutions for all strictly increasing and smooth nonlinearities close to the smoothened step function in  $C^1$ -norm. Hyperbolicity of the periodic orbits gained in the second step is of key importance.

The global attractor is described entirely only for special infinite dimensional systems, for example for gradient systems of parabolic equations [14]. By examining the unstable sets of the previous large-amplitude periodic orbits, Chapter 4 offers complete picture of the global attractor outside the spindles. Techniques developed for monotone nonlinearities are of great use in this chapter: the monotone property of the semiflow, a discrete Lyapunov functional, the Poincaré–Bendixson theorem and the theory of invariant manifolds are all necessary to arrive at the desired result.

The result in Chapter 3 implies the question whether we can guarantee the existence of more large-amplitude periodic orbits oscillating around the same equilibria. An analogous problem is solved for the negative feedback case in Chapter 5: for all  $\mu > 0$ , a locally Lipschitz continuous map  $f$  with  $xf(x) < 0$  for  $x \in \mathbb{R} \setminus \{0\}$  is constructed such that Eq. (1.1) has an infinite sequence of periodic orbits. All periodic solutions defining these orbits oscillate slowly around 0 in the sense that they admit at most one

sign change in each interval of length 1. In this example,  $f$  is close to an unbounded step function. Based on this property, an infinite sequence of contracting return maps is given, their fixed points being the initial segments of the periodic solutions. If  $f$  is continuously differentiable, then all periodic orbits are hyperbolic and orbitally asymptotically stable with asymptotic phase.

It is also an interesting task to extend theorems given for continuously differentiable and strictly monotone nonlinear maps to nonlinearities with weaker properties. Such feedback functions come up in several applications. Chapter 6 considers Eq. (1.1) with the piecewise linear Hopfield activation function  $f : \mathbb{R} \ni x \mapsto 0.5(|x+1| - |x-1|) \in [-1, 1]$  and analyzes the truth of a conjecture given by Györi and Hartung in [11]. The fact that the Hopfield activation function is neither strictly monotone nor smooth gives rise to nontrivial technical problems. In this case, the solution operator is neither injective nor differentiable everywhere. It is shown in this work that although most of the solutions converge to an equilibrium as  $t \rightarrow \infty$ , there is a periodic solution for certain choices of parameters. The proof projects the unstable set of the unstable equilibrium together with its closure to the two-dimensional plane and studies it with the help of the discrete Lyapunov functional. The periodic orbit is determined by the fixed point of a Poincaré return map defined on a subset of the two-dimensional plane.

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## 2 Delay Differential Equations: Theoretical Background

### 2.1 Basic theory

This chapter gives an overview of the basic theory applied in the dissertation. For more information, see the two monographs on functional-differential equations written by Diekmann, van Gils, Verduyn Lunel and Walther [7] and Hale, Verduyn Lunel [16].

#### Phase space, solution.

The natural *phase space* for

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \quad (1.1)$$

is the space  $C = C([-1, 0], \mathbb{R})$  of continuous real functions defined on  $[-1, 0]$  equipped with the supremum norm  $\|\varphi\| = \sup_{-1 \leq s \leq 0} |\varphi(s)|$ .

If  $I \subset \mathbb{R}$  is an interval,  $u : I \rightarrow \mathbb{R}$  is continuous, then for  $[t-1, t] \subset I$ , *segment*  $u_t$  is the element of  $C$  given by  $u_t(s) = u(t+s)$  for  $-1 \leq s \leq 0$ .

In the sequel we consider Eq. (1.1) with smooth and non-continuous (e.g. step function) nonlinearities and linear variational equations as well. This requires a slightly more general form of equation and a more general definition of solutions.

Consider the equation

$$\dot{y}(t) = g(t, y_t) \quad (2.1)$$

assuming that  $g : \mathbb{R} \times C \rightarrow \mathbb{R}$  satisfies the condition: for each interval  $I \subset \mathbb{R}$  and each continuous function  $u : I + [-1, 0] \rightarrow \mathbb{R}$ , the map  $I \ni t \mapsto g(t, u_t) \in \mathbb{R}$  is locally integrable (i.e., integrable on compact subintervals of  $I$ ). Then for given  $t_0 \in \mathbb{R}$  and  $0 < a \leq \infty$ , a function  $y : [t_0 - 1, t_0 + a) \rightarrow \mathbb{R}$  is called a *solution* of (2.1) on  $[t_0 - 1, t_0 + a)$  if  $y$  is continuous and

$$y(t) = y(t_0) + \int_{t_0}^t g(s, y_s) \, ds$$

holds for all  $t \in [t_0, t_0 + a)$ . A function  $y : \mathbb{R} \rightarrow \mathbb{R}$  is a solution of Eq. (2.1) on  $\mathbb{R}$  if it is a solution of (2.1) on  $[t_0 - 1, \infty)$  for all  $t_0 \in \mathbb{R}$ .

## 2 Delay Differential Equations: Theoretical Background

If  $y : [t_0 - 1, t_0 + a) \rightarrow \mathbb{R}$  is a solution of (2.1) on  $[t_0 - 1, t_0 + a)$  and for some  $(\alpha, \beta) \subset (t_0, t_0 + a)$ , the map  $(\alpha, \beta) \ni t \mapsto g(t, y_t) \in \mathbb{R}$  is continuous, then it is clear that  $y$  is continuously differentiable on  $(\alpha, \beta)$ , moreover, (2.1) holds for all  $t \in (\alpha, \beta)$ .

If  $y : [t_0 - 1, t_0 + a) \rightarrow \mathbb{R}$  is a solution of (2.1), then obviously  $y$  is absolutely continuous on  $[t_0, t_0 + a)$ , and (2.1) holds almost everywhere on  $[t_0, t_0 + a)$ .

If

$$g(t, \varphi) = -\mu\varphi(0) + h(t, \varphi(-1)), \quad (t, \varphi) \in \mathbb{R} \times C,$$

with some  $\mu \in \mathbb{R}$  and  $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  so that  $g$  satisfies the above local integrability condition, then for each  $\varphi \in C$  a unique solution  $y : [-1, \infty) \rightarrow \mathbb{R}$  with  $y_0 = \varphi$  can be given by the method of steps. Set  $y(t) = \varphi(t)$  for  $-1 \leq t \leq 0$ . Suppose that a continuous  $y : [-1, n] \rightarrow \mathbb{R}$  is already given for some  $n \geq 0$ . Then for  $t \in [n, n+1]$ , define

$$y(t) = e^{-\mu(t-n)}y(n) + \int_n^t e^{-\mu(t-s)}h(s, y(s-1))ds.$$

Then  $y|_{[n, n+1]}$  is absolutely continuous and (2.1) holds almost everywhere on  $[n, n+1]$ . It is easy to see that this construction gives the unique solution  $y^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  with  $y_0^\varphi = \varphi$ .

### Semiflow.

Suppose  $\mu \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Then the solutions of Eq. (1.1) define the *continuous semiflow*

$$\Phi : \mathbb{R}^+ \times C \ni (t, \varphi) \mapsto x_t^\varphi \in C. \quad (2.2)$$

All maps  $\Phi(t, \cdot) : C \rightarrow C$ ,  $t \geq 1$ , are compact. If in addition,  $f$  is strictly increasing, then all maps  $\Phi(t, \cdot) : C \rightarrow C$ ,  $t \geq 0$ , are injective. It follows that if  $f$  is strictly increasing, then for every  $\varphi \in C$  there is at most one solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) with  $x_0 = \varphi$ . Whenever such solution exists, we denote it also by  $x^\varphi$ .

A function  $\hat{\xi} \in C$  is an *equilibrium point* (or stationary point) of  $\Phi$  if  $\hat{\xi}(s) = \xi$  for all  $-1 \leq s \leq 0$  with  $\xi \in \mathbb{R}$  satisfying  $-\mu\xi + f(\xi) = 0$ .

A set  $M \subset C$  is called *positively invariant* under  $\Phi$  if  $\Phi(t, M) \subseteq M$  for all  $t \geq 0$ . A set  $M \subset C$  is said to be invariant if for any  $\varphi \in M$  there exists a solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  with  $x_0^\varphi = \varphi$  and  $x_t^\varphi \in M$  for all  $t \in \mathbb{R}$ .

### Limit sets, convergence.

If  $\varphi \in C$  and  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  is a bounded solution of Eq. (1.1), then the  $\omega$ -limit set

$$\begin{aligned} \omega(\varphi) = \{ & \psi \in C : \text{there exists a sequence } (t_n)_0^\infty \text{ in } [0, \infty) \\ & \text{with } t_n \rightarrow \infty \text{ and } \Phi(t_n, \varphi) \rightarrow \psi \text{ as } n \rightarrow \infty \} \end{aligned}$$



is nonempty, compact, connected and invariant. For a solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  such that  $x|_{(-\infty, 0]}$  is bounded, the  $\alpha$ -limit set

$$\alpha(x) = \{\psi \in C : \text{there exists a sequence } (t_n)_0^\infty \text{ in } \mathbb{R} \\ \text{with } t_n \rightarrow -\infty \text{ and } x_{t_n} \rightarrow \psi \text{ as } n \rightarrow \infty\}$$

is nonempty, compact, connected and invariant. If for some  $\varphi \in C$ , there is a unique solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  such that  $x|_{(-\infty, 0]}$  is bounded and  $x_0 = \varphi$ , then we may use notation  $\alpha(x) = \alpha(\varphi)$ . This is the case if  $f$  is strictly increasing and  $\varphi \in \omega(\psi)$  for some  $\psi \in C$ .

In Chapter 6 of [43] Smith introduces the partial order  $\leq_\nu$  on  $C$ :  $\varphi \leq_\nu \psi$  if and only if  $\varphi(s) \leq \psi(s)$  for all  $s \in [-1, 0]$  and  $(\psi(s) - \varphi(s))e^{\nu s}$  is nondecreasing on  $[-1, 0]$ . Whenever  $\varphi \leq_\nu \psi$  and  $\varphi \neq \psi$ , write  $\varphi <_\nu \psi$ . We intend to use the following theorem stated also in Chapter 6 of [43].

**Theorem 2.1.1.** *Consider the equation*

$$\dot{x}(t) = g(x_t), \tag{2.3}$$

where  $g : C \rightarrow \mathbb{R}$  is continuous and satisfies a Lipschitz condition on each compact subset of  $C$ . Suppose  $x^\varphi(t)$  is defined for all  $t \geq 0$ , and the following conditions hold for Eq. (2.3):

(T) Functional  $g$  maps bounded subsets of  $C$  to bounded subsets of  $\mathbb{R}$ . For each  $\varphi \in C$ ,  $x^\varphi$  is bounded for  $t \geq 0$ . For each compact subset  $A \subset C$ , there exists a closed and bounded subset  $B \subset C$  such that for each  $\varphi \in A$ ,  $x^\varphi(t) \in B$  for all large  $t$ .

(SM) There exists  $\nu \geq 0$  such that whenever  $\varphi, \psi \in C$  satisfy  $\varphi <_\nu \psi$ , then

$$\nu(\psi(0) - \varphi(0)) + g(\psi) - g(\varphi) > 0.$$

Then the set of convergent points (namely those  $\varphi \in C$  for which  $\lim_{t \rightarrow \infty} x^\varphi(t)$  exists and finite) contains an open and dense subset in  $C$ . In addition, if Eq. (2.3) has exactly two equilibrium points, then all solutions converge to one of these.

### Boundedness.

It is a direct consequence of the next proposition that if  $\mu > 0$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous and bounded map with  $\sup_{x \in \mathbb{R}} |f(x)| \leq M$ , in addition  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a periodic solution of (1.1) so that 0 is in the range of  $p$ , then  $\max_{t \in \mathbb{R}} |p(t)| < M/\mu$ .

**Proposition 2.1.2.** *If  $\mu > 0$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $\sup_{x \in \mathbb{R}} |f(x)| \leq M$  and  $x : [t_0 - 1, \infty) \rightarrow \mathbb{R}$  is a solution of (1.1) with  $x(t_0) = 0$ , then  $|x(t)| < M/\mu$  for all  $t > t_0$ .*

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*Proof.* Let  $u : \mathbb{R} \rightarrow \mathbb{R}$  be the solution of the initial value problem

$$\begin{cases} \dot{u}(t) = -\mu u(t) + M, & t \in \mathbb{R}, \\ u(t_0) = 0. \end{cases}$$

Then  $u(t) = M(1 - e^{-\mu(t-t_0)})/\mu$  for  $t \in \mathbb{R}$ . Clearly, if  $x : [t_0 - 1, \infty) \rightarrow \mathbb{R}$  is a solution of (1.1), then  $\dot{x}(t) \leq -\mu x(t) + M$  for all  $t \in \mathbb{R}$ . In consequence, Corollary 6.2 of Chapter I in [15] implies that for  $t > t_0$ ,  $x(t) \leq u(t) < M/\mu$ . The lower bound can be verified analogously.  $\square$

### The global attractor.

Assume  $\mu > 0$  and  $f$  is continuously differentiable. If the *global attractor*  $\mathcal{A}$  of the semiflow  $\Phi$  exists, it is a nonempty, compact set in  $C$ , it is positively invariant in the sense that  $\Phi(t, \mathcal{A}) = \mathcal{A}$  for all  $t \geq 0$ , and it attracts bounded sets in the sense that for every bounded set  $B \subset C$  and for every open set  $U \supset \mathcal{A}$ , there exists  $t \geq 0$  with  $\Phi([t, \infty) \times B) \subset U$ . Global attractors are uniquely determined [14]. We know several sufficient conditions for the existence of the global attractor, for example  $\mu > 0$  and  $\limsup_{|x| \rightarrow \infty} |f(x)/x| < \mu$ . In case  $\mathcal{A}$  exists, its structure contains all relevant information about the long term behavior of solutions. It can be shown that

$$\mathcal{A} = \{\varphi \in C : \text{there is a bounded solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1) so that } \varphi = x_0\},$$

see [25, 31, 40].

If in addition to smoothness,  $f$  is strictly increasing, the compactness of  $\mathcal{A}$ , its invariance property and the injectivity of the maps  $\Phi(t, \cdot) : C \rightarrow C$ ,  $t \geq 0$ , combined permit to verify that the map

$$[0, \infty) \times \mathcal{A} \ni (t, \varphi) \mapsto \Phi(t, \varphi) \in \mathcal{A}$$

extends to a continuous flow  $\Phi_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \rightarrow \mathcal{A}$ ; for every  $\varphi \in \mathcal{A}$  and for all  $t \in \mathbb{R}$  we have  $\Phi_{\mathcal{A}}(t, \varphi) = x_t$  with a uniquely determined solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) satisfying  $x_0 = \varphi$ .

Note that we have  $\mathcal{A} = \Phi(1, \mathcal{A}) \subset C^1$ ;  $\mathcal{A}$  is a closed subset of  $C^1$ . Using the flow  $\Phi_{\mathcal{A}}$  and the continuity of the map

$$C \ni \varphi \mapsto \Phi(1, \varphi) \in C^1,$$

one obtains that  $C$  and  $C^1$  define the same topology on  $\mathcal{A}$ .

**Linearization and unstable manifolds [7, 16].**

Set  $\mu > 0$ . If  $f$  is continuously differentiable, then  $\Phi(t, \cdot)$  is continuously differentiable for  $t \geq 0$ . Suppose  $\hat{\xi} \in C$  is an equilibrium. For each  $\varphi \in C$ , we have  $D_2\Phi(t, \hat{\xi})\varphi = y_t^\varphi$ , where  $y^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  is the solution of the linear variational equation

$$\dot{y}(t) = -\mu y(t) + f'(\xi) y(t-1)$$

with initial function  $y_0^\varphi = \varphi$ . The operators  $D_2\Phi(t, \hat{\xi}) : C \rightarrow C$ ,  $t \geq 0$ , form a strongly continuous semigroup. One gets information about the stability of the equilibrium and the oscillation frequencies in the stable and unstable sets of the equilibrium from the spectrum of the generator of the semigroup. The spectrum of the generator consists of eigenvalues given by the zeros of the *characteristic function*

$$C \ni \lambda \mapsto \lambda + \mu - f'(\xi) e^{-\lambda} \in \mathbb{C}.$$

If  $f'(\xi) > 0$ , then there is exactly one real eigenvalue  $\lambda_0$ , and the rest of the spectrum appears as a sequence of complex conjugate pairs  $(\lambda_j, \overline{\lambda_j})_1^\infty$  with

$$\lambda_0 > \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \dots > \operatorname{Re}\lambda_n > \dots,$$

$$\operatorname{Re}\lambda_j \rightarrow -\infty \quad j \rightarrow \infty,$$

and

$$(2j-1)\pi < \operatorname{Im}\lambda_j < 2j\pi \text{ for } 1 \leq j \in \mathbb{N},$$

see [7]. All the eigenvalues are simple. If  $0 < f'(\xi) < \mu$ , then  $\lambda_0 < 0$  and  $\hat{\xi}$  is stable and hyperbolic. If  $f'(\xi) > \mu > 0$ , then  $\lambda_0 > 0$  and  $\hat{\xi}$  is unstable. If  $\mu > 0$  and

$$f'(\xi) > \frac{\mu}{\cos \theta_\mu} \text{ for } \theta_\mu \in (3\pi/2, 2\pi) \text{ with } \theta_\mu = -\mu \tan \theta_\mu, \quad (2.4)$$

then  $\operatorname{Re}\lambda_1 > 0$ .

In case (2.4) let  $P$  be the 3-dimensional realified generalized eigenspace of the generator of the semigroup  $D_2\Phi(t, \hat{\xi}) : C \rightarrow C$ ,  $t \geq 0$ , associated with the spectral set  $\{\lambda_0, \lambda_1, \overline{\lambda_1}\}$ , and let  $Q$  be the realified generalized eigenspace of the generator associated with the remaining spectrum. Then  $C = P \oplus Q$ . Choose  $\beta > 1$  with  $e^{\operatorname{Re}\lambda_2} < \beta < e^{\operatorname{Re}\lambda_1}$ . According to Theorem I.4. in monograph [26], there is an open neighborhood  $N$  of  $\hat{\xi}$  such that

$$\mathcal{W}_{1,loc}^u(\hat{\xi}) = \left\{ \varphi \in \left\{ \hat{\xi} \right\} + N : \Phi(1, \cdot) \text{ has a trajectory } (\varphi_n)_{-\infty}^0 \text{ with } \varphi_0 = \varphi, \right.$$

$$\left. (\varphi_n - \hat{\xi}) \beta^{-n} \in N \text{ for all } n \leq 0, \text{ and } (\varphi_n - \hat{\xi}) \beta^{-n} \rightarrow 0 \text{ as } n \rightarrow -\infty \right\}$$

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is a 3-dimensional  $C^1$ -smooth local manifold with tangent space  $\{\hat{\xi}\} + P$  at  $\hat{\xi}$ .  $\mathcal{W}_{1,loc}^u(\hat{\xi})$  is called the *leading or fast unstable manifold* of  $\hat{\xi}$ , and contains segments of those solutions that are defined on  $(-\infty, 0]$  and approach  $\xi$  as  $t \rightarrow -\infty$  faster than  $t \mapsto \beta^t$  approaches 0. The forward extension

$$\mathcal{W}_1^u(\hat{\xi}) = \Phi([0, \infty) \times \mathcal{W}_{1,loc}^u(\hat{\xi}))$$

is called the *leading unstable set* of  $\hat{\xi}$ . For all  $\varphi \in \mathcal{W}_1^u(\hat{\xi})$ ,  $\varphi - \hat{\xi}$  has at most two sign changes on  $[-1, 0]$ , see [26].

As we need it later, we also note that if (2.4) holds, and  $\varphi \in \mathcal{A} \setminus \{\hat{\xi}\}$  belongs to the stable set

$$\mathcal{W}^s(\hat{\xi}) = \{\varphi : \omega(\varphi) \text{ exists and } \omega(\varphi) = \hat{\xi}\}$$

of  $\hat{\xi}$ , then  $\varphi - \hat{\xi}$  has at least three sign changes on  $[-1, 0]$ , see Lemma 3.9 in [40] for a proof.

More generally, if for some  $k \geq 1$ ,

$$f'(\xi) > \frac{\mu}{\cos \theta_\mu} \text{ for } \theta_\mu \in \left(2k\pi - \frac{\pi}{2}, 2k\pi\right) \text{ with } \theta_\mu = -\mu \tan \theta_\mu, \quad (2.5)$$

then  $\text{Re} \lambda_k > 0$ . Choose  $P_k$  to be the  $(2k+1)$ -dimensional realified generalized eigenspace of the generator associated with the spectral set  $\{\lambda_0, \lambda_1, \overline{\lambda_1}, \dots, \lambda_k, \overline{\lambda_k}\}$ , and let  $Q_k$  be the realified generalized eigenspace of the generator associated with the remaining spectrum. Then  $C = P_k \oplus Q_k$ . Set  $\beta$  such that  $e^{\text{Re} \lambda_{k+1}} < \beta < e^{\text{Re} \lambda_k}$ . Then there exists a  $(2k+1)$ -dimensional  $C^1$ -smooth local unstable manifold  $\mathcal{W}_{k,loc}^u(\hat{\xi})$  of  $\hat{\xi}$  with tangent space  $\{\hat{\xi}\} + P_k$  at  $\hat{\xi}$  [26]. It consists of segments of solutions that are defined on  $(-\infty, 0]$  and approach  $\xi$  as  $t \rightarrow -\infty$  faster than  $t \mapsto \beta^t$  approaches 0.

For  $f'(\xi) < 0$ , there is a sequence of complex conjugate pairs of simple eigenvalues  $(\lambda_j, \overline{\lambda_j})_1^\infty$  with

$$\text{Re} \lambda_1 > \text{Re} \lambda_2 > \dots > \text{Re} \lambda_n > \dots,$$

$$\text{Re} \lambda_j \rightarrow -\infty, \quad j \rightarrow \infty,$$

and

$$2j\pi < \text{Im} \lambda_j < (2j+1)\pi \text{ for } 1 \leq j \in \mathbb{N}.$$

In addition, there are exactly two eigenvalues in strip  $\{z \in \mathbb{C} : -\pi < \text{Im} z < \pi\}$ ; two reals  $\lambda_{00} \geq \lambda_0 > \text{Re} \lambda_1$  for  $-e^{-\mu-1} < f'(\xi) < 0$ , and a complex conjugate pair  $(\lambda_0, \overline{\lambda_0})$  with  $\text{Re} \lambda_0 > \text{Re} \lambda_1$  for  $f'(\xi) < -e^{-\mu-1}$  [7].

Suppose  $f'(\xi) < -e^{-\mu-1}$  and  $\text{Re} \lambda_k > 0$  with some  $k \geq 0$ . Similarly to the positive feedback case,  $C$  can be decomposed as  $C = P_k \oplus Q_k$  into the closed subspaces  $P_k$  and  $Q_k$ , where  $P_k$  is the  $(2k+2)$ -dimensional realified generalized eigenspace of the generator corresponding to eigenvalues  $\lambda_0, \overline{\lambda_0}, \dots, \lambda_k, \overline{\lambda_k}$ , and  $Q$  is the realified gener-

alized eigenspace corresponding to the rest of the spectrum. Choose  $e^{\operatorname{Re}\lambda_{k+1}} < \beta < e^{\operatorname{Re}\lambda_k}$ . Then the local unstable manifold, that contains segments of solutions defined on  $(-\infty, 0]$  approaching  $\xi$  as  $t \rightarrow -\infty$  faster than  $t \mapsto \beta^t$  approaches 0, is a  $(2k+2)$ -dimensional  $C^1$ -smooth manifold with tangent space  $\{\hat{\xi}\} + P_k$  at  $\hat{\xi}$ . It is also denoted by  $\mathcal{W}_{k,loc}^u(\hat{\xi})$ .

In both cases we use notation  $\mathcal{W}_k^u(\hat{\xi})$  for the forward extension

$$\Phi([0, \infty) \times \mathcal{W}_{k,loc}^u(\hat{\xi})).$$

If the global attractor  $\mathcal{A}$  exists, then  $\overline{\mathcal{W}_k^u(\hat{\xi})}$ , the closure of  $\mathcal{W}_k^u(\hat{\xi})$ , belongs to  $\mathcal{A}$ .

## 2.2 Results for monotone feedback

Although the form of Eq. (1.1) is quite simple, the dynamics generated by it can be very rich. This section focuses on the structure of solutions in case of monotone feedback.

We talk about *positive feedback* if the nonlinear map  $f$  is continuous,  $f(0) = 0$  and  $xf(x) > 0$  for all  $x \neq 0$ . In the *negative feedback* case  $f$  is continuous,  $f(0) = 0$  and  $xf(x) < 0$  for all  $x \neq 0$ .

Mallet-Paret and Sell has given a Poincaré–Bendixson type result in [33] for both cases. Assume  $f$  is continuously differentiable and strictly increasing. If for some  $\varphi \in C$ , solution  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  is bounded, then  $\omega(\varphi)$  is either a single non-constant periodic orbit, or for each  $\psi \in \omega(\varphi)$ ,  $\alpha(\psi) \cup \omega(\psi)$  is a subset of the set of equilibrium points. The proof of this widely cited theorem is based on a discrete Lyapunov functional introduced by the same authors and also presented in the next section.

Krisztin, Walther and Wu, among others, have given more detailed results.

### Positive feedback [20, 22, 23, 25, 26, 27].

Assume  $\mu > 0$ ,  $f$  is continuously differentiable and  $f'(x) > 0$  for all  $x \in \mathbb{R}$ . Suppose  $\xi_-, 0, \xi_+$  are three consecutive zeros of  $\xi \mapsto -\mu\xi + f(\xi)$  so that  $f'(\xi_-) < \mu$ , (2.4) holds for  $\xi = 0$ , and  $f'(\xi_+) < \mu$ . Then equilibria  $\hat{\xi}_-, \hat{\xi}_+$  defined by  $\xi_-, \xi_+$ , respectively, are stable and hyperbolic. Equilibrium point  $\hat{0}$  given by the 0 solution of  $-\mu\xi + f(\xi) = 0$  is unstable. In addition, assume that  $f(x)/x < \mu$  outside a bounded neighborhood of 0.

The monograph [26] of Krisztin, Walther and Wu gives a clear picture of  $\overline{\mathcal{W}_1^u(\hat{0})}$ , the closure of unstable set  $\mathcal{W}_1^u(\hat{0})$  of equilibrium  $\hat{0}$ . It contains the three equilibria  $\hat{\xi}_-, \hat{0}, \hat{\xi}_+$ , a unique periodic orbit  $\mathcal{O}_1$  and connecting orbits among them. The periodic solution  $p$  defining  $\mathcal{O}_1$  oscillates slowly, that is each segment of  $p$  has at most two sign changes. Set  $\overline{\mathcal{W}_1^u(\hat{0})}$  is homeomorphic to the closed unit ball in  $\mathbb{R}^3$ , and its boundary

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is homeomorphic to the unit sphere in  $\mathbb{R}^3$ . Also, there is a 2-dimensional smooth disk in  $\overline{\mathcal{W}_1^u(\hat{0})}$  with boundary  $\mathcal{O}_1$ . This disk contains  $\hat{0}$ , the orbit  $\mathcal{O}_1$  and heteroclinic connections from  $\hat{0}$  to  $\mathcal{O}_1$ . It separates  $\overline{\mathcal{W}_1^u(\hat{0})}$  into two halves, which subsets belong to the domain of attraction of  $\hat{\xi}_-$  and of  $\hat{\xi}_+$ . In the literature  $\overline{\mathcal{W}_1^u(\hat{0})}$  is called a spindle.

Under further conditions ( $f$  is odd, and  $(0, \infty) \ni \xi \mapsto \xi f'(\xi) / f(\xi)$  strictly decreases), the set  $\mathcal{W}_1^u(\hat{0})$  is the global attractor of restriction  $\Phi|_{[0, \infty) \times B}$ , where

$$B = \{\varphi \in C : \xi_- \leq \varphi(s) \leq \xi_+ \text{ for all } s \in [-1, 0]\},$$

see papers [23, 25] of Krisztin and Walther. Well-known examples are

$$f(x) = a \tanh(bx) \quad \text{and} \quad f(x) = a \tan^{-1}(bx)$$

with  $a \neq 0$  and  $b > 0$ . In other cases we cannot exclude the existence of further periodic solutions oscillating around 0.

If (2.5) holds with  $\xi = 0$  and  $k \geq 2$ , then  $\mathcal{W}_k^u(\hat{0})$  exists. The structure of  $\overline{\mathcal{W}_k^u(\hat{0})}$  is characterized by Krisztin and Wu [27]. It contains  $k$  periodic orbits  $\mathcal{O}_1, \dots, \mathcal{O}_k$  so that for all  $j \in \{1, \dots, k\}$ , segments of the periodic solution defining  $\mathcal{O}_j$  have  $2j - 1$  or  $2j$  sign changes. For each  $j$  and  $l$  in  $\{1, \dots, k\}$ , set

$$C_j^0 = \left\{ \varphi \in \overline{\mathcal{W}_k^u(\hat{0})} : \text{there is a solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1)} \right. \\ \left. \text{with } x_0 = \varphi, \alpha(x) = \{\hat{0}\}, \omega(\varphi) = \mathcal{O}_j \right\},$$

$$C_{\pm}^0 = \left\{ \varphi \in \overline{\mathcal{W}_k^u(\hat{0})} : \text{there is a solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1)} \right. \\ \left. \text{with } x_0 = \varphi, \alpha(x) = \{\hat{0}\}, \omega(\varphi) = \{\hat{\xi}_{\pm}\} \right\},$$

$$C_l^j = \left\{ \varphi \in \overline{\mathcal{W}_k^u(\hat{0})} : \text{there is a solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1)} \right. \\ \left. \text{with } x_0 = \varphi, \alpha(x) = \mathcal{O}_j, \omega(\varphi) = \mathcal{O}_l \right\},$$

$$C_{\pm}^j = \left\{ \varphi \in \overline{\mathcal{W}_k^u(\hat{0})} : \text{there is a solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1)} \right. \\ \left. \text{with } x_0 = \varphi, \alpha(x) = \mathcal{O}_j, \omega(\varphi) = \{\hat{\xi}_{\pm}\} \right\}.$$

Then

$$\begin{aligned} \overline{\mathcal{W}_k^u(\hat{0})} &= \{\hat{\xi}_-, \hat{0}, \hat{\xi}_+\} \cup \left( \bigcup_{j=1}^k \mathcal{O}_j \right) \cup \left( \bigcup_{j=1}^k C_j^0 \right) \cup C_-^0 \cup C_+^0 \\ &\cup \left( \bigcup_{1 \leq l < j \leq k} C_l^j \right) \cup \left( \bigcup_{j=1}^k C_-^j \right) \cup \left( \bigcup_{j=1}^k C_+^j \right). \end{aligned}$$

### Negative feedback [27, 45, 46, 47, 48, 49].

Suppose  $\mu > 0$ ,  $f$  is continuously differentiable,  $f(0) = 0$ ,  $f'(x) < 0$  for all  $x \in \mathbb{R}$ ,  $f'(0) < -e^{-\mu-1}$  and  $f$  is either bounded from above or bounded from below. Then  $\hat{0}$  is the unique equilibrium.

In [45] Walther has verified that if  $\operatorname{Re} \lambda_0 > 0$ , then  $\overline{\mathcal{W}_0^u(\hat{0})}$  is a 2-dimensional  $C^1$ -smooth submanifold of  $C$  with boundary, and it is homeomorphic to the 2-dimensional closed unit disk. The boundary of  $\mathcal{W}_0^u(\hat{0})$  is a slowly oscillatory periodic orbit (i.e. an orbit defined by a periodic solution having at most one sign change on each interval of length 1).

Under the above conditions Walther and Yebdri [47, 49] has confirmed that the set

$$\mathcal{W}_{so} = \{x_0 : x : \mathbb{R} \rightarrow \mathbb{R} \text{ is a bounded, slowly oscillatory solution of (1.1)}\} \cup \{\hat{0}\}$$

is the graph of a  $C^1$ -map defined on a subset  $D$  of  $P_0$ , moreover  $D$  is homeomorphic to the closed unit disk in  $\mathbb{R}^2$ , provided  $\mathcal{W}_{so} \neq \{\hat{0}\}$ . The manifold boundary of  $\mathcal{W}_{so}$  is a slowly oscillatory periodic orbit. Further slowly oscillatory periodic orbits may exist in  $\mathcal{W}_{so}$ . The nonperiodic orbits in  $\mathcal{W}_{so} \setminus \{0\}$  oscillate around 0 and make heteroclinic connections between periodic orbits or between  $\hat{0}$  and a periodic orbit. These results are of high importance as  $\mathcal{W}_{so}$  attracts all solutions starting from an open dense subset of  $C$  [34].

There is a Morse-decomposition similarly to the positive feedback case. For  $k \geq 0$ ,

$$\overline{\mathcal{W}_k^u(\hat{0})} = \{\hat{0}\} \cup \left( \bigcup_{j=0}^k \mathcal{O}_j \right) \cup \left( \bigcup_{j=0}^k C_j^0 \right) \cup \left( \bigcup_{0 \leq l < j \leq k} C_l^j \right),$$

where  $\mathcal{O}_j$ ,  $j \in \{0, \dots, k\}$ , is a periodic orbit with segments having  $2j$  or  $2j + 1$  sign changes, and connecting sets  $C_j^0$ ,  $C_l^j$ ,  $j, l \in \{0, \dots, k\}$ , are defined as above [27].

Open questions related to this field are drawn up in work [20] of Krisztin. For information about nonmonotone feedback, see papers cited within [20].

## 2.3 Key technical tools

### 2.3.1 A discrete Lyapunov functional

Mallet-Paret and Sell introduced *discrete Lyapunov functionals* in [32] for both positive and negative feedback case. These functionals proved to be fundamental technical tools. Combined with several other dynamical system methods, they permit to obtain a lot of information about the structure of the global attractor (e.g. a Poincaré–Bendixson type result [33]). Here we restrict attention to the positive feedback case.

For  $\varphi \in C \setminus \{0\}$ , set  $sc(\varphi) = 0$  if  $\varphi \geq 0$  or  $\varphi \leq 0$ , otherwise define

$$sc(\varphi) = \sup \left\{ k \in \mathbb{N} \setminus \{0\} : \text{there exists a strictly increasing sequence} \right.$$

$$(s_i)_0^k \subseteq [-1, 0] \text{ with } \varphi(s_{i-1})\varphi(s_i) < 0 \text{ for } i \in \{1, 2, \dots, k\} \left. \right\}.$$

Then set  $V : C \setminus \{0\} \rightarrow 2\mathbb{N} \cup \{\infty\}$  by

$$V(\varphi) = \begin{cases} sc(\varphi), & \text{if } sc(\varphi) \text{ is even or } \infty, \\ sc(\varphi) + 1, & \text{if } sc(\varphi) \text{ is odd.} \end{cases}$$

Also define

$$R = \left\{ \varphi \in C^1 : \begin{aligned} &\varphi(0) \neq 0 \text{ or } \dot{\varphi}(0)\varphi(-1) > 0, \\ &\varphi(-1) \neq 0 \text{ or } \dot{\varphi}(-1)\varphi(0) < 0, \\ &\text{all zeros of } \varphi \text{ are simple} \end{aligned} \right\}.$$

$V$  has the following lower semi-continuity and continuity property (for a proof, see [26, 33]).

**Lemma 2.3.1.** *For each  $\varphi \in C \setminus \{0\}$  and  $(\varphi_n)_0^\infty \subset C \setminus \{0\}$  with  $\varphi_n \rightarrow \varphi$  as  $n \rightarrow \infty$ ,  $V(\varphi) \leq \liminf_{n \rightarrow \infty} V(\varphi_n)$ . For each  $\varphi \in R$  and  $(\varphi_n)_0^\infty \subset C^1 \setminus \{0\}$  with  $\|\varphi_n - \varphi\|_{C^1} \rightarrow 0$  as  $n \rightarrow \infty$ ,  $V(\varphi) = \lim_{n \rightarrow \infty} V(\varphi_n) < \infty$ .*

The next result explains why  $V$  is called a Lyapunov functional.

**Lemma 2.3.2.** *Assume that  $J \subset \mathbb{R}$  is an interval,  $\alpha : J \rightarrow \mathbb{R}$  is locally Lebesgue integrable,  $\beta : J \rightarrow \mathbb{R}$  is nonnegative,  $z : J + [-1, 0] \rightarrow \mathbb{R}$  is continuous, and  $z$  is differentiable on  $J$ . Suppose that*

$$\dot{z}(t) = -\alpha(t)z(t) + \beta(t)z(t-1) \tag{2.6}$$

*holds for all  $t > \inf J$  in  $J$ . Then the following statements hold.*

*(i) If  $t_1, t_2 \in J$  with  $t_1 < t_2$  and  $z_{t_2} \neq 0$ , then  $V(z_{t_1}) \geq V(z_{t_2})$ .*



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- (ii) If  $t, t-2 \in J$ ,  $z(t-1) = z(t) = 0$  but  $z_t \neq 0$ , then either  $V(z_t) = \infty$  or  $V(z_{t-2}) > V(z_t)$ .
- (iii) If  $\beta$  is positive on  $J$ ,  $t \in J$ ,  $t-3 \in J$ ,  $z(t) \neq 0$  for some  $t \in J + [-1, 0]$  and  $V(z_{t-3}) = V(z_t) < \infty$ , then  $z_t \in R$ .
- (iv) If  $J = \mathbb{R}$ ,  $\alpha \equiv \mu \in \mathbb{R}$ ,  $\beta$  is bounded and measurable,  $z$  is bounded and  $z_t \neq 0$  for all  $t \in \mathbb{R}$ , then  $V(z_t) < \infty$  for all  $t \in \mathbb{R}$ .

*Proof.* For a positive and continuous  $\beta$  and constant  $\alpha$ , assertions (i), (ii) and (iii) are shown in [26] and [32]. The proof of Lemma VI.2 in [26] can be modified in a straightforward manner to cover our slightly more general case. Therefore the details are omitted here.

Statement (iv) is a corollary of Theorem 2.4 in [32] with  $\delta^* = 1$ ,  $N = 0$ ,  $f^0(t, u, v) = -\mu u + \beta(t)v$ . Property I of Theorem 2.4 in [32] holds as  $\beta$  is bounded.  $\square$

*Remark 2.3.3.* Notice that if  $\beta$  is positive and  $z : J + [-1, 0] \rightarrow \mathbb{R}$  satisfies (2.6) for all  $t \in J$ ,  $t > \inf J$ , moreover  $z(t) \neq 0$  for some  $t \in J + [-1, 0]$ , then  $z_t \neq 0$  for all  $t \in J$ .

If  $f$  is a  $C^1$ -smooth, nondecreasing function and  $x, \hat{x} : J + [-1, 0] \rightarrow \mathbb{R}$  are solutions of Eq. (1.1), then Lemma 2.3.2 (i) and Lemma 2.3.2 (ii) can be applied for  $z = x - \hat{x}$  with the constant function  $\alpha : J \ni t \mapsto \mu \in \mathbb{R}$  and the nonnegative continuous function

$$\beta : J \ni t \mapsto \int_0^1 f'(sx(t-1) + (1-s)\hat{x}(t-1)) ds \in [0, \infty).$$

If  $f'(x) > 0$  for all  $x \in \mathbb{R}$ , then  $\beta$  is positive, which condition is needed in Lemma 2.3.2 (iii).

We introduce the linear map  $\pi : C \rightarrow \mathbb{R}^2$  by  $\pi(\varphi) = (\varphi(0), \varphi(-1))$ . The following proposition holds.

**Proposition 2.3.4.** *Assume  $\mu \in \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing, bounded, and either it is continuously differentiable on  $\mathbb{R}$ , or there exist  $u_1 < u_2 < \dots < u_N$  with  $N \geq 1$  so that the restrictions of  $f$  to the intervals  $(-\infty, u_1]$ ,  $[u_1, u_2]$ , ...,  $[u_{N-1}, u_N]$ ,  $[u_N, \infty)$  are continuously differentiable. Let  $x : \mathbb{R} \rightarrow \mathbb{R}$  and  $\tilde{x} : \mathbb{R} \rightarrow \mathbb{R}$  be different periodic solutions of (1.1). Then  $t \mapsto V(x_t - \tilde{x}_t)$  is finite and constant. Furthermore,  $\pi(x_t - \tilde{x}_t) \neq (0, 0)$  for all  $t \in \mathbb{R}$ .*

*Proof.* The difference  $z = x - \tilde{x}$  satisfies equation (2.6) with  $\alpha(t) \equiv \mu$  and

$$\beta(t) = \begin{cases} \frac{f(x(t-1)) - f(\tilde{x}(t-1))}{x(t-1) - \tilde{x}(t-1)} & \text{if } x(t-1) \neq \tilde{x}(t-1), \\ D^+ f(x(t-1)) & \text{otherwise,} \end{cases}$$

where  $D^+ f$  denotes the right hand side derivative of  $f$ . Then  $\beta$  is bounded, measurable and nonnegative. Clearly,  $z_t \neq 0$  for all  $t \in \mathbb{R}$ . Lemma 2.3.2 (iv) implies  $V(z_t) < \infty$  for all  $t \in \mathbb{R}$ .

## 2 Delay Differential Equations: Theoretical Background

Let  $\omega$  and  $\tilde{\omega}$  denote the minimal periods of  $x$  and  $\tilde{x}$ , respectively. If  $\tilde{\omega} = 0$  or  $\omega/\tilde{\omega}$  is rational, then  $z$  is periodic. Thus Lemma 2.3.2 (i) yields that  $t \mapsto V(z_t)$  is constant. If  $\omega/\tilde{\omega}$  is irrational, then one may choose sequences  $(n_l)_1^\infty \subset \mathbb{Z}$  and  $(k_l)_1^\infty \subset \mathbb{Z}$  with  $n_l \rightarrow \infty$  and  $k_l \rightarrow \infty$  as  $l \rightarrow \infty$  so that  $n_l\omega/\tilde{\omega} - k_l \rightarrow 0$  as  $l \rightarrow \infty$ . Fix  $t \in \mathbb{R}$  arbitrarily. As for all  $s \in [-1, 0]$ ,

$$\begin{aligned} z_{t+n_l\omega}(s) &= x_{t+n_l\omega}(s) - \tilde{x}_{t+n_l\omega}(s) = x_t(s) - \tilde{x}_{t+n_l\omega-k_l\tilde{\omega}}(s) \\ &= x(t+s) - \tilde{x}\left(t + \tilde{\omega}\left(n_l\frac{\omega}{\tilde{\omega}} - k_l\right) + s\right), \end{aligned}$$

we see that  $z_{t+n_l\omega}(s)$  tends to  $z_t(s) = x(t+s) - \tilde{x}(t+s)$  as  $l \rightarrow \infty$  uniformly in  $s \in [-1, 0]$ . So Lemma 2.3.1 implies  $V(z_t) \leq \liminf_{l \rightarrow \infty} V(z_{t+n_l\omega})$  for all  $l \geq 0$ . As  $\mathbb{R} \ni u \mapsto V(z_u) \in 2\mathbb{N} \cup \{\infty\}$  is monotone nonincreasing by Lemma 2.3.2 (i), we obtain that  $V(z_t) = V(z_{t+u})$  for all  $u \geq 0$ . As  $t$  is arbitrary, we conclude that  $t \mapsto V(z_t)$  is constant.

The second statement now follows from Lemma 2.3.2 (ii).  $\square$

We mention that in the negative feedback case  $V(\varphi)$  counts the sign changes of  $\varphi \in C \setminus \{0\}$  if it is an odd number or infinity, otherwise  $V(\varphi)$  is the number of sign changes plus one. Then  $V(\varphi) \in \{1, 3, \dots\} \cup \{\infty\}$ . The analogue of Lemma 2.3.2 holds, in particular the map  $t \mapsto V(x_t)$  is monotone nonincreasing along the solutions of Eq. (1.1).

### 2.3.2 Poincaré return maps

Assume that  $\mu \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  in Eq. (1.1) is continuously differentiable. Let  $p : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic solution of Eq. (1.1), and  $\omega > 1$  be the minimal period of  $p$ . Let a closed linear subspace  $H \subset C$  of codimension 1 be given so that  $p_0 \in H$  and  $\dot{p}_0 \notin H$ . An application of the implicit function theorem yields a convex bounded open neighborhood  $N$  of 0 in  $H$ ,  $\nu \in (0, \omega)$  and a  $C^1$ -map  $\gamma : \{p_0\} + N \rightarrow (\omega - \nu, \omega + \nu)$  with  $\gamma(p_0) = \omega$  so that for each  $(t, \varphi) \in (\omega - \nu, \omega + \nu) \times (\{p_0\} + N)$ , segment  $x_t^\varphi$  belongs to  $H$  if and only if  $t = \gamma(\varphi)$  ([7], Appendix I in [26], [28]). The *Poincaré return map* is set

$$P : \{p_0\} + N \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in H.$$

Then  $P$  is continuously differentiable and has fixed point  $p_0$ . In addition,  $P$  depends smoothly on the right hand side of Eq. (1.1) [28].

Map  $DP(p_0) : H \rightarrow H$  is a compact operator. The spectrum  $\sigma$  of  $DP(p_0)$  is countable with one possible accumulation point at 0. All the nonzero points in  $\sigma$  are eigenvalues of finite multiplicity. The periodic orbit determined by solution  $p$  is said to be *hyperbolic* if  $p_0$  is a hyperbolic fixed point of  $P$ , that is  $DP(p_0)$  has no eigenvalues on the unit circle in  $\mathbb{C}$ . This hyperbolicity is the same as the one defined by the spectrum

of the monodromy operator [7, 26]. The nonzero points of  $\sigma$  and 1 are called *Floquet multipliers*.

The following proposition is a particular case of a more general result of Lani-Wayda [28]. It states that if Eq. (1.1) admits a hyperbolic periodic solution with minimal period greater than the delay, then small perturbations preserve the periodic solution.

**Theorem 2.3.5.** *Assume that  $f \in C^1(\mathbb{R}, \mathbb{R})$ , and  $p$  is a hyperbolic periodic solution of Eq. (1.1) with minimal period  $\omega > 1$ . Let  $D \subset \mathbb{R}$  be open with  $\{p(t) : t \in [0, \omega)\} \subset D$ . Then there exist an open ball  $B \subset C_b^1(D, \mathbb{R})$  centered at  $f$ , an open neighborhood  $V \subset N$  of 0 in  $H$  and a  $C^1$ -function  $\chi : B \rightarrow \{p_0\} + V \subset H$  with  $\chi(f) = p_0$  such that for  $g \in B$ , the solution  $x^{x(g)}$  of Eq. (1.1) with initial value  $\chi(g)$  is periodic (and therefore can be defined on  $\mathbb{R}$ ). The minimal period of  $x^{x(g)}$  is in  $(\omega - \nu, \omega + \nu)$ . If  $\varphi \in \{p_0\} + V$  is the initial segment of any periodic solution of  $\dot{x}(t) = -\mu x(t) + g(x(t-1))$  for some  $g \in B$  with minimal period in  $(\omega - \nu, \omega + \nu)$ , then  $\varphi = \chi(g)$ . If  $\|g - f\|_{C_b^1} \rightarrow 0$ , then  $\chi(g) \rightarrow \chi(f) = p_0$  in  $C$ .*

We apply this theorem in Chapter 3 with  $D = \mathbb{R}$ .

There are other standard techniques applying Poincaré return maps to detect periodic orbits. For example suppose that  $f$  in Eq. (1.1) is continuous,  $A$  is a nonempty, closed, convex subset of  $C$ , and a map  $P : A \rightarrow C$  is defined so that for all  $\varphi \in C$ ,  $P(\varphi) = \Phi(q, \varphi)$  with some  $q = q(\varphi) > 0$ . If  $P(A) \subset A$  and  $P$  is a strict contraction, then  $P$  has a fixed point, the initial segment of a periodic solution. If  $f$  is continuously differentiable, then the periodic orbit is necessarily hyperbolic and stable [46]. This argument is used in Chapter 5.

A third type of reasoning is presented in Chapter 6. To verify the existence of a periodic orbit in the closure  $\overline{\mathcal{W}_1^u}$  of the leading unstable set of an equilibrium, we project  $\overline{\mathcal{W}_1^u}$  to the 2-dimensional plane and define a suitable Poincaré return map on the image of  $\overline{\mathcal{W}_1^u}$ . Using the discrete Lyapunov functional and elementary topological arguments, we confirm that this Poincaré map has a fixed point, which implies the existence of a periodic orbit.

## 2.4 Notions and notations

Symbols  $\mathbb{R}$  and  $\mathbb{R}^+$  stand for the set of reals and nonnegative reals, respectively.  $\mathbb{Z}$  and  $\mathbb{N}$  denote the set of integers and positive integers, respectively.

$C$  is the Banach space of all real valued continuous functions defined on  $[-1, 0]$  with supremum norm  $\|\cdot\|$ . In addition,  $C^1$  is the space of all real valued continuously differentiable functions on  $[-1, 0]$  with norm  $\|\varphi\|_{C^1} = \|\varphi\| + \|\varphi'\|$ .

For  $D \subseteq \mathbb{R}$  open,  $C_b^1(D, \mathbb{R})$  denotes the space of bounded continuously differentiable functions  $g : D \rightarrow \mathbb{R}$  with bounded first derivative together with norm  $\|g\|_{C_b^1} = \sup_{x \in D} |g(x)| + \sup_{x \in D} |g'(x)|$ .

## 2 Delay Differential Equations: Theoretical Background

For Banach spaces  $E$  and  $F$  over  $\mathbb{R}$ , the space of bounded linear operators is denoted by  $\mathcal{L}(E, F)$ .

For  $\varphi, \psi \in C$ , we define

- $\varphi \leq \psi$  if  $\varphi(s) \leq \psi(s)$  for all  $s \in [-1, 0]$ ,
- $\varphi < \psi$  if  $\varphi \leq \psi$  and  $\varphi \neq \psi$ ,
- $\varphi \prec \psi$  if  $\varphi \leq \psi$  and  $\varphi(0) < \psi(0)$ ,
- $\varphi \ll \psi$  if  $\varphi(s) < \psi(s)$  for all  $s \in [-1, 0]$ .

Relations “ $\geq$ ”, “ $>$ ”, “ $\succ$ ” and “ $\gg$ ” are defined analogously.

For a simple closed curve  $c : [a, b] \rightarrow \mathbb{R}^2$ ,  $\text{int}(c)$  and  $\text{ext}(c)$  stand for the interior and exterior, i.e., the bounded and unbounded component of  $\mathbb{R}^2 \setminus c([a, b])$ , respectively.

If  $U$  is a subset of a topological space, then  $\text{bd}U$  is for the boundary of  $U$ ,  $\text{int}U$  is for the interior of  $U$ , and  $\overline{U}$  is for the closure of  $U$ .

For an interval  $I \subset \mathbb{R}$ , we define

$$I + [-1, 0] = \{t \in \mathbb{R} : t = t_1 + t_2 \text{ with } t_1 \in I, t_2 \in [-1, 0]\}.$$

If  $\xi \in \mathbb{R}$  is a zero of  $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$ , then a solution  $x : [-1, \infty) \rightarrow \mathbb{R}$  of Eq. (1.1) *oscillates* around  $\xi$  if the set of zeros of  $x - \xi$  is not bounded from above.

In the positive feedback case (i.e. when  $f$  is continuous and  $xf(x) > 0$  for  $x \neq 0$ ) a solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is called *slowly oscillatory* around  $\xi$  if  $V(x_t - \hat{\xi}) = 2$  for each  $t \in \mathbb{R}$ , where  $\hat{\xi}(s) = \xi$  for  $s \in [-1, 0]$ . A solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is *rapidly oscillatory* around  $\xi$  if  $V(x_t - \hat{\xi}) \geq 4$  for all  $t \in \mathbb{R}$ . Note that slow oscillation in the positive feedback case is different from the usual one used for equations with negative feedback condition [7, 47]. In the negative feedback case (i.e. when  $f$  is continuous and  $xf(x) < 0$  for  $x \neq 0$ ) a solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is called *slowly oscillatory* around  $\xi$  if the successive sign changes of  $x - \xi$  are spaced at distances larger than the delay 1. In both cases a slowly oscillatory solution is defined to be slowly oscillatory around 0. Slowly oscillatory periodic solutions are abbreviated as *SOP solutions*.

Assume  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a periodic solution of (1.1) with minimal period  $\omega$ . We say  $x$  is of *special symmetry* if relation  $x(t + \omega/2) = -x(t)$  holds for all  $t \in \mathbb{R}$ . Set  $t_0 < t_1 < t_0 + \omega$  so that  $x(t_0) = \min_{t \in \mathbb{R}} x(t)$  and  $x(t_1) = \max_{t \in \mathbb{R}} x(t)$ . Solution  $x$  is said to be of *monotone type* if  $x$  is nondecreasing on  $[t_0, t_1]$  and nonincreasing on  $[t_1, t_0 + \omega]$ .

Assume that 0 is in the range of a periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of (1.1). Then  $x$  is *normalized* if  $x(-1) = 0$  and  $x(s) > 0$  for all  $s \in (-1, -1 + \eta)$  with some  $\eta > 0$ .

## 3 Large-Amplitude Periodic Solutions for Monotone Positive Feedback

### 3.1 Introduction to the problem

In this chapter we consider the equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \quad (1.1)$$

and assume that the following hypothesis holds (see Fig. 3.1):

(H1)  $\mu > 0$ ,  $f \in C^1(\mathbb{R}, \mathbb{R})$  with  $f'(\xi) > 0$  for all  $\xi \in \mathbb{R}$ , and

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

are five consecutive zeros of  $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$  with  $f'(\xi_j) < \mu$  for  $j \in \{-2, 0, 2\}$  and  $f'(\xi_k) > \mu$  for  $k \in \{-1, 1\}$ .

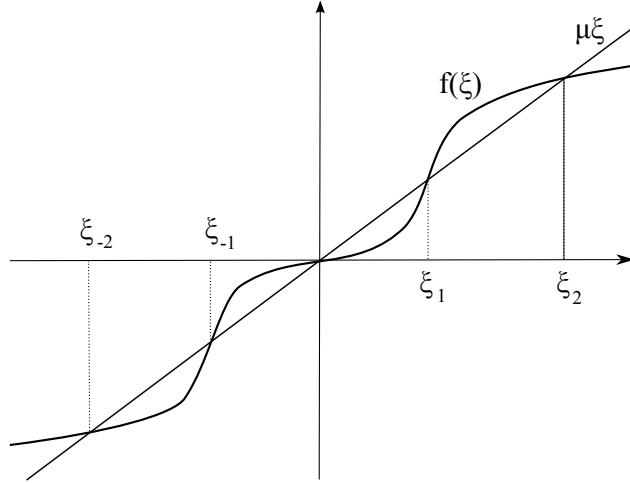


Figure 3.1: A feedback function satisfying condition (H1)

Under hypothesis (H1),  $\hat{\xi}_j \in C$  defined by  $\hat{\xi}_j(s) = \xi_j$ ,  $-1 \leq s \leq 0$ , is an equilibrium point of  $\Phi$  for  $j \in \{-2, -1, 0, 1, 2\}$ . In addition,  $\hat{\xi}_{-2}, \hat{\xi}_0, \hat{\xi}_2$  are stable and  $\hat{\xi}_{-1}, \hat{\xi}_1$  are unstable.

### 3 Large-Amplitude Periodic Solutions for Monotone Positive Feedback

By the monotone property of  $f$ , the subsets

$$C_{-2,2} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\},$$

$$C_{-2,0} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq 0 \text{ for all } s \in [-1, 0]\},$$

$$C_{0,2} = \{\varphi \in C : 0 \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\}$$

of the phase space  $C$  are positively invariant under the semiflow  $\Phi$ . The structures of the global attractors  $\mathcal{A}_{-2,0}$  and  $\mathcal{A}_{0,2}$  of the restrictions  $\Phi|_{[0,\infty) \times C_{-2,0}}$  and  $\Phi|_{[0,\infty) \times C_{0,2}}$ , respectively, are (at least partially) well understood, see [20, 22, 23, 25, 26, 27] and Section 2.2. In particular cases,  $\mathcal{A}_{-2,0}$  and  $\mathcal{A}_{0,2}$  have spindle-like structures described in [20, 25, 26, 27]:  $\mathcal{A}_{0,2}$  is the closure of the unstable set of  $\hat{\xi}_1$  containing the equilibrium points  $\hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2$ , periodic orbits in  $C_{0,2}$  and heteroclinic orbits among them; and analogously for  $\mathcal{A}_{-2,0}$ .

Let  $\mathcal{A}$  denote the global attractor of the restriction  $\Phi|_{[0,\infty) \times C_{-2,2}}$ . It is easy to see that if (H1) holds and  $\xi_{-2}, \xi_{-1}, 0, \xi_1, \xi_2$  are the only zeros of  $-\mu\xi + f(\xi)$ , then  $\mathcal{A}$  is the global attractor of  $\Phi$ . The problem, whether under hypothesis (H1) the equality

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \tag{3.1}$$

holds or not, arose in [26], see Fig. 3.2.

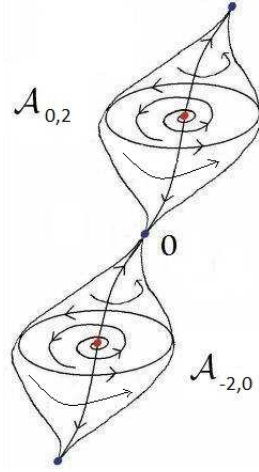


Figure 3.2:  $\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$

The main result of this chapter is that  $\mathcal{A}$  can be more complicated than given by (3.1). We construct examples so that Eq. (1.1) with assumption (H1) has periodic orbits in  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ .

A periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) with nonlinearity satisfying (H1) is called a large amplitude periodic solution if  $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ . As we have defined before, a solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is slowly oscillatory if for each  $t$ , the restriction  $x|_{[t-1, t]}$  has one or

### 3.1 Introduction to the problem

two sign changes. A large-amplitude slowly oscillatory periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  will be abbreviated as an LSOP solution. An LSOP solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is normalized if  $x(-1) = 0$ , and for some  $\eta > 0$ ,  $x(s) > 0$  for all  $s \in (-1, -1 + \eta)$ .

**Theorem 3.1.1.** *There exist  $\mu$  and  $f$  satisfying (H1) such that Eq. (1.1) has exactly two normalized LSOP solutions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$ . For the ranges of  $p$  and  $q$ ,  $p(\mathbb{R}) \subsetneq q(\mathbb{R})$  holds. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

*are hyperbolic and unstable with 2 and 1 Floquet multipliers outside the unit circle, respectively.*

In Theorem 3.1.1 the nonlinear map  $f$  is close to the step function  $f^{K,0}$  parametrized by  $K > 0$  and given by  $f^{K,0}(x) = 0$  for  $|x| \leq 1$ , and  $f^{K,0}(x) = K \operatorname{sgn}(x)$  for  $|x| > 1$ , see Fig 3.3. Equations with such nonlinearity model neural networks of identical neurons that do not react upon small feedback; the feedback has to reach a certain threshold value to have a considerable effect [12]. Our result may have interesting consequences for the dynamics of neural networks with the above property. See [3, 4, 5, 6, 50] for a bistable situation.

Suppose  $f$  is odd and satisfies (H1). It follows from results in [33] that if  $x : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution of Eq. (1.1) with minimal period  $\omega > 0$ , then the following statements hold.

- (i) The minimal period  $\omega$  belongs to interval  $(1, 2)$ .
- (ii) Solution  $x$  is of special symmetry meaning that relation  $x(t + \omega/2) = -x(t)$  holds for all  $t \in \mathbb{R}$ .
- (iii) Solution  $x$  is of monotone type in the following sense: if  $t_0 < t_1 < t_0 + \omega$  is set so that  $x(t_0) = \min_{t \in \mathbb{R}} x(t)$  and  $x(t_1) = \max_{t \in \mathbb{R}} x(t)$ , then  $x$  is nondecreasing on  $[t_0, t_1]$  and nonincreasing on  $[t_1, t_0 + \omega]$ .

This motivates the next definition. We say a periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) with feedback function  $f^{K,0}$ ,  $K > 0$ , is an LSOP solution if properties (i), (ii) and (iii) hold for  $x$ .

For Eq. (1.1) with  $\mu = 1$  and  $f = f^{K,0}$ , the LSOP solutions are described in Theorem 3.5.5: there is no such solution if  $K < K^* \approx 6.8653$  and there are exactly two for  $K > K^*$  (up to time translation). It can be also verified that there is exactly one LSOP solution for  $K = K^*$ . This is the starting point of our construction. The implicit function theorem and perturbations of Poincaré maps from [28] can be applied to find exactly two LSOP orbits of Eq. (1.1) for  $\mu = 1$  and nonlinearities that satisfy (H1) and are close to  $f^{K,0}$  with  $K > K^*$ . We verify only the case  $K = 7$ , which suffices for the proof of Theorem 3.1.1. Our results and numerical examples suggest that the LSOP orbits appear in a saddle-node-like bifurcation. However, it remains an open

### 3 Large-Amplitude Periodic Solutions for Monotone Positive Feedback

problem to understand this phenomenon.

The chapter is organized as follows.

Section 3.2 introduces a smooth approximation  $f^{K,\varepsilon}$ ,  $\varepsilon \in [0, 1)$ , of the step function  $f^{K,0}$ . The notion of LSOP solutions is extended for a slightly wider range of feedback functions including  $f^{K,\varepsilon}$ ,  $\varepsilon \in [0, 1)$ . Fix  $K > 3$ . We define an open set  $U^1$  in  $(0, 1)^3 \times [0, 1)$  and a continuous map  $\Sigma : U^1 \rightarrow C$  so that for  $\varepsilon > 0$  small,  $U_\varepsilon^1 \ni a \mapsto \Sigma(a, \varepsilon) \in C$  is smooth and injective (see Proposition 3.2.7), where  $U_\varepsilon^1$  denotes the set  $\{a \in (0, 1)^3 : (a, \varepsilon) \in U^1\}$ . Consequently,  $\Sigma(U_\varepsilon^1 \times \{\varepsilon\})$  is a 3-dimensional  $C^1$ -submanifold of  $C$ . There exists an open subset  $U^3$  of  $U^1$  such that if  $\mu = 1$  and  $f = f^{K,\varepsilon}$ , then for all  $(a, \varepsilon) \in U^3$ , the solution  $x^{\Sigma(a, \varepsilon)} : [-1, \infty) \rightarrow \mathbb{R}$  of Eq. (1.1) returns into  $\Sigma(U_\varepsilon^1 \times \{\varepsilon\})$ , i.e., there exists  $t > 0$  with  $x_t^{\Sigma(a, \varepsilon)} \in \Sigma(U_\varepsilon^1 \times \{\varepsilon\})$ . This induces a smooth map  $F : U^3 \rightarrow \mathbb{R}^3$  so that for all  $(a, \varepsilon) \in U^3$ , we have  $F(a, \varepsilon) = b$  if  $x_t^{\Sigma(a, \varepsilon)} = \Sigma(b, \varepsilon)$  for some  $t > 0$ . If  $F(a, \varepsilon) = a$  holds for some  $(a, \varepsilon) \in U^3$ , then the solution  $x^{\Sigma(a, \varepsilon)}$  of Eq. (1.1) with  $\mu = 1$  and  $f = f^{K,\varepsilon}$  is an LSOP solution. Therefore the problem of finding LSOP solutions is reduced to a 3-dimensional fixed point equation depending on parameter  $\varepsilon$ . Proposition 3.2.8 shows that there is  $K^* \approx 6.8653$  so that for  $K > K^*$ , equation  $F(a, 0) = a$  has a unique solution  $a^*$  in  $U_0^3 = \{a \in (0, 1)^3 : (a, 0) \in U^3\}$ . The fixed point  $a^*$  is hyperbolic; it is rigorously checked for  $K = 7$ . Then the implicit function theorem gives that if  $K = 7$ , then equation  $F(a, \varepsilon) = a$  has a solution  $a^*(\varepsilon)$  in  $U_\varepsilon^3 = \{a \in (0, 1)^3 : (a, \varepsilon) \in U^3\}$  for small  $\varepsilon > 0$  so that  $D_a F(a^*(\varepsilon), \varepsilon)$  is hyperbolic. Analogously to the above construction, Subsection 3.2.2 gives another LSOP solution of (1.1) with  $\mu = 1$  and  $f = f^{7,\varepsilon}$  for  $\varepsilon > 0$  small.

Other examples, in which the problem of finding periodic solutions is reduced to a finite dimensional fixed point problem, are found e.g in [28, 44, 46]. However, the corresponding return maps in [44, 46] are contractions, and the obtained periodic orbits are stable. This is not the case here, thus we cannot apply any contraction mapping theorem.

Section 3.3 shows that the hyperbolicity of the fixed points of the 3-dimensional maps of Section 3.2 guarantees the hyperbolicity of the corresponding LSOP orbits of Eq. (1.1) with  $\mu = 1$  and  $f = f^{7,\varepsilon}$ ,  $\varepsilon > 0$  small, see Proposition 3.3.3. The key fact toward the proof is that a small neighborhood of the fixed point  $\Sigma(a^*(\varepsilon), \varepsilon)$  in a hyperplane of  $C$  is mapped into the 3-dimensional submanifold  $\Sigma(U_\varepsilon^3 \times \{\varepsilon\})$  by a suitable Poincaré return map (Proposition 3.3.1). The hyperbolicity of these LSOP orbits together with a result in [28] guarantee the existence of LSOP solutions for all nonlinearities  $f$  satisfying (H1) that are close to  $f^{7,\varepsilon}$ ,  $\varepsilon > 0$  small, in  $C^1$ -norm. Thereby the existence of the two LSOP solutions in Theorem 3.1.1 is verified.

Section 3.4 contains preparatory results toward the exact number of LSOP solutions. Propositions 3.4.1 and 3.4.2 prove monotone and symmetry properties of periodic solutions of (1.1). The  $C^1$ -smoothness and strict monotonicity from [33] is weakened



slightly. The technical result of Proposition 3.4.3 shows that all LSOP solutions of (1.1) with  $\mu = 1$  and  $f = f^{7,\varepsilon}$ ,  $\varepsilon > 0$  small, have nice regulatory properties.

Section 3.5 studies the exact number of LSOP solutions for the step function  $f^{K,0}$  with  $K > 0$ , then for  $f^{7,\varepsilon}$  with  $\varepsilon > 0$  small, and finally for functions  $f$  close to  $f^{7,\varepsilon}$ . Summarizing the above results, Theorem 3.1.1 is obtained.

All numerical approximations presented in this chapter are generated with the aid of the CAPD program [1] using rigorous numerics. The author thanks Ferenc Bartha for giving these numerical results.

## 3.2 LSOP solutions for special nonlinearities

In the remaining part of the chapter we fix  $\mu = 1$ . The results can be easily modified for different values of  $\mu > 0$ .

Let  $\rho : \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$ -smooth function such that  $\rho(t) = 0$  for  $t \leq 0$ ,  $\rho(t) = 1$  for  $t \geq 1$  and  $\rho'(t) > 0$  for  $t \in (0, 1)$ . For given  $K > 0$  and  $\varepsilon \in (0, 1)$ , define  $f^{K,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$  (Fig. 3.3) by

$$f^{K,\varepsilon}(x) = K\rho\left(\frac{|x| - 1}{\varepsilon}\right) \operatorname{sgn}(x).$$

The function  $f^{K,0} : \mathbb{R} \rightarrow \mathbb{R}$  (Fig. 3.3) is given by

$$f^{K,0}(x) = \lim_{\varepsilon \rightarrow 0+} f^{K,\varepsilon}(x) = \begin{cases} -K & \text{if } x < -1, \\ 0 & \text{if } |x| \leq 1, \\ K & \text{if } x > 1. \end{cases}$$

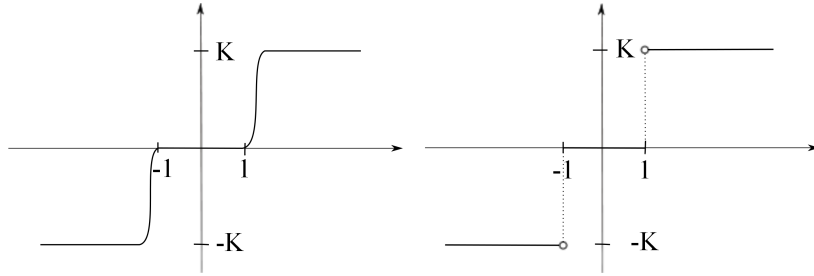


Figure 3.3: Plot of  $f^{K,\varepsilon}$  for  $\varepsilon > 0$  small and for  $\varepsilon = 0$

Consider the delay differential equation

$$\dot{x}(t) = -x(t) + f^{K,\varepsilon}(x(t-1)). \quad (3.2)$$

Set  $J_i^\varepsilon = (f^{K,\varepsilon})^{-1}(i)$  for  $i \in \{-K, 0, K\}$ .

If  $t_0 < t_1$  and  $x : [t_0 - 1, t_1] \rightarrow \mathbb{R}$  is a solution of Eq. (3.2) such that for some  $i \in \{-K, 0, K\}$ , we have  $x(t-1) \in J_i^\varepsilon$  for all  $t \in (t_0, t_1)$ , then Eq. (3.2) reduces to the

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ordinary differential equation

$$\dot{x}(t) = -x(t) + i$$

on the interval  $(t_0, t_1)$ , and thus

$$x(t) = i + (x(t_0) - i) e^{-(t-t_0)}, \quad t \in [t_0, t_1]. \quad (3.3)$$

We say that a function  $x : [t_0, t_1] \rightarrow \mathbb{R}$  is *of type (i) on  $[t_0, t_1]$*  for some  $i \in \{-K, 0, K\}$  if (3.3) holds. If  $x : [t_0 - 1, t_1] \rightarrow \mathbb{R}$  is a solution of Eq. (3.2) so that  $x$  is of type (i) on  $[t_0 - 1, t_1 - 1]$  for some  $i \in \{-K, 0, K\}$ , then with  $j = x(t_0 - 1)$  the equality

$$x(t) = x(t_0) e^{-(t-t_0)} + \int_0^{t-t_0} e^{-(t-t_0-s)} f^{K,\varepsilon}(i + (j - i) e^{-s}) ds \quad (3.4)$$

holds for all  $t \in [t_0, t_1]$ . This motivates the next definition. A function  $x : [t_0, t_1] \rightarrow \mathbb{R}$  is *of type (i, j) on  $[t_0, t_1]$*  with  $i \in \{-K, 0, K\}$  and  $j \in \mathbb{R}$  if (3.4) holds for all  $t \in [t_0, t_1]$ .

In the rest of the section assume that  $K > 3$ .

Let

$$T(\varepsilon) = \ln(1 + \varepsilon), \quad \hat{T}(\varepsilon) = \ln \frac{K - 1}{K - 1 - \varepsilon}, \quad \tilde{T}(\varepsilon) = \ln \frac{K + 1 + \varepsilon}{K + 1}$$

denote the times that a function of type (0) needs to decrease from  $1 + \varepsilon$  to 1 or to increase from  $-1 - \varepsilon$  to  $-1$ , a function of type  $(-K)$  needs to decrease from  $-1$  to  $-(1 + \varepsilon)$ , a function of type  $(-K)$  needs to decrease from  $1 + \varepsilon$  to 1, respectively. Clearly,  $T(0) = \hat{T}(0) = \tilde{T}(0) = 0$ .

We extend the notion of LSOP solutions to feedback functions that are not strictly monotone, in particular for  $f^{K,\varepsilon}$ ,  $\varepsilon > 0$ . In case  $f \in C^1(\mathbb{R}, \mathbb{R})$  with  $f'(\xi) \geq 0$  for all  $\xi \in \mathbb{R}$ , and  $\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$  are five consecutive zeros of  $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$ , a periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) is called a large amplitude periodic solution if  $x(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ . A large-amplitude slowly oscillatory periodic solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  is abbreviated as an LSOP solution.

Recall that this definition is modified for the step function  $f^{K,0}$  in the following way. Solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) with nonlinearity  $f = f^{K,0}$ ,  $K > 0$ , is a large-amplitude slowly oscillatory periodic (LSOP) solution if  $x$  is of monotone type, special symmetry, and the minimal period of  $x$  is in the open interval  $(1, 2)$ .

#### 3.2.1 The first construction

Define

$$U^1 = \left\{ (a, \varepsilon) \in (0, 1)^3 \times [0, 1) : a = (a_1, a_2, a_3), \quad a_1 + a_2 + a_3 + 2T(\varepsilon) + \hat{T}(\varepsilon) < 1 \right\}.$$

It is easy to see that  $U^1$  is open in  $(0, 1)^3 \times [0, 1)$ .

For given  $(a, \varepsilon) \in U^1$ , set

$$\begin{aligned}
 s_0 &= -1, \\
 s_1 &= s_0 + a_1 = -1 + a_1, \\
 s_1^* &= s_1 + T(\varepsilon) = -1 + a_1 + T(\varepsilon), \\
 s_2 &= s_1^* + a_2 = -1 + a_1 + T(\varepsilon) + a_2, \\
 s_2^* &= s_2 + \hat{T}(\varepsilon) = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon), \\
 s_3 &= s_2^* + a_3 = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3, \\
 s_3^* &= s_3 + T(\varepsilon) = -1 + a_1 + T(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3 + T(\varepsilon).
 \end{aligned}$$

Clearly  $s_i = s_i^*$ ,  $i \in \{1, 2, 3\}$ , for  $\varepsilon = 0$ .

Define  $h = h(a, \varepsilon) : \mathbb{R} \rightarrow \mathbb{R}$  (Fig. 3.4) by

$$h(t) = \begin{cases} K & \text{if } t < s_1, \\ f^{K, \varepsilon}((1 + \varepsilon) e^{-(t-s_1)}) & \text{if } s_1 \leq t < s_1^*, \\ 0 & \text{if } s_1^* \leq t < s_2, \\ f^{K, \varepsilon}(-K + (K - 1) e^{-(t-s_2)}) & \text{if } s_2 \leq t < s_2^*, \\ -K & \text{if } s_2^* \leq t < s_3, \\ f^{K, \varepsilon}(-(1 + \varepsilon) e^{-(t-s_3)}) & \text{if } s_3 \leq t < s_3^*, \\ 0 & \text{if } s_3^* \leq t. \end{cases}$$

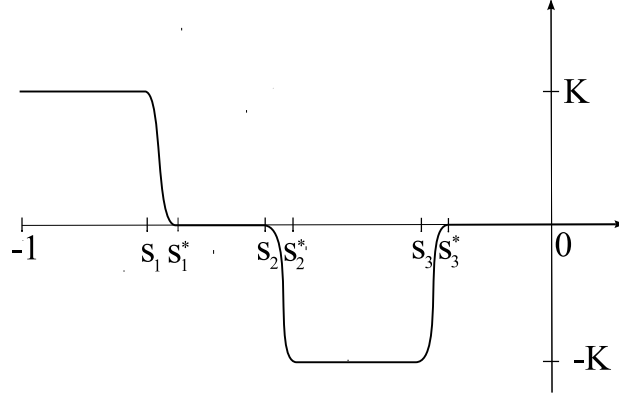


Figure 3.4: Function  $h(a, \varepsilon)$

Define the map  $\Sigma : U^1 \rightarrow C$  by

$$\Sigma(a, \varepsilon)(t) = e^{-t} \int_{-1}^t e^s h(a, \varepsilon)(s) ds \quad (-1 \leq t \leq 0). \quad (3.5)$$

We look for initial segments of LSOP solutions in the set  $\Sigma(U^1) \subset C$ .

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Notice that  $\Sigma(a, \varepsilon)$  is the unique solution of the initial value problem

$$\begin{cases} \dot{y}(t) = -y(t) + h(a, \varepsilon)(t) & (-1 \leq t \leq 0) \\ y(-1) = 0. \end{cases} \quad (3.6)$$

**Proposition 3.2.1.**  $\Sigma : U^1 \rightarrow C$  is continuous.

*Proof.* The continuity of the map  $U^1 \ni (a, \varepsilon) \mapsto h(a, \varepsilon)|_{[-1, 0]} \in L^1(0, 1)$  follows in a straightforward way from the definition of  $h(a, \varepsilon)$ . Applying formula (3.5), the continuity of  $\Sigma$  is obvious.  $\square$

For each fixed  $(a, \varepsilon) \in U^1 \cap (0, 1)^3 \times (0, 1)$ , the map  $[-1, 0] \ni t \mapsto h(a, \varepsilon)(t) \in \mathbb{R}$  is  $C^1$ -smooth with derivative  $h'(a, \varepsilon)(t)$ .

For given  $\varepsilon \in [0, 1)$ , define

$$U_\varepsilon^1 = \left\{ a \in (0, 1)^3 : (a, \varepsilon) \in U^1 \right\}.$$

Proposition 3.2.1 implies that  $U_\varepsilon^1$  is open.

If  $a \in U_\varepsilon^1$  and  $|\delta| < \frac{1}{2} \min \{a_1, a_2, a_3\}$ , then

$$\begin{aligned} h(a_1 + \delta, a_2, a_3, \varepsilon)(t) &= h(a, \varepsilon)(t - \delta) \quad \text{for } t \in [-1, 0], \\ h(a_1, a_2 + \delta, a_3, \varepsilon)(t) &= \begin{cases} h(a, \varepsilon)(t) & \text{for } t \in [-1, s_1^* + \frac{a_2}{2}] , \\ h(a, \varepsilon)(t - \delta) & \text{for } t \in [s_1^* + \frac{a_2}{2}, 0] , \end{cases} \\ h(a_1, a_2, a_3 + \delta, \varepsilon)(t) &= \begin{cases} h(a, \varepsilon)(t) & \text{for } t \in [-1, s_2^* + \frac{a_3}{2}] , \\ h(a, \varepsilon)(t - \delta) & \text{for } t \in [s_2^* + \frac{a_3}{2}, 0] . \end{cases} \end{aligned}$$

Now it is clear that we have

$$\frac{\partial}{\partial a_i} h(a, \varepsilon)(t) = \begin{cases} 0 & \text{for } t \in [-1, s_i] \\ -h'(a, \varepsilon)(t) & \text{for } t \in [s_i, 0] \end{cases}$$

for  $i \in \{1, 2, 3\}$ . Define  $\psi_i \in C$ ,  $i \in \{1, 2, 3\}$ , by

$$\psi_i(t) = \psi_i(a, \varepsilon)(t) = e^{-t} \int_{-1}^t e^s \frac{\partial}{\partial a_i} h(a, \varepsilon)(s) ds \quad (t \in [-1, 0]).$$

Obviously  $\psi_1$ ,  $\psi_2$  and  $\psi_3$  are linearly independent elements of  $C$ . With the above notation, we obtain the following  $C^1$ -smoothness property of  $\Sigma$ .

**Proposition 3.2.2.** For each fixed  $\varepsilon \in (0, 1)$ , the map  $U_\varepsilon^1 \ni a \mapsto \Sigma(a, \varepsilon) \in C$  is  $C^1$ -smooth with  $D_a \Sigma(a, \varepsilon)(b) = b_1 \psi_1 + b_2 \psi_2 + b_3 \psi_3$  for all  $a \in U_\varepsilon^1$  and  $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ .

*Proof.*  $\Sigma(a, \varepsilon)$  is the unique solution of the initial value problem (3.6). Hence the claim of the proposition follows from the differentiability of solutions of ordinary differential equations with respect to the parameters.  $\square$

Let

$$U^2 = \left\{ (a, \varepsilon) \in U^1 : \begin{aligned} &\Sigma(a, \varepsilon)(s) > 1 + \varepsilon \text{ for } s \in [s_1, s_1^*], \\ &|\Sigma(a, \varepsilon)(s)| < 1 \text{ for } s \in [s_2, s_2^*], \\ &\Sigma(a, \varepsilon)(s) < -1 - \varepsilon \text{ for } s \in [s_3, s_3^*] \end{aligned} \right\}.$$

Proposition 3.2.1 and the definition of  $U^2$  imply that  $U^2$  is open in  $(0, 1)^3 \times [0, 1)$ .

For  $(a, \varepsilon) \in U^2$ , consider the solution  $x = x^{(a, \varepsilon)} = x^{\Sigma(a, \varepsilon)} : [-1, \infty) \rightarrow \mathbb{R}$  of Eq. (3.2).

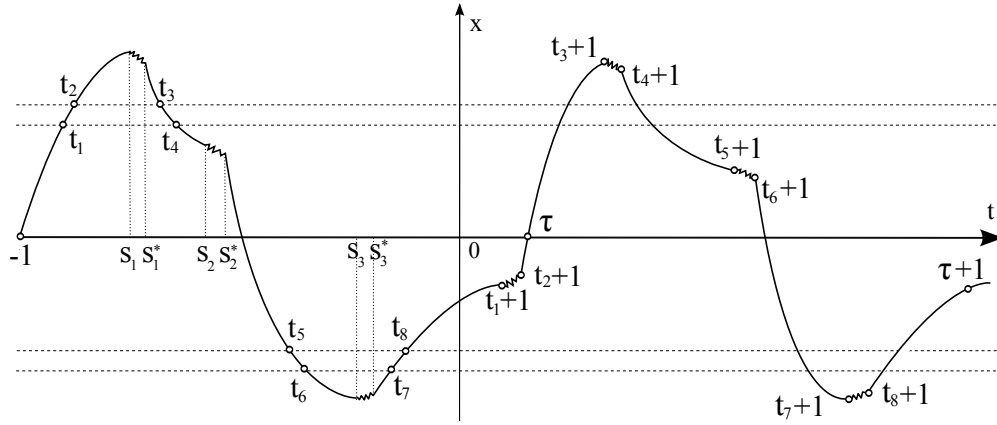


Figure 3.5: Solution  $x^{\Sigma(a,\varepsilon)}$  of Eq. (3.2)

By the definition of  $U^2$ , there exist  $t_1, t_2, \dots, t_6$  in  $[-1, 0]$  such that

$$-1 < t_1 \leq t_2 < s_1 \leq s_1^* < t_3 \leq t_4 < s_2 \leq s_2^* < t_5 \leq t_6 < s_3 \leq s_3^*$$

and

$$x(t_1) = 1, \ x(t_2) = 1 + \varepsilon, \ x(t_3) = 1 + \varepsilon, \ x(t_4) = 1, \ x(t_5) = -1, \ x(t_6) = -1 - \varepsilon,$$

see Fig. 3.5.

For  $\varepsilon \in (0, 1)$ , introduce

$$c_1(\varepsilon) = \int_0^{T(\varepsilon)} e^s f^{K,\varepsilon}((1+\varepsilon)e^{-s}) \, ds,$$

$$c_2(\varepsilon) = \int_0^{\hat{T}(\varepsilon)} e^s f^{K,\varepsilon}(K - (K-1)e^{-s}) \, ds.$$

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These integrals appear in the explicit evaluation of a return map. Observe that

$$c_1(\varepsilon) = \int_0^1 \frac{\varepsilon(1+\varepsilon)}{(1+\varepsilon u)^2} K \rho(u) du \quad \text{for } \varepsilon \in (0, 1),$$

and

$$c_2(\varepsilon) = \int_0^1 \frac{\varepsilon}{(K-1-\varepsilon u)^2} K(K-1) \rho(u) du \quad \text{for } \varepsilon \in (0, 1).$$

From the last two equalities it is elementary to show that with the extension  $c_1(0) = 0$ ,  $c_2(0) = 0$  of  $c_1, c_2$  from  $(0, 1)$  to  $[0, 1)$ , the functions  $c_1$  and  $c_2$  are  $C^1$ -smooth on  $[0, 1)$ .

We also need the following integrals:

$$\begin{aligned} I_1 &= \int_{-1}^{s_1} e^s h(a, \varepsilon)(s) ds = K(e^{s_1} - e^{-1}) = \frac{K}{e}(e^{a_1} - 1), \\ I_{1,*} &= \int_{-1}^{s_1^*} e^s h(a, \varepsilon)(s) ds = I_1 + \int_{s_1}^{s_1^*} e^s f^{K,\varepsilon}((1+\varepsilon)e^{-(s-s_1)}) ds \\ &= I_1 + e^{s_1} c_1(\varepsilon) = \frac{1}{e} [K(e^{a_1} - 1) + e^{a_1} c_1(\varepsilon)], \\ I_2 &= \int_{-1}^{s_2} e^s h(a, \varepsilon)(s) ds = I_{1,*}, \\ I_{2,*} &= \int_{-1}^{s_2^*} e^s h(a, \varepsilon)(s) ds = I_2 + \int_{s_2}^{s_2^*} e^s f^{K,\varepsilon}(-K + (K-1)e^{-(s-s_2)}) ds \\ &= I_2 - e^{s_2} c_2(\varepsilon) \\ &= \frac{1}{e} [K(e^{a_1} - 1) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)c_2(\varepsilon)], \\ I_3 &= \int_{-1}^{s_3} e^s h(a, \varepsilon)(s) ds = I_{2,*} + \int_{s_2^*}^{s_3} e^s (-K) ds = I_{2,*} + K e^{s_2^*} - K e^{s_3} \\ &= \frac{1}{e} \left[ K(e^{a_1} - 1) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)c_2(\varepsilon) \right. \\ &\quad \left. + e^{a_1+a_2}(1-e^{a_3}) \frac{(1+\varepsilon)K(K-1)}{K-1-\varepsilon} \right], \\ I_{3,*} &= \int_{-1}^{s_3^*} e^s h(a, \varepsilon)(s) ds = I_3 + \int_{s_3}^{s_3^*} e^s f^{K,\varepsilon}(-(1+\varepsilon)e^{-(s-s_3)}) ds \\ &= I_3 - e^{s_3} c_1(\varepsilon) \\ &= \frac{1}{e} \left[ K(e^{a_1} - 1) + e^{a_1} c_1(\varepsilon) - e^{a_1+a_2}(1+\varepsilon)c_2(\varepsilon) \right. \\ &\quad \left. + e^{a_1+a_2}(1-e^{a_3} - e^{a_3} c_1(\varepsilon)) \frac{(1+\varepsilon)K(K-1)}{K-1-\varepsilon} \right]. \end{aligned}$$

Notice that  $I_1, I_{1,*}, \dots, I_3, I_{3,*}$  are  $C^1$ -smooth functions from  $U^2$  into  $\mathbb{R}$ .

For  $t_1$  and  $t_2$ ,

$$e^{-t_1} \int_{-1}^{t_1} K e^s ds = 1 \text{ and } e^{-t_2} \int_{-1}^{t_2} K e^s ds = 1 + \varepsilon$$

hold, respectively. Hence

$$t_1 = \ln \frac{K}{K-1} - 1, \quad t_2 = \ln \frac{K}{K-1-\varepsilon} - 1 \text{ and } t_2 - t_1 = \ln \frac{K-1}{K-1-\varepsilon} = \hat{T}(\varepsilon). \quad (3.7)$$

**Proposition 3.2.3.** *The maps*

$$U^2 \ni (a, \varepsilon) \mapsto x^{(a, \varepsilon)}(t_1 + 1) = \frac{K-1}{K} I_{3,*} \in \mathbb{R},$$

$$U^2 \ni (a, \varepsilon) \mapsto x^{(a, \varepsilon)}(t_2 + 1) = \frac{K-1-\varepsilon}{K} I_{3,*} + \frac{K-1-\varepsilon}{K-1} c_2(\varepsilon) \in \mathbb{R},$$

$$U^2 \ni (a, \varepsilon) \mapsto x^{(a, \varepsilon)}(t_3 + 1) = K + \frac{K(1+\varepsilon)}{e(K-1-\varepsilon)I_{1,*}} \left( x^{(a, \varepsilon)}(t_2 + 1) - K \right) \in \mathbb{R}$$

are continuously differentiable.

*Proof.* Since  $I_{3,*}$ ,  $c_2$ ,  $I_{1,*}$  are  $C^1$ -smooth functions on  $U^2$ , one has to show only the stated equalities for  $x^{(a, \varepsilon)}(t_i + 1)$ ,  $i \in \{1, 2, 3\}$ . Set  $x = x^{\Sigma(a, \varepsilon)}$ .

From  $x(s) \in [0, 1]$ ,  $-1 \leq s \leq t_1$ , it follows that  $x$  is of type (0) on  $[0, t_1 + 1]$ . The definition of  $\Sigma(a, \varepsilon)$  gives that  $x$  is of type (0) on  $[s_1^*, 0]$  as well. Then

$$x(t) = e^{-(t-s_3^*)} x(s_3^*) \quad (s_3^* \leq t \leq t_1 + 1), \quad (3.8)$$

and using (3.5), (3.7) and the definitions of  $I_{3,*}$  and  $c_2(\varepsilon)$ , we get

$$x(t_1 + 1) = e^{-(t_1+1)} e^{s_3^*} x(s_3^*) = \frac{K-1}{K} I_{3,*}$$

and

$$\begin{aligned} x(t_2 + 1) &= e^{t_1-t_2} x(t_1 + 1) + e^{t_1-t_2} \int_0^{t_2-t_1} e^s f^{K, \varepsilon}(K - (K-1)e^{-s}) \, ds \\ &= \frac{K-1-\varepsilon}{K} I_{3,*} + \frac{K-1-\varepsilon}{K-1} c_2(\varepsilon). \end{aligned}$$

As  $x$  is of type  $(K)$  on  $[t_2 + 1, t_3 + 1]$ , we find that

$$x(t_3 + 1) = e^{t_2-t_3} (x(t_2 + 1) - K) + K. \quad (3.9)$$

From  $s_1^* < t_3 < s_2$ , (3.5) and  $h(a, \varepsilon)(t) = 0$  for  $t \in [s_1^*, t_3]$ ,  $x(t_3) = e^{-t_3} I_{1,*}$  follows. Since  $x(t_3) = 1 + \varepsilon$ , one concludes that

$$t_3 = \ln \frac{I_{1,*}}{1+\varepsilon}. \quad (3.10)$$

Substituting  $t_2$  and  $t_3$  from (3.7) and (3.10) into (3.9), the proof is complete.  $\square$

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Now we are in a position to define a further proper subset of  $U^1$ . Let

$$U^3 = \left\{ (a, \varepsilon) \in U^2 : x^{(a, \varepsilon)}(t_1 + 1) > -1, x^{(a, \varepsilon)}(t_2 + 1) < 0, x^{(a, \varepsilon)}(t_3 + 1) > 0 \right\}.$$

At this stage we do not know whether  $U^3 \neq \emptyset$ . However, Proposition 3.2.3 and the definition of  $U^3$  imply that  $U^3$  is open in  $(0, 1)^3 \times [0, 1)$ . A typical element of  $\Sigma(U^3)$  is presented in Fig. 3.5.

The next remark plays a prominent role in proving Theorem 3.1.1, as well as Remark 3.2.14 of the next subsection.

*Remark 3.2.4.* Observe that any  $\varphi \in \Sigma(U^3)$  can be characterized as follows: there exist  $\varepsilon \in [0, 1)$  and

$$-1 < s_1 \leq s_1^* < s_2 \leq s_2^* < s_3 \leq s_3^* < 0$$

with

$$s_1^* - s_1 = T(\varepsilon), \quad s_2^* - s_2 = \hat{T}(\varepsilon), \quad s_3^* - s_3 = T(\varepsilon)$$

so that  $\varphi \in C$  satisfies

- (i)  $\varphi(-1) = 0$ ,
- (ii)  $\varphi$  is of type  $(K)$  on  $[-1, s_1]$ ,
- (iii)  $\varphi$  is of type  $(0, 1 + \varepsilon)$  on  $[s_1, s_1^*]$ ,
- (iv)  $\varphi$  is of type  $(0)$  on  $[s_1^*, s_2]$ ,
- (v)  $\varphi$  is of type  $(-K, -1)$  on  $[s_2, s_2^*]$ ,
- (vi)  $\varphi$  is of type  $(-K)$  on  $[s_2^*, s_3]$ ,
- (vii)  $\varphi$  is of type  $(0, -1 - \varepsilon)$  on  $[s_3, s_3^*]$ ,
- (viii)  $\varphi$  is of type  $(0)$  on  $[s_3^*, 0]$ ,
- (ix)  $\varphi(s) > 1 + \varepsilon$  for  $s \in [s_1, s_1^*]$ ,
- (x)  $|\varphi(s)| < 1$  for  $s \in [s_2, s_2^*]$ ,
- (xi)  $\varphi(s) < -1 - \varepsilon$  for  $s \in [s_3, s_3^*]$ ,
- (xii) if  $-1 < t_1 < s_1$  with  $\varphi(t_1) = 1$ , then  $x^\varphi(t_1 + 1) > -1$ ,
- (xiii) if  $t_1 \leq t_2 < s_1$  with  $\varphi(t_2) = 1 + \varepsilon$ , then  $x^\varphi(t_2 + 1) < 0$ ,
- (xiv) if  $s_1^* < t_3 < s_2$  with  $\varphi(t_3) = 1 + \varepsilon$ , then  $x^\varphi(t_3 + 1) > 0$ .

Notice that (i)-(viii) characterize  $\varphi \in \Sigma(U^1)$ , and (i)-(xi) characterize  $\varphi \in \Sigma(U^2)$ .

If  $(a, \varepsilon) \in U^3$ , then for  $x = x^{(a, \varepsilon)}$  we have  $x(s_3^*) < -1 - \varepsilon$ ,  $x$  is of type  $(0)$  on  $[s_3^*, t_1 + 1]$  and  $x(t_1 + 1) > -1$ . So  $t_7$  and  $t_8$  can be uniquely defined by

$$s_3^* < t_7 \leq t_8 < t_1 + 1, \quad x(t_7) = -1 - \varepsilon, \quad x(t_8) = -1.$$

In addition, from  $(a, \varepsilon) \in U^3$  it follows that  $x$  has a zero in  $(t_2 + 1, t_3 + 1)$ . Since  $x$  is of type  $(K)$  on  $[t_2 + 1, t_3 + 1]$ , there is a unique zero. Let  $\tau$  denote the zero of  $x^{(a, \varepsilon)}$  in  $(t_2 + 1, t_3 + 1)$  (Fig. 3.5).



**Proposition 3.2.5.** *Suppose  $(a, \varepsilon) \in U^3$  and define  $t_1, t_2, \dots, t_8$  and  $\tau$  for  $x = x^{(a, \varepsilon)}$  as above. Then  $x_{\tau+1} \in \Sigma(U^1)$  and*

$$x_{\tau+1} = \Sigma(t_3 + 1 - \tau, t_5 - t_4, t_7 - t_6, \varepsilon).$$

*Proof.* Notice that  $\tau$  is the first positive zero of  $x$ . Indeed, we know that function  $x$  strictly increases on  $[s_3^*, t_1 + 1]$  from  $x(s_3^*) < -1 - \varepsilon$  to  $x(t_1 + 1) \in (-1, 0)$  and strictly increases on  $[t_2 + 1, t_3 + 1]$  from  $x(t_2 + 1) < 0$  to  $x(t_3 + 1) > 0$ . It remains to consider  $x$  on  $[t_1 + 1, t_2 + 1]$ , where it is of type  $(K, 1)$ , that is

$$x(t) = e^{-(t-t_1-1)}x(t_1 + 1) + \int_0^{t-t_1-1} e^{-(t-t_1-1-s)} f^{K, \varepsilon}(K + (1-K)e^{-s}) ds \quad (3.11)$$

for  $t_1 + 1 \leq t \leq t_2 + 1$ . The case  $\varepsilon = 0$  is evident. If  $\varepsilon > 0$  and  $z \in (t_1 + 1, t_2 + 1)$  is any zero of  $x$ , then

$$\dot{x}(z) = f^{K, \varepsilon}(x(z-1)) = f^{K, \varepsilon}(K - Ke^{-z}) > f^{K, \varepsilon}(K - Ke^{-t_1-1}) = f^{K, \varepsilon}(1) = 0.$$

Hence it is easy to see that the existence of a zero of  $x$  in  $(t_1 + 1, t_2 + 1)$  implies  $x(t_2 + 1) > 0$ , a contradiction. Thus  $x(t) < 0$  follows for all  $[0, \tau)$ .

From (3.11) one easily obtains that  $x(t_1 + 1) \leq x(t)$  for  $t \in [t_1 + 1, t_2 + 1]$ .

Now it should be clear that

- $x(\tau) = 0$ ,
- $x$  is of type  $(K)$  on  $[\tau, t_3 + 1]$ ,
- $x$  is of type  $(0, 1 + \varepsilon)$  on  $[t_3 + 1, t_4 + 1]$ ,
- $x$  is of type  $(0)$  on  $[t_4 + 1, t_5 + 1]$ ,
- $x$  is of type  $(-K, -1)$  on  $[t_5 + 1, t_6 + 1]$ ,
- $x$  is of type  $(-K)$  on  $[t_6 + 1, t_7 + 1]$ ,
- $x$  is of type  $(0, -1 - \varepsilon)$  on  $[t_7 + 1, t_8 + 1]$ ,
- $x$  is of type  $(0)$  on  $[t_8 + 1, \tau + 1]$ .

It remains to show that

$$t_4 - t_3 = T(\varepsilon), \quad t_6 - t_5 = \hat{T}(\varepsilon), \quad t_8 - t_7 = T(\varepsilon),$$

which relations are consequences of the definitions of  $t_3, t_4, t_5, t_6, t_7, t_8, T(\varepsilon), \hat{T}(\varepsilon)$  and the facts that  $x$  is of type  $(0)$  on  $[t_3, t_4]$  and on  $[t_7, t_8]$  and that  $x$  is of type  $(-K)$  on  $[t_5, t_6]$ . The proof is complete.  $\square$

We remark that if  $x_{\tau+1}^{(a, \varepsilon)} = \Sigma(a, \varepsilon)$  holds for some  $(a, \varepsilon) \in U^3$ , i. e.,

$$a_1 = t_3 + 1 - \tau, \quad a_2 = t_5 - t_4, \quad a_3 = t_7 - t_6,$$

then  $x$  is a periodic solution of Eq. (3.2) with minimal period  $\tau + 1$ . The dependence

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of  $t_3 + 1 - \tau$ ,  $t_5 - t_4$  and  $t_7 - t_6$  on  $(a, \varepsilon)$  is considered in the next result.

**Proposition 3.2.6.** *Suppose  $(a, \varepsilon) \in U^3$  and define  $t_3, t_4, t_5, t_6, t_7$  and  $\tau$  as in Proposition 3.2.5. Then*

$$\begin{aligned} t_3 + 1 - \tau &= 1 + \ln \frac{I_{1,*}}{1 + \varepsilon} - \ln \left( \frac{K}{K - 1 - \varepsilon} - \frac{I_{3,*}}{K} - \frac{c_2(\varepsilon)}{K - 1} \right), \\ t_5 - t_4 &= \ln \frac{I_{2,*} + K e^{s_2^*}}{(K - 1) I_{1,*}}, \\ t_7 - t_6 &= \ln \frac{-I_{3,*}(K - 1 - \varepsilon)}{(1 + \varepsilon)(I_{2,*} + K e^{s_2^*})}. \end{aligned}$$

In particular, if  $\varepsilon = 0$ , that is  $(a, 0) \in U^3$ , then

$$\begin{aligned} t_3 + 1 - \tau &= a_1 + \ln \frac{K(K - 1)(1 - e^{-a_1})}{K + (K - 1)e^{-1}(1 + e^{a_1+a_2+a_3} - e^{a_1} - e^{a_1+a_2})}, \\ t_5 - t_4 &= a_2 + \ln \frac{e^{-a_2}(1 - e^{-a_1}) + 1}{(K - 1)(1 - e^{-a_1})}, \\ t_7 - t_6 &= a_3 + \ln \left[ (K - 1) \left( \frac{e^{a_1+a_2}}{e^{a_1} + e^{a_1+a_2} - 1} - e^{-a_3} \right) \right]. \end{aligned}$$

*Proof.* Applying that  $x$  is of type  $(K)$  on  $[t_2 + 1, \tau]$ , an integration gives

$$0 = e^\tau x(\tau) = e^{t_2+1} x(t_2 + 1) + K(e^\tau - e^{t_2+1}).$$

Hence, using also Proposition 3.2.3,

$$\tau = \ln \left( \frac{K}{K - 1 - \varepsilon} - \frac{I_{3,*}}{K} - \frac{c_2(\varepsilon)}{K - 1} \right). \quad (3.12)$$

This formula combined with (3.10) yields the result for  $t_3 + 1 - \tau$ .

Obviously,  $t_4 = \ln I_{1,*}$ . Since  $s_2^* < t_5 < t_6 < s_3$ ,

$$\begin{aligned} -1 &= e^{-t_5} \left( I_{2,*} + \int_{s_2^*}^{t_5} e^s (-K) ds \right) = e^{-t_5} (I_{2,*} + K e^{s_2^*}) - K, \\ -1 - \varepsilon &= e^{-t_6} \left( I_{2,*} + \int_{s_2^*}^{t_6} e^s (-K) ds \right) = e^{-t_6} (I_{2,*} + K e^{s_2^*}) - K. \end{aligned}$$

So

$$t_5 = \ln \left( \frac{I_{2,*}}{K - 1} + \frac{K}{K - 1} e^{s_2^*} \right) \quad \text{and} \quad t_6 = \ln \left( \frac{I_{2,*}}{K - 1 - \varepsilon} + \frac{K}{K - 1 - \varepsilon} e^{s_2^*} \right).$$

Using that  $x$  is of type  $(0)$  on  $[s_3^*, t_1 + 1]$  and  $s_3^* < t_7 < t_1 + 1$ , we obtain that

$$-1 - \varepsilon = e^{-(t_7 - s_3^*)} x(s_3^*) = e^{-t_7} I_{3,*},$$

and

$$t_7 = \ln \frac{-I_{3,*}}{1 + \varepsilon}$$

follow. Therefore

$$\begin{aligned} t_5 - t_4 &= \ln \frac{I_{2,*} + Ke^{s_2^*}}{(K-1)I_{1,*}}, \\ t_7 - t_6 &= \ln \frac{-I_{3,*}(K-1-\varepsilon)}{(1+\varepsilon)(I_{2,*} + Ke^{s_2^*})}. \end{aligned}$$

The case  $\varepsilon = 0$  is an elementary exercise.  $\square$

The above results allow us to define the map  $F : U^3 \rightarrow \mathbb{R}^3$  by

$$F(a, \varepsilon) = (t_3 + 1 - \tau, t_5 - t_4, t_7 - t_6),$$

where  $t_3, t_4, t_5, t_6, t_7$  and  $\tau$  are uniquely determined by the solution  $x^{(a, \varepsilon)} = x^{\Sigma(a, \varepsilon)}$  of Eq. (3.2). An immediate consequence of the explicit representation of  $F(a, \varepsilon)$  in term of  $(a, \varepsilon)$  and the  $C^1$ -smoothness of the involved functions:

**Proposition 3.2.7.**  *$F$  is  $C^1$ -smooth.*

If  $(a, \varepsilon) \in U^3$  and  $F(a, \varepsilon) = a$ , then  $x^{(a, \varepsilon)}$  is a periodic solution of Eq. (3.2) with minimal period  $\tau + 1$ . A first step to find a solution of  $F(a, \varepsilon) = a$  in  $U^3$  is to consider the case  $\varepsilon = 0$ . Set

$$U_0^3 = \left\{ a \in \mathbb{R}^3 : (a, 0) \in U^3 \right\}.$$

Let  $K^*$  be the unique solution of  $w(K) = 1/e$  on  $(3, \infty)$ , where

$$w(K) = \frac{(K^2 - 2K - 1)^2}{(K-1)(K+1)^3}.$$

Then  $K^*$  is well-defined. Indeed,  $w(3) = 1/32$ ,  $\lim_{K \rightarrow \infty} w(K) = 1$ , and as  $K \mapsto 2K/(K^2 - 1)$  and  $K \mapsto (4K + 2)/(K + 1)^2$  are strictly decreasing functions on  $(3, \infty)$ ,

$$w(K) = \left(1 - \frac{2K}{K^2 - 1}\right) \left(1 - \frac{4K + 2}{(K + 1)^2}\right)$$

is strictly increasing on  $(3, \infty)$ . Evaluating  $w(6)$  and  $w(7)$ , one sees that  $K^* \in (6, 7)$ . We have the approximation  $K^* \approx 6.8653$ . Note that  $w(K) > 1/e$  for  $K > K^*$ .

**Proposition 3.2.8.** *For  $K \in (3, K^*]$ , equation  $F(a, 0) = a$  admits no solution in  $U_0^3$ . For  $K > K^*$ , there is a unique  $a^* \in U_0^3$  with  $F(a^*, 0) = a^*$ .*

*Proof.* Set  $K > 3$ . First assume that  $a \in \mathbb{R}^3$  is a solution of  $F(a, 0) = a$ . Using Proposition 3.2.6, it is a straightforward calculation to show that this is equivalent to

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the following:

$$a_2 = -\ln\left(K - 1 - \frac{1}{1 - e^{-a_1}}\right), \quad (3.13)$$

$$a_3 = \ln((K - 1)(e^{a_1} - 1)), \quad (3.14)$$

and  $g(a_1, K) = 1/e$ , where

$$g(u, K) = Ke^{-2u} \frac{[(K - 1)(1 - e^{-u}) - 1]^2}{(K - 1)^2 (1 - e^{-u})^3}.$$

Recall that by definition,  $a \in U_0^3$  if and only if

$$a_1 > 0, \ a_2 > 0, \ a_3 > 0, \ a_1 + a_2 + a_3 < 1,$$

$$x^{(a,0)}(s_1) > 1, \ |x^{(a,0)}(s_2)| < 1, \ x^{(a,0)}(s_3) < -1,$$

$$-1 < x^{(a,0)}(t_1 + 1) = x^{(a,0)}(t_2 + 1) < 0 \text{ and } x^{(a,0)}(t_3 + 1) > 0.$$

Not only  $a_2$  and  $a_3$  can be expressed as a function of  $K$  and  $a_1$ , but also  $a_1 + a_2 + a_3$ ,  $x^{(a,0)}(s_i)$  and  $x^{(a,0)}(t_i + 1)$ ,  $i = 1, 2, 3$ .

First of all, (3.13), (3.14) and  $g(a_1, K) = e^{-1}$  imply

$$a_1 + a_2 + a_3 = 1 + \ln\left(K - \frac{K}{(K - 1)(1 - e^{-a_1})}\right). \quad (3.15)$$

By (3.5) and the definition of  $I_1$ , we get  $x^{(a,0)}(s_1) = e^{-s_1} I_1 = K(1 - e^{-a_1})$ . Relations (3.5), (3.13), (3.14) and the definitions of  $I_2$  and  $I_3$  yield

$$\begin{aligned} x^{(a,0)}(s_2) &= e^{-s_2} I_2 = K[(K - 1)(1 - e^{-a_1}) - 1], \\ x^{(a,0)}(s_3) &= e^{-s_3} I_3 = -K(1 - e^{-a_1}) = -x^{(a,0)}(s_1). \end{aligned} \quad (3.16)$$

Also, (3.13), (3.14),  $g(a_1, K) = e^{-1}$ , Proposition 3.2.3 and the definitions of  $I_{1,*}$  and  $I_{3,*}$  give

$$\begin{aligned} x^{(a,0)}(t_1 + 1) = x^{(a,0)}(t_2 + 1) &= \frac{K - 1}{K} I_{3,*} = K[1 - (K - 1)(1 - e^{-a_1})] \\ &= -x^{(a,0)}(s_2), \\ x^{(a,0)}(t_3 + 1) &= K + \frac{K}{e(K - 1) I_{1,*}} (x^{(a,\varepsilon)}(t_1 + 1) - K) \\ &= K(1 - e^{-a_1}) = x^{(a,0)}(s_1). \end{aligned}$$

As one can check by elementary calculations, these relations imply that  $a \in \mathbb{R}^3$

satisfying  $F(a, 0) = a$  belongs to  $U_0^3$  if and only if

$$a_1 \in J_K = \left( \ln \frac{K-1}{K-2}, \ln \frac{K^2-K}{K^2-2K-1} \right).$$

Hence we get a unique solution  $a^* = (a_1^*, a_2^*, a_3^*)$  of  $F(a, 0) = a$  in  $U_0^3$  if there exist a unique  $a_1^* \in J_K$  with  $g(a_1^*, K) = e^{-1}$  and  $a_2^*$  and  $a_3^*$  are defined by (3.13) and (3.14).

We claim that  $g(\cdot, K)$  is strictly increasing on  $J_K$  for  $K > 3$ . Note that

$$\frac{\partial g(u, K)}{\partial u} = g(u, K) \frac{2 + e^{-u} + (K-1)(1 - e^{-u})e^{-u} - 2(K-1)(1 - e^{-u})}{[(K-1)(1 - e^{-u}) - 1](1 - e^{-u})}.$$

If  $u \in J_K$ , then  $(K-1)(1 - e^{-u}) - 1 \in (0, 1/K)$ . Hence it suffices to show that for  $K > 3$  and  $u \in J_K$ ,

$$2 + e^{-u} + (K-1)(1 - e^{-u})e^{-u} - 2(K-1)(1 - e^{-u}) > 0,$$

which inequality is equivalent to the second order inequality

$$(2K-4)z^2 - (3K-2)z + (K-1) < 0$$

with  $z = e^u$ . The solution formula gives that we have show that for  $K > 3$ ,  $J_K \subset (\ln z_1, \ln z_2)$ , where

$$z_1 = \frac{3K-2 - \sqrt{K^2 + 12(K-1)}}{2K-4} \text{ and } z_2 = \frac{3K-2 + \sqrt{K^2 + 12(K-1)}}{2K-4}.$$

As  $\sqrt{K^2 + 12(K-1)} > K$  for all  $K > 1$ , we see that

$$\ln z_1 < \ln \frac{K-1}{K-2} = \inf J_K.$$

The same estimate yields  $z_2 > (2K-1)/(K-2)$ , and it is easy to see that

$$\frac{2K-1}{K-2} > \frac{K^2-K}{K^2-2K-1}$$

that is

$$K > 2 + \frac{2}{K} - \frac{1}{K^2}$$

holds for  $K > 3$ . Hence  $\ln z_2 > \sup J_K$  and  $g'_u(u, K) > 0$  for  $K > 3$  and  $u \in J_K$ .

In addition,  $g(u, K) \rightarrow 0$  as  $u \rightarrow \inf J_K + 0$ . Also,

$$\lim_{u \rightarrow \sup J_K - 0} g(u, K) = w(K) \begin{cases} < \frac{1}{e}, & 3 < K < K^*, \\ = \frac{1}{e}, & K = K^*, \\ > \frac{1}{e} & K > K^*. \end{cases}$$

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Therefore the continuity and monotonicity of  $g$  implies that for  $K > 3$ , there exists  $a_1^* \in J_K$  with  $g(a_1^*, K) = e^{-1}$  if and only if  $K > K^*$ , and the solution is unique if it exists.  $\square$

Using a construction similar to the one given above, One may verify that for  $K = K^*$ ,  $F(\cdot, 0)$  has a fixed point on the boundary of  $U_0^3$ .

**Proposition 3.2.9.** *For  $K > K^*$ ,  $x^{\Sigma(a^*, 0)} : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution in the sense defined on page 23.*

*Proof.* Consider solution  $x = x^{\Sigma(a^*, 0)} : \mathbb{R} \rightarrow \mathbb{R}$ . It follows from the construction introduced above that the minimal period of  $x$  is  $\tau + 1$  with  $\tau > 0$ , and  $x$  is monotone nonincreasing on  $[s_1, s_3]$ . Therefore it suffices to prove that  $\tau < 1$ ,

$$2(s_3 - s_1) = \tau + 1 \quad (3.17)$$

and

$$x\left(t + \frac{\tau + 1}{2}\right) = -x(t) \quad (3.18)$$

for  $t \in [s_1, s_3]$ .

By (3.12), (3.16) and  $I_3^* = I_3 = x(s_3)e^{s_3}$ ,

$$\tau = \ln\left(\frac{K}{K-1} - \frac{x(s_3)e^{s_3}}{K}\right) = \ln\left(\frac{K}{K-1} + (1 - e^{-a_1^*})e^{s_3}\right)$$

Substituting result (3.15) into the right hand side, we get

$$\tau = \ln\left(K\left(1 - e^{-a_1^*}\right)\right). \quad (3.19)$$

So  $\tau < 1$  if and only if  $a_1^* < \ln K - \ln(K - e)$ . As  $a_1^* \in J_K$  (see the proof of Proposition 3.2.8), this bound holds.

Relations (3.13) and (3.14) imply

$$e^{2(s_3 - s_1)} = e^{2(a_2^* + a_3^*)} = e^{2a_1^*} \frac{(K-1)^2 (1 - e^{-a_1^*})^4}{\left[(K-1)(1 - e^{-a_1^*}) - 1\right]^2}$$

Using relation  $g(a_1, K) = e^{-1}$  from the proof of Proposition 3.2.8,

$$2(s_3 - s_1) = \ln\left(Ke\left(1 - e^{-a_1^*}\right)\right).$$

This result together with (3.19) give (3.17).

As  $x(s_1) = -x(s_3)$  by (3.16) and  $x$  is of type (0) on  $[s_1, s_2]$  and on  $[s_3, t_1 + 1]$ , the special symmetry follows for  $t \in [s_1, s_2]$  if  $s_2 - s_1 = t_1 + 1 - s_3$  holds. This equation is the direct consequence of (3.7), (3.13) and (3.15). In particular,  $x(s_2) = -x(t_1 + 1)$ .

### 3.2 LSOP solutions for special nonlinearities

As  $x$  is of type  $(-K)$  on  $[s_2, s_3]$  and of type  $(K)$  on  $[t_1 + 1, t_3 + 1]$ , special symmetry holds for  $t \in [s_2, s_3]$  if  $a_3 = s_3 - s_2 = t_3 - t_1$ . This result comes from (3.7), (3.10), the definition of  $I_{1,*}$  and (3.14). So (3.18) follows.

The proof is complete.  $\square$

*Remark 3.2.10.* A numerical study executed with the aid of the CAPD program [1] gives that for  $K = 7$ ,

$$a^* \in [0.2108, 0.2109] \times [0.3003, 0.3004] \times [0.3426, 0.3427].$$

It is shown that the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $D_a F(a^*, 0) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$  are real with  $\lambda_1 \in [0.7933, 0.7934]$ ,  $\lambda_2 \in [3.9187, 3.9188]$  and  $\lambda_3 \in [6.8362, 6.8363]$ .

Now we are capable of verifying the existence of an LSOP solution defined on page 26 for Eq. (3.2) with small  $\varepsilon > 0$ . In the sequel we fix  $K = 7$ , but the results below can be easily modified for any  $K > K^*$ . Since we look for an example with large amplitude periodic orbits, a particular  $K$  is sufficient.

**Proposition 3.2.11.** *Set  $K = 7$ . There exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in [0, \varepsilon_0)$ ,  $F(a, \varepsilon) = a$  has a solution  $a^*(\varepsilon)$  in  $U_\varepsilon^3 = \{a \in \mathbb{R}^3 : (a, \varepsilon) \in U^3\}$ , and  $x^{\Sigma(a^*(\varepsilon), \varepsilon)} : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution of Eq. (3.2) with nonlinearity  $f^{7, \varepsilon}$ .*

*Proof.* As  $U^3$  is open in  $\mathbb{R}^3 \times [0, 1)$ ,

$$U = \{(a, \varepsilon) : (a, |\varepsilon|) \in U^3\}$$

is open in  $\mathbb{R}^4$ . We extend the definition of  $F$  for  $\varepsilon < 0$  because we intend to use the implicit function theorem. Let  $G : U \rightarrow \mathbb{R}^3$  be given by

$$G(a, \varepsilon) = \begin{cases} F(a, \varepsilon) & \text{if } \varepsilon \geq 0, \\ 2F(a, 0) - F(a, -\varepsilon) & \text{if } \varepsilon < 0. \end{cases}$$

Then  $G$  is  $C^1$ -smooth and  $G(a^*, 0) - a^* = 0$ . As 1 is not an eigenvalue of  $D_a G(a^*, 0)$  by Remark 3.2.10, the implicit function theorem yields the existence of  $\varepsilon_0 > 0$ , a convex bounded open neighborhood  $N$  of  $a^*$  in  $\mathbb{R}^3$  and a  $C^1$  function  $a^* : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^3$  so that  $N \times (-\varepsilon_0, \varepsilon_0) \subset U$ ,  $a^*((-\varepsilon_0, \varepsilon_0)) \subset N$ ,  $a^*(0) = a^*$  and for every  $(a, \varepsilon) \in N \times (-\varepsilon_0, \varepsilon_0)$ ,  $G(a, \varepsilon) - a = 0$  if and only if  $a = a^*(\varepsilon)$ . That is  $F(a^*(\varepsilon), \varepsilon) = a^*(\varepsilon)$  for all  $\varepsilon \in [0, \varepsilon_0)$ .

Clearly,  $x^{\Sigma(a^*(\varepsilon), \varepsilon)} : [-1, \infty) \rightarrow \mathbb{R}$  is a periodic solution of Eq. (3.2) with feedback function  $f^{7, \varepsilon}$  for all  $\varepsilon \in [0, \varepsilon_0)$ , hence the solution  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  can be extended to  $\mathbb{R}$ . According to Proposition 3.2.9,  $x^{\Sigma(a^*, 0)}$  is an LSOP solution. For  $\varepsilon \in (0, \varepsilon_0)$ , Lemma 2.3.2 (i) and the periodicity of  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  gives  $V(x_t^{\Sigma(a^*(\varepsilon), \varepsilon)})$  is the same constant for all  $t \in \mathbb{R}$ . It follows from the construction that  $V(\Sigma(a^*(\varepsilon), \varepsilon)) = 2$ . Thus

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$V\left(x_t^{\Sigma(a^*(\varepsilon), \varepsilon)}\right) = 2$  for all  $t \in \mathbb{R}$ . In addition, it is clear that  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ . Hence  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  is an LSOP solution also for all  $\varepsilon \in (0, \varepsilon_0)$ .  $\square$

*Remark 3.2.12.*  $D_a F(a^*(\varepsilon), \varepsilon)$  has at most three distinct (possibly partly complex) eigenvalues. As  $F$  is smooth (see Proposition 3.2.7), they are close to the eigenvalues of  $D_a F(a^*, 0)$  in  $\mathbb{C}$  for all  $\varepsilon > 0$  small. Because of Remark 3.2.10, we may choose  $\varepsilon_0 > 0$  sufficiently small such that for  $\varepsilon \in [0, \varepsilon_0)$ , the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $D_a F(a^*(\varepsilon), \varepsilon)$  are real, simple and satisfy

$$0 < \lambda_1 < 0.9, \quad 3 < \lambda_2 < 5 < \lambda_3.$$

Consider the case  $\varepsilon = 0$ . As equation  $\dot{x}(t) = -x(t)$  admits no nontrivial periodic solution, any periodic solution  $x$  of Eq. (3.2) with initial function in  $\Sigma(U_0^1)$  necessarily satisfies  $x(s_1) > 1$  or  $x(s_3) < -1$ . However, condition  $x(s_2) < 1$  is not self-evident. This recognition leads to an alternative construction yielding a second LSOP solution of Eq. (3.2) for  $K > K^*$  and  $\varepsilon = 0$ , then for  $K = 7$  and  $\varepsilon > 0$  small. Next we introduce this construction but omit the detailed calculations as they are analogous to the previous ones.

#### 3.2.2 The second construction

For  $K > 3$ , define

$$\tilde{U}^1 = \left\{ (a, \varepsilon) \in (0, 1)^3 \times [0, 1] : a_1 + a_2 + a_3 + 2\tilde{T}(\varepsilon) + \hat{T}(\varepsilon) < 1 \right\}$$

and

$$\tilde{U}_\varepsilon^1 = \left\{ a \in \mathbb{R}^3 : (a, \varepsilon) \in \tilde{U}^1 \right\}, \quad \varepsilon \in [0, 1].$$

Note that  $\tilde{U}_0^1 = U_0^1$ .

For given  $(a, \varepsilon) \in \tilde{U}^1$ , set

$$\begin{aligned} s_0 &= -1, \\ s_1 &= s_0 + a_1 = -1 + a_1, \\ s_1^* &= s_1 + \tilde{T}(\varepsilon) = -1 + a_1 + \tilde{T}(\varepsilon), \\ s_2 &= s_1^* + a_2 = -1 + a_1 + \tilde{T}(\varepsilon) + a_2, \\ s_2^* &= s_2 + \hat{T}(\varepsilon) = -1 + a_1 + \tilde{T}(\varepsilon) + a_2 + \hat{T}(\varepsilon), \\ s_3 &= s_2^* + a_3 = -1 + a_1 + \tilde{T}(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3, \\ s_3^* &= s_3 + \tilde{T}(\varepsilon) = -1 + a_1 + \tilde{T}(\varepsilon) + a_2 + \hat{T}(\varepsilon) + a_3 + \tilde{T}(\varepsilon). \end{aligned}$$

Define the continuous map  $\tilde{\Sigma} : \tilde{U}^1 \rightarrow C$  by

$$\tilde{\Sigma}(a, \varepsilon)(t) = e^{-t} \int_{-1}^t e^{s\tilde{h}(a, \varepsilon)}(s) ds \quad (-1 \leq t \leq 0),$$



where  $\tilde{h} = \tilde{h}(a, \varepsilon) : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\tilde{h}(t) = \begin{cases} K & \text{if } t < s_1, \\ f^{K,\varepsilon}(-K + (K+1+\varepsilon)e^{-(t-s_1)}) & \text{if } s_1 \leq t < s_1^*, \\ 0 & \text{if } s_1^* \leq t < s_2, \\ f^{K,\varepsilon}(-K + (K-1)e^{-(t-s_2)}) & \text{if } s_2 \leq t < s_2^*, \\ -K & \text{if } s_2^* \leq t < s_3, \\ f^{K,\varepsilon}(K - (K+1+\varepsilon)e^{-(t-s_3)}) & \text{if } s_3 \leq t < s_3^*, \\ 0 & \text{if } s_3^* \leq t. \end{cases}$$

Note that for  $a \in \tilde{U}_0^1 = U_0^1$ ,  $\tilde{\Sigma}(a, 0) = \Sigma(a, 0)$ .

**Proposition 3.2.13.** *For each fixed  $\varepsilon \in (0, 1)$ , the map  $\tilde{U}_\varepsilon^1 \ni a \mapsto \tilde{\Sigma}(a, \varepsilon) \in C$  is  $C^1$ -smooth with*

$$D_a \tilde{\Sigma}(a, \varepsilon)(b) = b_1 \tilde{\psi}_1 + b_2 \tilde{\psi}_2 + b_3 \tilde{\psi}_3$$

for  $a \in \tilde{U}_\varepsilon^1$  and  $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ , where

$$\tilde{\psi}_i : [-1, 0] \ni t \mapsto e^{-t} \int_{-1}^t e^s \frac{\partial}{\partial a_i} \tilde{h}(a, \varepsilon)(s) ds \in \mathbb{R}, \quad i \in \{1, 2, 3\},$$

are linearly independent elements of  $C$ .

Now let

$$\tilde{U}^2 = \left\{ (a, \varepsilon) \in \tilde{U}^1 : \begin{aligned} &\tilde{\Sigma}(a, \varepsilon)(s) > 1 + \varepsilon \text{ for } s \in [s_1, s_1^*] \cup [s_2, s_2^*], \\ &\tilde{\Sigma}(a, \varepsilon)(s) < -1 - \varepsilon \text{ for } s \in [s_3, s_3^*] \end{aligned} \right\}.$$

If  $(a, \varepsilon) \in \tilde{U}^2$  and  $x : [-1, \infty) \rightarrow \mathbb{R}$  is the solution of Eq. (3.2) with initial function  $\tilde{\Sigma}(a, \varepsilon)$ , then there exist  $t_1, t_2, \dots, t_6$  in  $[-1, 0]$  such that

$$-1 < t_1 \leq t_2 < s_1 \leq s_1^* < s_2 \leq s_2^* < t_3 \leq t_4 < t_5 \leq t_6 < s_3 \leq s_3^*$$

and

$$x(t_1) = 1, \quad x(t_2) = 1 + \varepsilon, \quad x(t_3) = 1 + \varepsilon, \quad x(t_4) = 1, \quad x(t_5) = -1, \quad x(t_6) = -1 - \varepsilon,$$

see Fig. 3.6. A second subset of  $\tilde{U}^1$  is

$$\tilde{U}^3 = \left\{ (a, \varepsilon) \in \tilde{U}^2 : x^{\tilde{\Sigma}(a, \varepsilon)}(t_2 + 1) < -1 - \varepsilon, \quad x^{\tilde{\Sigma}(a, \varepsilon)}(t_3 + 1) > 0 \right\}.$$

One may show that  $\tilde{U}^3$  is open in  $(0, 1)^3 \times [0, 1)$ .

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Fig. 3.6 shows a typical element of set  $\tilde{\Sigma}(\tilde{U}^3)$ .

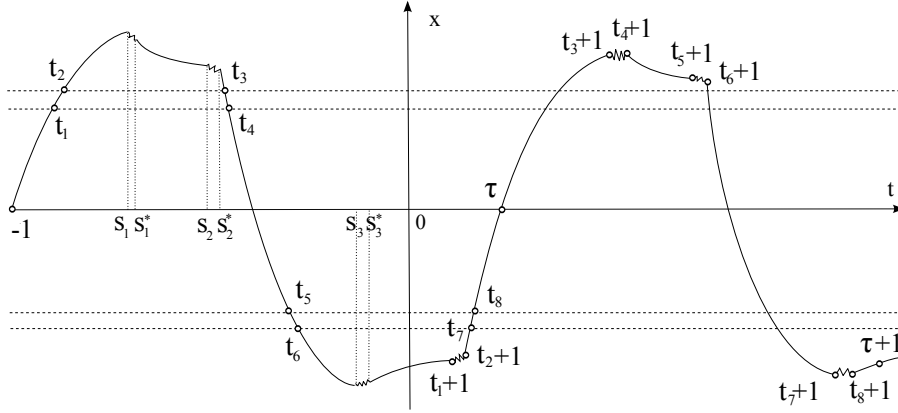


Figure 3.6: Solution  $x^{\tilde{\Sigma}(a, \epsilon)}$  of Eq. (3.2)

The following remark resembles Remark 3.2.4, and we are going to refer to this observation throughout the chapter.

*Remark 3.2.14.* Observe that any  $\varphi \in \tilde{\Sigma}(\tilde{U}^3)$  can be characterized as follows: there exist  $\varepsilon \in [0, 1)$  and

$$-1 < s_1 \leq s_1^* < s_2 \leq s_2^* < s_3 \leq s_3^* < 0$$

with

$$s_1^* - s_1 = \tilde{T}(\varepsilon), \quad s_2^* - s_2 = \hat{T}(\varepsilon), \quad s_3^* - s_3 = \tilde{T}(\varepsilon)$$

so that  $\varphi \in C$  and

- (i)  $\varphi(-1) = 0$ ,
- (ii)  $\varphi$  is of type  $(K)$  on  $[-1, s_1]$ ,
- (iii)  $\varphi$  is of type  $(-K, 1 + \varepsilon)$  on  $[s_1, s_1^*]$ ,
- (iv)  $\varphi$  is of type  $(0)$  on  $[s_1^*, s_2]$ ,
- (v)  $\varphi$  is of type  $(-K, -1)$  on  $[s_2, s_2^*]$ ,
- (vi)  $\varphi$  is of type  $(-K)$  on  $[s_2^*, s_3]$ ,
- (vii)  $\varphi$  is of type  $(K, -1 - \varepsilon)$  on  $[s_3, s_3^*]$ ,
- (viii)  $\varphi$  is of type  $(0)$  on  $[s_3^*, 0]$ ,
- (ix)  $\varphi(s) > 1 + \varepsilon$  for  $s \in [s_1, s_1^*] \cup [s_2, s_2^*]$ ,
- (x)  $\varphi(s) < -1 - \varepsilon$  for  $s \in [s_3, s_3^*]$ ,
- (xi) if  $-1 \leq t_2 < s_1$  with  $\varphi(t_2) = 1 + \varepsilon$ , then  $x^\varphi(t_2 + 1) < 0$ ,
- (xii) if  $s_1^* < t_3 < s_2$  with  $\varphi(t_3) = 1 + \varepsilon$ , then  $x^\varphi(t_3 + 1) > 0$ .

Note that (i)-(viii) characterize  $\varphi \in \tilde{\Sigma}(\tilde{U}^1)$  and (i)-(x) characterize  $\varphi \in \tilde{\Sigma}(\tilde{U}^2)$ .

For  $(a, \varepsilon) \in \tilde{U}^3$ , let  $\tau$  be the (unique) zero of  $x^{\tilde{\Sigma}(a, \varepsilon)}$  on  $[t_2 + 1, t_3 + 1]$ . If  $(a, \varepsilon) \in \tilde{U}^3$

### 3.2 LSOP solutions for special nonlinearities

and  $t_1, t_2, \dots, t_8, \tau$  are defined as in this subsection, then  $x_{\tau+1} \in \tilde{\Sigma}(\tilde{U}^1)$  and

$$x_{\tau+1} = \tilde{\Sigma}(t_3 + 1 - \tau, t_5 - t_4, t_7 - t_6, \varepsilon).$$

As in the previous subsection,  $\tau$  and  $t_i$ ,  $i \in \{1, \dots, 6\}$ , are  $C^1$ -smooth functions of  $(a, \varepsilon)$ . Therefore we may introduce the  $C^1$ -smooth map  $\tilde{F} : \tilde{U}^3 \rightarrow \mathbb{R}^3$ ,  $\tilde{F}(a, \varepsilon) = (t_3 + 1 - \tau, t_5 - t_4, t_7 - t_6)$ . In case  $\tilde{F}(a, \varepsilon) = a$  for  $(a, \varepsilon) \in \tilde{U}^3$ , then  $x^{\tilde{\Sigma}(a, \varepsilon)}$  is a periodic solution of Eq. (3.2).

Introduce notation

$$\tilde{U}_\varepsilon^3 = \left\{ a \in \mathbb{R}^3 : (a, \varepsilon) \in \tilde{U}^3 \right\}, \quad \varepsilon \in [0, 1),$$

and recall the definition of  $K^*$  from the previous subsection. We obtain the following results analogously to Proposition 3.2.8 and Proposition 3.2.11.

**Proposition 3.2.15.** *For  $K > K^*$ , there exists a unique  $\tilde{a} \in \tilde{U}_0^3$  with  $\tilde{F}(\tilde{a}, 0) = \tilde{a}$ . For  $K \in (3, K^*]$ ,  $\tilde{F}(a, 0) = a$  has no solution in  $\tilde{U}_0^3$ .*

It can be shown that for  $K = K^*$ ,  $\tilde{F}(\cdot, 0)$  has a fixed point on the boundary of  $\tilde{U}_0^3$ , and it equals the fixed point of  $F(\cdot, 0)$ .

**Proposition 3.2.16.** *For  $K > K^*$ ,  $x^{\tilde{\Sigma}(\tilde{a}, 0)} : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution.*

*Remark 3.2.17.* For  $K = 7$ , a numerical study executed with the aid of the CAPD program [1] gives that

$$\tilde{a} \in [0.2202, 0.2203] \times [0.2876, 0.2877] \times [0.3585, 0.3586].$$

In addition, it is shown that the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $D_a \tilde{F}(\tilde{a}, 0) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$  are real with  $\lambda_1 = 0$ ,  $\lambda_2 \in [-0.2415, 0.2347]$  and  $\lambda_3 \in [2.3226, 2.3227]$ .

**Proposition 3.2.18.** *For  $K = 7$ , there exists  $\tilde{\varepsilon}_0 > 0$  such that for all  $\varepsilon \in [0, \tilde{\varepsilon}_0)$ ,  $\tilde{F}(a, \varepsilon) = a$  has a solution  $\tilde{a}(\varepsilon)$  in  $\tilde{U}_\varepsilon^3$ , and  $x^{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)} : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution.*

*Remark 3.2.19.* It follows from the smoothness of  $\tilde{F}$  and Remark 3.2.17, that one may set  $\tilde{\varepsilon}_0 > 0$  so small that for  $\varepsilon \in [0, \tilde{\varepsilon}_0)$ , the eigenvalues  $\lambda_1, \lambda_2$  and  $\lambda_3$  of  $D_a \tilde{F}(\tilde{a}(\varepsilon), \varepsilon)$  satisfy

$$0 \leq |\lambda_1| \leq |\lambda_2| < .5, \quad 2 < \lambda_3.$$

Note that  $\lambda_3$  is necessarily real. Either both  $\lambda_1$  and  $\lambda_2$  is real, or  $\lambda_2 = \overline{\lambda_1}$ .

We can summarize our results regarding case  $\varepsilon = 0$  as follows. For  $K \in (3, K^*)$ , Eq. (3.2) admits no periodic solutions with initial function in  $\Sigma(U_0^3, 0) \cup \tilde{\Sigma}(\tilde{U}_0^3, 0)$ . For  $K > K^*$ , Eq. (3.2) has a unique periodic solution with initial segment in  $\Sigma(U_0^3, 0)$  and a

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unique periodic solution with initial segment in  $\tilde{\Sigma}(\tilde{U}_0^3, 0)$ . It can be shown that for  $K = K^*$ , there is a single periodic solution with initial function in  $\text{bd}\Sigma(U_0^3, 0) \cap \text{bd}\tilde{\Sigma}(\tilde{U}_0^3, 0)$ .

To give a more detailed picture of case  $\varepsilon = 0$ , we are going to show the following results in Section 3.5. For  $K > K^*$  and  $\varepsilon = 0$ ,  $x^{\Sigma(a^*, 0)} : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^{\tilde{\Sigma}(\tilde{a}, 0)} : \mathbb{R} \rightarrow \mathbb{R}$  are the only normalized LSOP solutions of Eq. (3.2) (see Proposition 3.5.4). For  $0 < K < K^*$  and  $\varepsilon = 0$ , Eq. (3.2) has no such nontrivial periodic solutions (see Corollary 3.5.2 and Proposition 3.5.4).

### 3.3 LSOP solutions for a monotone nonlinearity

Theorem 3.1.1 states that one may give a strictly increasing feedback function  $f$  so that (1.1) has exactly two LSOP solutions. In this section we discuss the existence of these LSOP solutions.

Let  $K = 7$  and  $\varepsilon \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$  be fixed, where  $\varepsilon_0$  and  $\tilde{\varepsilon}_0$  are given by Propositions 3.2.11 and 3.2.18, respectively. Proposition 3.2.11 implies that Eq. (3.2) has an LSOP solution with initial function  $\Sigma(a^*(\varepsilon), \varepsilon)$  and with minimal period  $\omega \in (1, 2)$ .

Observe that  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  is a normalized LSOP solution of (3.2) with

$$\Sigma(a^*(\varepsilon), \varepsilon) \in H = \{\varphi \in C : \varphi(-1) = 0\}, \quad \frac{d}{dt}\Sigma(a^*(\varepsilon), \varepsilon) \notin H.$$

Then a Poincaré return map can be defined on  $\{\Sigma(a^*(\varepsilon), \varepsilon)\} + N$ , where  $N$  is a convex bounded open neighborhood of 0 in  $H$ , see Subsection 2.3.2. As  $P$  is  $C^1$ -smooth and has fixed point  $\Sigma(a^*(\varepsilon), \varepsilon)$ , there exists a convex open neighborhood  $\hat{N} \subset N$  of 0 in  $H$  so that  $P^2 = P \circ P$  is defined on  $\{\Sigma(a^*(\varepsilon), \varepsilon)\} + \hat{N}$ . We have the following observation regarding the range of  $P^2$ .

**Proposition 3.3.1.** *There exists an open neighborhood  $V \subseteq \hat{N}$  of 0 in  $H$  so that if  $\varphi \in \{\Sigma(a^*(\varepsilon), \varepsilon)\} + V$ , then  $P^2(\varphi) \in \Sigma(U_\varepsilon^3 \times \{\varepsilon\})$ .*

*Proof.* If  $\varphi \in \{\Sigma(a^*(\varepsilon), \varepsilon)\} + V$ , with an appropriate open ball  $V$  centered at 0 in  $H$ , then  $x_1^\varphi$  and  $x_1^{\Sigma(a^*(\varepsilon), \varepsilon)}$  are close in  $C^1$ -norm, and there exist

$$-1 < \bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \bar{t}_4 < \bar{t}_5 < \bar{t}_6 < \bar{t}_7 < \bar{t}_8 < 0 < \tau_\varphi$$

such that

$$\begin{aligned} \varphi(\bar{t}_1) &= 1, \quad \varphi(\bar{t}_2) = 1 + \varepsilon, \quad \varphi(\bar{t}_3) = 1 + \varepsilon, \quad \varphi(\bar{t}_4) = 1, \\ \varphi(\bar{t}_5) &= -1, \quad \varphi(\bar{t}_6) = -1 - \varepsilon, \quad \varphi(\bar{t}_7) = -1 - \varepsilon, \quad \varphi(\bar{t}_8) = -1, \quad x^\varphi(\tau_\varphi) = 0, \\ \varphi(t) &\in (-1, 1) \quad \text{for } t \in [-1, \bar{t}_1], \\ \varphi(t) &> 1 + \varepsilon \quad \text{for } t \in (\bar{t}_2, \bar{t}_3), \\ \varphi(t) &\in (-1, 1) \quad \text{for } t \in (\bar{t}_4, \bar{t}_5), \end{aligned}$$

### 3.3 LSOP solutions for a monotone nonlinearity

$$\begin{aligned}\varphi(t) &< -1 - \varepsilon \quad \text{for } t \in (\bar{t}_6, \bar{t}_7), \\ x^\varphi(t) &\in (-1, 1) \quad \text{for } t \in (\bar{t}_8, \tau_\varphi],\end{aligned}$$

and the smallest positive zero  $\tau_\varphi$  of  $x^\varphi$  is simple and belongs to  $(\bar{t}_2 + 1, \bar{t}_3 + 1)$ . In consequence,  $P(\varphi) = x_{\tau_\varphi+1}^\varphi$ , and we have

$$\begin{aligned}P(\varphi)(-1) &= 0, \\ P(\varphi) &\text{ is of type (7) on } [-1, \bar{t}_3 - \tau_\varphi], \\ P(\varphi) &\text{ is of type (0) on } [\bar{t}_4 - \tau_\varphi, \bar{t}_5 - \tau_\varphi], \\ P(\varphi) &\text{ is of type (-7) on } [\bar{t}_6 - \tau_\varphi, \bar{t}_7 - \tau_\varphi], \\ P(\varphi) &\text{ is of type (0) on } [\bar{t}_8 - \tau_\varphi, 0].\end{aligned}$$

If the radius of  $V$  is small enough, then also

$$\begin{aligned}P(\varphi)(t) &> 1 + \varepsilon \text{ for } t \in [\bar{t}_3 - \tau_\varphi, \bar{t}_4 - \tau_\varphi], \\ |P(\varphi)(t)| &< 1 \text{ for } t \in [\bar{t}_5 - \tau_\varphi, \bar{t}_6 - \tau_\varphi] \\ \text{and } P(\varphi)(t) &< -1 - \varepsilon \text{ for } t \in [\bar{t}_7 - \tau_\varphi, \bar{t}_8 - \tau_\varphi].\end{aligned}$$

In this case it also follows that whenever  $P(\varphi)$  maps the disjoint subintervals  $J_1, J_2, J_3, J_4$  of  $[-1, 0]$  onto the intervals  $[1, 1 + \varepsilon], [1, 1 + \varepsilon], [-1 - \varepsilon, -1], [-1 - \varepsilon, -1]$ , respectively, then  $P(\varphi)$  is of type (7), (0), (-7), (0) on  $J_1, J_2, J_3, J_4$ , respectively, and therefore  $x^{P(\varphi)}$  is of type (7, 1), (0, 1 +  $\varepsilon$ ), (-7, -1), (0, -1 -  $\varepsilon$ ) on  $J_1 + 1, J_2 + 1, J_3 + 1, J_4 + 1$ , respectively. Using an argument similar to the one given above, now it is easy to see that if we take neighborhood  $V$  small enough, then  $P^2(\varphi)$  satisfies conditions (i)-(viii) of Remark 3.2.4 with some

$$-1 < \bar{s}_1 < \bar{s}_1^* < \bar{s}_2 < \bar{s}_2^* < \bar{s}_3 < \bar{s}_3^* < 0,$$

where

$$\bar{s}_1^* - \bar{s}_1 = T(\varepsilon), \quad \bar{s}_2^* - \bar{s}_2 = \hat{T}(\varepsilon), \quad \bar{s}_3^* - \bar{s}_3 = T(\varepsilon).$$

Using the smooth dependence of solutions on initial data and decreasing the radius of  $V$  further, we can achieve that  $P^2(\varphi)$  satisfies conditions (ix)-(xiv) of Remark 3.2.4 and thus  $P^2(\varphi) \in \Sigma(U_\varepsilon^3 \times \{\varepsilon\})$ .  $\square$

Note that for any small neighborhood  $V$  of 0 in  $H$ , there is  $\varphi \in \{x^{\Sigma(a^*(\varepsilon), \varepsilon)}\} + V$  so that  $P(\varphi)$  does not satisfy conditions (iii), (v) and (vii) of Remark 3.2.4. So we cannot state that  $P(\varphi) \in \Sigma(U_\varepsilon^3 \times \{\varepsilon\})$ .

Proposition 3.2.18 yields that Eq. (3.2) has another LSOP solution with initial segment  $\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$ . Then one may define a Poincaré return map  $P$  in a neighborhood of  $\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$  in  $H$  in an analogous fashion. The analogue of Proposition 3.3.1 holds.

**Proposition 3.3.2.** *There is an open neighborhood  $\tilde{V}$  of 0 in  $H$  such that*

$$\text{if } \varphi \in \{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)\} + \tilde{V}, \text{ then } P^2(\varphi) \in \tilde{\Sigma}(\tilde{U}_\varepsilon^3 \times \{\varepsilon\}).$$

We omit the proof.

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The hyperbolicity of the LSOP orbits is confirmed with the aid of the next proposition.

**Proposition 3.3.3.** *Suppose that  $X$  is a real Banach space,  $\mathcal{V}_0, \mathcal{V}_1$  and  $\mathcal{U}_0, \mathcal{U}_1$  are open subsets of  $X$  and  $\mathbb{R}^m$ , respectively,  $\mathcal{V}_1 \subset \mathcal{V}_0$ ,  $\mathcal{U}_1 \subset \mathcal{U}_0$ ,  $x_0 \in \mathcal{V}_1$ ,  $u_0 \in \mathcal{U}_1$ , the maps*

$$Q : \mathcal{U}_0 \rightarrow \mathbb{R}^m, R : \mathcal{U}_0 \rightarrow X, S : \mathcal{V}_0 \rightarrow X$$

*are  $C^1$ -smooth,  $Q(u_0) = u_0$ ,  $R(u_0) = x_0$ ,  $S(x_0) = x_0$ ,  $Q(\mathcal{U}_1) \subset \mathcal{U}_0$ ,  $R(\mathcal{U}_1) \subset \mathcal{V}_0$ ,  $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$ , moreover,  $DR(u_0) \in \mathcal{L}(\mathbb{R}^m, X)$  is injective and  $S(R(u)) = R(Q(u))$  for all  $u \in \mathcal{U}_1$ . Then*

$$\sigma(DS(x_0)) = \{0\} \cup \sigma(DQ(u_0)),$$

*and for each  $\lambda \in \sigma(DS(x_0)) \setminus \{0\}$ , the corresponding generalized eigenspaces of  $DS(x_0)$  and  $DQ(u_0)$  have the same dimension.*

*Proof.* By introducing the maps

$$u \mapsto Q(u + u_0) - Q(u_0), u \mapsto R(u + u_0) - R(u_0), x \mapsto S(x + x_0) - S(x_0),$$

we may assume that  $x_0 = 0$  and  $u_0 = 0$ .

By the injectivity of  $DR(0)$ , the set  $Y = \{DR(0)u : u \in \mathbb{R}^m\}$  is an  $m$ -dimensional subspace of  $X$  and

$$A : \mathbb{R}^m \ni u \mapsto DR(0)u \in Y$$

is a linear isomorphism. Let  $A^{-1}$  denote the inverse of  $A$ . Since  $Y$  is finite dimensional, there is a closed complementary subspace  $Z$  of  $Y$  in  $X$ , i.e.,  $X = Y \oplus Z$ . The set  $Y_0 = A(\mathcal{U}_0)$  is an open neighborhood of  $0 \in Y$ . Define the map

$$T : Y_0 + Z \ni y + z \mapsto R(A^{-1}(y)) + z \in X.$$

Clearly  $T$  is  $C^1$ -smooth,  $T(0) = 0$ ,  $DT(0) = id_X$  and  $T(Y_0) = R(\mathcal{U}_0)$ . The inverse mapping theorem shows that  $T$  is a local  $C^1$ -isomorphism at  $0 \in X$ .

If  $x$  is in a small neighborhood of  $0 \in X$  and  $x \in R(\mathcal{U}_1)$ , then there exist  $y \in Y_0$  and  $u \in \mathcal{U}_1$  so that  $x = R(u)$ ,  $y = T^{-1}(x)$ ,  $u = A^{-1}y$ . Then by applying  $S(R(u)) = R(Q(u))$ , we find

$$\begin{aligned} S(x) &= S(R(u)) = R(Q(u)) = R(A^{-1}(A(Q(u)))) = T(A(Q(u))) \\ &= T(A(Q(A^{-1}y))) = T \circ A \circ Q \circ A^{-1} \circ T^{-1}(x). \end{aligned} \quad (3.20)$$

In a sufficiently small open neighborhood of  $0 \in X$  define the  $C^1$ -smooth map  $s$  into  $X$  by

$$s(x) = T^{-1}(S(T(x))).$$

### 3.3 LSOP solutions for a monotone nonlinearity

If  $x$  is in the domain of  $s$  and  $T(x) \in \mathcal{V}_1$ , then by the assumption  $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$  there exists  $u \in \mathcal{U}_1$  so that

$$S(T(x)) = R(u) = R(A^{-1}(A(u))) = T(A(u)).$$

Hence for such an  $x$  we obtain that  $s(x) = Au \in Y$ . Therefore  $s$  maps a small neighborhood of  $0 \in X$  into  $Y$ . Consequently,  $Ds(0)(y+z) = By + Cz$  for all  $y \in Y$  and  $z \in Z$ , where  $B \in \mathcal{L}(Y, Y)$  is the derivative of  $s$  restricted to a neighborhood of  $0 \in Y$  and  $C \in \mathcal{L}(Z, Y)$  is the derivative of  $s$  restricted to a neighborhood of  $0 \in Z$ .

If  $y \in Y$  is in a sufficiently small neighborhood of  $0 \in Y$ , then there is  $u \in \mathcal{U}_1$  with  $y = Au$ ,

$$T(y) = T(A(u)) = R(A^{-1}(A(u))) = R(u) \in R(\mathcal{U}_1),$$

and consequently, by applying (3.20),

$$s(y) = T^{-1} \circ S \circ T(y) = T^{-1} \circ T \circ A \circ Q \circ A^{-1} \circ T^{-1} \circ T(y) = A \circ Q \circ A^{-1}(y).$$

Therefore  $B = A \circ DQ(0) \circ A^{-1}$ . From  $DT(0) = DT^{-1}(0) = id_X$  one gets  $DS(0) = Ds(0)$ . Thus

$$DS(0)(y+z) = (A \circ DQ(0) \circ A^{-1})y + Cz$$

for all  $y \in Y, z \in Z$ , with  $\text{range}(C) \subset Y$ , and the statements of the proposition follow in a straightforward way.  $\square$

**Proposition 3.3.4.** *The orbits defined by LSOP solutions  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  and  $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon), \varepsilon)}$  are hyperbolic with 2 and 1 Floquet multipliers outside the unit circle, respectively.*

*Proof.* First we prove that  $DP^2(\Sigma(a^*(\varepsilon), \varepsilon))$  has real eigenvalues  $\mu_1, \mu_2, \mu_3$  of multiplicity 1 with

$$0 < \mu_1 < 0.81, \quad 9 < \mu_2 < 25 < \mu_3.$$

For  $p_0 = \Sigma(a^*(\varepsilon), \varepsilon)$ , set  $X = H, m = 3, x_0 = p_0$  and  $u_0$  to be the fixed point  $a^*(\varepsilon)$  of  $F(\cdot, \varepsilon)$  in  $U_\varepsilon^3$  given by Proposition 3.2.11. Choose  $\mathcal{V}_0 = \{p_0\} + V$ , where open set  $V$  is given by Proposition 3.3.1. Set  $\mathcal{U}_0$  to be the open set on which  $F^2(\cdot, \varepsilon)$  is defined, that is  $\mathcal{U}_0 = \{a \in U_\varepsilon^3 : F(a, \varepsilon) \in U_\varepsilon^3\}$ . Let

$$\mathcal{U}_1 = \{a \in \mathcal{U}_0 : F^2(a, \varepsilon) \in \mathcal{U}_0 \text{ and } \Sigma(a, \varepsilon) \in \mathcal{V}_0\}.$$

Clearly  $\mathcal{U}_1 \subset \mathcal{U}_0$  is open and  $u_0 \in \mathcal{U}_1$ . Let  $\mathcal{V}_1 \subset \mathcal{V}_0$  be an open ball with  $p_0 \in \mathcal{V}_1$  and  $P^2(\mathcal{V}_1) \subset \Sigma(\mathcal{U}_1 \times \{\varepsilon\})$ . The latter set exists because  $p_0 \in \Sigma(\mathcal{U}_1 \times \{\varepsilon\})$ ,  $P^2$  is continuous and maps  $\mathcal{V}_0$  into  $\Sigma(U_\varepsilon^3 \times \{\varepsilon\})$  by Proposition 3.3.1.

Define  $Q = F^2 : \mathcal{U}_0 \rightarrow \mathbb{R}^3, R = \Sigma(\cdot, \varepsilon) : \mathcal{U}_0 \rightarrow H$  and  $S = P^2 : \mathcal{V}_0 \rightarrow H$ . Proposition 3.2.7 yields that  $Q$  is  $C^1$ -smooth, Proposition 3.2.2 gives that  $R$  is  $C^1$ -smooth and

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$DR(u_0)$  is injective. Map  $S$  is also smooth [28]. Clearly  $Q(u_0) = u_0$ ,  $R(u_0) = x_0$  and  $S(x_0) = x_0$ , moreover  $\mathcal{U}_1$  and  $\mathcal{V}_1$  are chosen so that  $Q(\mathcal{U}_1) \subset \mathcal{U}_0$ ,  $R(\mathcal{U}_1) \subset \mathcal{V}_0$  and  $S(\mathcal{V}_1) \subset R(\mathcal{U}_1)$  hold. It is easy to see that  $S(R(u)) = R(Q(u))$  for all  $u \in \mathcal{U}_1$ .

Remark 3.2.12 implies that the eigenvalues of  $DQ(u_0)$  are  $\mu_i$ ,  $i \in \{1, 2, 3\}$ , with  $0 < \mu_1 < 0.81$  and  $9 < \mu_2 < 25 < \mu_3$ . It follows from Proposition 3.3.3 that the eigenvalues of  $DP^2(p_0)$  are  $0, \mu_1, \mu_2, \mu_3$  with the above bounds, and  $\mu_i$ ,  $i \in \{1, 2, 3\}$ , are simple.

If  $\mu$  is an eigenvalue of  $DP(p_0)$ , then  $\mu^2$  is an eigenvalue of  $DP^2(p_0) = DP(p_0) \circ DP(p_0)$ , and the generalized eigenspace of  $DP(p_0)$  associated to  $\mu$  is clearly a subset of the generalized eigenspace of  $DP^2(p_0)$  associated to  $\mu^2$ . Consequently,  $DP(p_0)$  has two simple real eigenvalues outside the unit circle, and it has no eigenvalue with absolute value 1.

The statement for  $x^{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)}$  can be verified in a similar way.  $\square$

Choose  $D = \mathbb{R}$  and consider the Banach space  $C_b^1(D, \mathbb{R}) = C_b^1(\mathbb{R}, \mathbb{R})$ . Clearly  $f^{7, \varepsilon} \in C_b^1(\mathbb{R}, \mathbb{R})$  for all  $\varepsilon \in [0, 1)$ .

**Proposition 3.3.5.** *Set  $\mu = 1$ ,  $K = 7$ . Then for each  $\varepsilon \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$ , where  $\varepsilon_0$  and  $\tilde{\varepsilon}_0$  are given by Propositions 3.2.11 and 3.2.18, respectively, there exists  $\delta_0 = \delta_0(\varepsilon) > 0$  so that if  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfies (H1), and  $\|f - f^{7, \varepsilon}\|_{C_b^1} < \delta_0$ , then Eq. (1.1) admits two normalized LSOP solutions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  with  $p(\mathbb{R}) \subsetneq q(\mathbb{R})$ . The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

*are hyperbolic, and have 2 and 1 Floquet multipliers outside the unit circle, respectively.*

*Proof.* Consider nonlinearities  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfying hypothesis (H1). Then Theorem 2.3.5 and Proposition 3.3.4 imply that there exists  $\delta_0 = \delta_0(\varepsilon) > 0$  such that if  $\|f - f^{7, \varepsilon}\|_{C_b^1} < \delta_0$ , then Eq. (1.1) has two periodic solutions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  with  $p_0 \rightarrow \Sigma(a^*(\varepsilon), \varepsilon)$  and  $q_0 \rightarrow \tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$  in  $C$  as  $\|f - f^{K, \varepsilon}\|_{C_b^1} \rightarrow 0$ . As the initial segments  $p_0$  and  $q_0$  are arbitrarily close to  $\Sigma(a^*(\varepsilon), \varepsilon)$  and  $\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$ , respectively, and the periodic solutions are of monotone type, we get  $V(p_0) = V(q_0) = 2$  if  $\delta_0$  is small enough. In this case the periodicity of  $p$  and  $q$  and the monotonicity of  $V$  gives that  $V(p_t) = V(q_t) = 2$  for all  $t \in \mathbb{R}$ . In addition, it is easy to see that one may choose  $\delta_0$  so small that  $p(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$  and  $q(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ . Hence  $p$  and  $q$  are LSOP solutions of Eq. (1.1). Obviously we may assume that  $p$  and  $q$  are normalized. As  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}(\mathbb{R}) \subsetneq x^{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)}(\mathbb{R})$ , we have  $p(\mathbb{R}) \subsetneq q(\mathbb{R})$ .

As we have seen in Subsection 2.3.2, one may define a  $C^1$ -smooth Poincaré return map  $P$  in a small neighborhood of  $p_0$  in  $H = \{\varphi \in C : \varphi(-1) = 0\}$  with fixed point  $p_0$ . As the Poincaré return map depends smoothly on the right side of the equation



and as  $f$  is close to  $f^{7,\varepsilon}$  in  $C_b^1$ -norm, we may suppose using Proposition 3.3.4 that  $DP(p_0)$  has exactly two eigenvalues  $\lambda_1 > \lambda_2 > 1$  with absolute value not smaller than 1. So  $\mathcal{O}_p$  is hyperbolic with two Floquet multipliers outside the unit circle. Similarly, Proposition 3.3.4 implies  $\mathcal{O}_q$  is hyperbolic with exactly one Floquet multiplier outside the unit circle.  $\square$

The statement of the previous proposition holds even if we consider functions in  $C_b^1(D, \mathbb{R})$ , where  $D$  is chosen to be any open set containing

$$\left\{x^{\Sigma(a^*(\varepsilon), \varepsilon)}(t) : t \in \mathbb{R}\right\} \cup \left\{x^{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)}(t) : t \in \mathbb{R}\right\}.$$

To verify Theorem 3.1.1, we have to exclude the existence of more normalized LSOP solutions. The proof of Theorem 3.1.1 is completed at the end of Section 3.5.

### 3.4 Properties of periodic solutions

This section describes some useful properties of periodic solutions of Eq. (1.1). The next two results are well-known for the case when  $f$  is smooth and strictly increasing, see [26],[32] and [33]. The first proposition is analogous to Theorem 7.1 in [33] and the proof presented here is a slight modification of the proof of Theorem 7.1 in [33].

**Proposition 3.4.1. (Monotonicity)** *Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and bounded,  $f$  is either continuously differentiable or there exist  $u_1 < u_2 < \dots < u_N$  with  $N \geq 1$  so that  $f|_{(-\infty, u_1]}$ ,  $f|_{[u_1, u_2]}$ , ...,  $f|_{[u_N, \infty)}$  are continuously differentiable. If  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a nontrivial periodic solution of Eq. (1.1), then  $p$  is of monotone type.*

*Proof.* Set points  $t_0 < t_1 < t_0 + \omega$  so that  $p(t_0) = \min_{t \in \mathbb{R}} p(t)$  and  $p(t_1) = \max_{t \in \mathbb{R}} p(t)$ , where  $\omega$  is the minimal period of  $p$ . We have to verify that  $\dot{p}(t) \geq 0$  for  $t \in (t_0, t_1)$  and  $\dot{p}(t) \leq 0$  for  $t \in (t_1, t_0 + \omega)$ .

To prove the lemma indirectly, assume that  $\dot{p}(t) < 0$  for some  $t \in (t_0, t_1)$ .

Recall that  $\xi$  is a regular value of  $p$ , if for each  $t \in \mathbb{R}$  with  $p(t) = \xi$ ,  $\dot{p}(t) \neq 0$  holds. According to Sard's Lemma [42], we may choose  $\xi \in (p(t_0), p(t_1))$  so that  $\xi$  is a regular value of  $p$  and  $p(t^*) = \xi$ ,  $\dot{p}(t^*) < 0$  for some  $t^* \in (t_0, t_1)$ . Fix such  $\xi$  and  $t^*$ . Since  $p$  is continuously differentiable, one may give  $t_2 \in (t_0, t^*)$  and  $t_3 \in (t^*, t_1)$  so that  $p(t_2) = p(t_3) = \xi$ ,  $\dot{p}(t_2) > 0$ ,  $\dot{p}(t_3) > 0$  and for  $t \in (t_2, t_3) \setminus \{t^*\}$ ,  $p(t) \neq \xi$ .

Define the curve

$$\Gamma : [t_0, t_0 + \omega] \ni t \mapsto \pi p_t = (p(t), p(t-1)) \in \mathbb{R}^2$$

and

$$L : [0, 1] \ni s \mapsto (\xi, sp(t_2-1) + (1-s)p(t_3-1)) \in \mathbb{R}^2.$$

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We claim that  $\Gamma$  is a simple closed curve. If not, then there exist  $t_4, t_5$  with  $t_0 \leq t_4 < t_5 < t_0 + \omega$  so that  $\Gamma(t_4) = \Gamma(t_5)$ . With  $x(t) := p(t + t_4)$  and  $\tilde{x}(t) := p(t + t_5)$ , Proposition 2.3.4 implies  $\pi x_0 \neq \pi \tilde{x}_0$ , a contradiction. Thus curve  $\Gamma$  is simple.

Next we claim that if  $t \in [t_0, t_1]$  with  $p(t) = \xi$  and  $\dot{p}(t) < 0$ , then  $\Gamma(t) \notin L$ . Indeed, for such  $t$  we have  $f(p(t - 1)) = \dot{p}(t) + \xi < \xi$ , while  $f(p(t_i - 1)) = \dot{p}(t_i) + \xi > \xi$  for  $i \in \{2, 3\}$ . As  $f$  is monotone nondecreasing, the claim follows.

As a result,  $J = \Gamma|_{(t_2, t_3)} \cup L$  is a simple closed curve.

Since  $\dot{p}(t_2) > 0$ ,  $\dot{p}(t_3) > 0$  and  $\Gamma(t_2) \neq \Gamma(t_3)$ , there exist  $\varepsilon > 0$ ,  $C^1$ -maps  $\gamma_j : [\xi - \varepsilon, \xi + \varepsilon] \rightarrow \mathbb{R}$ , constants  $\delta_j^+ > 0$ ,  $\delta_j^- > 0$  for  $j \in \{2, 3\}$  so that

$$\{(u, \gamma_j(u)) : u \in [\xi - \varepsilon, \xi + \varepsilon]\} = \Gamma\left([t_j - \delta_j^-, t_j + \delta_j^+]\right), \quad j \in \{2, 3\},$$

and sets  $R^-$ ,  $R^+$  defined as

$$\{(u, v) : u \in (\xi - \varepsilon, \xi), v \text{ is in the open interval defined by } \gamma_2(u) \text{ and } \gamma_3(u)\},$$

$$\{(u, v) : u \in (\xi, \xi + \varepsilon), v \text{ is in the open interval defined by } \gamma_2(u) \text{ and } \gamma_3(u)\},$$

respectively, belong to different connected components of  $\mathbb{R}^2 \setminus J$  (since  $\Gamma(t) \notin L$  for all  $t \in (t_2, t_3)$ ). We have  $\Gamma(t_2 - \delta_2^-) \notin J$ ,  $\Gamma(t_3 + \delta_3^+) \notin J$  and  $\Gamma(t_2 - \delta_2^-) \in \overline{R^-}$ ,  $\Gamma(t_3 + \delta_3^+) \in \overline{R^+}$ . Combining the above facts, we conclude that  $\Gamma(t_2 - \delta_2^-)$  and  $\Gamma(t_3 + \delta_3^+)$  belong to different connected components of  $\mathbb{R}^2 \setminus J$ . Clearly,  $\Gamma(t_0)$  and  $\Gamma(t_1)$  belong to the exterior of  $J$ . Then in case  $\Gamma(t_2 - \delta_2^-) \in \text{int}(J)$  there exists  $t^{**} \in (t_0, t_2)$  such that  $\Gamma$  enters from  $\text{ext}(J)$  into  $\text{int}(J)$  through  $\Gamma(t^{**}) \in L$ . In this case  $R^+ \subset \text{ext}(J)$ ,  $R^- \subset \text{int}(J)$  and  $\dot{p}(t^{**}) < 0$  follows. This is a contradiction to the fact that if  $t \in [t_0, t_1]$  with  $p(t) = \xi$  and  $\dot{p}(t) < 0$ , then  $\Gamma(t) \notin L$ .

If  $\Gamma(t_3 + \delta_3^+) \in \text{int}J$ , then there is  $t^{**} \in (t_3, t_1)$  so that  $\Gamma$  enters from  $\text{int}(J)$  into  $\text{ext}(J)$  through  $\Gamma(t^{**}) \in L$ . We also have  $R^+ \subset \text{int}(J)$ ,  $R^- \subset \text{ext}(J)$  in this case and again  $\dot{p}(t^{**}) < 0$  follows, a contradiction.

The assumption that  $\dot{p}(t) > 0$  for some  $t \in (t_1, t_0 + \omega)$  leads to a contradiction analogously.  $\square$

The following statement resembles Theorem 7.2 in [33]. As we consider only scalar equations, the proof is elementary in our case.

**Proposition 3.4.2. (Symmetry)** *Assume the hypotheses of Proposition 3.4.1 and in addition suppose that  $f(0) = 0$ ,  $f$  is odd and 0 belongs to the range of  $p$ . Then  $p$  is of special symmetry.*

*Proof.* Let  $\omega$  denote the minimal period of  $p$ . Set points  $t_0 < t_1 < t_0 + \omega$  as in the previous proof, that is so that  $p(t_0) = \min_{t \in \mathbb{R}} p(t) < 0$  and  $p(t_1) = \max_{t \in \mathbb{R}} p(t) > 0$ .

### 3.4 Properties of periodic solutions

According to Proposition 3.4.1, the set of zeros of  $p$  in  $(t_0, t_1)$  is an interval:

$$[z_0, z_1] = \{t \in (t_0, t_1) : p(t) = 0\}$$

with  $t_0 < z_0 \leq z_1 < t_1$ . Similarly, one may set  $z_2$  and  $z_3$  so that  $[z_2, z_3] \subset (t_1, t_0 + \omega)$ ,  $p(t) = 0$  for  $t \in [z_2, z_3]$  and  $p(t) \neq 0$  for  $t \in (t_1, t_0 + \omega) \setminus [z_2, z_3]$ . Of course,  $z_0 = z_1$  or  $z_2 = z_3$  is possible.

Consider the curve  $\Gamma : [t_0, t_0 + \omega] \ni t \mapsto \pi p_t \in \mathbb{R}^2$ . As we have verified in the proof of Proposition 3.4.1,  $\Gamma$  is a simple closed curve. Setting  $x = p$  and  $\tilde{x} \equiv 0$ , Proposition 2.3.4 yields that  $\Gamma(t) \neq (0, 0)^{tr}$  for  $t \in [t_0, t_0 + \omega]$ .

Next we verify that  $(0, 0)^{tr} \in \text{int}(\Gamma)$ . For  $t \in (z_1, t_1]$ ,  $p(t) > 0$ ,  $\dot{p}(t) \geq 0$ , hence  $f(p(t-1)) = \dot{p}(t) + p(t) > 0$  and necessarily  $p(t-1) > 0$ . We claim that  $p(t-1) > 0$  holds also for  $t \in [z_0, z_1]$ . If not, then there exists  $z^* \in [z_0, z_1]$  so that  $p(z^* - 1) = 0$ , which contradicts  $\Gamma(z^*) \neq (0, 0)^{tr}$ . Therefore

$$\Gamma(t) \in \{(u, v) \in \mathbb{R}^2 : u \geq 0, v > 0\} \text{ for } t \in [z_0, t_1].$$

If  $t \in (z_3, t_0 + \omega]$ , then  $p(t) < 0$ ,  $\dot{p}(t) \leq 0$ , hence  $f(p(t-1)) = \dot{p}(t) + p(t) < 0$  and  $p(t-1) < 0$ . It can be verified in a similar manner that  $p(t-1) < 0$  holds for  $t \in [z_2, z_3]$  and thus

$$\Gamma(t) \in \{(u, v) \in \mathbb{R}^2 : u \leq 0, v < 0\} \text{ for } t \in [z_2, t_0 + \omega].$$

Since  $\Gamma$  is a simple closed curve and there exists no

$$t \in [t_0, t_0 + \omega] \setminus ([z_0, t_1] \cup [z_2, t_0 + \omega])$$

such that  $\Gamma(t)$  is on the ordinate-axis, we obtain that  $(0, 0)^{tr} \in \text{int}(\Gamma)$ .

Now take periodic function  $q : \mathbb{R} \ni t \mapsto -p(t) \in \mathbb{R}$  with minimal period  $\omega$  and consider curve  $\Gamma' : [t_0, t_0 + \omega] \ni t \mapsto \pi q_t \in \mathbb{R}^2$ . Since  $f$  is odd,  $q$  is a solution of Eq. (1.1). Clearly  $\Gamma'(t) = -\Gamma(t)$  for all  $t \in [t_0, t_0 + \omega]$ . Because  $(0, 0)^{tr} \in \text{int}(\Gamma)$ , curves  $\Gamma$  and  $\Gamma'$  intersect, that is  $t^* \in [t_0, t_0 + \omega]$  and  $t^{**} \in [t_0, t_0 + \omega]$  can be given with  $\Gamma(t^*) = \Gamma'(t^{**})$ . Set  $\tilde{q} : \mathbb{R} \ni t \mapsto p(t + t^* - t^{**}) \in \mathbb{R}$ . If  $q$  and  $\tilde{q}$  are different periodic solutions of Eq. (1.1), then Proposition 2.3.4 implies  $\pi q_{t^{**}} \neq \pi \tilde{q}_{t^{**}}$ , that is  $\Gamma(t^*) \neq \Gamma'(t^{**})$ , a contradiction. So  $p(t + t^* - t^{**}) = -p(t)$  for all  $t$ . Necessarily  $t^* - t^{**} = \omega/2$ .  $\square$

Note that we have an analogous result for special nonlinearity  $f^{K,0}$ ; it is shown in Section 3.2 that for  $K > K^*$ , periodic solutions  $x^{\Sigma(a^*,0)} : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^{\tilde{\Sigma}(\tilde{a},0)} : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (3.2) are of monotone type and special symmetry. We conjecture that for case  $\varepsilon = 0$ , all nontrivial periodic solutions of Eq. (3.2) are in possession of these properties.

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Let  $K_0 > 3$  and  $K_1 > K_0$  be fixed. Choose

$$\delta = \min_{K_0 \leq K \leq K_1} \frac{e^{1/K} - 1}{2(K+1)} > 0.$$

The next result is slightly more general than necessary in this chapter. The stated property uniformly holds for  $K$  in a compact interval.

**Proposition 3.4.3.** *Assume  $\mu = 1$ ,  $K \in [K_0, K_1]$ ,  $\varepsilon \in (0, \delta)$  and  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a normalized LSOP solution of Eq. (3.2) with minimal period  $\omega > 0$ . Then  $p$  is of monotone type and special symmetry, and the following assertions hold.*

- (i) *The zeros of  $p$  are simple.*
- (ii)  *$\omega \in \left(1 + \frac{1}{K}, 2 - \frac{1}{2K}\right)$ .*
- (iii)  *$\max_{t \in \mathbb{R}} p(t) > e^{1/K}$ .*
- (iv) *Choose  $t_{\max} \in (-1, 0)$  with  $p(t_{\max}) = \max_{t \in \mathbb{R}} p(t)$ . Let  $t_1$  be the largest  $t \in (-1, t_{\max})$  with  $p(t) = 1$ , and let  $t_4$  be the smallest  $t \in (t_{\max}, \infty)$  with  $p(t) = 1$ . Then*

$$\dot{p}(t) \geq K - 2 \text{ for all } t \in (t_1 - \delta, t_1 + \delta),$$

$$\dot{p}(t) \leq -\frac{1}{2} \text{ for all } t \in (t_4 - \delta, t_4 + \delta).$$

*Let  $t_2$  be the largest  $t \in (t_1, t_{\max})$  with  $p(t) = 1 + \varepsilon$ , and let  $t_3$  be the smallest  $t \in (t_{\max}, t_4)$  with  $p(t) = 1 + \varepsilon$ .*

- (v) *If  $t_2 + 2 < \omega$  and  $t_4 - t_1 < 1 - \omega/2$ , then  $p_0 \in \Sigma(U_\varepsilon^2, \varepsilon)$  with*

$$p_0 = \Sigma\left(t_3 + 2 - \omega, t_1 - t_4 + \frac{\omega}{2}, t_3 - t_2, \varepsilon\right).$$

- (vi) *If  $t_2 + 2 < \omega$  and  $t_3 - t_2 > 1 - \omega/2$ , then  $p_0 \in \tilde{\Sigma}(\tilde{U}_\varepsilon^2, \varepsilon)$  with*

$$p_0 = \tilde{\Sigma}\left(t_3 + 2 - \omega, t_1 - t_4 + \frac{\omega}{2}, t_3 - t_2, \varepsilon\right).$$

*Proof.* Assume  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a normalized LSOP solution of Eq. (3.2). By definition,  $V(p_t) = 2$  for all  $t \in \mathbb{R}$ . Proposition 3.4.1 and Proposition 3.4.2 imply  $p$  is of monotone type and special symmetry. Setting  $t_{\min} = t_{\max} + \omega/2$ , we have  $-p(t_{\max}) = p(t_{\min}) = \min_{t \in \mathbb{R}} p(t)$ , and  $p$  is monotone nondecreasing on intervals  $[t_{\min} + k\omega, t_{\max} + (k+1)\omega]$ , monotone nonincreasing on intervals  $[t_{\max} + k\omega, t_{\min} + k\omega]$ ,  $k \in \mathbb{Z}$ . By Proposition 2.3.4,  $(p(t-1), p(t)) \neq (0, 0)$  for all  $t \in \mathbb{R}$ .

We claim  $\omega \in (1, 2)$ . If  $\omega \geq 2$ , then  $t_{\min} = t_{\max} + \omega/2 > -1 + \omega/2 > 0$ . By the special symmetry,  $p(-1 + \omega/2) = p(-1) = 0$ . The monotone property yields  $p(s) \geq 0$  for  $s \in [-1, -1 + \omega/2]$ . Consequently  $V(p_0) = 0$ , a contradiction. Suppose  $\omega \leq 1$ . Then  $-1 < t_{\max} < t_{\min} < -1 + \omega \leq 0$ , and  $p(-1 + \omega) = 0$ ,  $p(-1 + \omega + s) > 0$  for all  $s \in (0, \eta)$  for some  $\eta > 0$ . Clearly there is an arbitrarily small  $s > 0$  with

$\dot{p}(-1 + \omega + s) > 0$ . Then from Eq. (3.2)

$$f^{K,\varepsilon}(p(-1 + \omega + s - 1)) = \dot{p}(-1 + \omega + s) + p(-1 + \omega + s) > 0$$

and  $p(-1 + \omega + s - 1) > 1$  follow. By continuity,  $p(-2 + \omega) \geq 1$ . Hence, by using  $-2 + \omega \leq -1$ ,  $p(-1) = 0$  and the monotone property of  $p$ , one obtains  $\mu \in (-2 + \omega, -1)$  with  $p(\mu) < 0$ . Then  $p$  has at least three sign changes on  $[-2 + \omega, -1 + \omega]$ , a contradiction. Therefore  $1 < \omega < 2$ .

We claim  $p(0) < 0$ . The equality  $p(0) = 0$  contradicts Proposition 2.3.4 since  $p(-1) = 0$ . If  $p(0) > 0$ , then by (3.2) and  $p(-1) = 0$ ,  $\dot{p}(0) < 0$ . The monotone property of  $p$  yields either  $p(s) \geq 0$  for all  $s \in [-1, 0]$  or  $\omega < 1$ , a contradiction. Thus  $p(0) < 0$ .

From  $p(0) < 0$ , by (3.2) and  $p(-1) = 0$ ,  $\dot{p}(0) > 0$  follows. Hence  $t_{\min} < 0$ .

Set  $\tau = \omega - 1 \in (0, 1)$ . It is easy to see that  $p(t) \leq 0$  for all  $t \in [0, \tau]$ , and  $p(t) > 0$  for all  $t \in (\tau, \tau + \eta)$  for some  $\eta > 0$ .

Define  $t_5 = t_1 + \omega/2$ ,  $t_8 = t_4 + \omega/2$ ,  $t_9 = t_1 + \omega$  and  $t_{12} = t_4 + \omega$ , see Fig. 3.7. Clearly  $t_9 > \tau$ . Note that  $z = -1 + \omega/2 = \frac{\tau-1}{2} \in (t_4, t_5)$  is also a negative zero of  $p$ .

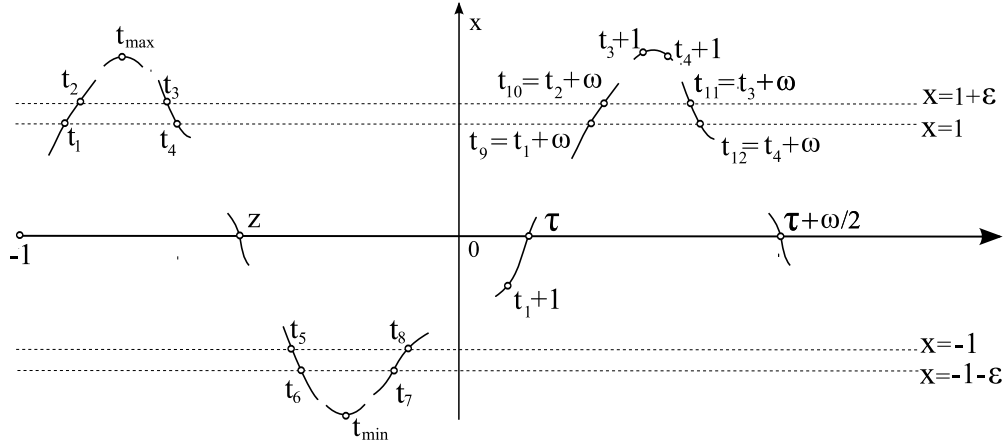


Figure 3.7: Plot of  $p$  in the proof of Proposition 3.4.3

Observe that  $0 < \varepsilon < \delta$  implies

$$\varepsilon < \frac{1}{K} \text{ and } \varepsilon < \frac{e^{\frac{1}{K}}}{2K}.$$

**Claim (i).**  $\tau \in (t_1 + 1, t_4 + 1)$ .

Proof. As  $p(0) < 0$  and  $p$  is of type (0) on  $[0, t_1 + 1]$ ,

$$p(t) = p(0)e^{-t} < 0 \text{ for all } t \in [0, t_1 + 1]. \quad (3.21)$$

So  $\tau > t_1 + 1$ . If  $p(t_4 + 1) < 0$ , then on the one hand  $p(t) < 0$  for all  $t \in [t_4 + 1, z + 1]$

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(as  $p$  is of type (0) on  $[t_4 + 1, z + 1]$ ), on the other hand

$$z + 1 = \frac{\omega}{2} \in \left( \tau, \tau + \frac{\omega}{2} \right)$$

and  $p(t) \geq 0$  for all  $t \in [\tau, \tau + \omega/2]$ , a contradiction. If  $p(t_4 + 1) = 0$ , then  $p(t) = 0$  for all  $t \in [t_4 + 1, z + 1]$ . By  $(\tau + 1)$ -periodicity,  $p(t) = 0$  follows for  $t \in [t_4 - \tau, z - \tau]$ . By the definitions of  $t_1, t_4$  and  $z$ , the minimal zero of  $p$  in  $(-1, z]$  is in  $(t_4, z]$ . As  $z = (\tau - 1)/2 > \tau - 1$  and thus  $z - \tau > -1$ , this a contradiction. Consequently,  $p(t_4 + 1) > 0$  and  $\tau \in (t_1 + 1, t_4 + 1)$ .  $\square$

Assertion (i) is a direct consequence of Claim (i). Note that if  $t \in (t_1 + 1, t_4 + 1)$  with  $p(t) = 0$ , then

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) = f^{K,\varepsilon}(p(t-1)) > 0.$$

Hence  $\tau$  is a simple zero of  $p$ , and it is the only zero in  $(t_1 + 1, t_4 + 1)$ . By the special symmetry of  $p$ , all zeros of  $p$  are simple, and  $-1, z, \tau$  are the only zeros in  $[-1, \tau]$ .

Assertion (ii) also follows from Claim (i). Indeed, for  $t \in [\tau, t_9]$ ,

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \leq f^{K,\varepsilon}(p(t-1)) \leq K.$$

Hence

$$t_9 - \tau = t_1 + \omega - \tau \geq 1/K. \quad (3.22)$$

Applying (3.22) and  $\tau > t_1 + 1$ , we get

$$\omega \geq \tau - t_1 + \frac{1}{K} = \tau - (t_1 + 1) + 1 + \frac{1}{K} > 1 + \frac{1}{K}.$$

For  $t \in \mathbb{R}$ ,  $|p(t)| \leq K$  by Proposition 2.1.2 and thus  $\dot{p}(t) \geq -2K$  by Eq. (3.2). Hence

$$t_4 \leq z - \frac{1}{2K} < -\frac{1}{2K},$$

and by Claim (i),

$$\omega = \tau + 1 < t_4 + 2 < 2 - \frac{1}{2K}.$$

**Claim (ii).**  $\max_{t \in \mathbb{R}} p(t) \geq e^{1/K}$ .

Proof. We have already shown that  $p(t_4 + 1) > 0$ . For  $t \in [t_4 + 1, t_5 + 1]$ ,  $p(t) = p(t_4 + 1)e^{-(t-t_4-1)}$ , thus  $p$  strictly decreases on  $[t_4 + 1, t_5 + 1]$ . So  $t_9 < t_4 + 1$ . As  $t_4 + (K + 1)/K < t_4 + \omega = t_{12}$ , we derive that  $[t_4 + 1, t_4 + (K + 1)/K] \subset [t_9, t_{12}]$  and thus  $p(t_4 + (K + 1)/K) \geq 1$ .

From (3.22),  $t_5 - t_4 \geq 1/K$  follows by special symmetry. So  $p$  is of type (0) on

$[t_4 + 1, t_4 + 1 + 1/K]$ , and thus

$$\max_{t \in \mathbb{R}} p(t) \geq p(t_4 + 1) = p\left(t_4 + \frac{K+1}{K}\right) e^{\frac{1}{K}} \geq e^{\frac{1}{K}}. \quad \square$$

As a consequence of Claim (ii),  $\max_{t \in \mathbb{R}} p(t) > 1 + \varepsilon$  and one may set  $t_2$  and  $t_3$  so that  $t_2$  is the maximal  $t \in (t_1, t_{\max})$  with  $p(t) = 1 + \varepsilon$  and  $t_3$  is the minimal  $t \in (t_{\max}, t_4)$  with  $p(t) = 1 + \varepsilon$ . Define  $t_6 = t_2 + \omega/2$ ,  $t_7 = t_3 + \omega/2$ ,  $t_{10} = t_2 + \omega$  and  $t_{11} = t_3 + \omega$ , see Fig. 3.7.

Note that it is also verified in the proof of the previous claim that

$$t_{12} - (t_4 + 1) > 1 + \frac{1}{K}. \quad (3.23)$$

**Claim (iii).**  $\dot{p}(t) \leq -1$  for  $t \in [t_4 + 1, t_{12}]$ , and thus  $t_4 - t_3 = t_{12} - t_{11} \leq \varepsilon$ .

**Proof.** First note that  $t_{12} < \tau + \omega/2 < \tau + 1$ . Hence for  $t \in [t_4 + 1, t_{12}]$ ,  $p(t) \geq 1$ ,  $p(t-1) \leq 1$ , and

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \leq -p(t) \leq -1,$$

which is our first assertion. In addition, using  $p(t_{12}) = 1$  and estimation (3.23), we obtain that  $p(t_{12} - s) \geq 1 + s$  for all  $0 \leq s \leq 1/K$ , hence  $t_{12} - t_{11} \leq \varepsilon$ .  $\square$

**Claim (iv).**  $1 + t_2 < t_9$  and  $t_{10} < t_3 + 1$ . In consequence,  $t_2 - t_1 \leq \varepsilon/(K-2)$ .

**Proof.** It follows from the previous claim that  $p(t) \geq 0$  for  $t \in [t_3 + 1, t_4 + 1]$ . Indeed,  $p(t) \geq p(t_4 + 1) - 2K\varepsilon > e^{1/K} - 2K\varepsilon > 0$  by the choice of  $\delta$  and  $\varepsilon \in (0, \delta)$ . Hence  $\dot{p}(t) \leq K$  for  $t \in [t_3 + 1, t_4 + 1]$ .

Suppose that  $t_3 + 1 \leq t_{10}$ , that is  $p(t_3 + 1) \leq 1 + \varepsilon$ . Applying the facts that  $t_4 - t_3 \leq \varepsilon$  and  $p$  strictly decreases on  $[t_4 + 1, t_{12}]$  (see Claim (iii)), we obtain that

$$\max_{t \in \mathbb{R}} p(t) = \max_{t \in [t_3 + 1, t_4 + 1]} p(t) \leq 1 + \varepsilon + K\varepsilon = 1 + (K+1)\varepsilon < e^{\frac{1}{K}}$$

by  $\varepsilon \in (0, \delta)$ , a contradiction to Claim (ii). Thus  $t_{10} < t_3 + 1$ .

If  $t_9 \leq t_2 + 1$ , then

$$t_9 \leq t_2 + 1 < t_2 + \frac{K+1}{K} < t_2 + \omega = t_{10} < t_3 + 1$$

and hence for  $t \in [t_2 + 1, t_2 + (K+1)/K]$ ,

$$\dot{p}(t) = -p(t) + K \geq -(1 + \varepsilon) + K.$$

Thus

$$1 + \varepsilon = p(t_{10}) \geq p\left(t_2 + \frac{K+1}{K}\right) \geq p(t_2 + 1) + \frac{K-1-\varepsilon}{K} \geq 1 + \frac{K-1-\varepsilon}{K},$$

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which contradicts  $\varepsilon < \delta$ . So  $1 + t_2 < t_9$ .

As a result,  $\dot{p}(t) = -p(t) + K \geq K - 2$  for all  $t \in [t_9, t_{10}]$ , and  $\varepsilon = \int_{t_9}^{t_{10}} \dot{p}(s) ds \geq (K - 2)(t_{10} - t_9)$ . As  $t_2 - t_1 = t_{10} - t_9$ , the third statement follows.  $\square$

*A lower bound for  $t_9 - (t_2 + 1)$ .* Applying  $p(t_1 + 1) < 0$  by (3.21),  $t_2 - t_1 \leq \varepsilon / (K - 2)$  by Claim (iv) and  $\dot{p}(t) \leq 2K$  for all  $t \in \mathbb{R}$ , we find  $p(t_2 + 1) < 2K\varepsilon / (K - 2)$ . Therefore

$$1 - \frac{2K}{K - 2}\varepsilon < p(t_9) - p(t_2 + 1) = \int_{t_2+1}^{t_9} \dot{p}(s) ds \leq 2K(t_9 - t_2 - 1),$$

and we obtain that

$$t_9 - (t_2 + 1) > \frac{1}{2K} - \frac{\varepsilon}{K - 2} > \delta. \quad (3.24)$$

*A lower bound for  $t_3 + 1 - t_9$ .* Claim (ii) implies  $\max p(t) \geq e^{1/K}$ . Claim (iv) gives that  $t_9 < t_3 + 1 < t_4 + 1 < t_{12}$ . For  $t \in [t_9, t_4 + 1]$ ,

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t - 1)) \leq -p(t) + K \leq K - 1.$$

In addition,  $p$  strictly decreases on  $[t_4 + 1, t_{12}]$  by Claim (iii), that is  $\max_{t \in \mathbb{R}} p(t) = \max_{t \in [t_9, t_4 + 1]} p(t)$ . So, by using Claim (iii) again,

$$\begin{aligned} e^{\frac{1}{K}} &\leq \max_{t \in \mathbb{R}} p(t) = \max_{t \in [t_9, t_4 + 1]} p(t) \leq 1 + (K - 1)(t_4 + 1 - t_9) \\ &= 1 + (K - 1)(t_3 + 1 - t_9 + t_4 - t_3) \\ &\leq 1 + (K - 1)(t_3 + 1 - t_9) + (K - 1)\varepsilon, \end{aligned}$$

from which

$$t_3 + 1 - t_9 \geq \frac{e^{1/K} - 1}{K - 1} - \varepsilon > \frac{e^{1/K} - 1}{2(K + 1)} \geq \delta \quad (3.25)$$

follows.

Also note that if  $t \geq t_{12}$ , then  $\dot{p}(t) \geq -2K$  and

$$p(t) \geq 1 - 2K(t - t_{12}).$$

Thus

$$p(t) \geq 1/2 \text{ for all } t \in \left[ t_{12}, t_{12} + \frac{1}{4K} \right]. \quad (3.26)$$

Then (3.24) and (3.25) imply that for  $t \in (t_9 - \delta, t_9 + \delta)$ ,  $t - 1 \in (t_2, t_3)$ . Also,  $p(t) \leq p(t_9) + \delta \max \dot{p}(t) \leq 1 + 2K\delta \leq 2$  for  $t \in (t_9 - \delta, t_9 + \delta)$ . Hence

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t - 1)) \geq -2 + K \text{ for } t \in (t_9 - \delta, t_9 + \delta).$$

As  $t_{12} - (t_4 + 1) = t_4 + \omega - (t_4 + 1) > 1/K > \delta$ , Claim (iii) clearly implies  $p(t) \geq 1$



### 3.5 There are two LSOP solutions

and  $\dot{p}(t) \leq -\frac{1}{2}$  for  $t \in (t_{12} - \delta, t_{12}]$ . At last note that as  $t_4 + 1/K \leq t_5 < 0$ ,

$$t_{12} + \delta - 1 = (t_4 + 1 + \tau) + \delta - 1 = \tau + t_4 + \delta \leq \tau + t_5 < \tau$$

that is  $t - 1 \in (t_4, \tau)$  for all  $t \in [t_{12}, t_{12} + \delta)$ . Thus using (3.26) we conclude that

$$\dot{p}(t) = -p(t) + f^{K,\varepsilon}(p(t-1)) \leq -\frac{1}{2} + 0 = -\frac{1}{2}, \quad t \in [t_{12}, t_{12} + \delta).$$

Statement (iv) follows by periodicity.

It remains to prove assertions (v) and (vi). Suppose  $t_2 + 2 < \omega$  and  $t_4 < t_1 + 1 - \omega/2$ . Then  $t_2 + 1 < \omega - 1 = \tau$ ,  $t_{10} < t_9 + \delta < t_3 + 1$ , and

$$t_4 + 1 < t_{11} < t_{12} = t_4 + \omega < t_1 + 1 + \omega/2 = t_5 + 1.$$

It follows that  $p$  is of type  $(K)$  on  $[\tau, t_3 + 1]$ , it is of type  $(0)$  on  $[t_4 + 1, t_5 + 1]$ . The periodicity of  $p$  and the fact that  $p$  is of type  $(0)$  on  $[t_3, t_4]$  imply  $p$  is of type  $(0, 1 + \varepsilon)$  on  $[t_3 + 1, t_4 + 1]$ . By periodicity and special symmetry,  $p$  is of type  $(-K)$  on  $[t_5, t_6]$ , and it is of type  $(-K, -1)$  on  $[t_5 + 1, t_6 + 1]$ . The special symmetry and monotonicity yield  $p_0 = p_\omega \in \Sigma(U_\varepsilon^2, \varepsilon)$  with

$$p_0 = \Sigma\left(t_3 + 2 - \omega, t_1 - t_4 + \frac{\omega}{2}, t_3 - t_2, \varepsilon\right).$$

The case  $t_2 + 2 < \omega$  and  $t_3 - t_2 > 1 - \omega/2$  is analogous. □

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Set  $\mu = 1$ . We study the exact number of LSOP solutions of Eq. (1.1) first for non-linearity  $f^{K,0}$  with  $K > 0$ , then for  $f^{7,\varepsilon}$  with  $\varepsilon > 0$  small, finally for those feedback functions, that are close to  $f^{7,\varepsilon}$  in  $C_b^1$ -norm. As a consequence, we prove Theorem 3.1.1. For simplicity, we use notations introduced in Section 3.2 - without repeating definitions.

### The number of periodic solutions for the step function

As a preliminary result, we show that  $K$  has to be sufficiently large so that Eq. (3.2) has periodic solutions of monotone type and special symmetry.

**Proposition 3.5.1.** *Suppose  $K > 0$ ,  $\varepsilon \in [0, 1)$ ,  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a nontrivial periodic solution of Eq. (3.2), and  $p$  is of monotone type and special symmetry. Then  $K > 1$  and*

$$\frac{\omega}{2} \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K},$$

where  $\omega > 0$  denotes the minimal period of  $p$ .

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*Proof.* Let  $p : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic solution of Eq. (3.2) of monotone type and special symmetry and with minimal period  $\omega > 0$ . It is no restriction to assume that  $p$  is normalized. Clearly,  $\max_{t \in \mathbb{R}} p(t) > 1$  as  $\dot{x}(t) = -x(t)$  has no periodic solutions. Then there exists  $(c_1, c_2, c_3) \in (0, 1)^3$  with  $c_1 + c_2 + c_3 = \omega/2$  so that  $p$  is nondecreasing on  $[-1, -1 + c_1]$  with range in  $[0, 1]$ ,  $p(t) > 1$  for  $t \in (-1 + c_1, -1 + c_1 + c_2)$  and  $p$  is nonincreasing on  $[-1 + c_1 + c_2, -1 + \omega/2]$  with range in  $[0, 1]$ .

As  $\dot{p}(t) \geq 0$  almost everywhere on  $[-1, -1 + c_1]$ ,

$$f^{K,\varepsilon}(p(t-1)) = \dot{p}(t) + p(t) > 0 \text{ for } t \in (-1, -1 + c_1],$$

that is  $p(t) > 1$  for  $t \in (-2, -2 + c_1]$ . Using special symmetry, we conclude that  $c_2 \geq c_1$ .

Obviously,  $(e^t p(t))' = e^t f^{K,\varepsilon}(p(t-1))$  almost everywhere on  $\mathbb{R}$ . Integrating on  $[-1, -1 + c_1]$ , we get

$$e^{-1+c_1} = \int_{-1}^{-1+c_1} e^s f^{K,\varepsilon}(p(s-1)) ds \leq K \int_{-1}^{-1+c_1} e^s ds = K [e^{-1+c_1} - e^{-1}],$$

therefore  $1 \leq K(1 - e^{-c_1})$ . As  $1 - e^{-c_1} < 1$ , necessarily  $K > 1$  and  $c_1 \geq \ln(K/(K-1))$ . Integrating on  $[-1 + c_1 + c_2, -1 + c_1 + c_2 + c_3]$ , we obtain that

$$-e^{-1+c_1+c_2} \geq -K \int_{-1+c_1+c_2}^{-1+c_1+c_2+c_3} e^s ds = -K [e^{-1+c_1+c_2+c_3} - e^{-1+c_1+c_2}],$$

hence  $1 \leq K(e^{c_3} - 1)$  and  $c_3 \geq \ln((K+1)/K)$ .

Therefore

$$\frac{\omega}{2} = c_1 + c_2 + c_3 \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K}.$$

□

**Corollary 3.5.2.** *For  $K \in (0, 3]$  and  $\varepsilon = 0$ , Eq. (3.2) admits no LSOP solutions.*

*Proof.* It is excluded by the previous proposition that we have LSOP solutions for  $K \in (0, 1]$  and  $\varepsilon = 0$ . Suppose  $K \in (1, 3]$ ,  $\varepsilon = 0$  and  $p : \mathbb{R} \rightarrow \mathbb{R}$  is an LSOP solution of Eq. (3.2). Assumption  $\omega/2 < 1$  and Proposition 3.5.1 give that

$$1 > \frac{\omega}{2} = c_1 + c_2 + c_3 \geq 2 \ln \frac{K}{K-1} + \ln \frac{K+1}{K} = \ln \frac{K(K+1)}{(K-1)^2},$$

that is

$$e > \frac{K(K+1)}{(K-1)^2} = 1 + \frac{3}{K-1} + \frac{2}{(K-1)^2}.$$

This is a second order inequality for  $z = 1/(K-1)$ , hence the solution formula gives that

$$z_1 = \frac{-3 - \sqrt{8e+1}}{4} < \frac{1}{K-1} < z_2 = \frac{-3 + \sqrt{8e+1}}{4}.$$

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The first inequality is clearly satisfied as  $K > 1$  and  $z_1 < 0$ . The second inequality implies  $K > 1 + 1/z_2 > 3$ , a contradiction.  $\square$

Recall from Remarks 3.2.4 and 3.2.14 that  $\varphi \in C$  is in  $\Sigma(U_0^1, 0) = \tilde{\Sigma}(\tilde{U}_0^1, 0)$  if and only if  $\varphi(-1) = 0$  and there exist  $-1 < s_1 < s_2 < s_3 < 0$  so that  $x^\varphi$  is of type  $(K)$  on  $[-1, s_1]$ , of type  $(0)$  on  $[s_1, s_2]$ , of type  $(-K)$  on  $[s_2, s_3]$  and of type  $(0)$  on  $[s_3, 0]$ .

**Proposition 3.5.3.** *Assume  $K > 3$ ,  $\varepsilon = 0$  and  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a nontrivial periodic solution of Eq. (3.2),  $x$  is of special symmetry and  $x_0 \in \Sigma(U_0^1, 0) = \tilde{\Sigma}(\tilde{U}_0^1, 0)$ . Then  $x(s_2) = 1$  implies  $K = K^*$ .*

*Proof.* Assume that  $x$  satisfies the conditions of the proposition with  $x(s_2) = 1$ .

Then using (3.5) and the definitions of  $I_1$  and  $I_2$ , we get

$$x(s_1) = e^{-s_1} I_1 = K(1 - e^{-a_1}) \quad (3.27)$$

and

$$e^{a_2} = e^{a_2} x(s_2) = e^{a_2} e^{-s_2} I_2 = K(1 - e^{-a_1}). \quad (3.28)$$

From (3.5), the definition of  $I_3$  and relation (3.28) it follows that

$$x(s_3) = e^{-s_3} I_3 = K(1 - e^{-a_1}) e^{-a_2 - a_3} + K(e^{-a_3} - 1) = e^{-a_3} + K(e^{-a_3} - 1). \quad (3.29)$$

Let  $-1 < t_1 = t_2 < t_3 = t_4 < \dots$  be the consecutive times for which  $x(t_i) \in \{-1, 1\}$ . As  $x$  strictly increases on  $[-1, s_1]$ , strictly decreases on  $[s_1, s_2]$ ,  $\max_{t \in \mathbb{R}} x(t) > 1$  and  $x(s_2) = 1$ , we obtain that  $-1 < t_1 < s_1$  and  $t_3 = s_2$ . Similarly,  $s_2 < t_5 < s_3$ . By special symmetry,  $x(s_3) = -x(s_1)$ , and  $x(s_2) = -x(t_1 + 1) = 1$ . So combining (3.27) and (3.29), we get

$$e^{a_3} = \frac{K+1}{K} e^{a_1}. \quad (3.30)$$

As in the proof of Proposition 3.2.3, we can show that

$$x(t_1 + 1) = \frac{K-1}{K} I_3.$$

Using  $x(t_1 + 1) = -1$ , the definition of  $I_3$ , relations (3.28) and (3.30), it follows that

$$-1 = \frac{K-1}{e} \left[ e^{a_1} - 1 + e^{a_1+a_2} (1 - e^{a_3}) \right] = \frac{-1}{e} (K^2 - 1) (e^{a_1} - 1)^2. \quad (3.31)$$

As  $x$  is periodic,  $a_2 = t_5 - t_4$  (see the remark preceding Proposition 3.2.6). One may show analogously to the proof of Proposition 3.2.6 that

$$a_2 = t_5 - t_4 = \ln \frac{I_2 + K e^{s_2}}{(K-1) I_1} = \ln \frac{e^{a_1} - 1 + e^{a_1+a_2}}{(K-1) (e^{a_1} - 1)}.$$

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Combining this relation with (3.28), we get that  $a_1$  is the following function of  $K$ :

$$a_1 = \ln \frac{K(K-1)}{K^2 - 2K - 1}.$$

Substituting the last result to (3.31), we obtain that equation

$$\frac{(K^2 - 1)(K + 1)^2}{(K^2 - 2K - 1)^2} = e$$

holds for  $K$ , which equation has a unique solution on  $(3, \infty)$  and that is  $K^*$  (see the definition of  $K^*$  before Proposition 3.2.8). So  $K = K^*$ .  $\square$

**Proposition 3.5.4.** *Assume  $K \in (3, \infty) \setminus \{K^*\}$ ,  $\varepsilon = 0$  and  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a normalized LSOP solution of Eq. (3.2). Then  $K > K^*$ , and either  $x_0 = \Sigma(a^*, 0)$ , or  $x_0 = \tilde{\Sigma}(\tilde{a}, 0)$ , where  $\Sigma(a^*, 0)$  and  $\tilde{\Sigma}(\tilde{a}, 0)$  are given in Section 3.2.*

*Proof.* Let  $\tau$  denote the smallest zero of  $x$  on  $[0, \infty)$  with the property that  $x > 0$  on  $(\tau, \tau + \eta)$  with some  $\eta > 0$  small. Since  $x$  is normalized periodic solution with minimal period in  $(1, 2)$ , and as it is of special symmetry and of monotone type, the minimal period is  $\omega = \tau + 1$  and  $x(0) \leq 0$ .

Set  $t_{\max} \in (-1, 0)$  so that  $x(t_{\max}) = \max_{t \in \mathbb{R}} x(t)$  and choose  $t_{\min} = t_{\max} + \omega/2$ . Clearly  $x(t_{\min}) = \min_{t \in \mathbb{R}} x(t)$ . As equation  $\dot{x}(t) = -x(t)$  has no periodic solution,  $x(t_{\max}) = -x(t_{\min}) > 1$ .

As  $x$  is of monotone type and  $x(t_{\max}) = -x(t_{\min}) > 1$ , there exists  $t_1 \in (-1, t_{\max})$  maximal with  $x(t) = 1$  and  $t_3 \in (t_{\max}, t_{\min})$  minimal with  $x(t) = 1$ . Then  $t_5 = t_1 + \omega/2$  is the maximal  $t \in (t_{\max}, t_{\min})$  with  $x(t) = -1$  and  $t_7 = t_3 + \omega/2$  is the minimal  $t \in (t_{\min}, \tau)$  with  $x(t) = -1$ .

Solution  $x$  must be piecewise of type  $(i)$  with  $i \in \{-K, 0, K\}$ . To be more precise,  $x$  is of type  $(0)$  on interval  $[0, t_1 + 1]$ , of type  $(K)$  on  $[t_1 + 1, t_3 + 1]$ , of type  $(0)$  on  $[t_3 + 1, t_5 + 1]$ , of type  $(-K)$  on  $[t_5 + 1, t_7 + 1]$  and of type  $(0)$  on  $[t_7 + 1, \tau + 1]$ . If  $t_1 + 1 \leq \tau < t_3 + 1$ , then

$$(t_3 + 1 - \tau, t_5 - t_3, t_7 - t_5, 0) \in (0, 1)^3 \times \{0\}$$

is in  $U^1 = \tilde{U}^1$  because  $t_3 + 1 - \tau + t_5 - t_3 + t_7 - t_5 = t_7 + 1 - \tau < 1$ , and

$$x_0 = x_{\tau+1} = \Sigma(t_3 + 1 - \tau, t_5 - t_3, t_7 - t_5, 0) = \tilde{\Sigma}(t_3 + 1 - \tau, t_5 - t_3, t_7 - t_5, 0)$$

by Remarks 3.2.4 and 3.2.14.

So we claim that  $\tau \in [t_1 + 1, t_3 + 1]$ . As  $x$  is of type  $(0)$  on  $[0, t_1 + 1]$ ,  $x(t) = x(0)e^{-t} \leq 0$  for  $t \in [0, t_1 + 1]$ . So  $\tau \geq t_1 + 1$ . Suppose for contradiction that  $x(t_3 + 1) \leq$

### 3.5 There are two LSOP solutions

0. Proposition 2.1.2 implies  $|x(t)| < K$ ,  $t \in \mathbb{R}$ . Then as

$$\dot{x}(t) = -x(t) + K > 0, \quad t_1 + 1 < t < t_3 + 1, \quad (3.32)$$

and as

$$x(t) = x(t_3 + 1) e^{-(t-t_3-1)}, \quad t_3 + 1 \leq t \leq t_5 + 1, \quad (3.33)$$

we get that  $x$  is nondecreasing and nonpositive on  $[t_1 + 1, t_5 + 1]$ . So  $x(t) \leq 0$  on  $[t_5, t_5 + 1]$ . On the other hand, for  $t_5 + \omega/2 \in [t_5, t_5 + 1]$  we have  $x(t_5 + \omega/2) = x(t_1 + \omega) = 1$ , a contradiction. Thus  $x(t_3 + 1) > 0$ ,  $\tau \in [t_1 + 1, t_3 + 1)$  and  $x_{\tau+1} \in \Sigma(U_0^1, 0) = \tilde{\Sigma}(\tilde{U}_0^1, 0)$ .

Equations (3.32) and (3.33) now imply that  $x$  strictly increases on  $[t_1 + 1, t_3 + 1]$  and strictly decreases on  $[t_3 + 1, t_5 + 1]$ . Thus  $x(t_3 + 1)$  is a local maximum of  $x$ . As  $x$  is of monotone type and  $\max_{t \in \mathbb{R}} x(t) > 1$ ,  $x(t_3 + 1) > 1$  follows. Also,  $x(t_5 + 1) > 0$  by (3.33). By special symmetry,  $x(t_7 + 1) = x(t_3 + \omega/2 + 1) = -x(t_3 + 1) < -1$ . Remarks 3.2.4 and 3.2.14 yield that if  $x(t_5 + 1) < 1$ , then  $x_0 = x_{\tau+1} \in \Sigma(U_0^2, 0)$ ; if  $x(t_5 + 1) > 1$ , then  $x_0 = x_{\tau+1} \in \tilde{\Sigma}(\tilde{U}_0^2, 0)$ . Case  $x(t_5 + 1) = 1$  is excluded by Proposition 3.5.3.

We have already verified that  $x(t_1 + 1) < 0$  and  $x(t_3 + 1) > 0$ . If  $x_0 \in \Sigma(U_0^2, 0)$ , then  $x(t_1 + 1) = -x(t_5 + 1) > -1$ , so Remark 3.2.4 yields that  $x_0 \in \Sigma(U_0^3, 0)$  and thus  $(t_3 + 1 - \tau, t_5 - t_3, t_7 - t_5)$  is a fixed point of  $F(\cdot, 0)$ . Proposition 3.2.8 implies  $K > K^*$  and  $x_0 = \Sigma(a^*, 0)$ . Similarly, if  $x_0 \in \tilde{\Sigma}(\tilde{U}_0^2, 0)$ , then  $x_0 \in \tilde{\Sigma}(\tilde{U}_0^3, 0)$ . By Proposition 3.2.18,  $K > K^*$  and  $x_0 = \tilde{\Sigma}(\tilde{a}, 0)$ .  $\square$

As a direct consequence of Corollary 3.5.2 and Proposition 3.5.4, we get the following.

**Theorem 3.5.5.** *Eq. (3.2) has no LSOP solutions for  $K \in (0, K^*)$  and  $\varepsilon = 0$ , and it admits exactly two normalized LSOP solutions for  $K > K^*$  and  $\varepsilon = 0$ .*

It can be also shown that in case  $K = K^*$  and  $\varepsilon = 0$ , there is exactly one normalized LSOP solution.

#### There are two LSOP solutions for $f^{K, \varepsilon}$ with $\varepsilon > 0$ , and for close nonlinearities

Recall that if  $K = 7$  and  $\varepsilon \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$ , where  $\varepsilon_0$  and  $\tilde{\varepsilon}_0$  are given by Propositions 3.2.11 and 3.2.18, respectively, then Eq. (3.2) admits two LSOP solutions with initial functions  $\Sigma(a^*(\varepsilon), \varepsilon)$  and  $\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$ .

**Proposition 3.5.6.** *Let  $K = 7$ . A threshold number  $\varepsilon_* \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$  can be given so that for  $\varepsilon \in (0, \varepsilon_*)$ ,  $x^{\Sigma(a^*(\varepsilon), \varepsilon)} : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^{\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)} : \mathbb{R} \rightarrow \mathbb{R}$  are the only normalized LSOP solutions of Eq. (3.2).*

### 3 Large-Amplitude Periodic Solutions for Monotone Positive Feedback

*Proof.* Suppose for contradiction that there is a sequence  $(\varepsilon^n)_1^\infty$  in  $(0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$  converging to 0 as  $n \rightarrow \infty$  and a sequence of functions  $(x^n)_0^\infty$  so that for  $n \geq 0$ ,  $x^n : \mathbb{R} \mapsto \mathbb{R}$  is a normalized LSOP solution of (3.2) with  $K = 7$  and  $\varepsilon = \varepsilon^n$ , and

$$x_0^n \notin \left\{ \Sigma(a^*(\varepsilon^n), \varepsilon^n), \tilde{\Sigma}(\tilde{a}(\varepsilon^n), \varepsilon^n) \right\}.$$

Let  $\omega_n > 0$  denote the minimal period of  $x^n$ . According to Proposition 3.4.3 (ii),  $\omega_n \in (8/7, 27/14)$  for all sufficiently large  $n$ .

For all  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ , Proposition 2.1.2 implies  $|x^n(t)| \leq 7$ , therefore Eq. (3.2) gives  $|\dot{x}^n(t)| \leq 14$ . Applying the Arzelà–Ascoli theorem and changing to a subsequence if necessary, we may assume that there are  $\omega \in [8/7, 27/14]$  and a continuous function  $x : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\omega_n \rightarrow \omega$  as  $n \rightarrow \infty$ , and  $x^n(t) \rightarrow x(t)$  as  $n \rightarrow \infty$  uniformly on all compact subsets of the real line. It is easy to see that  $x$  is periodic with minimal period  $\omega$ , it is of monotone type and special symmetry. In addition,  $x(-1) = x(-1 + \omega/2) = 0$  and  $x(t) \geq 0$  for  $t \in [-1, -1 + \omega/2]$ . By Proposition 3.4.3 (iii),

$$\max_{t \in \mathbb{R}} x(t) \geq \liminf_{n \rightarrow \infty} \max_{t \in \mathbb{R}} x^n(t) \geq e^{\frac{1}{7}}.$$

Proposition 3.4.3 (iv) gives that if  $t_0 \in \mathbb{R}$  and  $|x(t_0)| = 1$ , then

$$\liminf_{h \rightarrow 0} \left| \frac{x(t+h) - x(t)}{h} \right| \geq \frac{1}{2} \text{ for all } t \in (t_0 - \delta, t_0 + \delta)$$

with  $\delta_0 > 0$  defined before Proposition 3.4.3. Therefore there exist unique

$$t_1, t_4 \in [-1, -1 + \omega/2] \text{ with } 1 < t_1 < t_4 < -1 + \omega/2$$

such that  $x(t_1) = x(t_4) = 1$ . In addition, for all  $\gamma \in (0, \delta/2)$  fixed,  $|x(t) - 1| \geq \gamma$  for all  $t \in [-1, -1 + \omega/2]$  with  $|t - t_1| \geq 2\gamma$  and  $|t - t_4| \geq 2\gamma$ . Set

$$S_\gamma = \{s \in [-1, 0] : x(s) \in (-1 - \gamma, -1 + \gamma) \cup (1 - \gamma, 1 + \gamma)\}.$$

As  $x$  is the limit of LSOP solutions,  $S$  is the union of at most 4 intervals. Our previous observations and the special symmetry of  $x$  imply that for the Lebesgue measure  $\mu(S_\gamma)$  of  $S_\gamma$ , we have estimation  $\mu(S_\gamma) \leq 4 \cdot 4\gamma = 16\gamma$ . Similarly, the measure of

$$S_\gamma^n = \{s \in [-1, 0] : x^n(s) \in (-1 - \gamma, -1 + \gamma) \cup (1 - \gamma, 1 + \gamma)\}$$

is not larger than  $16\gamma$  for all sufficiently large  $n$  by Proposition 3.4.3 (iv).

We claim that for  $t \in [0, 1]$ ,

$$\lim_{n \rightarrow \infty} \int_0^t e^{-(t-s)} f^{7, \varepsilon^n}(x^n(s-1)) ds = \int_0^t e^{-(t-s)} f^{7, 0}(x(s-1)) ds,$$

### 3.5 There are two LSOP solutions

that is to each  $\eta > 0$  small, there corresponds  $n_* \geq 1$  so that for  $n \geq n_*$ , we have

$$\begin{aligned} & \left| \int_0^t e^{-(t-s)} \left[ f^{7,0}(x(s-1)) - f^{7,\varepsilon^n}(x^n(s-1)) \right] ds \right| \leq \\ & \int_0^t e^{-(t-s)} \left| f^{7,0}(x(s-1)) - f^{7,0}(x^n(s-1)) \right| ds \\ & + \int_0^t e^{-(t-s)} \left| f^{7,0}(x^n(s-1)) - f^{7,\varepsilon^n}(x^n(s-1)) \right| ds < \eta \end{aligned}$$

for all  $t \in [0, 1]$ . Set  $0 < \gamma < \min\{\delta/2, \eta/224\}$ . There exists  $n_1 = n_1(\gamma) \geq 1$  so that for  $n \geq n_1$ , we have

$$f^{7,0}(x(s-1)) - f^{7,0}(x^n(s-1)) = 0 \quad \text{for } s-1 \notin S_\gamma,$$

and

$$\left| f^{7,0}(x(-1+s)) - f^{7,0}(x^n(-1+s)) \right| \leq 7 \quad \text{for } s-1 \in S_\gamma.$$

Therefore the first term is not larger than  $7 \cdot 16\gamma t \leq 112\gamma < \eta/2$  for  $n \geq n_1$ . Also there is  $n_2 = n_2(\gamma) \geq 1$  so that for  $n \geq n_2$ , we have  $\varepsilon^n < \gamma$ . Then for  $s-1 \notin S_\gamma^n$ ,

$$f^{7,0}(x^n(-1+s)) - f^{7,\varepsilon^n}(x^n(-1+s)) = 0,$$

and for  $s-1 \in S_\gamma^n$ ,

$$\left| f^{7,0}(x^n(-1+s)) - f^{7,\varepsilon^n}(x^n(-1+s)) \right| \leq 7.$$

So the second term is not larger than  $7 \cdot 16\gamma t \leq 112\gamma < \eta/2$  if  $n \geq n_2$ . Set  $n_* = \min\{n_1, n_2\}$ . The claim is verified.

It follows that for all  $t \in [0, 1]$ ,

$$\begin{aligned} x(t) &= \lim_{n \rightarrow \infty} x^n(t) = \lim_{n \rightarrow \infty} \left( e^{-t} x^n(0) + \int_0^t e^{-(t-s)} f^{7,\varepsilon^n}(x^n(s-1)) ds \right) \\ &= e^{-t} x(0) + \int_0^t e^{-(t-s)} f^{7,0}(x(s-1)) ds, \end{aligned}$$

that is,  $x$  satisfies Eq. (3.2) with  $K = 7$  and  $\varepsilon = 0$  for all  $t \in [0, 1]$ . It is analogous to show that  $x$  satisfies the equation on  $[1, 2]$ . As  $x_\omega = x_0$ , we gain that  $x$  is a solution on  $\mathbb{R}$ .

Proposition 3.5.4 yields  $x_0 = \Sigma(a^*, 0)$  or  $x_0 = \tilde{\Sigma}(\tilde{a}, 0)$ . Suppose  $x_0 = \Sigma(a^*, 0)$  for example. Note that as  $x$  is of special symmetry, the construction of  $\Sigma(a^*, 0)$  gives  $a^* = (t_4 + 2 - \omega, t_1 - t_4 + \omega/2, t_4 - t_1)$ .

Proposition 3.4.3 gives that if  $n$  is large enough, then there exist uniquely defined

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$-1 < t_1^n < t_2^n < t_3^n < t_4^n < 0$  with

$$x^n(t_1^n) = 1, x^n(t_2^n) = 1 + \varepsilon^n, x^n(t_3^n) = 1 + \varepsilon^n, x^n(t_4^n) = 1.$$

Also,  $\lim_{n \rightarrow \infty} t_1^n = \lim_{n \rightarrow \infty} t_2^n = t_1$  and  $\lim_{n \rightarrow \infty} t_3^n = \lim_{n \rightarrow \infty} t_4^n = t_4$ .

It follows from the definition of  $U_0^3$ , that  $t_1 + 2 < \omega$  and  $t_4 < t_1 + 1 - \omega/2$ . Thus there exists  $n_{**} \geq 1$  so that for  $n \geq n_{**}$ , we have  $t_2^n + 2 < \omega^n$  and  $t_4^n < t_1^n + 1 - \omega^n/2$ . By Proposition 3.4.3 (v),  $x_0^n = \Sigma(a^n, \varepsilon^n)$  for  $n \geq n_{**}$ , where

$$a^n = \left( t_3^n + 2 - \omega^n, t_1^n - t_4^n + \frac{\omega^n}{2}, t_3^n - t_2^n \right)$$

is a fixed point of  $F(\cdot, \varepsilon^n)$ . According to the proof of Proposition 3.2.11, there is a neighborhood  $N$  of  $a^*$  in  $(0, 1)^3$  so that the fixed point of  $F(\cdot, \varepsilon)$  is unique in  $N$  for  $\varepsilon \in [0, \varepsilon_0)$ . As  $a^n$  is arbitrary close to  $a^*$ , we may suppose that  $a^n \in N$  and thus  $a^n = a^*(\varepsilon^n)$ , a contradiction to our initial assumption.

At last suppose  $x_0 = \tilde{\Sigma}(\tilde{a}, 0)$ . Then with the aid of Proposition (3.4.3) (vi), one can verify the existence of  $\tilde{n} \geq 1$  so that  $x_0^n = \tilde{\Sigma}(\tilde{a}(\varepsilon^n), \varepsilon^n)$  for  $n \geq \tilde{n}$ , which is a contradiction again.  $\square$

Consider  $K = 7$  and  $\varepsilon \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$ . Proposition 3.3.5 implies that there exists  $\delta_0 = \delta_0(\varepsilon) > 0$  so that if  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  with  $\|f - f^{7, \varepsilon}\|_{C_b^1} < \delta_0$ , and (H1) holds for  $f$ , then Eq. (1.1) with  $\mu = 1$  and nonlinearity  $f$  has two normalized LSOP solutions  $p = p(f) : \mathbb{R} \rightarrow \mathbb{R}$  and  $q = q(f) : \mathbb{R} \rightarrow \mathbb{R}$ .

**Proposition 3.5.7.** *Set  $\mu = 1$ . To each  $\varepsilon \in (0, \varepsilon_*)$ , where  $\varepsilon_* \in (0, \min(\varepsilon_0, \tilde{\varepsilon}_0))$  is given by Proposition 3.5.6, there corresponds  $\delta_1 = \delta_1(\varepsilon) \in (0, \delta_0(\varepsilon))$  such that if  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfies (H1), and  $\|f - f^{7, \varepsilon}\|_{C_b^1} < \delta_1$ , then Eq. (1.1) admits at most two normalized LSOP solutions.*

*Proof.* Suppose for contradiction that a sequence  $(f^n)_{n=0}^\infty$  exists in  $C_b^1(\mathbb{R}, \mathbb{R})$  with

$$\|f^n - f^{7, \varepsilon}\|_{C_b^1} < 1/n \text{ for } n \in \mathbb{N}$$

so that for  $n \in \mathbb{N}$ ,  $f^n$  satisfies (H1), and the equation

$$\dot{x}(t) = -x(t) + f^n(x(t-1)) \quad (3.34)$$

has a normalized LSOP solution  $x^n : \mathbb{R} \rightarrow \mathbb{R}$  with  $x_0^n \notin \{p_0(f^n), q_0(f^n)\}$ , where LSOP solutions  $p(f^n)$  and  $q(f^n)$  are given by Proposition 3.3.5. Note that  $\|f^n - f^{7, \varepsilon}\|_{C_b^1} < \delta_0$  for all large  $n$ , hence it is no restriction to assume that  $p(f^n)$  and  $q(f^n)$  exist for all  $n \geq 1$ . Let  $\omega^n \in (1, 2)$  denote the minimal period of  $x^n$ ,  $n \in \mathbb{N}$ . Since

$$\sup_{x \in \mathbb{R}} |f^n(x)| \leq \|f^n\|_{C_b^1} \leq \|f^{7, \varepsilon}\|_{C_b^1} + 1 < \infty, \quad n \in \mathbb{N},$$



### 3.6 Open questions

Proposition 2.1.2 yields that  $\|x_t^n\| \leq \|f^{7,\varepsilon}\|_{C_b^1} + 1$  and thus  $\|\dot{x}_t^n\| \leq 2\|f^{7,\varepsilon}\|_{C_b^1} + 2$  for all  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ . Applying the Arzelà–Ascoli theorem, we may suppose that  $\omega^n \rightarrow \omega \in [1, 2]$  as  $n \rightarrow \infty$ , and  $x^n$  converges to a continuous function  $x : \mathbb{R} \rightarrow \mathbb{R}$  uniformly on each compact subset of  $\mathbb{R}$ . Then it is easy to see that  $x$  is a solution of Eq. (3.2) with minimal period  $\omega \in [1, 2]$ . Proposition 2.3.4 excludes the possibility that the period is 1, Proposition 2.3.4 and Proposition 3.4.2 exclude the possibility that the period is 2. So  $\omega \in (1, 2)$ . As  $x$  is necessary of monotone type, this yields  $V(x_t) = V(x_0) = 2$  for all  $t \in \mathbb{R}$ . As  $x^n$ ,  $n \in \mathbb{N}$ , is an LSOP solution, it is also easy to see that  $x$  is of large amplitude. We conclude that  $x$  is an LSOP solution of (3.2). Hence Proposition 3.5.6 implies we may assume that  $x_0$  is either  $\Sigma(a^*(\varepsilon), \varepsilon)$  or  $\tilde{\Sigma}(\tilde{a}(\varepsilon), \varepsilon)$ . If  $n$  is chosen large enough, then  $f^n$  is arbitrarily close to  $f^{7,\varepsilon}$  in  $C_b^1$  norm,  $x_0^n \in \{x_0\} + V$  and  $\omega^n \in (\omega - \nu, \omega + \nu)$ , where  $V$  and  $\nu$  are given by Theorem 2.3.5. So Theorem 2.3.5 gives  $x^n$  equals  $p(f^n)$  or  $q(f^n)$ , a contradiction to our initial assumption.  $\square$

**The proof of Theorem 3.1.1.** Fix  $\mu = 1$ ,  $K = 7$  and  $\varepsilon \in (0, \varepsilon_*)$ . Choose a nonlinearity  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfying (H1) so that  $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1(\varepsilon) < \delta_0(\varepsilon)$ . Then Theorem 3.1.1 follows from Propositions 3.3.5 and 3.5.7.  $\square$

### 3.6 Open questions

By Theorem 3.5.5, Eq. (3.2) has no LSOP solutions for  $K \in (0, K^*)$  and  $\varepsilon = 0$ . For  $K > K^*$  and  $\varepsilon = 0$ , there are exactly two normalized LSOP solutions of Eq. (3.2), and they are determined by the fixed point of  $F(\cdot, 0)$  in  $U_0^3$  and by the fixed point of  $\tilde{F}(\cdot, 0)$  in  $\tilde{U}_0^3$ . Fig. 3.8 shows the graphs of the first components of the fixed points for  $K \geq K^*$  (as functions of  $K$ ).

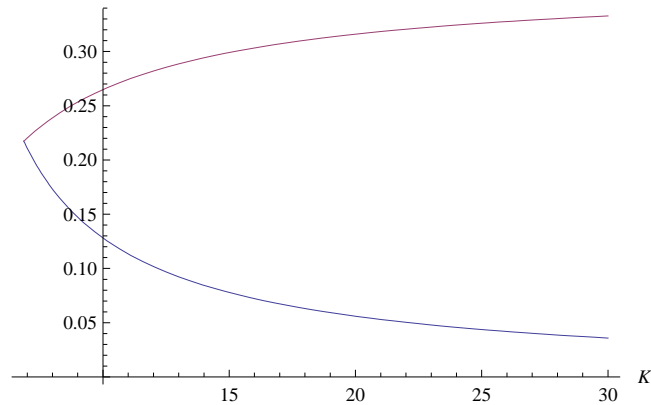


Figure 3.8: Plot of  $a_1^*$  and  $\tilde{a}_1$  vs.  $K \geq K^*$

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This suggests the following conjecture: for each fixed small  $\varepsilon > 0$  there exists  $K^*(\varepsilon)$  so that Eq. (3.2) undergoes a saddle-node-like bifurcation of periodic orbits at  $K = K^*(\varepsilon)$ .

Theorem 3.1.1 brings up a second question: for an arbitrary integer  $n > 1$ , is there a feedback function  $f$  with  $f(0) = 0$  and  $xf(x) > 0$  for  $x \neq 0$ , for which Eq. (1.1) admits  $2n$  slowly oscillatory periodic orbits? We conjecture that the answer is affirmative. To verify the conjecture (or at least a part of it), one could try to generalize the construction presented in this chapter. However, any generalization would be accompanied by several technical difficulties. The problem is solved in Chapter 5 for the negative feedback case.

## 4 The Global Attractor

### 4.1 Introduction

Under hypothesis (H1), the maps  $\Phi(t, \cdot) : C \rightarrow C$ ,  $t \geq 0$ , induced by Eq. (1.1) are monotone with respect to the pointwise ordering on  $C$  (see Proposition 4.4.1 in this chapter). Therefore the sets

$$C_{i,j} = \{\varphi \in C : \xi_i \leq \varphi(s) \leq \xi_j \text{ for all } s \in [-1, 0]\}, \quad i \in \{-2, 0\}, j \in \{0, 2\},$$

are positively invariant under the semiflow  $\Phi$ .

There exists a global attractor  $\mathcal{A} \subset C_{-2,2}$  of the restriction  $\Phi|_{[0,\infty) \times C_{-2,2}}$ , i.e., a nonempty, compact, positively invariant set, that attracts bounded sets in  $C_{-2,2}$ . It comes from general theory that

$$\begin{aligned} \mathcal{A} = \{ & \varphi \in C_{-2,2} : \text{there is a solution } x : \mathbb{R} \rightarrow \mathbb{R} \text{ of Eq. (1.1)} \\ & \text{with } x(\mathbb{R}) \subset [\xi_{-2}, \xi_2] \text{ and } \varphi = x_0 \}. \end{aligned}$$

The map  $[0, \infty) \times \mathcal{A} \ni (t, \varphi) \mapsto \Phi(t, \varphi) \in \mathcal{A}$  extends to a continuous flow  $\Phi_{\mathcal{A}} : \mathbb{R} \times \mathcal{A} \rightarrow \mathcal{A}$ ; for every  $\varphi \in \mathcal{A}$  and for all  $t \in \mathbb{R}$  we have  $\Phi_{\mathcal{A}}(t, \varphi) = x_t$  with a uniquely determined solution  $x : \mathbb{R} \rightarrow [\xi_{-2}, \xi_2]$  of Eq. (1.1) satisfying  $x_0 = \varphi$ .

In Chapter 3 we have shown that there exists nonlinearities for which  $\mathcal{A}$  contains  $\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$  as a proper subset, where  $\mathcal{A}_{-2,0} \subset C_{-2,0}$  and  $\mathcal{A}_{0,2} \subset C_{0,2}$  are the global attractors of the restrictions  $\Phi|_{[0,\infty) \times C_{-2,0}}$  and  $\Phi|_{[0,\infty) \times C_{0,2}}$ , respectively. In the situation of Theorem 3.1.1, Eq. (1.1) has two normalized LSOP solutions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  with  $\max_{t \in \mathbb{R}} q(t) > \max_{t \in \mathbb{R}} p(t)$ . We shall show in this chapter, that under further restriction on  $f$ , the dynamics in  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$  can be completely described.

By hypothesis (H1),  $\hat{\xi}_{-2}, \hat{\xi}_{-1}, \hat{0}, \hat{\xi}_1, \hat{\xi}_2$  are the only equilibrium points of  $\Phi$  in  $C_{-2,2}$ . In addition,  $\hat{\xi}_{-2}$ ,  $\hat{0}$  and  $\hat{\xi}_2$  are stable,  $\hat{\xi}_{-1}$  and  $\hat{\xi}_1$  are unstable. Note that there is no homoclinic orbit to  $\hat{\xi}_j$ ,  $j \in \{-2, 0, 2\}$  as they are stable. It follows from Proposition 3.1 in [23], that there exist no homoclinic orbits to  $\hat{\xi}_i$ ,  $i \in \{-1, 1\}$ .

The Poincaré–Bendixson theorem of Mallet-Paret and Sell [33] applied to this equation yields that if  $\varphi \in C_{-2,2}$ , then  $\omega(\varphi)$  is either a single nonconstant periodic orbit or for each  $\psi \in \omega(\varphi)$ ,  $\alpha(\psi) \cup \omega(\psi) \subseteq \{\hat{\xi}_{-2}, \hat{\xi}_{-1}, \hat{\xi}_0, \hat{\xi}_1, \hat{\xi}_2\}$ . An analogous result holds for  $\alpha(x)$  in case  $x$  is defined on  $\mathbb{R}$  and  $\{x_t : t \leq 0\} \subset C_{-2,2}$ .

#### 4 The Global Attractor

The nonlinearity  $f$  and the constant  $\mu$  in Theorem 3.1.1 are given so that there exist periodic solutions oscillating slowly around  $\xi_1$  and around  $\xi_{-1}$  with ranges in  $(0, \xi_2)$  and in  $(\xi_{-2}, 0)$ , respectively [26]. Among these periodic solutions there are  $x^1$  and  $x^{-1}$  so that the ranges  $x^1(\mathbb{R})$  and  $x^{-1}(\mathbb{R})$  are maximal in the sense that  $x^1(\mathbb{R}) \supset x(\mathbb{R})$  for all periodic solutions  $x$  oscillating slowly around  $\xi_1$  with range in  $(0, \xi_2)$ , analogously for  $x^{-1}$  (see Proposition 4.2.1). Set

$$\mathcal{O}_1 = \{x_t^1 : t \in \mathbb{R}\} \text{ and } \mathcal{O}_{-1} = \{x_t^{-1} : t \in \mathbb{R}\}.$$

Also, let  $\mathcal{W}^u(\mathcal{O}_p)$  and  $\mathcal{W}^u(\mathcal{O}_q)$  denote the unstable sets of  $\mathcal{O}_p = \{p_t : t \in \mathbb{R}\}$  and  $\mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$ , respectively. By definition,

$$\mathcal{W}^u(\mathcal{O}_*) = \{x_0 : x : \mathbb{R} \rightarrow \mathbb{R} \text{ is a solution of (1.1), } \alpha(x) \text{ exists and } \alpha(x) = \mathcal{O}_*\}$$

for  $* \in \{p, q\}$  [7, 26].

We are going to prove the next theorem.

**Theorem 4.1.1.** *One may set  $\mu$  and  $f$  satisfying (H1) such that the statement of Theorem 3.1.1 holds, and for the global attractor  $\mathcal{A}$  we have the equality*

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q).$$

Moreover, the dynamics on  $\mathcal{W}^u(\mathcal{O}_p)$  and  $\mathcal{W}^u(\mathcal{O}_q)$  is as follows.

For each  $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$ , the omega limit set  $\omega(\varphi)$  is either  $\{\hat{\xi}_{-2}\}$  or  $\{\hat{\xi}_2\}$ , and there exist heteroclinic connections from  $\mathcal{O}_q$  to  $\{\hat{\xi}_{-2}\}$  and to  $\{\hat{\xi}_2\}$ .

For each  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ ,  $\omega(\varphi)$  is one of the sets  $\{\hat{\xi}_{-2}\}$ ,  $\{\hat{0}\}$ ,  $\{\hat{\xi}_2\}$ ,  $\mathcal{O}_q$ ,  $\mathcal{O}_1$ ,  $\mathcal{O}_{-1}$ .

There are heteroclinic connections from  $\mathcal{O}_p$  to  $\{\hat{\xi}_{-2}\}$ ,  $\{\hat{0}\}$ ,  $\{\hat{\xi}_2\}$ ,  $\mathcal{O}_q$ ,  $\mathcal{O}_1$  and  $\mathcal{O}_{-1}$ .

The system of connecting orbits is represented in Fig. 4.1. The dashed arrows represent heteroclinic connections in  $\mathcal{A}_{-2,0}$  and in  $\mathcal{A}_{0,2}$ , the solid ones represent connecting orbits given by Theorem 4.1.1.

The chapter is organized as follows.

The next section verifies our statement that there are maximal periodic solutions among the periodic solutions oscillating slowly around  $\xi_1$  and  $\xi_{-1}$  with ranges in  $(0, \xi_2)$  and in  $(\xi_{-2}, 0)$ , respectively. It is also confirmed that the difference of  $p$  and any of these periodic solutions has at most two sign changes on each interval of length 1.

Section 4.3 excludes the existence of rapidly oscillatory periodic solutions.

$\mathcal{W}^u(\mathcal{O}_p)$  is the forward extension of  $\mathcal{W}^u(p_0)$ , the local unstable manifold of a Poincaré return map at its fixed point  $p_0$ . Similarly,  $\mathcal{W}^u(\mathcal{O}_q)$  is the forward extension of  $\mathcal{W}^u(q_0)$ , the local unstable manifold of a second Poincaré return map at fixed point  $q_0$ . Section 4.4 gives the characterizations of  $\mathcal{W}^u(p_0)$  and  $\mathcal{W}^u(q_0)$ , furthermore it shows that solutions with initial segments in  $\mathcal{W}^u(p_0)$  have nice oscillatory properties.

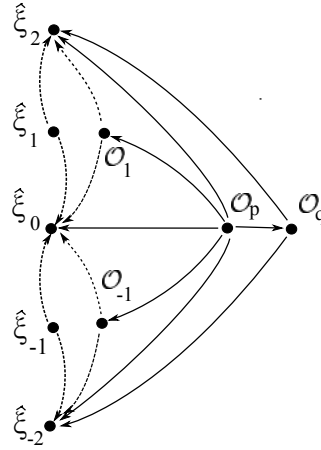


Figure 4.1: Connecting orbits

Section 4.5 completes the proof of Theorem 4.1.1. The existence of heteroclinic orbits from  $\mathcal{O}_p$  is based on the fact that the local unstable manifold  $\mathcal{W}^u(p_0)$  is 2-dimensional, and it is separated into two parts by the 1-dimensional leading unstable manifold  $\mathcal{W}_1^u(p_0)$ . Discrete Lyapunov functionals around  $\xi_{-1}, 0, \xi_1$ , the Poincaré–Bendixson theorem, the theory of invariant manifolds, the monotone property of the semiflow and elementary topological arguments yield the result.

## 4.2 Periodic solutions oscillating around $\hat{\xi}_{\pm 1}$

Set  $\mu = 1$ . We are going to use the following additional hypothesis.

- (H2) For  $j \in \{-1, 1\}$  and  $\theta \in (3\pi/2, 2\pi)$  with  $\theta = -\tan \theta$ , the inequality  $f'(\xi_j) > 1/\cos \theta$  holds.

Note that (H2) is simply condition (2.4) for  $\xi = \xi_1$  and  $\xi = \xi_{-1}$ .

It is shown in [26] that if (H1) and (H2) hold, then at least one periodic solution appears with the following two properties: it has range in  $(0, \xi_2)$ , and it is slowly oscillatory around  $\xi_1$ . Analogously, there is at least one periodic solution, that is slowly oscillatory around  $\xi_{-1}$  and has range in  $(\xi_{-2}, 0)$ . We emphasize that the existence of more solutions with the above properties is not excluded. It follows from Proposition 3.4.1 that the minimal periods of the slowly oscillatory solutions are in  $(1, 2)$ . The following proposition holds.

**Proposition 4.2.1.** *If conditions (H1) and (H2) are satisfied by  $f$ , then there exist periodic solutions  $x^1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^{-1} : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) oscillating slowly around  $\xi_1$  and  $\xi_{-1}$  with ranges in  $(0, \xi_2)$  and  $(\xi_{-2}, 0)$ , respectively, so that the ranges  $x^1(\mathbb{R})$  and  $x^{-1}(\mathbb{R})$  are maximal in the sense that  $x^1(\mathbb{R}) \supset x(\mathbb{R})$  for all periodic solutions  $x$  oscillating slowly around  $\xi_1$  with ranges in  $(0, \xi_2)$ ; and analogously for  $x^{-1}$ .*

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*Proof.* Lemma 2.3.2 (ii) and Proposition 2.3.4 easily imply that for two periodic solutions  $\hat{x}$  and  $\tilde{x}$  of Eq. (1.1) oscillating around  $\xi_1$ , either  $\hat{x}(\mathbb{R}) \supseteq \tilde{x}(\mathbb{R})$  or  $\hat{x}(\mathbb{R}) \subseteq \tilde{x}(\mathbb{R})$  holds. Suppose for contradiction that there is no periodic solution oscillating slowly around  $\xi_1$  with the stated properties. Then there exists a sequence of periodic solutions  $x^n : \mathbb{R} \rightarrow \mathbb{R}$  of (1.1) with minimal period  $\omega_n \in (1, 2)$ ,  $1 \leq n \in \mathbb{N}$ , so that  $x^n$  is slowly oscillatory around  $\xi_1$ ,  $x^n(\mathbb{R}) \subseteq x^{n+1}(\mathbb{R}) \subset (0, \xi_2)$  for  $n \geq 1$ , and there exists no solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  oscillating slowly around  $\xi_1$  with  $x^n(\mathbb{R}) \subseteq x(\mathbb{R}) \subset (0, \xi_2)$  for each  $n \geq 1$ .

As  $x^n(t) \in (0, \xi_2)$  for all  $t \in \mathbb{R}$  and  $f$  is bounded on  $(0, \xi_2)$ , Eq. (1.1) gives a uniform upper bound for  $|\dot{x}^n|$  on  $\mathbb{R}$ ,  $n \geq 1$ . Applying the Arzelà–Ascoli theorem and choosing a subsequence if necessary, we obtain that there exist  $\omega_* \in [1, 2]$  and a continuous function  $x^* : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\omega_n \rightarrow \omega_*$  and  $x^n$  converges to  $x^*$  as  $n \rightarrow \infty$  uniformly on each compact subset of the real line. It is easy to see that  $x^*$  is periodic with period  $\omega_*$ . Also, we find that

$$\dot{x}^n(t) \rightarrow -x^*(t) + f(x^*(t-1)) \text{ as } n \rightarrow \infty$$

uniformly on each compact subinterval of the real line. It follows that  $x^*$  is differentiable and satisfies Eq. (1.1) for all  $t \in \mathbb{R}$ .

As  $x^n(\mathbb{R}) \subseteq x^{n+1}(\mathbb{R}) \subset (0, \xi_2)$  for all  $n \geq 1$ , necessarily

$$0 \leq \min_{t \in \mathbb{R}} x^*(t) \leq \min_{t \in \mathbb{R}} x^n(t) < \xi_1 < \max_{t \in \mathbb{R}} x^n(t) \leq \max_{t \in \mathbb{R}} x^*(t) \leq \xi_2$$

for all  $n \geq 1$ . We claim that  $\min_{t \in \mathbb{R}} x^*(t) > 0$ . Indeed, if  $t_{\min} \in \mathbb{R}$  is chosen so that  $x^*(t_{\min}) = \min_{t \in \mathbb{R}} x^*(t) = 0$ , then

$$f(x^*(t_{\min} - 1)) = \dot{x}^*(t_{\min}) + \mu x^*(t_{\min}) = 0$$

and hypothesis (H1) implies  $x^*(t_{\min} - 1) = 0$ , a contradiction to Lemma 2.3.2 (ii), (iv) and the periodicity of  $x^*$ . Similarly,  $\max_{t \in \mathbb{R}} x^*(t) < \xi_2$ .

Proposition 2.3.4 implies  $t \mapsto V(x_t^* - \hat{\xi}_1)$  is finite and constant. It follows from Lemma 2.3.1 that

$$V(x_t^* - \hat{\xi}_1) \leq \liminf_{n \rightarrow \infty} V(x_t^n - \hat{\xi}_1) = 2$$

for all  $t \in \mathbb{R}$  and  $n \geq 1$ . However,  $V(x_t^* - \hat{\xi}_1) > 0$  as function  $x^* - \xi_1$  has sign changes. So  $V(x_t^* - \hat{\xi}_1) = 2$  for all  $t \in \mathbb{R}$ .

We conclude that solution  $x^*$  is periodic, slowly oscillatory around  $\xi_1$ , has range in  $(0, \xi_2)$ , and  $x^n(\mathbb{R}) \subseteq x(\mathbb{R}) \subset (0, \xi_2)$  for each  $n \geq 1$ , a contradiction to our initial assumption.

The proof is analogous for  $x^{-1}$ . □

#### 4.2 Periodic solutions oscillating around $\hat{\xi}_{\pm 1}$

If  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  is close to  $f^{7,\varepsilon}$  in  $C_b^1$ -norm with  $\varepsilon > 0$  small, then we may assume that  $f$  satisfies hypothesis (H2), see Remark 4.5.1.

Let  $p^0$  denote the periodic solution of Eq. (3.2) with  $K = 7$  and  $\varepsilon = 0$  determined by the unique fixed point  $a^*$  of  $F(\cdot, 0)$  in  $U_0^3$ .

Recall that if  $\mu = 1$  and  $K = 7$ , then for each  $\varepsilon \in (0, \varepsilon_*)$ , there exists  $\delta_1(\varepsilon) > 0$  such that if a nonlinearity  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfies (H1), and  $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1(\varepsilon)$ , then the statement of Theorem 3.1.1 holds for  $f$ . Without loss of generality, we may assume that  $\delta_1(\varepsilon) \rightarrow 0+$  as  $\varepsilon \rightarrow 0+$ . Hence we may assume that  $\max_{-1 \leq t \leq 2} |p(t) - p^0(t)| \rightarrow 0$  as  $\varepsilon \rightarrow 0+$ . We also have  $\xi_1 \rightarrow 1$  and  $\xi_2 \rightarrow 7$  as  $\varepsilon \rightarrow 0+$ .

**Proposition 4.2.2.** *Let  $r : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic solution of Eq. (1.1) either with range in  $(0, \xi_2)$  and with  $V(r_t - \hat{\xi}_1) = 2$  for all  $t \in \mathbb{R}$ , or with range in  $(\xi_{-2}, 0)$  and with  $V(r_t - \hat{\xi}_{-1}) = 2$  for all  $t \in \mathbb{R}$ . If  $\varepsilon > 0$  is sufficiently small, then  $V(p_t - r_s) = 2$  for all  $t \in \mathbb{R}$  and  $s \in \mathbb{R}$ .*

*Proof.* We consider the case when  $r$  has range in  $(0, \xi_2)$  and  $V(r_t - \hat{\xi}_1) = 2$  for all  $t \in \mathbb{R}$ . The other case is analogous.

By Proposition 2.3.4,  $V(p_t - r_s)$  is the same constant for all  $t \in \mathbb{R}$  and  $s \in \mathbb{R}$ . Thus it is sufficient to find a pair  $(t, s) \in \mathbb{R} \times \mathbb{R}$  with  $V(p_t - r_s) = 2$ .

Let  $\omega^0, \bar{\omega}, \rho$  denote the minimal periods of  $p^0, p, r$ , respectively. Define  $t_1, s_1, t_4, s_2, \tau = \omega^0 - 1$  for  $p^0$  as in Section 3.2. Set  $z = -1 + \omega/2$ . Then  $p^0$  strictly increases on  $[-1, s_1]$ , decreases on  $[s_1, z]$ ,  $p^0(t) < 0$  for  $t \in (z, \tau)$ ,  $p^0(-1) = p^0(z) = p^0(\tau) = 0$  and  $p^0(t_1) = p^0(t_4) = 1$ . There exist  $\bar{t}_1, \bar{s}_1, \bar{t}_4, \bar{z}, \bar{\tau} = \bar{\omega} - 1$  such that  $p$  strictly increases on  $[-1, \bar{s}_1]$ , decreases on  $[\bar{s}_1, \bar{z}]$ ,  $p(t) < 0$  for  $t \in (\bar{z}, \bar{\tau})$ ,  $p(-1) = p(\bar{z}) = p(\bar{\tau}) = 0$  and  $p(\bar{t}_1) = p(\bar{t}_4) = \xi_1$ . We have  $\bar{t}_1 \rightarrow t_1, \bar{s}_1 \rightarrow s_1, \bar{t}_4 \rightarrow t_4, \bar{z} \rightarrow z, \bar{\tau} \rightarrow \tau$  as  $\varepsilon \rightarrow 0+$ .

From Section 3.2 we know that  $\tau - 1 > t_1$  and  $\omega^0 > 1$ .

**Claim.** There exist  $\varepsilon_0 > 0$  and  $\delta_0 \in (0, \min\{\tau - 1 - t_1, \omega^0 - 1, t_4 - s_1\})$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ ,

- (i) if  $r(t_0 + \sigma) = \xi_1$  for some  $t_0 \in [\bar{t}_1 + \delta_0, \bar{s}_1]$  and  $\sigma \in \mathbb{R}$ , then  $r(\sigma + s) < p(s)$  for all  $s \in [t_0, \bar{s}_1]$ ,
- (ii) if  $r(t_0 + \sigma) = \xi_1$  for some  $t_0 \in [\bar{t}_4 + \delta_0, \bar{z}]$  and  $\sigma \in \mathbb{R}$ , then  $r(\sigma + s) > p(s)$  for all  $s \in [t_0, \bar{z}]$ ,
- (iii) if  $r(t_0 + \sigma) = \xi_1$  for some  $t_0 \in [\bar{s}_1, \bar{t}_4 - \delta_0]$  and  $\sigma \in \mathbb{R}$ , then  $r(\sigma + s) < p(s)$  for all  $s \in [\bar{s}_1, t_0]$ .

*Proof of (i).* Clearly,  $\xi_1 \rightarrow 1, \xi_2 \rightarrow 7, \bar{t}_1 + \delta_0 \rightarrow t_1 + \delta_0, \bar{s}_1 \rightarrow s_1$  and

$$\max_{-1 \leq t \leq 2} |p(t) - p^0(t)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0+.$$

For  $r$ , the differential inequality

$$\dot{r}(t) \leq -r(t) + f(\xi_2)$$

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holds for all  $t \in \mathbb{R}$ . Hence

$$r(\sigma + s) \leq \xi_1 e^{t_0-s} + (1 - e^{t_0-s}) f(\xi_2) \text{ for } s \geq t_0.$$

For a fixed  $t_0 \in [t_1 + \delta_0, s_1]$ , the right side of the inequality tends to  $7 - 6e^{t_0-s}$  as  $\varepsilon \rightarrow 0+$ . Using  $p^0(s) = 7 - 6e^{t_1-s}$ ,  $s \in [t_1, s_1]$ , one finds

$$\min_{s \in [t_0, s_1]} (p^0(s) - (7 - 6e^{t_0-s})) = 6(1 - e^{t_1-t_0}) \min_{s \in [t_0, s_1]} e^{t_0-s} \geq 6(e^{\delta_0} - 1)e^{t_1-s_1} > 0.$$

Since the derivatives of  $p$  and  $r$  are uniformly bounded for all  $t \in \mathbb{R}$  and for small  $\varepsilon > 0$ , the statement is obtained as  $\varepsilon \rightarrow 0+$ .

Assertions (ii) and (iii) of the Claim can be shown analogously.  $\square$

Let  $u_i$ ,  $i \in \{0, 1, 2, 3, 4\}$ , be given so that  $u_0 < u_1 < u_2 < u_3 < u_4$ ,  $u_4 = u_0 + \rho$ ,  $r(u_0) = r(u_2) = \xi_1$ ,  $r(u_1) = \min_{t \in \mathbb{R}} r(t) > 0$  and  $r(u_3) = \max_{t \in \mathbb{R}} r(t) < \xi_2$ . Propositions 3.4.2, 4.3.2 and Theorem 7.1 in [33] guarantees the existence of  $u_0, \dots, u_4$  and the fact that  $r$  strictly increases on  $[u_1, u_3]$  and strictly decreases on  $[u_3, u_5]$  with  $u_5 = u_1 + \rho$ .

We distinguish 3 cases.

Case 1:  $u_4 - u_2 \geq \bar{\tau} - \bar{t}_4$ . As  $\tau - t_4 > \tau - z = \omega^0/2 > 1/2$ , we may assume  $\bar{\tau} - \bar{t}_4 > 1/2$ . Then  $u_4 - u_3 < \bar{\tau} - \bar{t}_4$  or  $u_3 - u_2 < \bar{\tau} - \bar{t}_4$  holds because  $u_4 - u_2 < 1$ .

In case  $u_4 - u_3 < \bar{\tau} - \bar{t}_4$  set  $y(t) = r(t - \bar{\tau} + u_4)$ . Then  $y(\bar{\tau}) = \xi_1$ ,  $y$  decreases on  $[\bar{\tau} - u_4 + u_3, \bar{\tau}]$ ,  $y$  increases on  $[\bar{\tau} - u_4 + u_2, \bar{\tau} - u_4 + u_3]$ ,  $y(\bar{\tau} - u_4 + u_2) = \xi_1$ . If  $\bar{\tau} - u_4 + u_2 \in [\bar{s}_1, \bar{t}_4]$ , then  $p - y$  has one sign change on  $[\bar{\tau} - 1, \bar{\tau}]$  since  $y(t) < \xi_1 < p(t)$  for  $t \in [\bar{\tau} - 1, \bar{\tau} - u_4 + u_2]$ ,  $p$  decreases on  $[\bar{\tau} - u_4 + u_2, \bar{t}_4]$ ,  $y$  increases on  $[\bar{\tau} - u_4 + u_2, \bar{t}_4]$ , and  $y(t) > \xi_1 > p(t)$  for  $t \in (\bar{t}_4, \bar{\tau})$ . If  $\bar{\tau} - u_4 + u_2 < \bar{s}_1$ , then  $\bar{\tau} - u_4 + u_2 \in (\bar{\tau} - 1, \bar{s}_1)$ . Then for sufficiently small  $\varepsilon > 0$ ,  $\bar{\tau} - u_4 + u_2 \in (\bar{t}_1 + \delta_0, \bar{s}_1)$  will be satisfied, and Claim (i) can be applied to get  $y(t) < p(t)$  for all  $t \in [\bar{\tau} - u_4 + u_2, \bar{s}_1]$ . Clearly,  $y(t) < \xi_1$  for all  $t \in [\bar{\tau} - 1, \bar{\tau} - u_4 + u_2]$ . Now it is obvious that  $p - y$  has exactly 1 sign change on  $[\bar{\tau} - 1, \bar{\tau}]$ .

The case  $u_3 - u_2 < \bar{\tau} - \bar{t}_4$  is analogous.

Case 2:  $\bar{z} - \bar{t}_4 \leq u_4 - u_2 < \bar{\tau} - \bar{t}_4$ . Consider

$$y_\sigma(t) = y(t - \bar{t}_4 + \sigma) \text{ for } \sigma \in [u_2, u_4 + \bar{t}_4 - \bar{\tau}].$$

It can be shown that  $p - y_\sigma$  has at most two sign changes on the interval

$$[\bar{\tau} - 1, \bar{t}_4 + 1] \cap [u_1 + \bar{t}_4 - \sigma, u_5 + \bar{t}_4 - \sigma]$$

for all  $\sigma \in [u_2, u_4 + \bar{t}_4 - \bar{\tau}]$ . It is not difficult to show that there is a  $\sigma$  in interval



$[u_2, u_4 + \bar{t}_4 - \bar{\tau}]$  so that the length of

$$[\bar{\tau} - 1, \bar{t}_4 + 1] \cap [u_1 + \bar{t}_4 - \sigma, u_5 + \bar{t}_4 - \sigma]$$

is at least one.

Case 3:  $u_4 - u_2 < \bar{z} - \bar{t}_4$ . Consider

$$y_\sigma(t) = y(t - \bar{t}_4 - \delta_0 + \sigma) \text{ for } \sigma \in [u_2, u_4 + \bar{t}_4 + \delta_0 - \bar{\tau}].$$

Claim (ii) can be applied to show that  $p - y$  has at most two sign changes on the interval  $[\bar{\tau} - 1, \bar{t}_4 + \bar{\omega}] \cap [u_1 + \bar{t}_4 + \delta_0 - \sigma, u_5 + \bar{t}_4 + \delta_0 - \sigma]$  for all  $\sigma \in [u_2, u_4 + \bar{t}_4 + \delta_0 - \bar{\tau}]$ . A continuity argument yields that for some  $\sigma$ , the length of the interval is at least one.  $\square$

### 4.3 Rapidly oscillatory periodic solutions

We give conditions for the nonexistence of rapidly oscillatory solutions. First we go back to special nonlinearity  $f^{K,\varepsilon}$ .

**Proposition 4.3.1.** *For  $K \leq 8$  and  $\varepsilon \in (0, 1)$ , Eq. (3.2) has no periodic solution  $p : \mathbb{R} \rightarrow \mathbb{R}$  with  $V(p_t) \geq 4$  for  $t \in \mathbb{R}$ .*

*Proof.* If  $p : \mathbb{R} \rightarrow \mathbb{R}$  is a periodic solution of Eq. (3.2), then it is of monotone type and of special symmetry, see Propositions 3.4.1 and 3.4.2. Proposition 3.5.1 give Eq. (3.2) has no periodic solutions for  $K \in (0, 1]$ . Set  $K > 1$  and  $\varepsilon \in (0, 1)$ . If  $V(p_t) \geq 4$ , then  $3\omega/2 < 1$ , where  $\omega > 0$  is the minimal period of  $p$ . Proposition 3.5.1 gives that

$$1 > \frac{3}{2}\omega \geq 6 \ln \frac{K}{K-1} + 3 \ln \frac{K+1}{K} > 9 \ln \frac{K+1}{K},$$

that is

$$\begin{aligned} \frac{1}{K} < e^{\frac{1}{9}} - 1 &= \frac{1}{9} + \frac{\left(\frac{1}{9}\right)^2}{2!} + \frac{\left(\frac{1}{9}\right)^3}{3!} + \dots \\ &< \frac{1}{9} \left(1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \dots\right) = \frac{1}{9} \frac{1}{1 - \frac{1}{9}} = \frac{1}{8}. \end{aligned}$$

Thus  $K > 8$ , and the statement is verified.  $\square$

We also need the following simple observation regarding periodic solutions.

**Proposition 4.3.2.** *Assume  $\mu = 1$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable, nondecreasing, and*

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

*are five consecutive zeros of  $\xi \mapsto -\xi + f(\xi)$  with  $f'(\xi_j) < 1 < f'(\xi_k)$  for  $j \in \{-2, 0, 2\}$  and  $k \in \{-1, 1\}$ . Suppose  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a nontrivial periodic solution of Eq. (1.1) with*

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$x(t) \in [\xi_{-2}, \xi_2]$  for all  $t \in \mathbb{R}$ . Then the following statements hold. If  $\max_{t \in \mathbb{R}} x(t) > 0$ , then  $\xi_1 < \max_{t \in \mathbb{R}} x(t) < \xi_2$ . If  $\max_{t \in \mathbb{R}} x(t) < 0$ , then  $\xi_{-1} < \max_{t \in \mathbb{R}} x(t) < 0$ . If  $\min_{t \in \mathbb{R}} x(t) > 0$ , then  $0 < \min_{t \in \mathbb{R}} x(t) < \xi_1$ . If  $\min_{t \in \mathbb{R}} x(t) < 0$ , then  $\xi_{-2} < \min_{t \in \mathbb{R}} x(t) < \xi_{-1}$ .

*Proof.* Assume  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a periodic solution of Eq. (1.1) with  $x(t) \in [\xi_{-2}, \xi_2]$  for all  $t \in \mathbb{R}$  and  $\max_{t \in \mathbb{R}} x(t) > 0$ . Choose  $t^* \in \mathbb{R}$  so that  $x(t^*) = \max_{t \in \mathbb{R}} x(t)$ . In case  $x(t^*) < \xi_1$  use the fact that  $f(x) < x$  for  $x \in (0, \xi_1)$  to derive that

$$0 = \dot{x}(t^*) = -x(t^*) + f(x(t^* - 1)) \leq -x(t^*) + f(x(t^*)) < 0,$$

a contradiction. If  $x(t^*) = \xi_1$ , then Proposition 2.3.4 implies  $x(t^* - 1) < x(t^*)$ . As  $f$  is strictly increasing in a neighborhood of  $\xi_1$ , we get that

$$0 = \dot{x}(t^*) = -x(t^*) + f(x(t^* - 1)) < -x(t^*) + f(x(t^*)) = 0,$$

a contradiction. Hence  $x(t^*) > \xi_1$ . One may deduce that  $\max_{t \in \mathbb{R}} x(t) < \xi_2$  in the same way. We leave the verification of the rest of the statements also to the reader.  $\square$

Note that the conditions of the previous proposition are fulfilled if (H1) holds for  $f$ .

Recall that a threshold number  $\varepsilon_* > 0$  can be given so that to each  $\varepsilon \in (0, \varepsilon_*)$  there corresponds  $\delta_1 = \delta_1(\varepsilon) > 0$  with the following properties: if  $\mu = 1$ ,  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfies (H1), and  $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_1$ , then Eq. (1.1) admits exactly two normalized LSOP solutions. Now we are able to prove even more.

**Proposition 4.3.3.** *To each  $\varepsilon \in (0, \varepsilon_*)$ , there corresponds  $\delta_2 = \delta_2(\varepsilon) > 0$  such that if  $\mu = 1$ ,  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfies hypothesis (H1), and  $\|f - f^{7,\varepsilon}\|_{C_b^1} < \delta_2$ , then Eq. (1.1) with  $\mu = 1$  and nonlinearity  $f$  has no periodic solutions oscillating rapidly around 0.*

*Proof.* Suppose for contradiction that there is a sequence  $(f^n)_1^\infty$  in  $C_b^1(\mathbb{R}, \mathbb{R})$  with  $\|f - f^{7,\varepsilon}\|_{C_b^1} \rightarrow 0$  as  $n \rightarrow \infty$  so that for  $n \in \mathbb{N}$ , (H1) holds for  $f^n$ , and  $\dot{x}(t) = -x(t) + f^n(x(t-1))$  has a periodic solution  $p^n : \mathbb{R} \rightarrow \mathbb{R}$ , with  $V(p_t^n) > 2$  for all  $t \in \mathbb{R}$ . Applying the Arzelà–Ascoli theorem, we get that there exists a continuous function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that  $p^n, \dot{p}^n$  converge to  $p, \dot{p}$  uniformly on compact subsets of  $\mathbb{R}$ , respectively. Then  $p$  is a periodic solution of Eq. (3.2) with feedback function  $f^{7,\varepsilon}$ .

It follows from Proposition 4.3.2 that  $\max_{t \in \mathbb{R}} p^n(t) > \xi_1 > 1$  for all  $n \geq 1$ . Hence  $\max_{t \in \mathbb{R}} p(t) > 1$ . Similarly,  $\min_{t \in \mathbb{R}} p(t) < -1$ . Thus Proposition 4.3.2 implies  $p(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ .

As  $p$  is periodic,  $V(p_t)$  is the same constant for all  $t \in \mathbb{R}$ . As  $p$  oscillates around 0,  $V(p_t) \geq 2$  for all  $t \in \mathbb{R}$ . If  $V(p_t) \equiv 2$ , then  $p$  is an LSOP solution, and it is either  $x^{\Sigma(a^*(\varepsilon), \varepsilon)}$  or  $x^{\widetilde{\Sigma}(\widetilde{a}(\varepsilon), \varepsilon)}$  up to time translation. Thus the zeros of  $p$  are simple. As  $p^n \rightarrow p$  and  $\dot{p}^n \rightarrow \dot{p}$  uniformly on compact subsets of  $\mathbb{R}$ , we obtain that  $V(p_t^n) \equiv 2$  for all large

$n$ , a contradiction to the choice of  $p^n$ . So  $V(p_t) > 4$ , which contradicts Proposition 4.3.1. The proof is complete.  $\square$

## 4.4 Unstable manifolds

This section assumes that we are in the situation of Theorem 3.1.1, namely  $\mu = 1$ ,  $f \in C^1$  satisfies (H1), furthermore  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  are the normalized LSOP solutions of Eq. (1.1) with  $\max_{t \in \mathbb{R}} q(t) > \max_{t \in \mathbb{R}} p(t)$ .

Consider the  $C^1$ -smooth Poincaré return map  $P$  defined in a small neighborhood of  $p_0$  in  $H = \{\varphi \in C : \varphi(-1) = 0\}$  with fixed point  $p_0$ . Theorem 3.1.1 states that  $p_0$  is hyperbolic and  $DP(p_0)$  has exactly two eigenvalues  $\lambda_1 > \lambda_2$  with absolute value greater than 1. Let  $H_s$  and  $H_u$  be the closed subspaces of  $H$  chosen so that  $H = H_s \oplus H_u$ ,  $H_s$  and  $H_u$  are invariant under  $L = DP(p_0)$ , and the spectra  $\sigma_s$  and  $\sigma_u$  of the induced maps  $H_s \ni x \mapsto Lx \in H_s$  and  $H_u \ni x \mapsto Lx \in H_u$  are contained in  $\{\mu \in \mathbb{C} : |\mu| < 1\}$  and in  $\{\mu \in \mathbb{C} : |\mu| > 1\}$ , respectively. Then  $H_u$  is 2-dimensional (Appendix VII in [26]).

*The unstable manifold.* According to Appendix I in [26], there exist convex bounded neighborhoods  $N_s, N_u$  of 0 in  $H_s, H_u$ , respectively, and a  $C^1$ -map  $w : N_u \rightarrow H_s$  with range in  $N_s$  so that  $w(0) = 0$ ,  $Dw(0) = 0$ , and subset

$$\mathcal{W}^u(p_0) = \{p_0 + x + w(x) : x \in N_u\}$$

of  $C$  is equal to

$$\left\{x \in p_0 + N_s + N_u : \text{there is a trajectory } (x_n)_{-\infty}^0 \text{ of } P \text{ in } p_0 + N_s + N_u \text{ with } x_0 = x \text{ and } x_n \rightarrow p_0 \text{ as } n \rightarrow -\infty\right\}.$$

$\mathcal{W}^u(p_0)$  is the (2-dimensional) local unstable manifold of  $P$  at  $p_0$ .

*The leading unstable manifold.* Let  $H_u^1, H_u^2$  be the linear subspaces in  $H_u$  generated by  $v_1, v_2$ , the eigenvectors corresponding to  $\lambda_1, \lambda_2$ , respectively. Then  $H_u = H_u^1 \oplus H_u^2$ . Set  $\beta$  so that  $1 < \lambda_2 < \beta < \lambda_1$ . There exist  $\delta_0 > 0$  and a  $C^1$ -map  $\tilde{w} : (-\delta_0, \delta_0) v_1 \rightarrow H_u^2 \oplus H_s$  with  $\tilde{w}(0) = 0$  and  $D\tilde{w}(0) = 0$  such that for  $\delta^* \in (-\delta_0, \delta_0)$ , there is a trajectory  $(x_n)_{-\infty}^0$  of  $P$  with  $x_0 = p_0 + \tilde{w}(\delta^* v_1) + \delta^* v_1$  and with  $\beta^{-n}(x_n - p_0) \rightarrow 0$  as  $n \rightarrow -\infty$ . Moreover,  $x_n$  belongs to

$$\mathcal{W}_1^u(p_0) = \{p_0 + \tilde{w}(\delta v_1) + \delta v_1 : |\delta| < \delta_0\}$$

for  $n \leq 0$ .  $\mathcal{W}_1^u(p_0)$  is the leading unstable manifold of  $P$  at  $p_0$ . It is a 1-dimensional submanifold of  $\mathcal{W}^u(p_0)$ .

Similarly, there is a Poincaré map (also denoted by  $P$ ) with fixed point  $q_0$ . By Theorem 3.1.1,  $DP(q_0)$  has exactly one eigenvalue with absolute value greater than 1.  $\mathcal{W}^u(q_0)$  denotes the (1-dimensional) unstable manifold of  $P$  at  $q_0$ . The characterization

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of  $\mathcal{W}^u(q_0)$  is analogous to the one given for  $\mathcal{W}_1^u(p_0)$ .

The unstable set of  $\mathcal{O}_p = \{p_t : t \in \mathbb{R}\}$  is defined as

$$\mathcal{W}^u(\mathcal{O}_p) = \{x_0 : x : \mathbb{R} \rightarrow \mathbb{R} \text{ is a solution of (1.1), } \alpha(x) \text{ exists and } \alpha(x) = \mathcal{O}_p\}.$$

It is the forward extension of  $\mathcal{W}^u(p_0)$ :

$$\mathcal{W}^u(\mathcal{O}_p) = \{x_t^\varphi : \varphi \in \mathcal{W}^u(p_0), t \geq 0\}.$$

Set  $\mathcal{W}^u(\mathcal{O}_q)$  can be described analogously. We also introduce the leading unstable set  $\mathcal{W}_1^u(\mathcal{O}_p) = \{x_t^\varphi : \varphi \in \mathcal{W}_1^u(p_0), t \geq 0\}$ .

Recall that  $\varphi \leq \psi$  for  $\varphi, \psi \in C$  if  $\varphi(s) \leq \psi(s)$  for all  $s \in [-1, 0]$ . Relation  $\varphi < \psi$  holds if  $\varphi \leq \psi$  and  $\varphi \neq \psi$ . In addition,  $\varphi \ll \psi$  if  $\varphi(s) < \psi(s)$  for all  $s \in [-1, 0]$ . Relations “ $\geq$ ”, “ $>$ ” and “ $\gg$ ” are defined analogously.

The semiflow  $\Phi$  is monotone in the following sense.

**Proposition 4.4.1.** *Suppose  $\varphi, \psi \in C$  with  $\varphi \neq \psi$ . Then  $x_t^\varphi \neq x_t^\psi$  for all  $t \geq 0$ . If  $\varphi < \psi$  ( $\varphi > \psi$ ), then  $x_t^\varphi \ll x_t^\psi$  ( $x_t^\varphi \gg x_t^\psi$ ) for all  $t > 1$ . In addition, if  $\varphi \ll \psi$  ( $\varphi \gg \psi$ ), then  $x_t^\varphi \ll x_t^\psi$  ( $x_t^\varphi \gg x_t^\psi$ ) for all  $t \geq 0$ .*

The assertion follows easily from the variation-of-constant formula. For a proof we refer to [43].

Proposition 4.3.2 implies that  $p(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$  and  $q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$ . Hence

$$\mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q) \subset \mathcal{A}$$

by Proposition 4.4.1. Consequently,  $\{x_t^\varphi : t \in \mathbb{R}\}$  is precompact for each  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$ .

We need a few more propositions before proving Theorem 4.1.1.

**Proposition 4.4.2.** *Assume  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a solution of Eq. (1.1) with initial function  $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$  such that  $x$  oscillates around  $\xi \in \{\xi_{-1}, 0, \xi_1\}$ . Then  $V(x_t - \hat{\xi}) = 2$  for all  $t \in \mathbb{R}$ , where  $\hat{\xi} \in C$  is the equilibrium  $\hat{\xi}(s) = \xi$ ,  $s \in [-1, 0]$ . In addition,  $V(x_{t+u} - p_t) = 2$  for all  $u, t \in \mathbb{R}$  and  $V(x_{t+u} - x_t) = 2$  for all  $u \in \mathbb{R} \setminus \{0\}$  and  $t \in \mathbb{R}$ . If  $x$  oscillates around  $\xi_i$  with  $i \in \{-1, 1\}$ , then  $V(x_{t+u} - x_t^i) = 2$  for all  $u, t \in \mathbb{R}$ , where  $x^i : \mathbb{R} \rightarrow \mathbb{R}$  is given by Proposition 4.2.1.*

*Proof.* Let  $x$  be a solution of Eq. (1.1) oscillating around  $\xi \in \{\xi_{-1}, 0, \xi_1\}$  with  $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$ . Clearly,  $x_0 \neq \hat{\xi}$ , hence  $x_t \neq \hat{\xi}$  for  $t \in \mathbb{R}$  by Proposition 4.4.1.

Since  $x_0 \in \mathcal{W}^u(p_0)$ , there exists  $(t_n)_0^\infty \subset \mathbb{R}$  so that  $t_n \rightarrow -\infty$  as  $n \rightarrow \infty$ ,  $x_{t_n} \in \mathcal{W}^u(p_0)$  for  $n \geq 0$  and  $x_{t_n} \rightarrow p_0$  in  $C$  as  $n \rightarrow \infty$ . Clearly,  $p_0 \in \mathcal{A}$  and  $x_t \in \mathcal{A}$  for all  $t \in \mathbb{R}$ . The norms  $\|\cdot\|$  and  $\|\cdot\|_1$  are equivalent on  $\mathcal{A}$ . Thus  $x_{t_n} \rightarrow p_0$  as  $n \rightarrow \infty$  also in  $C^1$ -norm.

Let  $\omega \in (1, 2)$  be the minimal period of  $p$ . Clearly,  $V(p_t - \hat{\xi}) = 2$  for all  $t \in [0, \omega)$ , hence Lemma 2.3.2 (iii) gives  $p_t - \hat{\xi} \in R$  for all  $t \in [0, \omega)$ , where function class  $R$  is defined in Subsection 2.3.1. Now Lemma 2.3.1 implies

$$2 = V(p_0 - \hat{\xi}) = \lim_{n \rightarrow \infty} V(x_{t_n} - \hat{\xi}).$$

Hence by Lemma 2.3.2 (i),  $V(x_t - \hat{\xi}) \leq 2$  for all real  $t$ . If  $V(x_{t^*} - \hat{\xi}) = 0$  for some  $t^* \in \mathbb{R}$ , that is  $x_{t^*} < \hat{\xi}$  or  $x_{t^*} > \hat{\xi}$ , then Proposition 4.4.1 implies  $x_t \ll \hat{\xi}$  or  $x_t \gg \hat{\xi}$  for all  $t > t^* + 1$ , respectively. This is a contradiction as  $x$  oscillates around  $\xi$ . So  $V(x_t - \hat{\xi}) = 2$  for all  $t \in \mathbb{R}$ .

It is easy to deduce from the monotone property of  $p$  that  $V(p_{t+\tau} - p_{t+\sigma}) = 2$  in case  $t \in \mathbb{R}$ ,  $\tau, \sigma \in [0, \omega)$  and  $\sigma \neq \tau$ . In consequence  $p_{t+\tau} - p_{t+\sigma} \in R$  all for  $t \in \mathbb{R}$  and  $\sigma \neq \tau$ .

Now choose any  $u \in \mathbb{R}$ . Using the continuity of the flow  $\Phi_A$ , we obtain that  $x_{t_n+u} \rightarrow p_u$  in  $C^1$ -norm as  $n \rightarrow \infty$ . By compactness, we may assume the existence of  $\sigma \in [0, \omega)$  such that  $p_{t_n} \rightarrow p_\sigma$  in  $C^1$ -norm as  $n \rightarrow \infty$ . If  $\sigma \neq u$ , then Lemma 2.3.1 implies

$$2 = V(p_u - p_\sigma) = \lim_{n \rightarrow \infty} V(x_{t_n+u} - p_{t_n}),$$

and Lemma 2.3.2 (i) gives  $V(x_{t+u} - p_t) \leq 2$  for all real  $t$ . In case  $\sigma = u$ , observe that  $x_{t_n+u+\varepsilon} \rightarrow p_\varepsilon \neq p_\sigma$  for any small  $\varepsilon > 0$ , thus we may use our previous result and Lemma 2.3.1 to get

$$V(x_{t+u} - p_t) \leq \liminf_{\varepsilon \rightarrow 0+} V(x_{t+u+\varepsilon} - p_t) \leq 2$$

for all real  $t$ .

Now assume that  $V(x_{t^*+u} - p_{t^*}) = 0$  for some  $t^* \geq 0$ , that is  $x_{t^*+u} \leq p_{t^*}$  or  $x_{t^*+u} \geq p_{t^*}$ . Suppose  $x_{t^*+u} \leq p_{t^*}$  for example. As  $x_0 \notin \mathcal{O}_p$ , Proposition 4.4.1 gives  $x_{t^*+u} \neq p_{t^*}$  and thus  $x_{t^*+u+2} \ll p_{t^*+2}$ . By Theorem 2.1.1, the set of those functions  $\varphi$  for which  $x_t^\varphi$  converges to an equilibrium as  $t \rightarrow \infty$  is dense in  $C$ . Consequently there exists  $\eta \in C$  so that  $x_t^\eta$  tends to one of the equilibrium points as  $t \rightarrow \infty$ , and  $x_{t^*+u+2} \ll \eta \ll p_{t^*+2}$ . As  $x_{t+t^*+u+2} \ll x_t^\eta \ll p_{t+t^*+2}$  for all  $t \geq 0$  again by Proposition 4.4.1, this equilibrium point is necessarily  $\hat{\xi}_{-2}$  contradicting to the fact that  $x$  oscillates around  $\xi$ . One comes to the same conclusion assuming that  $x_{t^*+u} \geq p_{t^*}$ .

The argument confirming the rest of the claim is similar, so we leave it to the reader. To prove the last assertion, use Proposition 4.2.2.  $\square$

In this chapter, a second essential technical tool besides the Lyapunov functional is the linear map  $\pi : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$  introduced in Subsection 2.3.1. From paper [33] of Mallet-Paret and Sell we know that  $\pi$  maps nontrivial periodic orbits of Eq. (1.1) into simple closed curves in  $\mathbb{R}^2$ , and the images of different periodic orbits are

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disjoint curves in  $\mathbb{R}^2$ . So

$$O_p : \mathbb{R} \ni t \mapsto \pi p_t \in \mathbb{R}^2, \quad O_q : \mathbb{R} \ni t \mapsto \pi q_t \in \mathbb{R}^2,$$

$$O_1 : \mathbb{R} \ni t \mapsto \pi x_t^1 \in \mathbb{R}^2 \text{ and } O_{-1} : \mathbb{R} \ni t \mapsto \pi x_t^{-1} \in \mathbb{R}^2$$

are simple closed curves and disjoint. Here solutions  $x^1$  and  $x^{-1}$  are the periodic solutions given by Proposition 4.2.1.

As  $q(\mathbb{R}) \supsetneq p(\mathbb{R})$ ,  $O_q \subset \text{ext}(O_p)$ . Also,  $\pi \hat{0}, O_1, O_{-1} \in \text{int}(O_p)$ , and  $\pi \hat{\xi}_{-2}, \pi \hat{\xi}_2$  belong to  $\text{ext}(O_q)$ . For the images of the unstable equilibria, we have  $\pi \hat{\xi}_{-1} \in \text{int}(O_{-1})$  and  $\pi \hat{\xi}_1 \in \text{int}(O_1)$ . If  $x : \mathbb{R} \rightarrow \mathbb{R}$  is periodic solution oscillating slowly around  $\xi_{-1}$  with  $x(\mathbb{R}) \subset (\xi_{-2}, 0)$ , then either  $\{\pi x_t : t \in \mathbb{R}\} = O_{-1}$  or  $\{\pi x_t : t \in \mathbb{R}\} \subsetneq O_{-1}$  by Proposition 4.2.1. Similarly, for a periodic solution  $x$  oscillating slowly around  $\xi_1$  with range in  $(0, \xi_2)$ , either  $\{\pi x_t : t \in \mathbb{R}\} = O_1$  or  $\{\pi x_t : t \in \mathbb{R}\} \subsetneq O_1$ .

Note that as  $p(-1) = q(-1) = 0$ ,  $p(0) < 0$ ,  $q(0) < 0$  and  $O_q \subset \text{ext}(O_p)$ , we have  $q(0) < p(0) < 0$ .

**Corollary 4.4.3.** *Let  $x : \mathbb{R} \rightarrow \mathbb{R}$  be a solution of Eq.(1.1) with initial data  $x_0 \in \mathcal{W}^u(p_0) \setminus p_0$  such that  $x$  oscillates around  $\xi \in \{\xi_{-1}, 0, \xi_1\}$ . Then curve  $S : \mathbb{R} \ni t \mapsto \pi x_t \in \mathbb{R}^2$  is simple and does not intersect  $O_p$ .*

*Proof.* Proposition 4.4.2 yields  $t \mapsto V(x_{t+u} - x_t)$  is finite and constant for all  $u \in \mathbb{R} \setminus \{0\}$ . If there exist  $t \in \mathbb{R}$  and  $u \in \mathbb{R} \setminus \{0\}$  such that  $\pi x_t = \pi x_{t+u}$ , then by Lemma 2.3.2 (ii),  $V(x_{t+u} - x_t) < V(x_{t+u-2} - x_{t-2})$ , a contradiction. So  $S$  is simple. It follows from Proposition 4.4.2 and Lemma 2.3.2 (ii) in a similar way that  $S$  and  $O_p$  are disjoint.  $\square$

### 4.5 The proof of Theorem 4.1.1.

Set  $\mu = 1$ ,  $K = 7$  and  $\varepsilon \in (0, \varepsilon_*)$ , where  $\varepsilon_*$  is given by Proposition 3.5.6. Choose non-linearity  $f \in C_b^1(\mathbb{R}, \mathbb{R})$  satisfying hypothesis (H1) so that  $\|f - f^{7,\varepsilon}\|_{C_b^1} < \min\{\delta_1, \delta_2\}$ . Then the conditions of Propositions 3.3.5, 3.5.7 and 4.3.3 are satisfied by  $f$ , which means that the statement of Theorem 3.1.1 holds, and Eq.(1.1) admits no rapidly oscillatory solutions.

*Remark 4.5.1.* We may assume that  $f$  satisfies hypothesis (H2) introduced in Section 4.2. As  $f$  is close to  $f^{7,\varepsilon}$  in  $C_b^1$ -norm, it suffices to verify this statement for  $f^{7,\varepsilon}$  with  $\varepsilon \in (0, \varepsilon_*)$ . Recall that  $f^{7,\varepsilon}$  is defined by

$$f^{7,\varepsilon}(x) = 7\rho\left(\frac{|x| - 1}{\varepsilon}\right) \text{sgn}(x)$$

for all  $\varepsilon \in [0, 1)$ , where  $\rho \in C^\infty$ ,  $\rho(t) = 0$  for  $t \leq 0$ ,  $\rho(t) = 1$  for  $t \geq 1$  and  $\rho'(t) > 0$  for

#### 4.5 The proof of Theorem 4.1.1.

all  $t \in (0, 1)$ . Set interval  $I_\varepsilon = \rho^{-1} [1/7, (1 + \varepsilon)/7]$ . Clearly,

$$\min_{t \in I_\varepsilon} \rho'(t) \geq \min_{t \in \rho^{-1} [\frac{1}{7}, \frac{2}{7}]} \rho'(t) = m > 0.$$

As (H1) holds and  $\xi_1 \in (1, 1 + \varepsilon)$ , there exists  $t_0 \in I_\varepsilon$  such that  $t_0 = (\xi_1 - 1)/\varepsilon$  and  $\rho(t_0) = \xi_1/7$ . We obtain that

$$(f^{7,\varepsilon})'(\xi_1) = \frac{7}{\varepsilon} \rho'(t_0) \geq \frac{7m}{\varepsilon} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0+.$$

Similarly,  $(f^{7,\varepsilon})'(\xi_{-1}) \rightarrow \infty$  as  $\varepsilon \rightarrow 0+$ . So we may assume that  $\varepsilon_* > 0$  is chosen so small that (H2) holds for  $f^{7,\varepsilon}$  provided  $\varepsilon \in (0, \varepsilon_*)$ .

Theorem 4.1.1 is the direct consequence of Claims 4.5.2-4.5.8 below.

*Claim 4.5.2.*  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}) = \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$ .

*Proof.* Clearly,  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}) \supseteq \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$ . Suppose  $x : \mathbb{R} \rightarrow \mathbb{R}$  is a solution of (1.1) with  $x_0 \in \mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ . Then  $\alpha(x)$  contains no stable equilibrium point, as in this case  $x_0$  would be the stable equilibrium itself. If  $\hat{\xi}_1 \in \alpha(x)$ , then Proposition 4.4.1 implies  $x_t \in C_{0,2}$  for all  $t \in \mathbb{R}$ , a contradiction to  $x_0 \notin \mathcal{A}_{0,2}$ . Similarly,  $\hat{\xi}_{-1} \notin \alpha(x)$ . As  $x$  is necessarily bounded, the Poincaré–Bendixson theorem implies  $\alpha(x)$  is a periodic orbit. Theorem 4.3.3 gives there are no periodic orbits in  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$  besides  $\mathcal{O}_p$  and  $\mathcal{O}_q$ . So  $x_0 \in \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$ .  $\square$

*Claim 4.5.3.* There exist connecting orbits from  $\mathcal{O}_p$  and  $\mathcal{O}_q$  to the equilibrium points  $\hat{\xi}_{-2}$  and  $\hat{\xi}_2$ . Moreover, for each  $\varphi \in \mathcal{W}_1^u(\mathcal{O}_p) \setminus \mathcal{O}_p$  and for each  $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$ ,  $\omega(\varphi)$  is either  $\hat{\xi}_{-2}$  or  $\hat{\xi}_2$ .

*Proof.* First consider the 1-dimensional leading unstable manifold  $\mathcal{W}_1^u(p_0)$ . By Appendix VII in [26], the eigenfunction  $v_1$  corresponding to the greatest positive eigenvalue  $\lambda_1$  of  $DP(p_0)$  is strictly positive. Choose  $\delta_1$  so small that  $\|D\tilde{w}(\delta v_1)\| < 1/2$  for  $|\delta| < \delta_1$ , where  $\tilde{w}$  is the  $C^1$ -map introduced on page 75. Observe that

$$\tilde{w}(\delta v_1) + \delta v_1 = \int_0^1 D\tilde{w}(s\delta v_1) \delta v_1 ds + \delta v_1 \gg 0$$

if  $\delta \in (0, \delta_1)$ , and  $\tilde{w}(\delta v_1) + \delta v_1 \ll 0$  if  $\delta \in (-\delta_1, 0)$ . Setting

$$\eta_1 = p_0 + \frac{\delta_1}{2} v_1 + \tilde{w}\left(\frac{\delta_1}{2} v_1\right) \text{ and } \eta_2 = p_0 - \frac{\delta_1}{2} v_1 + \tilde{w}\left(-\frac{\delta_1}{2} v_1\right),$$

we get  $\eta_1, \eta_2 \in \mathcal{W}_1^u(p_0)$  and  $\eta_2 \ll p_0 \ll \eta_1$ . According to Theorem 2.1.1, there exist  $\eta_1^+, \eta_1^-, \eta_2^+, \eta_2^- \in C$  such that

$$\eta_2^- \ll \eta_2 \ll \eta_2^+ \ll p_0 \ll \eta_1^- \ll \eta_1 \ll \eta_1^+,$$

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and for  $i = 1, 2$ , solutions  $x_t^{\eta_i^-}$  and  $x_t^{\eta_i^+}$  converge to one of the equilibrium points as  $t \rightarrow \infty$ . Since  $\max_{t \in \mathbb{R}} p(t) > \xi_1$ ,  $\min_{t \in \mathbb{R}} p(t) < \xi_{-1}$  and

$$x_t^{\eta_2^-} \ll x_t^{\eta_2^+} \ll p_t \ll x_t^{\eta_1^-} \ll x_t^{\eta_1^+} \quad \text{for } t \geq 0$$

by Proposition 4.4.1, we obtain that

$$x_t^{\eta_2^-} \rightarrow \hat{\xi}_{-2}, \quad x_t^{\eta_2^+} \rightarrow \hat{\xi}_{-2}, \quad x_t^{\eta_1^-} \rightarrow \hat{\xi}_2 \text{ and } x_t^{\eta_1^+} \rightarrow \hat{\xi}_2 \text{ as } t \rightarrow \infty.$$

Using Proposition 4.4.1 again, we get  $x_t^{\eta_2} \rightarrow \hat{\xi}_{-2}$  and  $x_t^{\eta_1} \rightarrow \hat{\xi}_2$  as  $t \rightarrow \infty$ .

For each  $\varphi \in \mathcal{W}_1^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ , there is a solution  $x : \mathbb{R} \rightarrow \mathbb{R}$  of Eq. (1.1) and a sequence  $(t_n)_0^\infty$  such that  $x_0 = \varphi$ ,  $x_{t_n} \in \mathcal{W}_1^u(p_0) \setminus p_0$  for all  $n \geq 0$  and  $x_{t_n} \rightarrow p_0$  as  $n \rightarrow \infty$ . Hence there exist  $\delta \in (-\delta_1, 0) \cup (0, \delta_1)$  and  $n^* \geq 0$  so that  $x_{t_{n^*}} = p_0 + \tilde{w}(\delta v_1) + \delta v_1$ . The above reasoning shows that if  $\delta < 0$ , then  $\omega(\varphi) = \hat{\xi}_{-2}$ , and if  $\delta > 0$ , then  $\omega(\varphi) = \hat{\xi}_2$ .

Since  $\mathcal{W}^u(q_0)$  is a 1-dimensional unstable manifold as well, and  $\mathcal{W}^u(\mathcal{O}_q)$  is the forward extension of  $\mathcal{W}^u(q_0)$ , it is analogous to show that for each  $\varphi \in \mathcal{W}^u(\mathcal{O}_q) \setminus \mathcal{O}_q$ ,  $\omega(\varphi)$  is either  $\hat{\xi}_{-2}$  or  $\hat{\xi}_2$ , moreover these connections indeed exist.  $\square$

It remains to describe  $\mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{W}_1^u(\mathcal{O}_p)$ .

*Claim 4.5.4.* Suppose that for  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ , the limit set  $\omega(\varphi)$  is a non-constant periodic orbit. Then the subsequent assertions are true. If solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around 0, then  $\omega(\varphi) = \mathcal{O}_q$ . Otherwise  $\omega(\varphi)$  is either  $\mathcal{O}_{-1}$  or  $\mathcal{O}_1$ .

*Proof.* Suppose  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ , and  $\omega(\varphi)$  is a non-constant periodic orbit  $\{r_t : t \in \mathbb{R}\}$ .

First let us examine the case when  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around 0. Then as  $\mathcal{W}^u(\mathcal{O}_p)$  is the forward extension of  $\mathcal{W}^u(p_0)$ , Proposition 4.4.2 implies  $V(x_t^\varphi) = 2$  for all  $t \in \mathbb{R}$ . For any  $t \in \mathbb{R}$  fixed, there exists  $(t_n)_0^\infty$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  so that  $r_t$  is the limit of  $x_{t_n}^\varphi$  in  $C$ . As we have seen before, this implies convergence also in  $C^1$ -norm. As the segments of any periodic solution belong to  $R$ , Lemma 2.3.1 gives  $V(r_t) = \lim_{n \rightarrow \infty} V(x_{t_n}^\varphi) = 2$ . In addition, Proposition 4.3.2 yields  $r(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$ . Therefore  $r$  equals  $p$  or  $q$  apart from shift by Theorem 3.1.1. We claim that  $\omega(\eta_3) \neq \mathcal{O}_p$ . Indeed, Corollary 4.4.3 implies  $\mathbb{R} \ni t \mapsto \pi x_t^\varphi \in \mathbb{R}^2$  is a simple curve winding around  $(0, 0)$ . This fact and the assumption that  $\text{dist}(\pi x_t^\varphi, \pi \mathcal{O}_p) \rightarrow 0$  as  $t \rightarrow \pm\infty$  give a contradiction by the Jordan curve theorem. So we obtain that if  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around 0, then  $\omega(\varphi) = \mathcal{O}_q$ .

Now assume that  $x^\varphi$  is not oscillating around 0, that is there exists  $t_* \in \mathbb{R}$  such that  $x_{t_*}^\varphi > 0$  or  $x_{t_*}^\varphi < 0$ . Suppose  $x_{t_*}^\varphi > 0$  for example. Then  $x_t^\varphi \gg 0$  for all  $t > t_* + 1$ . Necessarily  $r(t) > 0$  for all  $t \in \mathbb{R}$ , and Proposition 4.3.2 gives that

$$0 < \min_{t \in \mathbb{R}} r(t) < \xi_1 < \max_{t \in \mathbb{R}} r(t) < \xi_2.$$

As  $\omega(\varphi) = \{r_t : t \in \mathbb{R}\}$ , solution  $x^\varphi$  is also oscillatory around  $\xi_1$ . Thus  $V(x_t^\varphi - \hat{\xi}_1) = 2$



#### 4.5 The proof of Theorem 4.1.1.

for all  $t \in \mathbb{R}$  by Proposition 4.4.2. For each  $t \in \mathbb{R}$ , there corresponds a sequence  $(t_n)_0^\infty \subset \mathbb{R}$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $x_{t_n}^\varphi \rightarrow r_t$  in  $C$  (and thus in  $C^1$ ) as  $n \rightarrow \infty$ . Hence  $V(r_t - \hat{\xi}_1) = 2$  for all  $t \in \mathbb{R}$  by Lemma 2.3.1. We obtain that  $r$  is a slowly oscillatory periodic solution around  $\xi_1$  and has range in  $(0, \xi_2)$ . Recall from Proposition 4.2.1 that the periodic solution  $x^1 : \mathbb{R} \rightarrow \mathbb{R}$  is set so that it oscillates slowly around  $\xi_1$  with  $x^1(\mathbb{R}) \subset (0, \xi_2)$ , and the range  $x^1(\mathbb{R})$  is maximal in the sense that  $x^1(\mathbb{R}) \supset x(\mathbb{R})$  for all periodic solutions  $x$  oscillating slowly around  $\xi_1$  with ranges in  $(0, \xi_2)$ . Therefore  $\{\pi r_t : t \in \mathbb{R}\}$  either equals  $O_1$  or belongs to  $\text{int}(O_1)$ . Proposition 4.4.2 implies  $V(x_{t+u}^\varphi - x_t^1) = 2$  for all  $u, t \in \mathbb{R}$ . With Lemma 2.3.2 (ii), this yields that curve  $S : \mathbb{R} \ni t \mapsto \pi x_t^\varphi \in \mathbb{R}^2$  does not intersect  $O_1$ . So necessarily  $r$  equals  $x^1$  apart from shift and  $\omega(\varphi) = O_1$ . In case there is  $t_* \in \mathbb{R}$  such that  $x_{t_*}^\varphi < 0$ , we deduce in a similar way that  $\omega(\varphi) = O_{-1}$ .  $\square$

*Claim 4.5.5.* Assume that for  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ , limit set  $\omega(\varphi)$  is not a non-constant periodic orbit. Then it is a stable equilibrium.

*Proof.* As for all  $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ , orbit  $\{x_t^\varphi : t \geq 0\}$  is bounded, the Poincaré–Bendixson theorem can be applied. If  $\omega(\varphi)$  is not a non-constant periodic orbit, then for each  $\psi \in \omega(\varphi)$ , we have  $\alpha(\psi) \cup \omega(\psi) \subset \{\hat{\xi}_i : i = -2, -1, 0, 1, 2\}$  (see Section 4.1).

If  $\hat{\xi}_0$  is in  $\omega(\varphi)$ , then  $\hat{\xi}_0 = \omega(\varphi)$  as this equilibrium is stable. Similarly for  $\hat{\xi}_{-2}$  and  $\hat{\xi}_2$ .

Suppose for contradiction that  $\omega(\varphi)$  contains no stable equilibrium point. If  $\varphi$  is in the stable set of  $\hat{\xi}_i$  with  $i \in \{-1, 1\}$ , then as (H2) holds,  $V(x_t^\varphi - \hat{\xi}_i) > 2$  for all  $t \in \mathbb{R}$  (see Section 2.1), a contradiction to Proposition 4.4.2. So there exists  $\psi \in \omega(\varphi)$  such that  $\psi$  is not an equilibrium. Then  $\alpha(\psi) \cup \omega(\psi) \subset \{\hat{\xi}_{-1}, \hat{\xi}_1\}$ . As it is already mentioned, there exists no homoclinic orbit to  $\hat{\xi}_1$  and to  $\hat{\xi}_{-1}$ . Hence  $\alpha(\psi) \neq \omega(\psi)$ . If  $\alpha(\psi) = \hat{\xi}_{-1}$ , then there exists  $t^* \in \mathbb{R}$  with  $x_{t^*}^\psi \ll \hat{\xi}_0$ . By Proposition 4.4.1,  $x_t^\psi \ll \hat{\xi}_0$  for each  $t > t^*$ , a contradiction to  $\omega(\psi) = \hat{\xi}_1$ . One comes to the same conclusion assuming that  $\alpha(\psi) = \hat{\xi}_1$  and  $\omega(\psi) = \hat{\xi}_{-1}$ .

So  $\omega(\varphi)$  is a stable equilibrium.  $\square$

We have to show that the above connections indeed exist.

Recall that the unstable space

$$H_u = \{c_1 v_1 + c_2 v_2 : c_1, c_2 \in \mathbb{R}\}$$

of  $DP(p_0)$  is 2-dimensional, where  $v_1$  is a positive eigenfunction corresponding to the leading eigenvalue  $\lambda_1$  and  $v_2$  is the eigenfunction corresponding to the second eigenvalue  $\lambda_2$  greater than one. Then for the solution  $x_t^{v_2} : \mathbb{R} \rightarrow \mathbb{R}$  of the linear variational equation

$$\dot{x}(t) = -x(t) + f'(p(t-1))x(t-1) \quad (4.1)$$

#### 4 The Global Attractor

with initial segment  $v_2$ , we have  $V(x_t^{v_2}) = 2$  for all real  $t$  [26]. Clearly  $v_2(-1) = 0$  and so  $v_2(0) \neq 0$  by Lemma 2.3.2. Either  $v_2(0) > 0$  or  $v_2(0) < 0$  is possible. Assume eigenfunction  $v_2$  is chosen so that  $v_2(0) > 0$ . Also, we may set  $\|v_1\| = \|v_2\| = 1$ .

For  $n \geq 0$ , let  $S_n = \{\varphi \in C : \|\varphi - p_0\| = 1/n\}$  denote the sphere in  $C$  centered at  $p_0$  with radius  $1/n$ . As  $\mathcal{W}^u(p_0)$  and  $\mathcal{W}_1^u(p_0)$  are 2-dimensional and 1-dimensional local manifolds tangent to  $\{p_0\} + H_u$  and  $\{p_0\} + H_u^1$  at  $p_0$ , respectively, there exists  $n_0 \geq 0$  such that for  $n \geq n_0$ ,  $S_n \cap \mathcal{W}^u(p_0)$  is homeomorphic to  $S^1$ , and  $S_n \cap \mathcal{W}_1^u(p_0)$  consists of two points  $\eta_1^n \in H$  and  $\eta_2^n \in H$ . It is easy to see from the proof of Claim 4.5.3 that  $\eta_1^n \ll p_0 \ll \eta_2^n$  for each  $n \geq n_0$ .

For each  $n \geq n_0$ , let  $C_n : [-1, 1] \rightarrow S_n \cap \mathcal{W}^u(p_0)$  be a simple closed curve with  $C_n(-1) = C_n(1) = \eta_1^n$  and  $C_n(0) = \eta_2^n$  oriented so that  $Pr_{H_u}(C_n(-1, 0) - p_0)$  intersects  $\{cv_2 : c < 0\} \subset H_u^2$  and  $Pr_{H_u}(C_n(0, 1) - p_0)$  intersects  $\{cv_2 : c > 0\} \subset H_u^2$ , see Fig. 4.2. This choice is possible. Obviously,  $C_n(s) \neq p_0$  for  $n \geq n_0$  and  $s \in [-1, 1]$ .

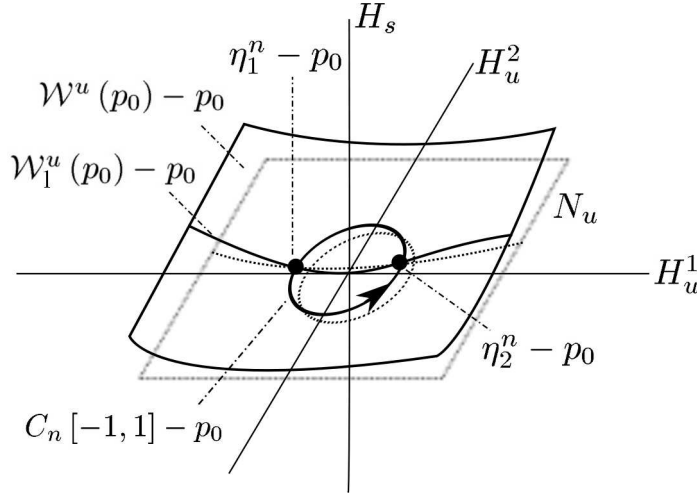


Figure 4.2: The unstable manifold

To prove the existence of the heteroclinic connections, we are going to apply the next assertion.

*Claim 4.5.6.* To each  $\xi \in \{\xi_{-1}, \xi_0, \xi_1\}$ , there correspond initial functions  $\varphi \in \mathcal{W}^u(p_0)$  and  $\psi \in \mathcal{W}^u(p_0)$  with

$$q(0) < \varphi(0) < p(0) < \psi(0) < 0$$

such that solutions  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^\psi : \mathbb{R} \rightarrow \mathbb{R}$  oscillate around  $\xi$ .

*Proof.* Set  $\xi \in \{\xi_{-1}, \xi_0, \xi_1\}$  and define

$$A_+ = \left\{ \eta \in \mathcal{W}^u(p_0) : x_t^\eta \gg \hat{\xi} \text{ for some } t \geq 0 \right\}$$

and

$$A_- = \left\{ \eta \in \mathcal{W}^u(p_0) : x_t^\eta \ll \hat{\xi} \text{ for some } t \geq 0 \right\}.$$

#### 4.5 The proof of Theorem 4.1.1.

Clearly  $\eta_1^n \in A_-$  and  $\eta_2^n \in A_+$  for all  $n \geq n_0$ . Then sets  $A_+ \cap C_n[-1, 0]$  and  $A_- \cap C_n[-1, 0]$  are disjoint, open and nonempty in  $C_n[-1, 0]$  for all  $n \geq n_0$ . It follows from connectedness that there exists  $s_n \in (-1, 0)$  with  $C_n(s_n) \notin (A_+ \cup A_-)$ , that is  $x^{C_n(s_n)} : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around  $\xi$ .

For  $n \geq n_0$ , function  $y^n : \mathbb{R} \rightarrow \mathbb{R}$  with

$$y^n(t) = \frac{x^{C_n(s_n)}(t) - p(t)}{\|C_n(s_n) - p_0\|}, \quad t \in \mathbb{R},$$

satisfies equation  $\dot{y}^n(t) = -y^n(t) + a^n(t)y^n(t-1)$ , where

$$a^n : \mathbb{R} \ni t \mapsto \int_0^1 f'(\theta x^{C_n(s_n)}(t-1) + (1-\theta)p(t-1)) d\theta \in \mathbb{R}.$$

Because of the choice of curves  $C_n$ ,  $a^n(t) \rightarrow f'(p(t-1))$  as  $n \rightarrow \infty$  uniformly on compact subsets of  $[0, \infty)$ .

Since  $C_n(s_n) \in \mathcal{W}^u(p_0) \setminus \{p_0\}$  for all  $n \geq n_0$ , since  $C_n(s_n) \rightarrow p_0$  as  $n \rightarrow \infty$  and as  $\mathcal{W}^u(p_0)$  is tangent to  $\{p_0\} + H_u$  at  $p_0$ , we may suppose that  $y_0^n \rightarrow z_0 \in C$  as  $n \rightarrow \infty$ , where  $z_0 \in H_u$ . Since  $\|y_0^n\| = 1$  for all  $n \geq n_0$ ,  $\|z_0\| = 1$ . Let  $z : [-1, \infty) \rightarrow \mathbb{R}$  be the solution of the linear variational equation (4.1) with initial data  $z_0$ . Then  $y^n \rightarrow z$  uniformly on compact subsets of  $[-1, \infty)$ .

We claim that  $z_0 = -v_2$ . Assume that  $z_0 = c_1 v_1 + c_2 v_2$  with  $c_1 \neq 0$ . As  $v_1$  is a positive eigenfunction corresponding to the leading eigenvalue  $\lambda_1 > 1$ , there exists  $t^* = t^*(c_1)$  such that  $z_{t^*} \gg 0$  (or  $z_{t^*} \ll 0$ ) and thus  $y_{t^*}^n \gg 0$  (or  $y_{t^*}^n \ll 0$ ) for some  $n \geq n_0$ . This is impossible by Proposition 4.4.2. So  $z_0 = c_2 v_2$  with  $c_2 \in \mathbb{R}$ . The definition of  $C_n$  and the fact that  $s_n \in (-1, 0)$  implies  $c_2 \leq 0$ . Also,  $|c_2| = 1$  as  $\|z_0\| = \|v_2\| = 1$ . So  $c_2 = -1$ .

As  $v_2(0) > 0$ , we conclude that  $z_0(0) < 0$ . Since  $y_0^n \rightarrow z_0$  and  $C_n(s_n) \rightarrow p_0$  as  $n \rightarrow \infty$ , there exist  $n_1 \in \mathbb{N}$  so that for  $n \geq n_1$ ,  $q_0(0) < C_n(s_n)(0) < p_0(0)$ . Accordingly set  $\varphi = C_{n_1}(s_{n_1})$ .

Similarly, there exists  $t_n \in (0, 1)$  so that solution  $x^{C_n(t_n)} : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around  $\xi$ . The same reasoning carried out for  $(C_n(t_n))_{n_0}^\infty$  instead of  $(C_n(s_n))_{n_0}^\infty$  implies that  $p_0(0) < C_n(t_n)(0) < 0$  for  $n \geq n_2$  with some  $n_2 \in \mathbb{N}$ . So choose  $\psi = C_{n_2}(t_{n_2})$ .

Clearly  $\varphi$  and  $\psi$  are in possession of the required properties.  $\square$

*Claim 4.5.7.* There exist heteroclinic connections from  $\mathcal{O}_p$  to  $\hat{0}$  and to  $\mathcal{O}_q$ .

*Proof.* Claim 4.5.6 gives that there exists  $\eta_3, \eta_4 \in \mathcal{W}^u(p_0)$  with

$$q(0) < \eta_3(0) < p(0) < \eta_4(0) < 0$$

such that solutions  $x^{\eta_3} : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^{\eta_4} : \mathbb{R} \rightarrow \mathbb{R}$  oscillate around 0. Claim 4.5.5 gives that  $\omega(\eta_i)$ ,  $i \in \{3, 4\}$ , is either a periodic orbit or a stable equilibrium. If

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$\omega(\eta_3) = \hat{\xi}_2$ , then by the monotone property of the semiflow  $\Phi$  (see Proposition 4.4.1) there is  $t_0 > 0$  such that  $x_t^{\eta_3} \gg 0$  for  $t > t_0$ , a contradiction. Similarly,  $\omega(\eta_3) \neq \hat{\xi}_{-2}$  and  $\omega(\eta_4) \notin \{\hat{\xi}_{-2}, \hat{\xi}_2\}$ . We prove that  $\omega(\eta_3) = \mathcal{O}_q$  and  $\omega(\eta_4) = \hat{0}$ .

Consider curves

$$S_3 : \mathbb{R} \ni t \mapsto \pi x_t^{\eta_3} \in \mathbb{R}^2 \quad \text{and} \quad S_4 : \mathbb{R} \ni t \mapsto \pi x_t^{\eta_4} \in \mathbb{R}^2.$$

By Corollary 4.4.3,  $S_3$  and  $S_4$  are simple, furthermore they have no points in common with  $O_p$ .

Function  $\eta_3$  is selected so that  $S_3(0) = (\eta_3(0), \eta_3(-1)) \in \text{ext}(O_p)$ . Thus  $S_3(t) \in \text{ext}(O_p)$  for all  $t \in \mathbb{R}$ . As a consequence,  $\hat{0}$  is not in  $\omega(\eta_3)$ . Note that all the other stable equilibria have already been excluded, hence it follows from Claim 4.5.5 that  $\omega(\eta_3) = \{r_t : t \in \mathbb{R}\}$ , where  $r$  is a nontrivial periodic solution of Eq.(1.1). As  $x^{\eta_3}$  oscillates around 0,  $\omega(\eta_3) = \mathcal{O}_q$  by Claim 4.5.4.

Similarly, Claim 4.5.4 yields that if  $\omega(\eta_4)$  is a non-constant periodic orbit, then  $\omega(\eta_4) = \mathcal{O}_q$ . However, the choice of  $\eta_4$  implies  $S_4(0) = (\eta_3(0), \eta_3(-1)) \in \text{int}(O_p)$ , hence  $S_4(t) \in \text{int}(O_p)$  for all  $t \in \mathbb{R}$ . It follows immediately that  $\omega(\eta_4) \neq \mathcal{O}_q$ . So  $\omega(\eta_4)$  is a stable equilibrium by Claim 4.5.5. As  $\hat{\xi}_{-2}$  and  $\hat{\xi}_2$  have been excluded at the beginning of this proof, necessarily  $\omega(\eta_4) = \hat{0}$ .  $\square$

*Claim 4.5.8.* There are heteroclinic connections from  $\mathcal{O}_p$  to the orbits  $\mathcal{O}_1$  and  $\mathcal{O}_{-1}$ .

*Proof.* According to Claim 4.5.6, there exists  $\eta_5 \in \mathcal{W}^u(p_0)$  with  $0 > \eta_5(0) > p(0)$  such that solution  $x^{\eta_5} : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around  $\xi_1$ . Curve  $S_5 : \mathbb{R} \ni t \mapsto \pi x_t^{\eta_5} \in \mathbb{R}^2$  does not intersect  $O_p$ . Hence  $S(t) \in \text{int}(O_p)$  for all  $t \in \mathbb{R}$  and  $\omega(\eta_5) \neq \mathcal{O}_q$ . Also,  $\omega(\eta_5)$  is not a stable equilibrium or  $\mathcal{O}_{-1}$  as  $x^{\eta_5}$  oscillates around  $\xi_1$ . So  $\omega(\eta_5) = \mathcal{O}_1$ , see Claim 4.5.4.

Finally, set  $\eta_6 \in \mathcal{W}^u(p_0)$  with  $0 > \eta_6(0) > p(0)$  so that  $x^{\eta_6} : \mathbb{R} \rightarrow \mathbb{R}$  oscillates around  $\xi_{-1}$ . This is possible by Claim 4.5.6. An analogous argument verifies that  $\omega(\eta_6) = \mathcal{O}_{-1}$ .  $\square$

# 5 Slowly Oscillatory Periodic Solutions for Negative Feedback

## 5.1 Introduction to the problem

Consider Eq. (1.1) in the negative feedback case, i.e.

$$\dot{x}(t) = -\mu x(t) - f(x(t-1)) \quad (5.1)$$

where  $\mu > 0$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function with  $f(0) = 0$  and  $xf(x) > 0$  for all  $x \in \mathbb{R} \setminus \{0\}$ .

In [46] Walther has given a class of Lipschitz continuous nonlinearities  $f$  for which Eq. (5.1) admits an SOP solution (that is a periodic solution with successive sign changes spaced at distances larger than 1). A nonlinearity  $f$  in the function class considered is close to  $a \cdot \operatorname{sgn}(x)$  outside a small neighborhood of 0; the Lipschitz constant for  $f$  is sufficiently small on  $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ ,  $\varepsilon > 0$  small. Hence the associated return map is a contraction, and a periodic solution arises as the fixed point of the return map. In case  $f$  is  $C^1$ -smooth, the corresponding periodic orbit is hyperbolic and stable. In a subsequent paper [38], Ou and Wu have verified that the same result holds for a wider class of nonlinearities.

In case  $f$  in Eq. (5.1) is continuously differentiable with  $f'(x) > 0$  for  $x \in \mathbb{R}$ , Cao [2] and Krisztin [22] have given sufficient conditions for the uniqueness of the SOP solution. In these works,  $x \mapsto f(x)/x$  is strictly decreasing on  $(0, \infty)$ .

In this chapter we follow the technique used by Walther in [46] to show that one may guarantee the existence of an arbitrary number of SOP solutions. For the nonlinearity  $f$  in the next theorem,  $x \mapsto f(x)/x$  is not monotone on  $(0, \infty)$ .

**Theorem 5.1.1.** *Assume  $\mu > 0$ . There exists a locally Lipschitz continuous odd nonlinear map  $f$  satisfying  $xf(x) > 0$  for all  $x \in \mathbb{R} \setminus \{0\}$ , for which Eq. (5.1) admits an infinite sequence of SOP solutions  $(p^n)_{n=1}^\infty$  with  $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$  for  $n \geq 0$ . If  $f$  is continuously differentiable, then the corresponding periodic orbits are stable and hyperbolic.*

## 5 Slowly Oscillatory Periodic Solutions for Negative Feedback

We point out that a similar result appears in paper [37] of Nussbaum for the case  $\mu = 0$ . Although the construction of Nussbaum is different from the one presented here,  $x \mapsto f(x)/x$  is likewise not monotone for the nonlinear map  $f$  given by him.

Suppose  $f$  in Theorem 5.1.1 is smooth with  $f'(x) > 0$  for  $x \in \mathbb{R}$ . Based on [47], it can be confirmed that for the hyperbolic and stable SOP solutions  $p^n, p^{n+1}$  with ranges  $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$ , there exists an SOP solution  $p^*$  with range  $p^n(\mathbb{R}) \subsetneq p^*(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$ . Also, we have a Poincaré–Bendixson type result. For each globally defined bounded slowly oscillating solution (i.e., for each bounded solution defined on  $\mathbb{R}$  with at most 1 sign change on each interval of length 1), the  $\omega$ -limit set is either  $\{0\}$  or a single periodic orbit defined by an SOP solution. Analogously for the  $\alpha$ -limit set. Moreover, the subset

$$\{x_0 : x : \mathbb{R} \rightarrow \mathbb{R} \text{ is a bounded, slowly oscillating solution of Eq. (1.1)}\} \cup \{0\}$$

of the phase space  $C = C([-1, 0], \mathbb{R})$  is homeomorphic to the 2-dimensional plane.

The nonlinear map in Theorem 5.1.1 is close to the odd step function  $f^*$  with

$$f^*(x) = \begin{cases} 0 & \text{for all } x \in [0, 1], \\ Kr^n & \text{for all } n \geq 0 \text{ and } x \in (r^n, r^{n+1}]. \end{cases}$$

We conjecture that with similar nonlinearities, equation  $\dot{x}(t) = -\mu x(t) + f(x(t-1))$  also admits an infinite number of periodic solutions oscillating slowly around zero in the sense that they have no 3 different zeros in any interval of length 1.

## 5.2 Periodic solutions for step functions

Fix  $\mu > 0$  and

$$K > \mu \frac{e^\mu + \sqrt{2e^{2\mu} - 2e^\mu + 1}}{e^\mu - 1} \quad (5.2)$$

in this chapter. As a starting point we look for periodic solutions of

$$\dot{x}(t) = -\mu x(t) - f^R(x(t-1)), \quad (5.3)$$

where  $R > 0$  and

$$f^R(x) = \begin{cases} -KR & \text{if } x < -R, \\ 0 & \text{if } |x| \leq R, \\ KR & \text{if } x > R. \end{cases} \quad (5.4)$$

Step function  $f^R$  is the same as  $f^{KR,0}$  defined in Chapter 3. However, the more simple notation  $f^R$  fits our purposes better in this chapter, so we change to this notation. This should not confuse the reader.

## 5.2 Periodic solutions for step functions

*Remark 5.2.1.* For each  $R > 0$  and  $x \in \mathbb{R}$ ,  $f^R(x) = Rf^1(x/R)$ . Hence all solutions of Eq. (5.3) are of the form  $Rx(t)$ , where  $x(t)$  is a solution of

$$\dot{x}(t) = -\mu x(t) - f^1(x(t-1)). \quad (5.5)$$

In particular, all periodic solutions of Eq. (5.3) are of the form  $Rx(t)$ , where  $x(t)$  is a periodic solution of Eq. (5.5). Thus the study of Eq. (5.3) is reduced to the investigation of Eq. (5.5).

Set  $R = 1$  and  $J_i = (f^1)^{-1}(i)$  for  $i \in \{-K, 0, K\}$ .

If  $t_0 < t_1$  and  $x : [t_0 - 1, t_1] \rightarrow \mathbb{R}$  is a solution of Eq. (5.5) such that for some  $i \in \{-K, 0, K\}$ , we have  $x(t-1) \in J_{-i}$  for all  $t \in (t_0, t_1)$ , then Eq. (5.5) reduces to the ordinary differential equation

$$\dot{x}(t) = -\mu x(t) + i$$

on the interval  $(t_0, t_1)$ , and thus

$$x(t) = \frac{i}{\mu} + \left(x(t_0) - \frac{i}{\mu}\right) e^{-\mu(t-t_0)} \quad \text{for } t \in [t_0, t_1]. \quad (5.6)$$

In coherence with Chapter 3, we say that a function  $x : [t_0, t_1] \rightarrow \mathbb{R}$  is of type  $(i/\mu)$  on  $[t_0, t_1]$  with  $i \in \{-K, 0, K\}$  if (5.6) holds.

It is an easy calculation to show that if  $\mu > 0$ , and  $K$  satisfy (5.2), then  $K > 2\mu$ . As we shall see later, condition (5.2) comes from assumptions

$$K > 0 \quad \text{and} \quad \frac{K^2 - 2K\mu - \mu^2}{K^2 - \mu^2} > e^{-\mu}. \quad (5.7)$$

As for any  $\mu > 0$  fixed, the second inequality is of second order in  $K$ , the solution formula gives (5.2) and (5.7) are equivalent.

Fix  $\varphi \in C$  with  $\varphi(s) > 1$  for  $s \in [-1, 0)$  and  $\varphi(0) = 1$ . This choice implies that solution  $x = x^\varphi : [-1, \infty) \mapsto \mathbb{R}$  of Eq. (5.3) is of type  $(-K/\mu)$  on  $[0, 1]$ , that is

$$x(t) = -\frac{K}{\mu} + \left(1 + \frac{K}{\mu}\right) e^{-\mu t} \quad \text{for } t \in [0, 1]. \quad (5.8)$$

Clearly,  $x$  is strictly decreasing on  $[0, 1]$ . We claim that

$$x(1) = -\frac{K}{\mu} + \left(1 + \frac{K}{\mu}\right) e^{-\mu} \quad (5.9)$$

is smaller than  $-1$ , that is  $e^{-\mu} < (K - \mu) / (K + \mu)$ . Indeed, (5.7) (which condition is

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equivalent to the initial assumption (5.2)) gives

$$e^{-\mu} < \frac{K^2 - 2K\mu - \mu^2}{K^2 - \mu^2} < \frac{(K - \mu)^2}{K^2 - \mu^2} = \frac{K - \mu}{K + \mu}.$$

Therefore equation  $x(t) = -1$  has a unique solution  $\tau$  in  $(0, 1)$ . It comes from (5.8) that

$$\tau = \frac{1}{\mu} \ln \frac{K + \mu}{K - \mu}. \quad (5.10)$$

Note that  $x$  maps  $[0, \tau]$  onto  $[-1, 1]$ . Hence  $x$  is of type (0) on  $[1, \tau + 1]$ . Relations (5.6) and (5.9) yield

$$x(t) = x(1)e^{-\mu(t-1)} = -\frac{K}{\mu}e^{-\mu(t-1)} + \left(1 + \frac{K}{\mu}\right)e^{-\mu t} \text{ for } t \in [1, \tau + 1]. \quad (5.11)$$

In particular,

$$x(\tau + 1) = \frac{K - \mu}{\mu} \left( e^{-\mu} - \frac{K}{K + \mu} \right) \quad (5.12)$$

by (5.10).

Assumption (5.7) implies  $x(\tau + 1) < -1$ . In addition,  $x(1) < -1$  and (5.11) give that  $x$  is strictly increasing on  $[1, \tau + 1]$ . So  $x(t) < -1$  for  $t \in [1, \tau + 1]$ . Also,  $x(t) < -1$  for  $t \in (\tau, 1)$  because  $x(\tau) = -1$ ,  $\tau \in (0, 1)$ , and  $x$  strictly decreases on  $[0, 1]$ .

In consequence,  $x$  is of type  $(K/\mu)$  on  $[\tau + 1, \tau + 2]$ . Then (5.6), (5.10) and (5.12) imply

$$x(t) = \frac{K}{\mu} + \frac{1}{\mu} \left( K + \mu - \frac{2K^2 e^\mu}{K - \mu} \right) e^{-\mu t} \text{ for } t \in [\tau + 1, \tau + 2], \quad (5.13)$$

and

$$x(\tau + 2) = \frac{1}{\mu} \left( K - \frac{2K^2}{K + \mu} e^{-\mu} + (K - \mu) e^{-2\mu} \right).$$

We claim  $x(\tau + 2) > -1$ . This statement is equivalent to

$$(e^\mu - 1)^2 K^2 + 2\mu e^{2\mu} K + \mu^2 (e^{2\mu} - 1) > 0.$$

So it suffices to show that

$$K > K_0(\mu) = \mu \frac{-e^{2\mu} + \sqrt{e^{4\mu} - (e^\mu - 1)^2 (e^{2\mu} - 1)}}{(e^\mu - 1)^2}.$$

This condition is clearly fulfilled, as  $K > 0$  and  $K_0(\mu) < 0$  for all  $\mu > 0$ . Hence  $x(\tau + 2) > -1$ .

Hypothesis (5.7) implies

$$K + \mu - \frac{2K^2 e^\mu}{K - \mu} < 0,$$

thus  $x$  is strictly increasing on  $[\tau + 1, \tau + 2]$  by formula (5.13). This result and  $x(\tau + 1) <$



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$-1 < x(\tau + 2)$  yield that there exists a unique  $z \in (\tau + 1, \tau + 2)$  with  $x(z) = -1$ . From (5.13) we get

$$z = 1 + \frac{1}{\mu} \ln \left( \frac{2K^2}{K^2 - \mu^2} - e^{-\mu} \right). \quad (5.14)$$

Clearly,  $2 < \tau + 2$ . We show that  $z < 2$ . Indeed,  $z < 2$  is equivalent to

$$\mu \frac{\sqrt{e^{2\mu} + 1}}{e^\mu - 1} < K,$$

which relation is a direct consequence of (5.2). So the monotonicity of  $x$  on  $[\tau + 1, \tau + 2]$  gives  $x(2) > -1$ .

It follows from the definition of  $z$ , from the estimate  $x(t) < -1$  for  $t \in (\tau, z)$  and from  $z - \tau > 1$  that

$$x_z(s) < -1 \text{ for } s \in [-1, 0), \text{ and } x_z(0) = -1.$$

For odd nonlinearities  $f$ , we have the following simple observation concluding from the variation-of-constants formula.

*Remark 5.2.2.* If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is odd, i.e.  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$ , then for all  $\varphi \in C$  and  $t \geq -1$ ,  $x^{-\varphi}(t) = -x^\varphi(t)$ .

Remark 5.2.2 and the previous argument give

$$x_{2z}(s) = x_z^{x_z}(s) > 1 \text{ for } s \in [-1, 0), \text{ and } x_{2z}(0) = x_z^{x_z}(0) = 1.$$

Hence  $x$  can be extended to a periodic solution of Eq. (5.5) on  $\mathbb{R}$ . Let  $u^1 : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic function with minimal period  $2z$ , and with

$$u^1(t) = \begin{cases} x(t), & t \in [0, z], \\ -x(t - z), & t \in (z, 2z). \end{cases}$$

Then  $u^1$  satisfies Eq. (5.5) for  $t \in \mathbb{R}$ .

Note that for all  $\varphi \in C$  with  $\varphi(s) > 1$  for  $s \in [-1, 0)$  and  $\varphi(0) = 1$ , we have  $x_t^\varphi = u_t^1$  for all  $t \geq 1$ .

By Remark 5.2.1, our reasoning gives the following result for Eq. (5.3).

**Proposition 5.2.3.** *Assume  $R > 0$ ,  $\mu > 0$ , and  $K$  is chosen such that (5.2) holds. Let  $\tau \in (0, 1)$  and  $z \in (\tau + 1, 2)$  be given by (5.10) and (5.14), respectively. Then Eq. (5.3) admits a periodic solution  $u^R : \mathbb{R} \rightarrow \mathbb{R}$  with the following properties.*

- (i) *The minimal period of  $u^R$  is  $2z$ .*
- (ii)  *$u^R(0) = -u^R(\tau) = -u^R(z) = R$ .*
- (iii)  *$u^R(t) > R$  on  $[-1, 0)$ ,  $u^R(t) \in (-R, R)$  on  $(0, \tau)$ ,  $u^R(t) < -R$  on  $(\tau, z)$  and  $u^R(t) > -R$  for all  $t \in (z, 2]$ .*

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(iv)  $u^R$  strictly decreases on  $[0, 1]$ , and it strictly increases on  $[1, 2]$ .

(v)  $u^R(t) = Ru^1(t)$  for all  $t \in \mathbb{R}$ .

In consequence,

(vi)  $\max_{t \in \mathbb{R}} |u^R(t)| = R \max_{t \in \mathbb{R}} |u^1(t)|$ , where

$$\max_{t \in \mathbb{R}} |u^1(t)| = -u^1(1) = \frac{K}{\mu} - \frac{K + \mu}{\mu} e^{-\mu} \in \left(1, \frac{K}{\mu}\right).$$

Proposition 5.2.3 is applied in the next section with  $R = r^n$ , where  $r > 1$  is fixed and  $n \geq 0$ . We are going to construct a feedback function  $f$  so that Eq. (5.1) has an SOP solution close to  $u^{r^n}$  in a sense to be clarified.

For technical purposes, we need the following notation. For  $\xi \in (0, 1)$ , set  $T_i(\xi) > 0$ ,  $i \in \{1, 2, 3\}$ , so that  $T_1(\xi)$ ,  $T_2(\xi)$ ,  $T_3(\xi)$  is the time needed by a function of type  $(-K/\mu)$  to decrease from 1 to  $1 - \xi$ , from  $-1 + \xi$  to  $-1$ , and from  $-1$  to  $-1 - \xi$ , respectively.

Using (5.6), one gets

$$T_1(\xi) = \frac{1}{\mu} \ln \left( 1 + \frac{\mu\xi}{K + \mu(1 - \xi)} \right).$$

As  $\ln(1 + x) < x$  for all  $x > 0$ , we obtain

$$T_1(\xi) < \frac{\xi}{K + \mu(1 - \xi)} < \frac{\xi}{K}. \quad (5.15)$$

Similarly,

$$T_2(\xi) < \frac{\xi}{K - \mu} \text{ and } T_3(\xi) < \frac{\xi}{K - 2\mu}. \quad (5.16)$$

As  $u^1$  is of type  $(-K/\mu)$  on  $[0, 1]$  (see (5.8)), and  $u^R(t) = Ru^1(t)$  for all  $R > 0$  and  $t \in \mathbb{R}$ , the definition of  $T_i(\xi)$ ,  $i \in \{1, 2\}$ , clearly gives

$$u^R(T_1(\xi)) = R(1 - \xi) \text{ and } u^R(\tau - T_2(\xi)) = -R(1 - \xi)$$

for  $R > 0$ ,  $\xi \in (0, 1)$  and  $\tau$  defined by (5.10). Analogously,  $u^R(\tau + T_3(\xi)) = -R(1 + \xi)$  for  $R > 0$  and  $\xi \in (0, \min\{1, |u^1(1) + 1|\})$ .

## 5.3 Slowly oscillatory solutions for continuous nonlinearities

Now we turn attention to continuous nonlinearities. In addition to parameters  $\mu > 0$  and  $K$  satisfying condition (5.2), fix a constant  $M > K$ .

For  $r > 1$ ,  $\varepsilon \in (0, r - 1)$  and  $\eta \in (0, M - K)$ , let  $N = N(r, \varepsilon, \eta)$  be the set of all

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continuous odd functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with

$$|f(x)| < \eta \text{ for } x \in [0, 1],$$

$$\left| \frac{f(x)}{r^n} \right| < M \text{ for all } x \in (r^n, r^n(1 + \varepsilon)) \text{ and } n \geq 0$$

and

$$\left| \frac{f(x)}{r^n} - K \right| < \eta \text{ for all } x \in [r^n(1 + \varepsilon), r^{n+1}] \text{ and } n \geq 0.$$

Elements of  $N$  restricted to  $[-r^n, r^n]$ ,  $n \geq 1$ , can be viewed as perturbations of  $f^{r^{n-1}}$  introduced in the previous section.

Observe that

$$\max_{f \in N(r, \varepsilon, \eta), x \in [-r^n, r^n]} |f(x)| < Mr^{n-1} \text{ for all } n \geq 1. \quad (5.17)$$

For  $f \in N(r, \varepsilon, \eta)$ , we look for SOP solutions of Eq. (5.1) with initial functions in the nonempty closed convex sets  $A_n = A_n(r, \varepsilon)$  defined as

$$A_n = \left\{ \varphi \in C : r^n(1 + \varepsilon) \leq \varphi(s) \leq r^{n+1} \text{ for } s \in [-1, 0], \varphi(0) = r^n(1 + \varepsilon) \right\}, \quad n \geq 0.$$

Solutions of Eq. (5.1) with  $f \in N(r, \varepsilon, \eta)$  and with initial segment in  $A_n(r, \varepsilon)$  converge to  $u^{r^n}$  on  $[0, 2]$  as  $r \rightarrow \infty$ ,  $\varepsilon \rightarrow 0+$  and  $\eta \rightarrow 0+$  in the following sense.

**Proposition 5.3.1.** *For each  $\delta > 0$  there are  $r_0 = r_0(\delta) > 1$ ,  $\varepsilon_0 = \varepsilon_0(\delta) > 0$  and  $\eta_0 = \eta_0(\delta) > 0$  such that for all  $r > r_0$ ,  $\varepsilon \in (0, \varepsilon_0)$ ,  $\eta \in (0, \eta_0)$  and  $n \geq 0$ ,*

$$\sup_{f \in N(r, \varepsilon, \eta), \varphi \in A_n(r, \varepsilon), t \in [0, 2]} |x^\varphi(t) - u^{r^n}(t)| < \delta r^n.$$

*Proof.* Fix  $\delta > 0$  arbitrarily. Set  $r, \varepsilon, \eta$  as in the definition of  $N(r, \varepsilon, \eta)$ , and choose  $r$  to be greater than  $-u^1(1)$ . In addition, assume that

$$\varepsilon + \eta < r + u^1(1), \text{ and } 2\varepsilon + \eta < \min \left\{ 1, \left| u^1(1) + 1 \right| \right\}. \quad (5.18)$$

This is clearly possible. Fix any  $n \geq 0$ ,  $\varphi \in A_n(r, \varepsilon)$  and  $f \in N(r, \varepsilon, \eta)$ . As usual, let  $x^\varphi$  denote the solution of Eq. (5.1) with feedback function  $f$  and initial segment  $\varphi$ .

1. By Proposition 5.2.3 (iii),  $u^{r^n}(t) > r^n$  for  $t \in [-1, 0)$ . Hence the definition of  $f^{r^n}$ , the definitions of the function classes  $N(r, \varepsilon, \eta)$  and  $A_n(r, \varepsilon)$  and the variation-of-

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constants formula give that

$$\begin{aligned}
\left| x^\varphi(t) - u^{r^n}(t) \right| &\leq \left| x^\varphi(0) - u^{r^n}(0) \right| e^{-\mu t} \\
&\quad + \left| \int_0^t e^{-\mu(t-s)} f(\varphi(s-1)) ds - \int_0^t e^{-\mu(t-s)} f^{r^n}(u^{r^n}(s-1)) ds \right| \\
&\leq \varepsilon r^n e^{-\mu t} + \int_0^t e^{-\mu(t-s)} |f(\varphi(s-1)) - f^{r^n}(u^{r^n}(s-1))| ds \\
&\leq r^n (\varepsilon + \eta)
\end{aligned} \tag{5.19}$$

for  $t \in [0, 1]$ .

2. Similarly, for  $t \in [1, 2]$  we have

$$\begin{aligned}
\left| x^\varphi(t) - u^{r^n}(t) \right| &\leq \left| x^\varphi(1) - u^{r^n}(1) \right| e^{-\mu(t-1)} \\
&\quad + \int_1^t e^{-\mu(t-s)} \left| f(x^\varphi(s-1)) - f^{r^n}(u^{r^n}(s-1)) \right| ds \\
&\leq \|x_1^\varphi - x_1^{r^n}\| + \int_0^1 \left| f(x^\varphi(s)) - f^{r^n}(u^{r^n}(s)) \right| ds.
\end{aligned} \tag{5.20}$$

By the previous step,  $\|x_1^\varphi - x_1^{r^n}\| < r^n (\varepsilon + \eta)$ . Since  $|u^{r^n}(t)| \leq r^n |u^1(1)|$  holds for all real  $t$  by Proposition 5.2.3 (vi) and since  $\varepsilon + \eta < r + u^1(1)$  holds, it follows that

$$|x^\varphi(t)| \leq |u^{r^n}(t)| + r^n (\varepsilon + \eta) \leq r^n (-u^1(1) + \varepsilon + \eta) < r^{n+1} \quad \text{for } t \in [0, 1]. \tag{5.21}$$

We give an upper estimate for the integral on the right hand side in (5.20).

2.a. First we consider interval  $[0, \tau]$ , where  $\tau \in (0, 1)$  is defined by (5.10). Recall from Proposition 5.2.3 (iii) that  $u^{r^n}(t) \in [-r^n, r^n]$ , thus  $f^{r^n}(u^{r^n}(t)) = 0$  for  $t \in [0, \tau]$ .

Parameters  $\varepsilon, \eta$  are set so that  $0 < \varepsilon + \eta < 1$ , therefore  $T_i(\varepsilon + \eta)$ ,  $i \in \{1, 2\}$ , is defined, and  $T_1(\varepsilon + \eta) < \tau - T_2(\varepsilon + \eta)$ . By the monotonicity property of  $u^{r^n}$  on  $[0, 1]$  (see Proposition 5.2.3 (iv)) and the definitions of  $T_i$ ,  $i \in \{1, 2\}$ , we have

$$\left| u^{r^n}(t) \right| \leq r^n - r^n (\varepsilon + \eta) \quad \text{for } t \in [T_1(\varepsilon + \eta), \tau - T_2(\varepsilon + \eta)].$$

So with  $T_1 = T_1(\varepsilon + \eta)$  and  $T_2 = T_2(\varepsilon + \eta)$ , the estimate given in the first step implies

$$|x^\varphi(t)| \leq |u^{r^n}(t)| + r^n (\varepsilon + \eta) \leq r^n \quad \text{for } t \in [T_1, \tau - T_2].$$

In case  $n \geq 1$ , property (5.17) yields

$$\left| f(x^\varphi(t)) - f^{r^n}(u^{r^n}(t)) \right| = |f(x^\varphi(t))| < \frac{M}{r} r^n, \quad t \in [T_1, \tau - T_2].$$

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For  $n = 0$ ,

$$\left| f(x^\varphi(t)) - f^1(u^1(t)) \right| = |f(x^\varphi(t))| < \eta r^0, \quad t \in [T_1, \tau - T_2],$$

by the definition of the function class  $N(r, \varepsilon, \eta)$ . As  $0 < \tau - T_1 - T_2 < 1$ , it follows that

$$\int_{T_1}^{\tau - T_2} \left| f(x^\varphi(s)) - f^{r^n}(u^{r^n}(s)) \right| ds < \max \left\{ \frac{M}{r}, \eta \right\} r^n \quad (5.22)$$

for each  $n \geq 0$ .

For  $t \in [0, T_1) \cup (\tau - T_2, \tau]$ , we have  $|x^\varphi(t)| < r^{n+1}$  by (5.21). Hence (5.15), (5.16) and (5.17) imply

$$\begin{aligned} \left( \int_0^{T_1} + \int_{\tau - T_2}^\tau \right) \left| f(x^\varphi(s)) - f^{r^n}(u^{r^n}(s)) \right| ds &= \left( \int_0^{T_1} + \int_{\tau - T_2}^\tau \right) |f(x^\varphi(s))| ds \\ &\leq M r^n (T_1 + T_2) \leq \frac{2M}{K - \mu} (\varepsilon + \eta) r^n. \end{aligned} \quad (5.23)$$

2.b. Estimates for the interval  $(\tau, 1]$ . For each  $t \in (\tau, 1]$ ,  $u^{r^n}(t) < -r^n$ , hence  $f^{r^n}(u^{r^n}(t)) = -K r^n$ .

Parameters  $\varepsilon, \eta$  are fixed so that  $0 < 2\varepsilon + \eta < \min\{1, |u^1(1) + 1|\}$  holds, thus  $T_3(2\varepsilon + \eta)$  is defined and  $\tau + T_3(2\varepsilon + \eta) < 1$ . The fact that  $u^{r^n}$  strictly decreases on  $[0, 1]$  and the definition of  $T_3$  give that

$$u^{r^n}(t) \leq -r^n - r^n(2\varepsilon + \eta) \quad \text{for } t \in [\tau + T_3(2\varepsilon + \eta), 1].$$

Hence

$$x^\varphi(t) \leq u^{r^n}(t) + r^n(\varepsilon + \eta) \leq -r^n(1 + \varepsilon) \quad \text{for } t \in [\tau + T_3, 1],$$

where  $T_3 = T_3(2\varepsilon + \eta)$ . Also,  $x^\varphi(t) > -r^{n+1}$  for  $t$  in this interval. It follows from the definition of  $N(r, \varepsilon, \eta)$  that

$$\left| f(x^\varphi(t)) - f^{r^n}(u^{r^n}(t)) \right| = |f(x^\varphi(t)) - (-K r^n)| < r^n \eta$$

for  $t \in [\tau + T_3, 1]$ , and

$$\int_{\tau + T_3}^1 \left| f(x^\varphi(s)) - f^{r^n}(u^{r^n}(s)) \right| ds \leq (1 - \tau - T_3) r^n \eta \leq r^n \eta. \quad (5.24)$$

It remains to consider the interval  $(\tau, \tau + T_3)$ . From (5.16), (5.17) and (5.21) we

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obtain that

$$\begin{aligned} \int_{\tau}^{\tau+T_3} \left| f(x^\varphi(s)) - f^{r^n}(u^{r^n}(s)) \right| ds &\leq \int_{\tau}^{\tau+T_3} \left( |f(x^\varphi(s))| + |f^{r^n}(u^{r^n}(s))| \right) ds \\ &\leq T_3 (M + K) r^n \leq \frac{M + K}{K - 2\mu} (2\varepsilon + \eta) r^n. \end{aligned} \quad (5.25)$$

Set  $r_0, \varepsilon_0, \eta_0$  as in the definition of  $N(r, \varepsilon, \eta)$  with  $r_0 > -u^1(1)$  and  $M/r_0 < \delta/2$ . If necessary, decrease  $\varepsilon_0 > 0$  and  $\eta_0 > 0$  so that (5.18) holds for  $r_0, \varepsilon_0, \eta_0$ , and

$$(\varepsilon_0 + \eta_0) + \eta_0 + \frac{2M}{K - \mu} (\varepsilon_0 + \eta_0) + \eta_0 + \frac{M + K}{K - 2\mu} (2\varepsilon_0 + \eta_0) < \frac{\delta}{2}.$$

Then summing up the estimates (5.19), (5.20) and (5.22)-(5.25), we conclude that

$$\left| x^{\varphi, f}(t) - u^{r^n}(t) \right| < \delta r^n \text{ on } [0, 2]$$

for all  $r > r_0$ ,  $\varepsilon \in (0, \varepsilon_0)$ ,  $\eta \in (0, \eta_0)$ ,  $n \geq 0$ ,  $\varphi \in A_n(r, \varepsilon)$  and  $f \in N(r, \varepsilon, \eta)$ .  $\square$

Fix any  $w \in (\tau, z - 1)$ . Then  $w + 1 \in (\tau + 1, z)$ , and  $u^{r^n}(t) < -r^n$  on  $[w, w + 1]$  for all  $n \geq 0$  by Proposition 5.2.3 (iii).

In the subsequent result, we apply Proposition 5.3.1 and confirm that with an appropriate choice of parameters  $r, \varepsilon$  and  $\eta$ , we have  $x^\varphi(t) < -r^n(1 + \varepsilon)$  on  $[w, w + 1]$  for all  $f \in N(r, \varepsilon, \eta)$ ,  $n \geq 0$  and  $\varphi \in A_n(r, \varepsilon)$ . The same proposition and  $u^{r^n}(2) > -r^n$  guarantee  $x^\varphi(2) > -r^n$ . Hence there exists  $q \in (w + 1, 2)$  with  $x_q^\varphi \in -A_n(r, \varepsilon)$ .

Before reading the proof, recall that  $u^{r^n}(t) = r^n u^1(t)$ ,  $t \in \mathbb{R}$ , and

$$\frac{K}{\mu} > \left| u^1(1) \right| \geq u^1(2) > -1 > u^1(1).$$

**Proposition 5.3.2.** *There exist  $r_1 > 1$ ,  $\varepsilon_1 > 0$  and  $\eta_1 > 0$  so that for each  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$ ,  $\eta \in (0, \eta_1)$ ,  $n \geq 0$ ,  $f \in N(r, \varepsilon, \eta)$  and  $\varphi \in A_n(r, \varepsilon)$ , the solution  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  of Eq. (5.1) with nonlinearity  $f$  has the following properties.*

- (i)  $-r^{n+1} < x^\varphi(t) < r^{n+1}$  for  $t \in [0, 2]$ .
- (ii)  $x^\varphi(t) < -r^n(1 + \varepsilon)$  for  $t \in [w, w + 1]$ , and  $x^\varphi(2) > -r^n$ .
- (iii)  $\dot{x}^\varphi(t) < 0$  for  $t \in (0, 1)$ , and  $\dot{x}^\varphi(t) > 0$  for  $t \in (w + 1, 2]$ .
- (iv) If  $q = q(\varphi, f) \in (1 + w, 2)$  is set so that  $x^\varphi(q) = -r^n(1 + \varepsilon)$ , then  $q$  is unique, and  $x_q^\varphi \in -A_n(r, \varepsilon)$ .
- (v) If in addition  $\psi \in A_n(r, \varepsilon)$ , then for the semiflow (2.2) the equality  $\Phi(1 + w, \psi) = \Phi(1 + w, \varphi)$  implies  $q(\psi, f) = q(\varphi, f)$ .

*Proof.* Assume

$$0 < \delta < \min \left\{ \frac{1}{2} \left( \frac{K}{\mu} + u^1(1) \right), -\frac{1}{2} \left( \max_{t \in [w, w+1]} u^1(t) + 1 \right), 1 + u^1(2) \right\}.$$

### 5.3 Slowly oscillatory solutions for continuous nonlinearities

Note that all expressions on the right hand side are positive.

Choose  $r_1 = \max \{K/\mu, r_0(\delta)\}$ ,

$$\varepsilon_1 = \min \left\{ \varepsilon_0(\delta), -\frac{1}{2} \left( \max_{t \in [w, w+1]} u^1(t) + 1 \right) \right\}, \quad \eta_1 = \min \left\{ \eta_0(\delta), \frac{1}{2} (K + \mu u^1(1)) \right\},$$

where  $r_0(\delta)$ ,  $\varepsilon_0(\delta)$  and  $\eta_0(\delta)$  are given by Proposition 5.3.1. Consider  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$ ,  $\eta \in (0, \eta_1)$ ,  $n \geq 0$ ,  $f \in N(r, \varepsilon, \eta)$  and  $\varphi \in A_n(r, \varepsilon)$ .

(i) For  $t \in [0, 2]$ , it follows from Proposition 5.2.3 (vi) and Proposition 5.3.1, that

$$|x^\varphi(t)| \leq u^{r^n}(t) + r^n \delta \leq r^n (|u^1(1)| + \delta).$$

As we chose  $\delta$  to be smaller than  $K/\mu + u^1(1) \leq r + u^1(1)$ , we deduce that  $|x^\varphi(t)| < r^{n+1}$ .

(ii) For  $t \in [w, w+1]$  we get

$$x^\varphi(t) \leq u^{r^n}(t) + r^n \delta \leq r^n \left( \max_{t \in [w, w+1]} u^1(t) + \delta \right) < -r^n(1 + \varepsilon)$$

because  $\delta + \varepsilon < -\max_{t \in [w, w+1]} u^1(t) - 1$ . For  $t = 2$  we obtain that

$$x^\varphi(2) \geq u^{r^n}(2) - r^n \delta \geq r^n (u^1(2) - \delta) > -r^n,$$

as  $\delta < 1 + x^1(2)$ .

(iii) For  $t \in (0, 1)$ ,

$$\begin{aligned} \dot{x}^\varphi(t) &= -\mu x^\varphi(t) - f(\varphi(t-1)) \\ &\leq -\mu (u^{r^n}(t) - r^n \delta) - r^n (K - \eta) \\ &\leq r^n (-\mu u^1(1) + \mu \delta - K + \eta) < 0, \end{aligned}$$

as the parameters are set so that

$$\delta + \frac{\eta}{\mu} < \frac{K}{\mu} + u^1(1).$$

For  $t \in (w+1, 2]$ , we have  $t-1 \in (w, 1]$ . Thus  $-r^{n+1} < x^\varphi(t-1) < -r^n(1 + \varepsilon)$  by assertions (i) and (ii) of this proposition, and

$$\begin{aligned} \dot{x}^\varphi(t) &= -\mu x^\varphi(t) - f(x^\varphi(t-1)) \\ &\geq -\mu (u^{r^n}(t) + r^n \delta) + r^n (K - \eta) \\ &\geq r^n (-\mu u^1(2) - \mu \delta + K - \eta) > 0, \end{aligned}$$

since

$$\delta + \frac{\eta}{\mu} < \frac{K}{\mu} + u^1(1) < \frac{K}{\mu} - u^1(2).$$

Assertion (iv) now follows immediately.

(v) If  $\psi \in A_n(r, \varepsilon)$  and  $\Phi(1+w, \psi) = \Phi(1+w, \varphi)$ , then  $x^\psi(t) = x^\varphi(t)$  for  $t \geq 1+w$ . As  $q(\psi, f) > 1+w$  and  $q(\varphi, f) > 1+w$ ,  $q(\psi, f) = q(\varphi, f)$  follows.  $\square$

## 5.4 Lipschitz continuous return maps

Recall that  $\mu > 0$ , and (5.2) holds in this chapter. In addition, from now on we assume that  $K > \mu e^\mu$ .  $M > K$  is fixed as before.

Set  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$  and  $\eta \in (0, \eta_1)$  in this section, where  $r_1$ ,  $\varepsilon_1$  and  $\eta_1$  are specified by Proposition 5.3.2. Following Walther [46] and based on the results of Proposition 5.3.2, we introduce the Lipschitz continuous return map

$$R_f^n : A_n(r, \varepsilon) \ni \varphi \mapsto -\Phi(q(\varphi, f), \varphi) \in A_n(r, \varepsilon)$$

for each  $f \in N(r, \varepsilon, \eta)$  and  $n \geq 0$ . As it is discussed in [46], the fixed point of  $R_f^n$ ,  $n \geq 0$ , is the initial segment of a periodic solution  $p^n$  of Eq. (5.1) with minimal period  $2q$  and special symmetry  $p^n(t) = -p^n(t+q)$ ,  $t \in \mathbb{R}$ . As  $p^n$  has at most 1 zero on  $[0, q]$  and  $q > 1$ , the special symmetry property implies that  $p^n$  is an SOP solution.

In order to verify the Lipschitz continuity of  $R_f^n$ , we define the map

$$s_f^n : \Phi(1+w, A_n(r, \varepsilon)) \ni \psi \mapsto q(\varphi, f) - 1 - w \in (0, 1-w), \text{ with } \psi = \Phi(1+w, \varphi),$$

for each  $n \geq 0$  and  $f \in N(r, \varepsilon, \eta)$ . Also, set

$$F_1^n : A_n(r, \varepsilon) \ni \varphi \mapsto \Phi(1, \varphi) \in C,$$

$$F_w^n : \Phi(1, A_n(r, \varepsilon)) \ni \varphi \mapsto \Phi(w, \varphi) \in C,$$

$$S_f^n : \Phi(1+w, A_n(r, \varepsilon)) \ni \varphi \mapsto -\Phi(s_f^n(\varphi), \varphi) \in A_n(r, \varepsilon)$$

for all  $f \in N(r, \varepsilon, \eta)$  and  $n \geq 0$ . Proposition 5.3.2 implies that  $s_f^n$  and  $S_f^n$  are well-defined. Then  $R_f^n$  is the composite of  $F_1^n$ , followed by  $F_w^n$ , then by  $S_f^n$ .

We give Lipschitz constants for the maps above. As next result we state Proposition 3.1 of [46] without proof.

**Proposition 5.4.1.** *Set  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$  and  $\eta \in (0, \eta_1)$ . Assume  $n \geq 0$ , and  $f \in N(r, \varepsilon, \eta)$  is locally Lipschitz continuous. If  $L^n = L^n(f)$  and  $L_*^n = L_*^n(f)$  are Lipschitz constants for the restrictions  $f|_{[-r^{n+1}, r^{n+1}]}$  and  $f|_{[r^n(1+\varepsilon), r^{n+1}]}$ , respectively, then  $L_*^n$  is a Lipschitz constant for  $F_1^n$ , and  $1+wL^n$  is a Lipschitz constant for  $F_w^n$ .*



#### 5.4 Lipschitz continuous return maps

The following result is analogous to Proposition 3.2 in [46], and the proof needs only slight modifications.

**Proposition 5.4.2.** *Let  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$ ,  $\eta \in (0, \eta_1)$  and  $n \geq 0$ . Assume in addition that*

$$K - \eta > (1 + \varepsilon) \mu e^\mu.$$

*If  $f|_{[r^n(1+\varepsilon), r^{n+1}]}$  is Lipschitz continuous with Lipschitz constant  $L_*^n = L_*^n(f)$ , then  $s_f^n$  is Lipschitz continuous with Lipschitz constant*

$$L(s_f^n) = \frac{1 + e^\mu L_*^n}{r^n [K - \eta - \mu e^\mu (1 + \varepsilon)]},$$

*and  $S_f^n$  is Lipschitz continuous with Lipschitz constant  $L(s_f^n) (\mu r + M) r^n + 1 + L_*^n$ .*

*Proof.* Choose  $\varphi, \bar{\varphi} \in \Phi(1 + w, A_n(r, \varepsilon))$ . With  $s = s_f^n(\varphi) \in (0, 1 - w) \subset (0, 1)$  and  $\bar{s} = s_f^n(\bar{\varphi}) \in (0, 1 - w) \subset (0, 1)$ , we have

$$-(1 + \varepsilon) r^n = \varphi(0) e^{-\mu s} - \int_0^s e^{-\mu(s-\xi)} f(\varphi(\xi - 1)) d\xi,$$

and

$$-(1 + \varepsilon) r^n = \bar{\varphi}(0) e^{-\mu \bar{s}} - \int_0^{\bar{s}} e^{-\mu(\bar{s}-\xi)} f(\bar{\varphi}(\xi - 1)) d\xi.$$

Hence

$$\begin{aligned} (1 + \varepsilon) r^n |e^{\mu s} - e^{\mu \bar{s}}| &\geq \left| \int_0^s e^{\mu \xi} f(\varphi(\xi - 1)) d\xi - \int_0^{\bar{s}} e^{\mu \xi} f(\varphi(\xi - 1)) d\xi \right| \\ &\quad - |\varphi(0) - \bar{\varphi}(0)| \\ &\quad - \left| \int_0^{\bar{s}} e^{\mu \xi} \{f(\varphi(\xi - 1)) - f(\bar{\varphi}(\xi - 1))\} d\xi \right| \\ &\geq \left| \int_{\bar{s}}^s e^{\mu \xi} f(\varphi(\xi - 1)) d\xi \right| \\ &\quad - \|\varphi - \bar{\varphi}\| \\ &\quad - \left| \int_0^{\bar{s}} e^{\mu \xi} \{f(\varphi(\xi - 1)) - f(\bar{\varphi}(\xi - 1))\} d\xi \right|. \end{aligned}$$

Since  $-r^{n+1} < \varphi(t) < -r^n(1 + \varepsilon)$  and  $-r^{n+1} < \bar{\varphi}(t) < -r^n(1 + \varepsilon)$  for each  $t \in [-1, 0]$ , we conclude that

$$(1 + \varepsilon) r^n |e^{\mu s} - e^{\mu \bar{s}}| \geq |s - \bar{s}| r^n (K - \eta) - \|\varphi - \bar{\varphi}\| - e^\mu L_*^n \|\varphi - \bar{\varphi}\|.$$

On the other hand,  $|e^{\mu s} - e^{\mu \bar{s}}| \leq \mu e^\mu |s - \bar{s}|$ . Thus

$$|s - \bar{s}| \leq \frac{1 + e^\mu L_*^n}{r^n [K - \eta - \mu e^\mu (1 + \varepsilon)]} \|\varphi - \bar{\varphi}\|,$$

and the proof of the first assertion is complete.

## 5 Slowly Oscillatory Periodic Solutions for Negative Feedback

If  $\varphi = \Phi(1+w, \psi)$  with  $\psi \in A_n(r, \varepsilon)$ , then for  $t \in [-1, 0]$ ,

$$\begin{aligned} \Phi(\bar{s}, \varphi)(t) - \Phi(s, \varphi)(t) &= x_{1+w+\bar{s}}^\psi(t) - x_{1+w+s}^\psi(t) \\ &= \int_{1+w+s}^{1+w+\bar{s}} \dot{x}^\psi(\xi) d\xi \\ &= \int_{1+w+s}^{1+w+\bar{s}} \left\{ -\mu x^\psi(\xi) - f(x^\psi(\xi-1)) \right\} d\xi. \end{aligned}$$

So Proposition 5.3.2 (i) and (5.17) imply

$$|\Phi(\bar{s}, \varphi)(t) - \Phi(s, \varphi)(t)| \leq |s - \bar{s}| (\mu r + M) r^n \leq L(s_f^n) (\mu r + M) r^n \|\varphi - \bar{\varphi}\|$$

for  $t \in [-1, 0]$ . Also, it is easy to see using  $\bar{s} \in (0, 1)$ ,  $-r^{n+1} < \varphi(t)$ ,  $\bar{\varphi}(t) < -r^n(1 + \epsilon)$ ,  $t \in [-1, 0]$ , the oddness of  $f$  and the variation-of-constants formula, that

$$\|\Phi(\bar{s}, \varphi) - \Phi(\bar{s}, \bar{\varphi})\| \leq (1 + L_*^n) \|\varphi - \bar{\varphi}\|.$$

Hence

$$\begin{aligned} \|S(\varphi) - S(\bar{\varphi})\| &\leq \|\Phi(s, \varphi) - \Phi(\bar{s}, \varphi)\| + \|\Phi(\bar{s}, \varphi) - \Phi(\bar{s}, \bar{\varphi})\| \\ &\leq \left\{ \frac{1 + e^\mu L_*^n}{K - \eta - \mu e^\mu (1 + \varepsilon)} (\mu r + M) + 1 + L_*^n \right\} \|\varphi - \bar{\varphi}\|, \end{aligned}$$

and the proof is complete.  $\square$

It follows that under the assumptions of the last two propositions,  $R_f^n$  is Lipschitz continuous, and

$$L(R_f^n) = L_*^n (1 + w L^n) \left( \frac{1 + e^\mu L_*^n}{K - \eta - \mu e^\mu (1 + \varepsilon)} (\mu r + M) + 1 + L_*^n \right)$$

is a Lipschitz constant for  $R_f^n$ . Clearly, if  $L(R_f^n) < 1$ , then  $R_f^n$  is a strict contraction with a unique fixed point in  $A_n(r, \varepsilon)$ , and Eq. (5.1) has an SOP solution with initial function in  $A_n(r, \varepsilon)$ .

## 5.5 The proof of Theorem 5.1.1

**Proof of Theorem 5.1.1.** Choose  $r > r_1$ ,  $\varepsilon \in (0, \varepsilon_1)$  and  $\eta \in (0, \eta_1)$  with

$$K - \eta > (1 + \varepsilon) \mu e^\mu.$$

We give a nonlinearity  $f \in N(r, \varepsilon, \eta)$  so that  $R_f^n$  is a contraction for each  $n \geq 0$ . The function  $f$  is defined recursively on  $[-r^n, r^n]$  for  $n \geq 1$ .

First step. Let  $f : [-1 - \varepsilon, 1 + \varepsilon] \rightarrow \mathbb{R}$  be a Lipschitz continuous odd function

with  $|f(x)| < \eta$  for  $x \in [0, 1]$ ,  $|f(x)| < M$  for all  $x \in (1, 1 + \varepsilon)$  and  $f(1 + \varepsilon) \in (K - \eta, K + \eta)$ . Let  $L_{**}^0$  be a Lipschitz constant for  $f|_{[-1-\varepsilon, 1+\varepsilon]}$ . Extend the definition of  $f$  to domain  $[-r, r]$  so that  $f$  remains odd,  $|f(x) - K| < \eta$  for  $x \in [1 + \varepsilon, r]$ , and  $f|_{[1+\varepsilon, r]}$  is Lipschitz continuous with Lipschitz constant  $L_*^0$  satisfying

$$L_*^0 \left( 1 + w \max \{L_*^0, L_{**}^0\} \right) \left( \frac{1 + e^\mu L_*^0}{K - \eta - \mu e^\mu (1 + \varepsilon)} (\mu r + M) + 1 + L_*^0 \right) < 1.$$

This is possible by choosing  $L_*^0$  sufficiently small. Then  $L^0 = \max \{L_*^0, L_{**}^0\}$  is a Lipschitz constant for  $f|_{[-r, r]}$ , and  $R_f^0$  is a strict contraction.

Recursive step. If  $f$  is defined for  $[-r^n, r^n]$  with some  $n \geq 1$ , extend the definition of  $f$  to the domain  $[-r^{n+1}, r^{n+1}]$  so that  $f$  remains odd, Lipschitz continuous,

$$\left| \frac{f(x)}{r^n} \right| < M \text{ for all } x \in (r^n, r^n(1 + \varepsilon)),$$

$$\left| \frac{f(x)}{r^n} - K \right| < \eta \text{ for all } x \in [r^n(1 + \varepsilon), r^{n+1}],$$

and if  $L_{**}^n$  is a Lipschitz constant for  $f|_{(r^n, r^n(1+\varepsilon))}$ , then  $f|_{[r^n(1+\varepsilon), r^{n+1}]}$  has a Lipschitz constant  $L_*^n$  with

$$L_*^n \left( 1 + w \max_{0 \leq k \leq n} \{L_*^k, L_{**}^k\} \right) \left( \frac{1 + e^\mu L_*^n}{K - \eta - \mu e^\mu (1 + \varepsilon)} (\mu r + M) + 1 + L_*^n \right) < 1.$$

Then  $L^n = \max_{0 \leq k \leq n} \{L_*^k, L_{**}^k\}$  is a Lipschitz constant for  $f|_{[-r^{n+1}, r^{n+1}]}$ , and  $R_f^n$  is a strict contraction.

Thereby we obtain a locally Lipschitz continuous odd function  $f$  for which  $R_f^n$  is a strict contraction for all  $n \geq 0$ . For such  $f$ , Eq. (5.1) has an infinite sequence of SOP solutions with initial segments in  $A_n(r, \varepsilon)$ ,  $n \geq 0$ . It is clear that one may set  $f$  in this construction so that  $xf(x) > 0$  holds for all  $x \in \mathbb{R} \setminus \{0\}$ .

It follows from Section 4 in [46], that if  $f$  is continuously differentiable, then the corresponding periodic orbits are stable and hyperbolic.  $\square$

## 5.6 A possible modification

As before, set  $K > 0$  satisfying condition (5.2) and choose  $M > K$ . For  $r > 1$ ,  $\varepsilon \in (0, r - 1)$  and  $\eta \in (0, M - K)$ , let  $\tilde{N}(r, \varepsilon, \eta)$  be the set of all continuous odd functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with

$$\left| \frac{f(x)}{r^n} \right| < M \text{ for all } x \in (r^n, r^n(1 + \varepsilon)) \text{ and } n \in \mathbb{Z}$$

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and with

$$\left| \frac{f(x)}{r^n} - K \right| < \eta \text{ for all } x \in [r^n(1 + \varepsilon), r^{n+1}] \text{ and } n \in \mathbb{Z}.$$

Then minor modifications of our results in Section 3 and in Section 4 yield the subsequent theorem.

**Theorem 5.6.1.** *Assume  $\mu > 0$ . There exists a locally Lipschitz continuous odd non-linear map  $f \in \tilde{N}(r, \varepsilon, \eta)$  satisfying  $xf(x) > 0$  for all  $x \in \mathbb{R} \setminus \{0\}$ , for which Eq. (1.1) admits a two-sided infinite sequence of SOP solutions  $(p^n)_{-\infty}^{\infty}$  with*

$$\lim_{n \rightarrow -\infty} \max_{x \in \mathbb{R}} |p^n(x)| = 0, \quad \lim_{n \rightarrow \infty} \max_{x \in \mathbb{R}} |p^n(x)| = \infty,$$

and with  $p^n(\mathbb{R}) \subsetneq p^{n+1}(\mathbb{R})$  for  $n \in \mathbb{Z}$ .

It is easy to see that the elements of  $\tilde{N}(r, \varepsilon, \eta)$  are not differentiable at  $x = 0$ . Hence the hyperbolicity and stability of the periodic orbits given by the theorem does not follow directly from paper [45] of Walther. Still we conjecture that these periodic orbits are hyperbolic and stable.

## 6 Dynamics for the Hopfield Activation Function

### 6.1 The Györi-Hartung conjecture

This chapter studies Eq.(1.1) with the piecewise linear Hopfield activation function defined by

$$f : \mathbb{R} \ni x \mapsto \frac{1}{2} (|x+1| - |x-1|) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1. \end{cases} \quad (6.1)$$

Motivated by paper [11] of Györi and Hartung and by paper [17] of Heiden, Mackey and Walther, we consider the following more general form:

$$\dot{x}(t) = -\mu x(t) + af(x(t)) + bf(x(t-1)) + I, \quad (6.2)$$

where

$$a, b, \mu, I \in \mathbb{R}, \mu > 0, b \neq 0 \quad (6.3)$$

and  $f$  is the Hopfield activation function (6.1).

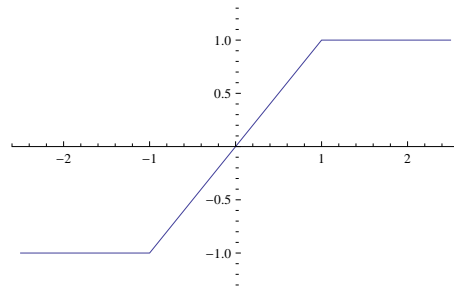


Figure 6.1: Plot of the Hopfield activation function

It is easy to check that there is a single equilibrium point if  $\mu > a + b - |I|$ . If  $\mu \leq a + b - |I|$ , Eq. (6.2) has more equilibria, which makes the asymptotic behavior of the solutions more interesting.

The dynamics of (6.1)-(6.3) was described by Györi and Hartung in [11] for certain choices of parameters. They have proved that the unique equilibrium point is globally

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asymptotically stable provided  $\mu > a + |b| + |I|$ . Conditions  $b > 0$  and  $a + b - |I| < \mu \leq a + b + |I|$  yield the same result. Based on numerical studies, Györi and Hartung have drawn up the conjecture that in case  $b > 0$  all solutions of (6.2) are convergent as  $t \rightarrow \infty$ . In this chapter we disprove this conjecture and intend to give a description of the dynamics in the remaining case, namely if

$$b > 0 \text{ and } 0 < \mu = a + b - |I| \quad (6.4)$$

and if

$$b > 0 \text{ and } 0 < \mu < a + b - |I|. \quad (6.5)$$

In the first place, the conjecture is sustained under condition (6.4) (see Theorem 6.3.1). Under hypothesis (6.5), most of the solutions of Eq. (6.2) still tend to a constant equilibrium, namely there exists a one-codimensional submanifold  $S$  of the phase space  $C$  such that all solutions with initial functions in  $C \setminus S$  are convergent. To describe this latter case in detail, we distinguish two subcases according to whether  $b > L(a, \mu)$  or  $b < L(a, \mu)$ , where

$$L(a, \mu) = \begin{cases} \frac{\mu - a}{\cos \theta} & \text{with } \theta \in (\pi, 2\pi) \text{ and } \theta = (a - \mu) \tan \theta & \text{if } \mu \neq a \\ \frac{3\pi}{2} & & \text{if } \mu = a. \end{cases}$$

The main purpose of this chapter is to show that condition  $b > L(a, \mu)$  implies the existence of a periodic solution of Eq. (6.2) (see Theorem 6.3.2). In case  $b < L(a, \mu)$  the description of the long-term behavior of solutions is not complete yet. We suspect that the conjecture is true for this choice of parameters. Assumption  $b = L(a, \mu)$  serves as an easy counterexample to the conjecture, as a continuum of periodic solutions appear in this case.

As we have mentioned in Section 2.2, if the feedback function  $f$  is smooth and strictly increasing, the so-called Poincaré–Bendixson theorem confirmed by Mallet-Paret and Sell [33] shows that all bounded solutions of (6.2)–(6.3) are convergent or asymptotically periodic. However, the fact that the Hopfield activation function is neither strictly monotone nor smooth gives rise to nontrivial technical problems: the solution operator is neither injective nor differentiable everywhere. Thereby the techniques of the Poincaré–Bendixson theorem cannot be used here.

It also appears to be evident to approximate the Hopfield function with a sequence  $(f_n)_0^\infty$  of smooth and strictly increasing feedback functions, and then apply either the Poincaré–Bendixson theorem or the results of Krisztin, Walter and Wu for the equation with feedback function  $f_n$ . In the special case when  $b > 0$ ,  $a = I = 0$  and  $f$  is a "good" strictly increasing smooth feedback function, the global dynamics was completely depicted by Krisztin, Walter and Wu, see Section 2.2, and the references therein. However, the method of approximation is not as beneficial as one would expect,

since the global attractor is only upper semicontinuous [14].

We use another approach in this chapter to describe (6.1)–(6.3) and we focus only on conditions (6.4) and (6.5).

## 6.2 Notations and preliminary results

Assume (6.1)–(6.3) and  $b > 0$ . As one can easily verify, there are three possible equilibrium points  $\hat{\xi}_+, \hat{\xi}_-, \hat{\xi}_0$  of (6.2) given by

$$\xi_+ = \frac{a+b+I}{\mu}, \quad \xi_- = \frac{-a-b+I}{\mu} \quad \text{and} \quad \xi_0 = \frac{I}{\mu-a-b} \quad (6.6)$$

if  $\mu \neq a+b$  in the third case. It is obvious that  $\hat{\xi}_+, \hat{\xi}_-$  and  $\hat{\xi}_0$  are equilibrium points of Eq. (6.2) if and only if  $1 \leq \xi_+, \xi_- \leq -1$  and  $-1 \leq \xi_0 \leq 1$ .

The following lemma is stated in [11] and holds without assumption  $b > 0$ .

**Lemma 6.2.1.** *Let  $\hat{\xi}_+, \hat{\xi}_-$  and  $\hat{\xi}_0$  be defined by (6.6). The following statements hold.*

(i) *If  $0 < \mu = a+b$  and  $I = 0$ , then any number  $\xi \in [-1, 1]$  defines an equilibrium of Eq. (6.2), and Eq. (6.2) has no other equilibria.*

(ii) *If  $0 < \mu = a+b-|I|$  and  $|I| \neq 0$ , then Eq. (6.2) has two equilibrium points:*

- *if  $I > 0$ , then  $\hat{\xi}_+ > 1$  and  $\hat{\xi}_- = \hat{\xi}_0 = -1$  are equilibria,*
- *if  $I < 0$ , then  $\hat{\xi}_+ = \hat{\xi}_0 = 1$  and  $\hat{\xi}_- < -1$  are equilibria.*

(iii) *If  $0 < \mu < a+b-|I|$ , then  $\hat{\xi}_+ > 1$ ,  $\hat{\xi}_- < -1$  and  $-1 < \hat{\xi}_0 < 1$  are the equilibrium points of Eq. (6.2).*

The phase space for Eq. (6.2) is  $C = C([-1, 0], \mathbb{R})$ . As in case Eq. (1.1), each  $\varphi \in C$  uniquely determines a solution  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  of Eq. (6.2) so that  $x_0^\varphi = \varphi$ .

The map  $\Phi : \mathbb{R}^+ \times C \ni (t, \varphi) \mapsto x_t^\varphi \in C$  is a continuous semiflow also in this case with three possible stationary points  $\hat{\xi}_+, \hat{\xi}_-$  and  $\hat{\xi}_0$ . As the Hopfield activation function is neither strictly increasing nor smooth,  $\Phi$  is neither injective nor differentiable everywhere.

Define  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$  (Fig. 6.2) by

$$\gamma(u, v) = \begin{cases} 1, & |u| < 1 \text{ and } |v| < 1, \\ \frac{1-v \operatorname{sgn} u}{(u-v) \operatorname{sgn} u}, & |u| \geq 1 \text{ and } |v| < 1, \\ \frac{1-u \operatorname{sgn} v}{(v-u) \operatorname{sgn} v}, & |u| < 1 \text{ and } |v| \geq 1, \\ \frac{2}{|u|+|v|}, & u \geq 1 \text{ and } v \leq -1 \text{ or } u \leq -1 \text{ and } v \geq 1, \\ 0, & u \geq 1 \text{ and } v \geq 1 \text{ or } u \leq -1 \text{ and } v \leq -1. \end{cases} \quad (6.7)$$

Then  $\gamma$  is a nonnegative and continuous function on the set  $\mathbb{R}^2 \setminus \{(-1, -1), (1, 1)\}$ .

## 6 Dynamics for the Hopfield Activation Function

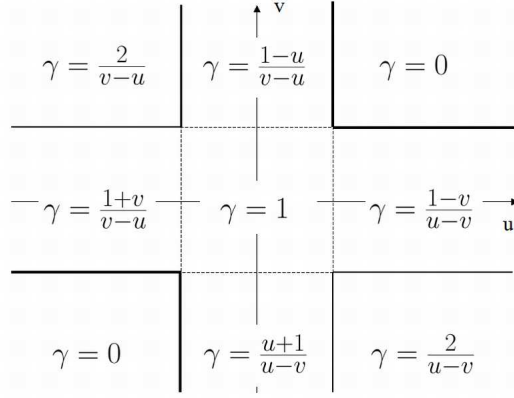


Figure 6.2: The definition of  $\gamma$

For each  $(u, v) \in \mathbb{R}^2 \setminus \{(-1, -1), (1, 1)\}$  we have

$$f(u) - f(v) = \int_0^1 f'(su + (1-s)v) ds (u - v) = \gamma(u, v)(u - v).$$

Therefore,  $f(u) - f(v) = \gamma(u, v)(u - v)$  for all  $(u, v) \in \mathbb{R}^2$ .

If  $u, v : [a, b] \rightarrow \mathbb{R}$  are continuous functions, then it is also easy to see that  $[a, b] \ni t \mapsto \gamma(u(t), v(t)) \in [0, 1]$  is Lebesgue integrable.

Let  $J$  be an interval. Setting

$$\alpha : J \ni t \mapsto \mu - a\gamma(x(t), \hat{x}(t)) \quad (6.8)$$

and

$$\beta : J \ni t \mapsto b\gamma(x(t-1), \hat{x}(t-1)) \quad (6.9)$$

where  $x : [-1, 0] + J \rightarrow \mathbb{R}$  and  $\hat{x} : [-1, 0] + J \rightarrow \mathbb{R}$  are solutions of Eq. (6.2), we find  $z := x - \hat{x}$  satisfies

$$\dot{z}(t) = -\alpha(t)z(t) + \beta(t)z(t-1) \quad (6.10)$$

for  $t \in J, t > \inf J$ . Notice that  $\alpha$  is locally Lebesgue integrable,  $\beta$  is nonnegative, and  $\beta$  positive provided  $\hat{x} \equiv \hat{\xi}_0$  and  $-1 < \hat{\xi}_0 < 1$  (see Lemma 6.2.1). Hence Lemma 2.3.2 can be applied for  $z = x - \hat{x}$ . By Remark 2.3.3, if  $x_0 \neq \hat{\xi}_0$ , then  $x_t \neq \hat{\xi}_0$  for all  $t$ . This fact plays an important role later in this chapter.

Recall that for  $\varphi, \psi \in C$ , we have  $\varphi \leq \psi$  if  $\varphi(s) \leq \psi(s)$  for all  $s \in [-1, 0]$ ,  $\varphi \prec \psi$  if  $\varphi \leq \psi$  and  $\varphi(0) < \psi(0)$ , and in addition  $\varphi \ll \psi$  if  $\varphi(s) < \psi(s)$  for all  $s \in [-1, 0]$ .

**Proposition 6.2.2.** *Let  $\varphi$  and  $\psi$  be elements of  $C$  with  $\varphi \leq \psi$  ( $\varphi \prec \psi$ ). Then  $x^\varphi(t) \leq x^\psi(t)$  ( $x^\varphi(t) < x^\psi(t)$ ) for all  $t \geq 0$ .*

*Proof.* Assume that  $\varphi \in C$  and  $\psi \in C$  with  $\varphi \leq \psi$  ( $\varphi \prec \psi$ ). Set  $y = x^\psi - x^\varphi$ . Then  $y$  satisfies Eq. (6.10) for  $t > 0$ , where  $\alpha$  and  $\beta$  are given by (6.8) and (6.9) with  $x = x^\psi$



and  $\hat{x} = x^\varphi$ . As  $t \mapsto \int_0^t \alpha(u) du$  is absolutely continuous, one obtains

$$y(t) = e^{-\int_0^t \alpha(u) du} y(0) + \int_0^t e^{-\int_s^t \alpha(u) du} \beta(s) y(s-1) ds \geq 0 \quad (> 0)$$

for  $t \in [0, 1]$ . The proof can be completed by the method of steps.  $\square$

Recall from Lemma 6.2.1 that if conditions (6.4) and  $|I| \neq 0$  are fulfilled, or (6.5) holds, then  $\hat{\xi}_+ \geq 1$  and  $\hat{\xi}_- \leq -1$  are equilibria.

**Proposition 6.2.3.** *Assume conditions (6.4) and  $|I| \neq 0$  are satisfied, or (6.5) holds. If  $x : [-1, \infty) \rightarrow \mathbb{R}$  is a solution of Eq. (6.2), then*

$$\min \left\{ -\|x_0\|, \hat{\xi}_- \right\} \leq x(t) \leq \max \left\{ \|x_0\|, \hat{\xi}_+ \right\} \quad \text{for all } t \geq -1.$$

*Proof.* Assume that there exist  $\epsilon > 0$  and  $t > 0$  so that  $x(t) > \max \left\{ \|x_0\|, \hat{\xi}_+ \right\} + \epsilon$ . Then there is a minimal  $t_0 > 0$  with  $x(t_0) = \max \left\{ \|x_0\|, \hat{\xi}_+ \right\} + \epsilon$ . Necessarily  $\dot{x}(t_0) \geq 0$ . On the other hand,

$$\dot{x}(t_0) \leq -\mu x(t_0) + a + b + I < 0,$$

which is a contradiction. One can prove analogously, that  $\min \left\{ -\|x_0\|, \hat{\xi}_- \right\} \leq x(t)$  for all  $t \geq -1$ .  $\square$

Observe that in case (6.4) and  $|I| \neq 0$  hold or if hypothesis (6.5) is satisfied, then Proposition 6.2.3 ensures that condition (T) introduced in Section 2.1 is valid for Eq. (6.2). If we choose  $\nu$  to be greater than  $\mu + |a|$ , then (SM) is also fulfilled. Hence the set of convergent points contains an open and dense subset of  $C$  in these cases by Theorem 2.1.1.

## 6.3 The main results of the chapter

First, we prove the truth of the Györi-Hartung conjecture in case (6.4).

**Theorem 6.3.1.** *Consider (6.1)–(6.3). If (6.4) holds, then every solution of Eq. (6.2) tends to an equilibrium as  $t \rightarrow \infty$ .*

As more complicated structures appear if  $\mu < a + b - |I|$ , the greatest part of the chapter deals with that case. Assume (6.1)–(6.3) and (6.5). Recall that  $\Phi$  has three stationary points:  $1 \ll \hat{\xi}_+$ ,  $\hat{\xi}_- \ll -1$  and  $-1 \ll \hat{\xi}_0 \ll 1$ .

Let interval  $J \subseteq \mathbb{R}$  be given. If  $x$  is a solution of Eq. (6.2) with  $|x(t)| < 1$  for  $t \in J + [-1, 0]$ , then (6.2) becomes linear, and for  $y := x - \hat{\xi}_0$  we get

$$\dot{y}(t) = (-\mu + a)y(t) + by(t-1) \tag{6.11}$$

## 6 Dynamics for the Hopfield Activation Function

for all  $t \in J$ ,  $t > \inf J$ . For this reason it is evident to examine (6.11). Let  $y^\psi : [-1, \infty) \rightarrow \mathbb{R}$  denote the solution of (6.11) with initial function  $\psi$ . The solution operator  $T(t) : C \rightarrow C$ ,  $t \geq 0$ , defined by relation  $T(t)\psi = y_t^\psi$ , is a strongly continuous semigroup of linear operators. The spectrum of the generator of the semigroup consists of eigenvalues that coincide with the zeros of the characteristic function

$$\mathbb{C} \ni \lambda \mapsto -\mu + a + be^{-\lambda} \in \mathbb{C}. \quad (6.12)$$

As discussed in Section 2.1, there is one real eigenvalue  $\lambda_0$ , the others form a sequence of complex conjugate pairs  $(\lambda_k, \overline{\lambda_k})$ ,  $1 \leq k \in \mathbb{N}$ , with  $(2k-1)\pi < \text{Im}\lambda_k < 2k\pi$ ,  $\lambda_0 > \text{Re}\lambda_k > \text{Re}\lambda_{k+1}$  for all  $k \geq 1$  and  $\text{Re}\lambda_k \rightarrow -\infty$  as  $k \rightarrow \infty$ . The real eigenvalue  $\lambda_0$  is positive if and only if  $\mu - a < b$ .

Set  $L(a, \mu) = (\mu - a)/\cos\theta$ , where  $\theta \in (\pi, 2\pi)$  with  $\theta = (a - \mu)\tan\theta$  if  $\mu \neq a$ , otherwise set  $L(a, \mu) = 3\pi/2$ . An elementary calculation yields that condition  $\mu - a < b < L(a, \mu)$  implies  $\text{Re}\lambda_1 < 0 < \lambda_0$ , while  $b > L(a, \mu)$  is equivalent to  $0 < \text{Re}\lambda_1 < \lambda_0$ . As we shall see, in the latter case interesting structures appear.

The asymptotic behavior of a solution of Eq. (6.2) also depends on whether the initial function belongs to the set

$$S = \left\{ \varphi \in C : x^\varphi - \hat{\xi}_0 \text{ has arbitrarily large zeros} \right\}.$$

The set  $S$  is a 1-codimensional Lipschitz submanifold of  $C$  (see Proposition 6.5.2), which also plays a role in the following theorem, the main result of this chapter.

**Theorem 6.3.2.** *Consider (6.1)–(6.3) and (6.5).*

- (i) *Most of the solutions are convergent. That is, if  $\varphi$  is an element of  $C \setminus S$ , then  $x_t^\varphi \rightarrow \hat{\xi}_+$  or  $x_t^\varphi \rightarrow \hat{\xi}_-$  as  $t \rightarrow \infty$ .*
- (ii) *Condition  $b > L(a, \mu)$  implies the existence of a periodic solution  $p : \mathbb{R} \rightarrow \mathbb{R}$  with minimal period  $\omega \in (1, 2)$ .*

Recently Garab and Krisztin [10] have shown for the case  $a = I = 0$  that if exactly  $2k + 1$  eigenvalues of the generator have positive real parts, then exactly  $k$  different periodic orbits exist. In particular the existence of a periodic orbits is precluded for  $b < L(a, \mu)$ . The proof of the existence of the periodic orbits is based on Theorem 6.3.2 (ii).

It remains an open problem whether the global attractor of Eq. (6.2) can be described as in [22] for the equation  $\dot{x}(t) = -\mu x(t) + f(x(t-1))$  with feedback function  $f(x) = \alpha \tanh(\beta x)$ . We suspect that the Györi–Hartung conjecture, stating that all solutions tend to a constant equilibrium, is true if  $b < L(a, \mu)$ .

Before moving on, it is worth mentioning case  $b = L(a, \mu)$ , as it serves as an easy counterexample to the conjecture. Condition  $b = L(a, \mu)$  is equivalent to  $\text{Re}\lambda_1 = 0$ ,

and in consequence the periodic function

$$\mathbb{R} \ni t \mapsto \hat{\xi}_0 + a \cos(\operatorname{Im} \lambda_1 t) + b \sin(\operatorname{Im} \lambda_1 t) \in \mathbb{R}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}$$

is a solution of Eq. (6.2) provided  $|a|$  and  $|b|$  are small enough.

## 6.4 The proof of Theorem 6.3.1

Theorem 6.3.1 follows immediately from results of Smith, Györi and Hartung.

**Proof of Theorem 6.3.1.** 1. Case  $I = 0$ . Evoke that if  $0 < \mu = a + b$  and  $I = 0$ , then any number  $\xi \in [-1, 1]$  defines an equilibrium of (6.2). Györi and Hartung have shown in [11] that if  $1 \ll \varphi$  ( $\varphi \ll -1$ ), then  $\lim_{t \rightarrow \infty} x^\varphi(t) = 1$  ( $\lim_{t \rightarrow \infty} x^\varphi(t) = -1$ ). As a consequence of these facts and monotonicity,  $\omega(\varphi)$  is a subset of  $C([-1, 0], [-1, 1])$  for all  $\varphi \in C$ .

First we prove that whenever  $\varphi \in C$  satisfies  $-1 \leq \varphi \leq 1$ ,  $x^\varphi$  tends to a constant as  $t \rightarrow \infty$ . In this case  $x^\varphi(t) \in [-1, 1]$  for all  $t \geq -1$ , and therefore  $x^\varphi : [-1, \infty) \rightarrow \mathbb{R}$  is the solution of the linear equation

$$\dot{x}(t) = -bx(t) + bx(t-1). \quad (6.13)$$

Using Lyapunov functionals, Haddock and Terjéki have shown in [13] that all solutions of Eq. (6.13) converge. See also Krisztin [21]. We come to the same conclusion using the theory of linear autonomous equations. Examining the characteristic equation

$$\lambda = -b + be^{-\lambda},$$

it is easy to see that the only real root is 0, that is simple, and all the others have negative real part. Therefore  $C \ni \varphi = \varphi_1 + \varphi_2$  and  $x_t^\varphi = x_t^{\varphi_1} + x_t^{\varphi_2}$ , where  $x_t^{\varphi_1} = \varphi_1$  is a constant function and  $x_t^{\varphi_2} \rightarrow 0$  as  $t \rightarrow \infty$  [16]. Consequently, if  $\varphi \in C$  with  $-1 \leq \varphi \leq 1$ , then the solution of equation (6.2) with initial function  $\varphi$  tends to a constant equilibrium.

Now suppose  $\varphi \in C$ ,  $t_n \rightarrow \infty$  and  $x_{t_n}^\varphi \rightarrow \psi \in C$  as  $n \rightarrow \infty$ . As  $\psi \in C([-1, 0], [-1, 1])$ , there exists an equilibrium point  $\hat{\xi}$  such that for any  $\varepsilon > 0$ ,  $\|x_T^\psi - \hat{\xi}\| < \varepsilon/2$  with some  $T = T(\varepsilon) > 0$ . Since  $x_{t_n+T}^\varphi \rightarrow x_T^\psi$  as  $n \rightarrow \infty$ , we have  $\|x_{t_n+T}^\varphi - \hat{\xi}\| < \varepsilon$  for  $n$  large enough. Combining monotonicity and this result, we get  $\|x_t^\varphi - \hat{\xi}\| < \varepsilon$  for all  $t$  large enough. As  $\varepsilon > 0$  was arbitrary,  $\omega(\varphi) = \hat{\xi}$  follows.

2. Case  $|I| \neq 0$  follows immediately from Theorem 2.1.1. Proposition 6.2.3 ensures that (T) holds for (6.2). If we choose  $\nu$  to be greater than  $\mu + |a|$ , then (SM) is also fulfilled. Additionally, Eq. (6.2) has exactly two equilibrium points if  $0 < \mu = a + b - |I|$  and  $|I| \neq 0$  by Lemma 6.2.1. Therefore all solutions converge to one of these.  $\square$

## 6.5 The separatrix and the leading unstable set of $\hat{\xi}_0$

We proceed to verify Theorem 6.3.2, so from now on consider Eq. (6.2) with feedback function (6.1) and parameters satisfying (6.3) and (6.5).

By Lemma 6.2.1 (iii) and Proposition 6.2.2, the closed and convex sets

$$K_{\geq} = \left\{ \varphi \in C : \hat{\xi}_0 \leq \varphi \right\}, \quad K_{\leq} = \left\{ \varphi \in C : \varphi \leq \hat{\xi}_0 \right\}$$

are positively invariant. The separatrix

$$S = \left\{ \varphi \in C : x^{\varphi} - \hat{\xi}_0 \text{ has arbitrarily large zeros} \right\}$$

is also positively invariant. Observe that

$$C \setminus S = \bigcup_{t \geq 0} \Phi(t, \cdot)^{-1} (\text{int} K_{\geq} \cup \text{int} K_{\leq}),$$

which shows that  $S$  is closed.

The next statement is analogous to Proposition 3.1 in Krisztin, Walter and Wu [26]. However, a different order is considered on  $C$ , as we examine a different type of equation.

**Proposition 6.5.1.** *For each  $\varphi, \psi \in C$  with  $\varphi \prec \psi$ , either  $\varphi \in C \setminus S$  or  $\psi \in C \setminus S$ .*

*Proof.* Provided there are  $\varphi \in S$  and  $\psi \in S$  with  $\varphi \prec \psi$ , we may suppose  $\varphi \ll \psi$  using Proposition 6.2.2 and the positive invariance of  $S$ . Theorem 2.1.1 ensures that we find  $\varphi^* \in C$  and  $\psi^* \in C$  with  $\varphi \ll \varphi^* \ll \psi^* \ll \psi$  so that  $x_t^{\varphi^*} \rightarrow \xi^{\varphi^*}$  and  $x_t^{\psi^*} \rightarrow \xi^{\psi^*}$  as  $t \rightarrow \infty$ , where  $\xi^{\varphi^*} \in \{\hat{\xi}_+, \hat{\xi}_-, \hat{\xi}_0\}$  and  $\xi^{\psi^*} \in \{\hat{\xi}_+, \hat{\xi}_-, \hat{\xi}_0\}$ . Then  $y = x^{\psi^*} - x^{\varphi^*}$  is positive and satisfies

$$\dot{y}(t) = (-\mu + a\hat{\gamma}(t))y(t) + b\hat{\gamma}(t-1)y(t-1)$$

for  $t > 0$ , where  $\hat{\gamma}(t) = \gamma(x^{\psi^*}(t), x^{\varphi^*}(t))$  and  $\gamma$  is defined by (6.7).

Suppose  $x_t^{\varphi^*}$  and  $x_t^{\psi^*}$  both converge to  $\hat{\xi}_0$  as  $t \rightarrow \infty$ . Then  $\hat{\gamma}(t) \rightarrow 1$  as  $t \rightarrow \infty$ , and there exists a  $t_0 \geq 0$  such that  $|\hat{\gamma}(t) - 1| < \frac{1}{2}$  for all  $t > t_0$ . Therefore

$$\frac{d}{dt} \left( \exp \left( \mu t - a \int_0^t \hat{\gamma}(s) ds \right) y(t) \right) = b \exp \left( \mu t - a \int_0^t \hat{\gamma}(s) ds \right) \hat{\gamma}(t-1)y(t-1) > 0$$

for  $t > t_0 + 1$ , and thus

$$\begin{aligned} y(t-1) &\leq \exp \left( \mu - a \int_{t-1}^t \hat{\gamma}(s) ds \right) y(t) \\ &\leq \exp \left( \max \left\{ \mu, \mu - \frac{3a}{2} \right\} \right) y(t) = e^c y(t) \quad \text{for } t > t_0 + 2, \end{aligned}$$

### 6.5 The separatrix and the leading unstable set of $\hat{\xi}_0$

where  $c = \max\{\mu, \mu - 3a/2\} > 0$ . Choose

$$\epsilon = 0.5(a + b - \mu)(e^c + 1)^{-1}.$$

Then  $\epsilon$  is positive since  $\mu < a + b$ . As  $\hat{\gamma}(t) \rightarrow 1$  as  $t \rightarrow \infty$ , there exists  $t_1 > t_0 + 2$  so that

$$|\hat{\gamma}(t) - 1| < \frac{\epsilon}{\max\{|a|, b\}} \quad \text{for } t > t_1.$$

It follows that for  $t > t_1$ ,

$$\dot{y}(t) \geq (-\mu + a - \epsilon)y(t) + (b - \epsilon)y(t - 1) \geq -(\mu - a + \epsilon(1 + e^c))y(t) + by(t - 1).$$

The choice of  $\epsilon$  ensures the existence of a positive constant  $\lambda$  such that

$$\lambda = -(\mu - a + \epsilon(1 + e^c)) + be^{-\lambda}.$$

Choose  $\delta > 0$  so that  $y(t) > \delta e^{\lambda t}$  on  $[t_1 - 1, t_1]$ . Function  $z(t) = \delta e^{\lambda t}$  is a solution of the equation

$$\dot{z}(t) = -(\mu - a + \epsilon(1 + e^c))z(t) + bz(t - 1).$$

Set  $u = y - z$ . Then  $0 \ll u_{t_1}$  and

$$\dot{u}(t) \geq -(\mu - a + \epsilon(1 + e^c))u(t) + bu(t - 1) \text{ for all } t > t_1.$$

Assume there exists a  $t_2 > t_1$  so that  $u(t_2) = 0$  and  $u$  is positive on  $[t_1 - 1, t_2]$ . Clearly  $\dot{u}(t_2) \leq 0$ . On the other hand, the inequality for  $u$  combined with the facts that  $u(t_2) = 0$  and  $u(t_2 - 1) > 0$  yields  $\dot{u}(t_2) > 0$ , which is a contradiction. So  $u(t) = y(t) - z(t) = y(t) - \delta e^{\lambda t} > 0$  for all  $t \geq t_1 - 1$ , which contradicts the boundedness of  $y$ .

Hence either  $\xi^{\varphi*} \in \{\hat{\xi}_+, \hat{\xi}_-\}$  or  $\xi^{\psi*} \in \{\hat{\xi}_+, \hat{\xi}_-\}$ . If  $x_t^{\psi*} \rightarrow \hat{\xi}_+$  as  $t \rightarrow \infty$ , then there exists  $t_0 > 0$  such that  $\hat{\xi}_0 \ll x_{t_0}^{\psi*}$ . By Proposition 6.2.2,  $\hat{\xi}_0 \ll x_{t_0}^{\psi*} \ll x_{t_0}^{\psi}$  and  $x_t^{\psi} \in \text{int}K_{\geq}$  for all  $t \geq t_0$ , a contradiction to  $\psi \in S$ . If  $x_t^{\psi*} \rightarrow \hat{\xi}_- \ll \hat{\xi}_0$  as  $t \rightarrow \infty$ , there exists  $t_0 > 0$  so that  $x_{t_0}^{\psi*} \ll \hat{\xi}_0$ . As  $x_{t_0}^{\varphi} \ll x_{t_0}^{\psi*} \ll \hat{\xi}_0$ , segment  $x_t^{\varphi} \in \text{int}K_{\leq}$  for  $t \geq t_0$ , a contradiction to  $\varphi \in S$ . The assumption that  $\omega(\varphi^*) \cap \{\hat{\xi}_+, \hat{\xi}_-\} \neq \emptyset$  also leads to a contradiction.  $\square$

Recall that the realified generalized eigenspace  $P$  of the generator associated with  $\lambda_0$  is 1-dimensional and is given by 0 and the segments of the solution  $\mathbb{R} \ni t \mapsto e^{\lambda_0 t} \in \mathbb{R}$  of Eq. (6.11). With notation  $\chi_0 : [-1, 0] \ni t \mapsto e^{\lambda_0 t} \in \mathbb{R}$ ,  $P = \mathbb{R}\chi_0$ . If the 1-codimensional realified generalized eigenspace of the generator associated with the rest of the spectrum is denoted by  $Q$ , then  $C = P \oplus Q$ .

We claim that  $S$  is a Lipschitz manifold of codimension 1.

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**Proposition 6.5.2.** *There exists a map  $Sep : Q \rightarrow P$  with  $\|Sep(\chi) - Sep(\tilde{\chi})\| \leq e^{\lambda_0} \|\chi - \tilde{\chi}\|$  for all  $\chi, \tilde{\chi} \in Q$  such that*

$$S = \left\{ \hat{\xi}_0 + \chi + Sep(\chi) : \chi \in Q \right\}.$$

Proposition 6.5.2 is a direct consequence of Proposition 6.5.1. It can be verified as Proposition 3.2 in [26], therefore the proof is omitted here.

Assume that  $b > L(a, \mu)$  in the rest of this section. Condition  $b > L(a, \mu)$  is equivalent to  $0 < \operatorname{Re}\lambda_1 < \lambda_0$ , which means the linear unstable space of the generator of the semigroup is at least 3-dimensional. Note that as Eq. (6.2) is linear in a neighborhood of  $\hat{\xi}_0$ , the leading unstable manifold  $\mathcal{W}_{1,loc}^u(\hat{\xi}_0) \subset C$  can be chosen so that it consists of those functions  $\varphi$  for which  $\|\varphi\| < 1$ , and  $\varphi - \hat{\xi}_0$  is an element of the realified generalized eigenspace of the generator given by  $\lambda_0$  and  $\lambda_1, \bar{\lambda}_1$ . So set

$$\begin{aligned} \mathcal{W}_{1,loc}^u(\hat{\xi}_0) = \left\{ \varphi \in C : \|\varphi\| < 1, \text{ there exist } a_0, a_1, a_2 \in \mathbb{R} \text{ so that for } t \in [-1, 0], \right. \\ \left. \varphi(t) = \hat{\xi}_0 + e^{\lambda_0 t} + e^{\operatorname{Re}\lambda_1 t} (a_1 \cos(\operatorname{Im}\lambda_1 t) + a_2 \sin(\operatorname{Im}\lambda_1 t)) \right\}. \end{aligned}$$

The forward extension  $\Phi(\mathbb{R}^+ \times \mathcal{W}_{1,loc}^u(\hat{\xi}_0))$  is denoted simply by  $\mathcal{W}$  in this case. Every  $\varphi$  in  $\mathcal{W}$  determines at least one solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  with  $x_t^\varphi \in \mathcal{W}$  for all  $t \in \mathbb{R}$  and  $x^\varphi(t) \rightarrow \hat{\xi}_0$  as  $t \rightarrow -\infty$ .

The next assertion is an easy application of the discrete Lyapunov functional presented in Subsection 2.3.1.  $\overline{\mathcal{W}}$  denotes the closure of  $\mathcal{W}$ .

**Proposition 6.5.3.** *Let  $\varphi \in \overline{\mathcal{W}}$  and  $\psi \in \overline{\mathcal{W}}$  with  $\varphi \neq \psi$ . Then  $V(\varphi - \psi) \leq 2$ .*

*Proof.* Since  $V$  is lower semicontinuous, it is sufficient to verify that  $V(\varphi - \psi) \leq 2$  for all  $\varphi \in \mathcal{W}, \psi \in \mathcal{W}$  with  $\varphi \neq \psi$ . By the definition of  $\mathcal{W}$ , there are solutions  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  and  $x^\psi : \mathbb{R} \rightarrow \mathbb{R}$  with initial functions  $\varphi$  and  $\psi$ , respectively, and there exists  $t_0 \leq 0$  so that  $x_t^\varphi \in \mathcal{W}_{1,loc}^u$  and  $x_t^\psi \in \mathcal{W}_{1,loc}^u$  if  $t < t_0$ . Then

$$\begin{aligned} x^\varphi(t) - x^\psi(t) &= a_0 e^{\lambda_0 t} + a_1 e^{\operatorname{Re}\lambda_1 t} \cos(\operatorname{Im}\lambda_1(t + b_1)) \\ &= e^{\operatorname{Re}\lambda_1 t} \left[ a_0 e^{\lambda_0 t - \operatorname{Re}\lambda_1 t} + a_1 \cos(\operatorname{Im}\lambda_1(t + b_1)) \right] \end{aligned}$$

for  $t < t_0$  with real constants  $a_0, a_1$  and  $b_1$ . As  $\lambda_0 > \operatorname{Re}\lambda_1$  and  $\operatorname{Im}\lambda_1 \in (\pi, 2\pi)$ , there is  $t_1 \leq t_0$  so that  $V(x_t^\varphi - x_t^\psi) \leq 2$  for  $t < t_1$ . Lemma 2.3.2 ensures that the function  $t \mapsto V(x_t^\varphi - x_t^\psi)$  is monotone decreasing, therefore  $V(\varphi - \psi) \leq 2$ .  $\square$

Our next goal is to describe set  $\overline{\mathcal{W} \cap S}$ .

**Proposition 6.5.4.** *The set  $\overline{\mathcal{W} \cap S}$  is compact and invariant.*

### 6.5 The separatrix and the leading unstable set of $\hat{\xi}_0$

*Proof.* Whenever  $\varphi \in \mathcal{W}$ , there exists a solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\hat{\xi}_- \ll x_t^\varphi \ll \hat{\xi}_+$  for some  $t \leq -1$ . Proposition 6.2.2 yields that  $\hat{\xi}_- \ll \varphi \ll \hat{\xi}_+$  for  $\varphi \in \mathcal{W}$ , consequently  $\hat{\xi}_- \leq \varphi \leq \hat{\xi}_+$  for  $\varphi \in \overline{\mathcal{W}}$ . It is clear that all  $\varphi \in \mathcal{W}$  is continuously differentiable. From Eq. (6.2) we get a bound for  $\dot{\varphi}$ ,  $\varphi \in \overline{\mathcal{W}}$ . By the Arzelà–Ascoli theorem,  $\overline{\mathcal{W}}$  is compact, and  $\overline{\mathcal{W} \cap S} \subset \overline{\mathcal{W}}$  is also compact.

Looking at the definition of  $\mathcal{W}$  and  $S$ , it is clear that  $\mathcal{W} \cap S$  is invariant. Let  $\varphi \in \overline{\mathcal{W} \cap S}$  and choose a sequence  $(\varphi_n)_{n=0}^\infty$  in  $\mathcal{W} \cap S$  converging to  $\varphi$  as  $n \rightarrow \infty$ . For  $t \geq 0$ ,

$$\Phi(t, \varphi) = \Phi\left(t, \lim_{n \rightarrow \infty} \varphi_n\right) = \lim_{n \rightarrow \infty} \Phi(t, \varphi_n),$$

and as  $\Phi(t, \varphi_n) \in \mathcal{W} \cap S$  for all  $n \in \mathbb{N}$ , we get  $\Phi(t, \varphi) \in \overline{\mathcal{W} \cap S}$ . We also have to show there exists a solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  with  $x_t^\varphi \in \overline{\mathcal{W} \cap S}$  for all  $t < 0$ . Consider solutions  $x^{\varphi_n} : \mathbb{R} \rightarrow \mathbb{R}$  for which it is true that  $x_t^{\varphi_n} \in \mathcal{W} \cap S$  for all  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ . By compactness, the sequence  $(x_{-1}^{\varphi_n})_{n=0}^\infty$  has a convergent subsequence  $(x_{-1}^{\varphi_{n_k}})_{n=0}^\infty$  with some limit  $\psi_{-1}$  in  $\overline{\mathcal{W} \cap S}$ . Using the continuity of  $\Phi(1, \cdot)$ , we get  $\Phi(1, \psi_{-1}) = \varphi$ . The sequence  $(x_{-2}^{\varphi_{n_k}})_{n=0}^\infty$  also has a convergent subsequence with limit  $\psi_{-2} \in \overline{\mathcal{W} \cap S}$ . Again  $\psi_{-1} = \Phi(1, \psi_{-2})$ . Repeating this procedure, we get a sequence  $(\psi_k)_{k < 0} \subset \overline{\mathcal{W} \cap S}$  with  $\Phi(1, \psi_{k-1}) = \psi_k$  for all  $k \in \mathbb{Z}$ ,  $k < 0$ . Consequently we get a solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  with  $x_k^\varphi = \psi_k$  for  $k \in \mathbb{Z}$ ,  $k < 0$  and  $x_t^\varphi \in \overline{\mathcal{W} \cap S}$  for all  $t \in \mathbb{R}$ .  $\square$

**Proposition 6.5.5.** *If  $\varphi \in \overline{\mathcal{W} \cap S} \setminus \{\hat{\xi}_0\}$  and  $x = x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a solution with  $x_t \in \overline{\mathcal{W} \cap S}$  for all  $t \in \mathbb{R}$ , then  $V(\varphi - \hat{\xi}_0) = 2$  and there exists a sequence  $(t_n)_{n=-\infty}^\infty$  so that for all  $n \in \mathbb{Z}$ ,*

$$\begin{aligned} t_{n+1} - t_n &< 1, \quad t_{n+2} - t_n > 1, \\ x(t_n) &= \hat{\xi}_0, \quad \dot{x}(t_{2n}) > 0, \quad \dot{x}(t_{2n+1}) < 0, \\ x(t) &> \hat{\xi}_0 \text{ if } t \in (t_{2n}, t_{2n+1}), \\ x(t) &< \hat{\xi}_0 \text{ if } t \in (t_{2n-1}, t_{2n}). \end{aligned}$$

*Proof.* Suppose  $\varphi \in \overline{\mathcal{W} \cap S} \setminus \{\hat{\xi}_0\}$  and  $x = x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a solution with  $x_t \in \overline{\mathcal{W} \cap S}$  for all real  $t$ . By a previous remark in Section 6.2,  $x_t \neq \hat{\xi}_0$  for all  $t \in \mathbb{R}$ . As  $\hat{\xi}_0 \in \overline{\mathcal{W}}$ , Proposition 6.5.3 gives that  $V(x_t - \hat{\xi}_0) \leq 2$  for all  $t \in \mathbb{R}$ . Assume there exists  $t_0 \in \mathbb{R}$  with  $V(x_{t_0} - \hat{\xi}_0) = 0$ . Since  $t \mapsto V(x_t - \hat{\xi}_0)$  is monotone decreasing, the function  $x(t) - \hat{\xi}_0$  has no sign change on  $[t_0 - 1, \infty)$ . On the other hand, there exists  $t_1 \in [t_0, t_0 + 1]$  with  $x(t_1) \neq \hat{\xi}_0$ . We get  $\hat{\xi}_0 \prec x_{t_1}$  or  $x_{t_1} \prec \hat{\xi}_0$ , which contradicts Proposition 6.5.1. So  $V(x_t - \hat{\xi}_0) = 2$  for all  $t \in \mathbb{R}$ .

Assertion (ii) of Lemma 2.3.2 implies that if  $x(t) = \hat{\xi}_0$  for some  $t$ , then  $x(t - 1) \neq \hat{\xi}_0$ . Eq. (2.6) shows the zeros of  $x - \hat{\xi}_0$  are all simple, hence there exists a strictly increasing sequence  $(t_n)_{n=-\infty}^\infty$  such that  $x(t_n) = \hat{\xi}_0$ ,  $\dot{x}(t_{2n}) > 0$ ,  $\dot{x}(t_{2n+1}) < 0$ ,  $x(t) > \hat{\xi}_0$  if  $t \in (t_{2n}, t_{2n+1})$  and  $x(t) < \hat{\xi}_0$  if  $t \in (t_{2n-1}, t_{2n})$  for all  $n \in \mathbb{Z}$ . As we have seen before,

$sc(x_t - \hat{\xi}_0) \in \{1, 2\}$  for all  $t$ . It is certainly evident that in this case  $sc(x_t - \hat{\xi}_0)$  is given by the number of zeros in  $(t-1, t)$ . Thus  $t_{n+1} - t_n < 1$  and  $t_{n+2} - t_n \geq 1$  for all  $n \in \mathbb{Z}$ . Lemma 2.3.2 rules out the possibility that  $t_{n+2} - t_n = 1$ , so  $t_{n+2} - t_n > 1$  for all  $n \in \mathbb{Z}$ .  $\square$

## 6.6 The proof of Theorem 6.3.2

**Proof of Theorem 6.3.2 (i).** Suppose conditions (6.1)–(6.3) and (6.5) are satisfied. Let  $\varphi$  be an element of  $C \setminus S$ . Then by the definition of  $S$ , there exists  $T \in \mathbb{R}$  so that  $\hat{\xi}_0 \ll x_T^\varphi$  or  $x_T^\varphi \ll \hat{\xi}_0$ . Assume  $\hat{\xi}_0 \ll x_T^\varphi$ . We show that in this case  $x^\varphi(t) \rightarrow \hat{\xi}_+$  as  $t \rightarrow \infty$ .

On the one hand, Theorem 2.1.1 yields an initial function  $\psi_1 \in C$  with  $\hat{\xi}_+ \ll \psi_1$  and  $\varphi \ll \psi_1$  such that  $x^{\psi_1}$  converges one of the equilibria as  $t \rightarrow \infty$ . By monotonicity, this equilibrium point is necessarily  $\hat{\xi}_+$ .

On the other hand, set  $\psi_2 = \hat{\xi}_0 + ae^{\lambda_0 t}$  with  $a > 0$  chosen to be so small that  $\|\psi_2\| < 1$  holds. Then  $x^{\psi_2}(t) = \hat{\xi}_0 + ae^{\lambda_0 t}$  for  $t \leq 0$  and  $x^{\psi_2}(t) \rightarrow \hat{\xi}_0$  as  $t \rightarrow -\infty$ . We claim that  $x^{\psi_2}(t) \rightarrow \hat{\xi}_+$  as  $t \rightarrow \infty$ . Notice that Proposition 6.2.2 and the fact that  $x_t^{\psi_2} \rightarrow \hat{\xi}_0$  as  $t \rightarrow -\infty$  imply  $x^{\psi_2}(t) < \hat{\xi}_+$  for all  $t \in \mathbb{R}$ . Clearly  $x_s^{\psi_2} \ll x_t^{\psi_2}$  for  $s < t \leq 0$ , thus  $x_s^{\psi_2} \ll x_t^{\psi_2}$  for all  $s < t$  by Proposition 6.2.2. Solution  $x^{\psi_2}$  is injective, because otherwise there exists  $s \neq t$  with  $x^{\psi_2}(s) = x^{\psi_2}(t)$ , and  $x_t^{\psi_2} - x_s^{\psi_2}$  has a zero at 0, which contradicts  $x_s^{\psi_2} \ll x_t^{\psi_2}$ . Therefore  $x^{\psi_2}$  is strictly monotone. Since  $x_t^{\psi_2} \rightarrow \hat{\xi}_0$  as  $t \rightarrow -\infty$ , we get  $x^{\psi_2}$  is strictly increasing. It follows that  $x^{\psi_2}(t)$  converges to some  $\xi \in (\hat{\xi}_0, \hat{\xi}_+]$  as  $t \rightarrow \infty$ . By Eq. (6.2),  $\dot{x}^{\psi_2}(t) \rightarrow -\mu\xi + af(\xi) + bf(\xi) + I$  as  $t \rightarrow \infty$ . If  $-\mu\xi + af(\xi) + bf(\xi) + I \neq 0$ , then  $\dot{x}^\varphi$  is bounded away from 0 on an unbounded interval, contradicting Proposition 6.2.3. For this reason  $-\mu\xi + af(\xi) + bf(\xi) + I = 0$  and  $[-1, 0] \ni t \mapsto \xi$  is a constant solution of Eq. (6.2), which means that  $\xi = \hat{\xi}_+$ . The claim is verified.

The choice of  $\psi_1$  and  $\psi_2$  ensures that  $x_s^{\psi_2} \ll x_T^\varphi \ll x_T^{\psi_1}$  with some  $s \in \mathbb{R}$  and  $T$  given above. Since  $x^{\psi_i}(t) \rightarrow \hat{\xi}_+$  as  $t \rightarrow \infty$  for  $i = 1, 2$ , we get  $x^\varphi(t) \rightarrow \hat{\xi}_+$  as  $t \rightarrow \infty$ .

One can confirm analogously that  $x^\varphi(t) \rightarrow \hat{\xi}_-$  as  $t \rightarrow \infty$  if  $x_T^\varphi \ll \hat{\xi}_0$  for some  $T \in \mathbb{R}$ .  $\square$

Assume again that not only conditions (6.2)–(6.1) and (6.5) are satisfied, but also  $b > L(a, \mu)$  holds. Recall the properties of the set  $\overline{\mathcal{W} \cap \mathcal{S}}$  from the previous section. For the sake of simplicity, if  $\varphi$  belongs to  $\overline{\mathcal{W} \cap \mathcal{S}}$ , let  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$  denote any solution of Eq. (6.2) for which it is true that  $x_0^\varphi = \varphi$  and  $x_t^\varphi \in \overline{\mathcal{W} \cap \mathcal{S}}$  for all  $t \in \mathbb{R}$ . We move on to show the existence of a unique periodic solution with segments in  $\overline{\mathcal{W} \cap \mathcal{S}}$ .

We need the continuous map

$$\pi_2 : C \ni \varphi \mapsto (\varphi(0) - \hat{\xi}_0, \varphi(-1) - \hat{\xi}_0) \in \mathbb{R}^2.$$



Map  $\pi_2$  is a modification of map  $\pi$  introduced in Subsection 2.3.1.

The rest of the chapter makes great use of the following assertion.

**Proposition 6.6.1.** *If  $\varphi \in \overline{\mathcal{W} \cap \mathcal{S}}$ ,  $\psi \in \overline{\mathcal{W} \cap \mathcal{S}}$  and  $\pi_2 \varphi = \pi_2 \psi$ , then  $x^\varphi(t) = x^\psi(t)$  for all  $t \geq 0$ .*

*Proof.* Consider  $\varphi \in \overline{\mathcal{W} \cap \mathcal{S}}$  and  $\psi \in \overline{\mathcal{W} \cap \mathcal{S}}$  with  $\pi_2 \varphi = \pi_2 \psi$ . If  $\varphi = \psi$ , there is nothing to prove, so suppose that  $\varphi \neq \psi$ . Lemma 2.3.2 and Proposition 6.5.3 yield  $V(\varphi - \psi) < V(x_{-2}^\varphi - x_{-2}^\psi) \leq 2$ . Consequently  $V(\varphi - \psi) = 0$  and  $\varphi \leq \psi$  or  $\psi \leq \varphi$ . Assume  $\psi \leq \varphi$  for example. Then  $x^\psi(t) \leq x^\varphi(t)$  for all  $t \geq -1$  by monotonicity. If there exists a  $t_0 > 0$  with  $x^\psi(t_0) < x^\varphi(t_0)$ , then  $x_{t_0}^\psi \prec x_{t_0}^\varphi$ , which contradicts Proposition 6.5.1. Necessarily  $x^\psi(t) = x^\varphi(t)$  for all  $t \geq 0$ .  $\square$

For  $\varphi \in \overline{\mathcal{W} \cap \mathcal{S}}$ , the curve

$$\chi : \mathbb{R} \ni t \mapsto \pi_2 x_t^\varphi \in \mathbb{R}^2$$

is  $C^1$ -smooth and has its range in  $\pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$ . It is called the canonical curve associated with solution  $x^\varphi : \mathbb{R} \rightarrow \mathbb{R}$ . The range of two different canonical curves may have points in common. If  $\chi$  and  $\hat{\chi}$  are such canonical curves, that is there exists a  $t_0 \in \mathbb{R}$  with  $\chi(t_0) = \hat{\chi}(t_0)$ , then  $\chi(t_0 + s) = \hat{\chi}(t_0 + s)$  for all  $s \geq 1$  by Proposition 6.6.1.

The images of the closed hyperplane

$$H = \{\varphi \in C : \varphi(0) = \hat{\xi}_0\}$$

and its subsets

$$H^+ = \{\varphi \in H : \varphi(-1) > \hat{\xi}_0\}, \quad H^- = \{\varphi \in H : \varphi(-1) < \hat{\xi}_0\}$$

under  $\pi_2$  are

$$\{(u, v) \in \mathbb{R}^2 : u = 0\},$$

$$v_+ = \{(0, v) \in \mathbb{R}^2 : v > 0\} \text{ and } v_- = \{(0, v) \in \mathbb{R}^2 : v < 0\},$$

respectively. Evoke function  $\gamma$  defined by (6.7). If  $\pi_2 x_t^\varphi \in v_+(v_-)$  for some  $\varphi \in \overline{\mathcal{W} \cap \mathcal{S}}$  and  $t \in \mathbb{R}$ , then

$$\dot{x}^\varphi(t) = 0 + \beta(t) (x^\varphi(t-1) - \hat{\xi}_0) > 0 (< 0),$$

where  $\beta : \mathbb{R} \ni t \mapsto b\gamma(x^\varphi(t-1), \hat{\xi}_0) \in \mathbb{R}$  is a positive function. Therefore the canonical curves intersect  $v_- \cup v_+$  transversally.

For  $\varphi \in \overline{\mathcal{W} \cap \mathcal{S}} \setminus \{\hat{\xi}_0\}$  and  $n \geq 0$ , let  $z_n = z_n(\varphi)$  denote the  $n$ th zero of the function  $x^\varphi - \hat{\xi}_0$  on  $[0, \infty)$ . Proposition 6.5.5 ensures the existence of  $z_n$ . Next we introduce a

## 6 Dynamics for the Hopfield Activation Function

Poincaré return map

$$\mathcal{P} : (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}}) \ni \chi^0 \mapsto \pi_2 \Phi(z_2, \varphi) \in (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}}),$$

where  $\varphi$  is any element of  $\pi_2^{-1}(\chi^0) \subset (H^+ \cup H^-) \cap \overline{\mathcal{W} \cap \mathcal{S}}$ . Since  $z_2 > 1$ , Proposition 6.6.1 shows that map  $\mathcal{P}$  is well-defined. Proposition 6.5.5 also yields that

$$\mathcal{P}(v_- \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})) = v_- \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$$

and

$$\mathcal{P}(v_+ \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})) = v_+ \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}}).$$

One would expect  $\mathcal{P}$  to be continuous. We can verify only the following weaker property.

**Proposition 6.6.2.** *For each  $\chi^0 \in (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$  and sequence  $(\chi^n)_{n=1}^\infty \subset (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$  with  $\chi^n \rightarrow \chi^0$  as  $n \rightarrow \infty$ , there exists a subsequence  $(\chi^{n_k})_{k=1}^\infty$  so that  $\mathcal{P}(\chi^{n_k}) \rightarrow \mathcal{P}(\chi^0)$  as  $k \rightarrow \infty$ .*

*Proof.* Assume that  $(v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}}) \ni \chi^n \rightarrow \chi^0 \in (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$  as  $n \rightarrow \infty$ . Let  $(\varphi^n)_{n=1}^\infty$  be a sequence in  $\overline{\mathcal{W} \cap \mathcal{S}}$  with  $\pi_2(\varphi^n) = \chi^n$  for  $n \in \mathbb{N}$ . Since  $\overline{\mathcal{W} \cap \mathcal{S}}$  is compact,  $(\varphi^n)_{n=1}^\infty$  has a convergent subsequence  $(\varphi^{n_k})_{k=1}^\infty$ . Let  $\varphi^0$  denote the limit of this subsequence. Then necessarily  $\varphi^0 \in (H^+ \cup H^-) \cap \overline{\mathcal{W} \cap \mathcal{S}}$  and  $\pi_2(\varphi^0) = \chi^0$ . We show that  $\pi_2 \Phi(z_2, \varphi^{n_k}) \rightarrow \pi_2 \Phi(z_2, \varphi^0)$ , that is  $\mathcal{P}(\chi^{n_k}) \rightarrow \mathcal{P}(\chi^0)$  as  $k \rightarrow \infty$ . As  $\pi_2$  and  $\Phi$  are continuous, it remains to show that the function  $z_2 : (H^+ \cup H^-) \cap \overline{\mathcal{W} \cap \mathcal{S}} \rightarrow \mathbb{R}$  giving the second smallest zero is also continuous. Let  $(\psi_n)_{n=0}^\infty \subset (H^+ \cup H^-) \cap \overline{\mathcal{W} \cap \mathcal{S}}$  be a convergent sequence with limit  $\psi$  in  $(H^+ \cup H^-) \cap \overline{\mathcal{W} \cap \mathcal{S}}$ . Since  $\Phi$  is continuous,  $x^{\psi_n} \rightarrow x^\psi$  uniformly on compact subsets of  $[-1, \infty)$ . Using (6.2), we conclude that  $\dot{x}^{\psi_n} \rightarrow \dot{x}^\psi$  also uniformly on compact subsets of  $[0, \infty)$ . Proposition 6.5.5 yields  $z_2(\psi_n) \rightarrow z_2(\psi)$  as  $n \rightarrow \infty$ .  $\square$

Further notations are needed.

As usual, let  $\mathcal{P}^n$  denote the map  $\mathcal{P}^n = \mathcal{P} \circ \mathcal{P}^{n-1}$  for  $n \geq 2$ .

Let  $(\chi^n)_{n=1}^\infty \subset v_+ \cap \pi_2(\overline{\mathcal{W} \cap \mathcal{S}})$  be a trajectory of  $\mathcal{P}$ , and let  $\chi$  be a canonical curve associated with solution  $x : \mathbb{R} \rightarrow \mathbb{R}$ . The sequence  $(\chi^n)_{n=1}^\infty$  is the trajectory of  $\mathcal{P}$  associated with solution  $x$ , if

$$\{\chi^n : n \in \mathbb{Z}\} = v_+ \cap \chi((-\infty, \infty)).$$

At last we define a relation  $<^2$  on set  $\{(0, v) : v \in \mathbb{R}\}$ . Let  $\chi^0 <^2 \chi^1$  if the second component of  $\chi^0 \in \mathbb{R}^2$  is smaller than that of  $\chi^1$ .

The following argument is analogous to Proposition 7.1 in [26]. However, we have a meaningful difference in the statement as well as in its proof, since in [26] the map  $\pi_2|_{\overline{\mathcal{W} \cap \mathcal{S}}}$  is invertible, which may not hold here.

**Proposition 6.6.3.** *If  $\chi^0, \hat{\chi}^0 \in (v_- \cup v_+) \cap \pi_2(\overline{\mathcal{W} \cap S})$  with  $\chi^0 <^2 \hat{\chi}^0$ , then*

$$\mathcal{P}(\chi^0) <^2 \mathcal{P}(\hat{\chi}^0) \text{ or } \mathcal{P}^n(\chi^0) = \mathcal{P}^n(\hat{\chi}^0) \text{ for all } n \geq 2.$$

*Proof.* Assertion is obviously true if  $\chi^0 \in v_-$  and  $\hat{\chi}^0 \in v_+$ , as  $\mathcal{P}(\chi^0) \in v_-$  and  $\mathcal{P}(\hat{\chi}^0) \in v_+$ . For this reason suppose  $\chi^0 \in v_+$ ,  $\hat{\chi}^0 \in v_+$  and  $\chi^0 <^2 \hat{\chi}^0$ . Let  $x : \mathbb{R} \rightarrow \mathbb{R}$  and  $\hat{x} : \mathbb{R} \rightarrow \mathbb{R}$  be any solutions of (6.2) with segments in  $\overline{\mathcal{W} \cap S}$  such that  $\pi_2 x_0 = \chi^0$  and  $\pi_2 \hat{x}_0 = \hat{\chi}^0$ . As mentioned before,  $\mathcal{P}(\chi^0)$  and  $\mathcal{P}(\hat{\chi}^0)$  are independent of the choice of solutions  $x$  and  $\hat{x}$ . By definition, canonical curves  $\chi$  and  $\hat{\chi}$  associated with  $x$  and  $\hat{x}$  satisfy  $\chi(t) = (x(t) - \hat{\xi}_0, x(t-1) - \hat{\xi}_0)$  and  $\hat{\chi}(t) = (\hat{x}(t) - \hat{\xi}_0, \hat{x}(t-1) - \hat{\xi}_0)$  for  $t \in \mathbb{R}$ . Also  $\chi(0) = \chi^0$  and  $\hat{\chi}(0) = \hat{\chi}^0$ . Clearly  $\hat{\xi}_0 = x(0) = \hat{x}(0)$ ,  $\hat{\xi}_0 < x(-1) < \hat{x}(-1)$ , so  $\dot{x}(0) > 0$ ,  $\dot{\hat{x}}(0) > 0$ , and for the smallest positive zeros  $z_1$  of  $x - \hat{\xi}_0$  and  $\hat{z}_1$  of  $\hat{x} - \hat{\xi}_0$  we have  $x > \hat{\xi}_0$  on  $(0, z_1)$  and  $\hat{x} > \hat{\xi}_0$  on  $(0, \hat{z}_1)$ . According to Proposition 6.5.5,  $\dot{x}(z_1) < 0$ ,  $x(z_1 - 1) < \hat{\xi}_0$ ,  $\dot{\hat{x}}(\hat{z}_1) < 0$  and  $\hat{x}(\hat{z}_1 - 1) < \hat{\xi}_0$ .

Next we verify that the restriction  $\chi|_{[0, z_1]}$  and the line segment

$$\lambda : [0, 1] \ni s \mapsto \chi^0 + s(\chi(z_1) - \chi^0) \in \mathbb{R}^2$$

form a simple closed curve  $\zeta$ . It is obvious that  $\chi|_{(0, z_1)}$  has no points with the line segment in common, as  $x > \hat{\xi}_0$  in  $(0, z_1)$ . Suppose there exist  $t_1, t_2 \in (0, z_1)$ ,  $t_1 < t_2$  with  $\chi(t_1) = \chi(t_2)$ . According to Proposition 6.6.1, this implies  $x(t_1 + s) = x(t_2 + s)$  for  $s \geq 0$ . With  $s = z_1 - t_2 > 0$  we get  $\hat{\xi}_0 = x(z_1) = x(z_1 - t_2 + t_1)$ , a contradiction to  $x > \hat{\xi}_0$  in  $(0, z_1)$ . Consequently  $\zeta$  is a simple closed curve.

One can easily see that the set

$$\left\{ (u, v)^{tr} : u < 0 \text{ or } u = 0 \text{ and } v < x(z_1 - 1) \text{ or } u = 0 \text{ and } v > x(-1) \right\}$$

belongs to  $\text{ext}(\zeta)$ , in particular  $\hat{\chi}(0) \in \text{ext}(\zeta)$ .

We have to distinguish two cases.

1. If there exist  $t_0 \in [0, z_1]$  and  $\hat{t}_0 \in [0, \hat{z}_1]$  with  $\chi(t_0) = \hat{\chi}(\hat{t}_0)$ , then by Proposition 6.6.1,  $x(t_0 + s) = \hat{x}(\hat{t}_0 + s)$  for  $s \geq 0$  and  $\chi(t_0 + s) = \hat{\chi}(\hat{t}_0 + s)$  for  $s \geq 1$ . Proposition 6.5.5 yields  $z_{l+2} - z_l > 1$  for all  $l \in \mathbb{N}$ , hence  $\chi(z_n) = \hat{\chi}(\hat{z}_n)$  for  $n \geq 3$  and  $\mathcal{P}^n(\chi^0) = \mathcal{P}^n(\hat{\chi}^0)$  for  $n \geq 2$ .

2. Now suppose that  $\hat{\chi}([0, \hat{z}_1]) \cap \chi([0, z_1]) = \emptyset$ . Using  $\hat{x} > \hat{\xi}_0$  on  $(0, \hat{z}_1)$  and  $\hat{\chi}(0) \in \text{ext}(\zeta)$ , we have  $\hat{\chi}([0, \hat{z}_1]) \subset \mathbb{R}^2 \setminus |\zeta|$ , and therefore  $\hat{\chi}([0, \hat{z}_1]) \subset \text{ext}(\zeta)$ . Assume  $\chi(z_1) <^2 \hat{\chi}(\hat{z}_1)$ . Then  $\hat{\chi}(\hat{z}_1) = (0, \hat{x}(\hat{z}_1 - 1)) \in \lambda((0, 1))$ . Since  $\dot{\hat{x}}(\hat{z}_1) < 0$ ,  $\hat{\chi}((\hat{z}_1 - \varepsilon, \hat{z}_1)) \subset \text{int}(\zeta)$  with some  $\varepsilon > 0$ , which contradicts the fact that  $\hat{\chi}([0, \hat{z}_1]) \subset \text{ext}(\zeta)$ . As  $\hat{\chi}(\hat{z}_1) \neq \chi(z_1)$ , we infer  $\hat{\chi}(\hat{z}_1) <^2 \chi(z_1)$ .

By inverting the role of  $\chi^0$  and  $\hat{\chi}^0$  in the argument above, we come to the same conclusion if  $\chi^0, \hat{\chi}^0 \in v_-$  with  $\chi^0 <^2 \hat{\chi}^0$ .

## 6 Dynamics for the Hopfield Activation Function

In case  $\hat{\chi}([0, \hat{z}_1]) \cap \chi([0, z_1]) = \emptyset$  apply the same argument to  $\chi(z_1) \in v_-$  and  $\hat{\chi}(\hat{z}_1) \in v_-$  in order to deduce that  $\mathcal{P}(\chi^0) <^2 \mathcal{P}(\hat{\chi}^0)$  or  $\mathcal{P}^2(\chi^0) = \mathcal{P}^2(\hat{\chi}^0)$ .  $\square$

**Proof of Theorem 6.3.2 (ii).** Let  $\psi$  be an element of  $\mathcal{W} \cap S \setminus \{\hat{\xi}_0\}$ . Proposition 6.5.5 yields the existence of a  $z_0 \geq 0$  with  $x_{z_0}^\psi \in H^+ \cap \mathcal{W} \cap S$ . Put  $\chi^0 = \pi_2 x_{z_0}^\psi$  and take trajectory  $(\chi^n)_{-\infty}^\infty \subset v_+ \cap \pi_2(\mathcal{W} \cap S)$  of  $\mathcal{P}$  associated with solution  $x^\psi : \mathbb{R} \rightarrow \mathbb{R}$ . Clearly  $\chi(z_{2n}) = \chi^n$ , where  $(z_n)_{-\infty}^\infty$  is the sequence of zeros of  $x^\psi - \hat{\xi}_0$ . Since  $\psi \in \mathcal{W}$ ,  $\chi^n \rightarrow 0$  as  $n \rightarrow -\infty$ . By Proposition 6.6.3,  $(\chi^n)_{-\infty}^\infty$  is either strictly increasing according to the order introduced on  $v_+$ , or there exists  $n_0 \in \mathbb{N}$  so that  $(\chi^n)_{-\infty}^\infty$  is constant for  $n \geq n_0$ . As  $\pi_2(\overline{\mathcal{W} \cap S})$  is compact,  $\chi_+ = \lim_{n \rightarrow \infty} \chi^n \in v_+ \cap \pi_2(\overline{\mathcal{W} \cap S})$  exists. Clearly  $\chi_+ \neq 0$ , and Proposition 6.6.2 ensures the existence of a subsequence  $(\chi^{n_k})_{k=0}^\infty$  so that

$$\chi^{n_k+1} = \mathcal{P}(\chi^{n_k}) \rightarrow \mathcal{P}(\chi_+) \text{ as } k \rightarrow \infty.$$

Necessarily  $\mathcal{P}(\chi_+) = \chi_+$ . Choose  $\eta \in H^+ \cap \overline{\mathcal{W} \cap S}$  so that  $\pi_2 \eta = \chi_+$ . With  $q(t) = x^\eta(t)$ ,  $t \in \mathbb{R}$ , we get  $\pi_2 \Phi(z_2, q_0) = \mathcal{P}(\chi_+) = \chi_+$  that is  $\pi_2 q_{z_2} = \pi_2 q_0$ . Using Proposition 6.6.1 we conclude  $q(t) = q(t + z_2)$  for  $t \geq 0$ . Let  $p$  be the periodic extension of  $q$  to  $\mathbb{R}$ .

Obviously  $p_t \in \overline{\mathcal{W} \cap S}$  for all  $t \in \mathbb{R}$  and Proposition 6.5.5 gives  $V(p_t - \hat{\xi}_0) = 2$  for all real  $t$ . The statement related to the minimal period  $\omega$  follows also from Proposition 6.5.5 and  $\mathcal{P}(\chi_+) = \chi_+$ .  $\square$

As one can suspect from the proof of Theorem 6.3.2 (ii),  $x_t^\varphi \rightarrow \{p_t : t \in [0, \omega]\}$  as  $t \rightarrow \infty$  for all initial functions  $\varphi$  in  $\mathcal{W} \cap S \setminus \{\hat{\xi}_0\}$ , where  $p$  denotes the unique periodic solution with segments in  $\overline{\mathcal{W} \cap S}$ .

## 7 Summary

The dissertation studies the scalar delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \quad (1.1)$$

with parameter  $\mu > 0$  and nonlinear feedback function  $f$ . Both continuously differentiable and nonsmooth, both monotone increasing and monotone decreasing nonlinearities are considered. The goal is to detect periodic orbits and describe the global attractor as thoroughly as possible under a wide variety of conditions.

Such equations appear in artificial neural networks. Some examples motivating this work are listed in the introduction of the dissertation, and book [50] of Wu contains a more detailed description of applications.

Chapter 2 gives a general theoretical overview. First, basic concepts (phase space, solution, semiflow, global attractor, unstable manifold, etc.) are introduced, then Section 2.2 summarizes the most relevant results within the field of monotone nonlinearities. Krisztin, Walther and Wu have described the asymptotic behavior of the solutions in case of positive feedback, i.e. when  $f$  is continuous and  $xf(x) > 0$  for all  $x \neq 0$  [20, 22, 23, 25, 26, 27], whereas in case of negative feedback ( $f$  is continuous and  $xf(x) < 0$  for all  $x \neq 0$ ) the works [45]-[49] of Walther and Yebdri have provided the main references. This present thesis is based on their results and on paper [11] of Györi and Hartung. Section 2.3 places a special emphasis on the most important tools applied in the subsequent chapters. Mallet-Paret and Sell [32] have introduced a discrete Lyapunov functional  $V$  counting the sign changes of the elements of the phase space  $C$ . Though most of their findings cannot be applied directly, a straightforward generalization of their theorems proves to be an efficient tool in understanding the dynamics of the equation. Poincaré return maps also play an essential role as their fixed points yield the initial segments of the periodic solutions. According to the Floquet theory, the spectrum of the derivative of a Poincaré map at its fixed point determines the stability of the associated periodic solution. The work of Lani-Wayda [28] is also applied, as it shows that small perturbations of the feedback function preserve these periodic orbits, provided they are hyperbolic.

Chapters 3 and 4 examine the positive feedback case. In these chapters, a strictly increasing, continuously differentiable feedback function  $f$  is considered so that  $\xi \mapsto$

## 7 Summary

$-\mu\xi + f(\xi)$  has 5 consecutive zeros

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2,$$

the equilibria  $\hat{\xi}_{-2}, \hat{\xi}_0, \hat{\xi}_2$  defined by  $\xi_{-2}, \xi_0, \xi_2$  are stable, and the equilibria  $\hat{\xi}_{-1}, \hat{\xi}_1$  defined by  $\xi_{-1}, \xi_1$  are unstable. The monotonicity of  $f$  implies that the subsets

$$C_{-2,0} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq 0 \text{ for all } s \in [-1, 0]\},$$

$$C_{0,2} = \{\varphi \in C : 0 \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\}$$

of the phase space  $C = C([-1, 0], \mathbb{R})$  are positively invariant. The restrictions of the semiflow to  $C_{-2,0}$  and to  $C_{0,2}$  have global attractors  $\mathcal{A}_{-2,0}$  and  $\mathcal{A}_{0,2}$ , respectively. Sets  $\mathcal{A}_{-2,0}$  and  $\mathcal{A}_{0,2}$  have spindle-like structures according to the Krisztin, Walther and Wu characterization. The question whether the equality  $\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$  holds for the global attractor  $\mathcal{A}$  of the semiflow restricted to

$$C_{-2,2} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1, 0]\}$$

has already been drawn up in [26].

Theorem 3.1.1 in Chapter 3 shows that the structure of  $\mathcal{A}$  can be more complicated: a smooth, strictly increasing nonlinear map  $f$  is given so that there are exactly two periodic orbits  $\mathcal{O}_p$  and  $\mathcal{O}_q$  in  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ , which are unstable with 2 and 1 Floquet multipliers in  $\{z \in \mathbb{C} : |z| > 1\}$ . The solutions  $p$  and  $q$  defining these periodic orbits are so-called LSOP solutions: on the one hand they are of large amplitude in the sense that  $p(\mathbb{R}) \supsetneq (\xi_{-1}, \xi_1)$  and  $q(\mathbb{R}) \supsetneq (\xi_{-1}, \xi_1)$ , and on the other hand they are slowly oscillatory in the sense that each segment of them has one or two sign changes. Note that such solutions cannot appear via local bifurcation, hence it is a challenging task to verify their existence. The nonlinear map  $f$  in Theorem 3.1.1 is "close" to the step function  $f^{K,0}$  parametrized by  $K > 0$  and given by

$$f^{K,0}(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ K \operatorname{sgn}(x) & \text{if } |x| > 1. \end{cases}$$

The starting point of the proof is to form explicit periodic solutions for Eq. (1.1) with  $\mu = 1$  and  $f = f^{K,0}$ , which is a finite dimensional problem and, therefore, a manageable one. Then the implicit function theorem and perturbations of Poincaré maps can be applied in order to find exactly two LSOP orbits for  $\mu = 1$  and nonlinearities close to  $f^{7,0}$ .

Chapter 4 analyzes the structure of the solutions in the situation of Theorem 3.1.1. By Theorem 4.1.1, one may set  $f$  so that Theorem 3.1.1 holds, and for the global

attractor  $\mathcal{A}$ , we have equality

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q),$$

where  $\mathcal{W}^u(\mathcal{O}_p)$  and  $\mathcal{W}^u(\mathcal{O}_q)$  denote the unstable sets of the LSOP orbits  $\mathcal{O}_p$  and  $\mathcal{O}_q$ , respectively. Sets  $\mathcal{W}^u(\mathcal{O}_p)$  and  $\mathcal{W}^u(\mathcal{O}_q)$  are also described, using discrete Lyapunov functionals around  $\xi_{-1}$ ,  $0$ ,  $\xi_1$ , the Poincaré–Bendixson theorem, information on the unstable sets of the fixed points of the Poincaré maps and elementary topological arguments, among others. Characterizing the global attractor is of key importance as this is the subset of the phase space  $C$  that determines the asymptotic behavior of all solutions in  $C_{-2,2}$ . This work is done only for a small class of infinite dimensional systems [14].

Chapter 5 turns to the negative feedback case. A locally Lipschitz continuous map  $f$  with  $xf(x) < 0$  for  $x \in \mathbb{R} \setminus \{0\}$  is constructed for all  $\mu > 0$  (see Theorem 5.1.1) such that Eq. (1.1) has an infinite number of periodic orbits. All periodic solutions defining these orbits oscillate slowly around  $0$  in the sense that their sign changes are spaced at distances larger than delay  $1$ . Moreover, if  $f$  is continuously differentiable, then the periodic orbits are hyperbolic and stable. In this example,  $f$  is "close" to the odd step function  $f^*$  set so that

$$f^*(x) = \begin{cases} 0 & \text{for } x \in [0, 1], \\ Kr^n & \text{for } n \geq 0 \text{ and } x \in (r^n, r^{n+1}], \end{cases}$$

where  $K$  and  $r$  are chosen to be large. Based on this property, an infinite sequence of contracting Poincaré return maps is given. Their fixed points are the initial segments of the periodic solutions. The construction can be easily modified to give a locally Lipschitz continuous map  $f$  such that Eq. (1.1) has a two-sided infinite sequence of slowly oscillatory periodic orbits.

In Chapter 6 a nonsmooth and not strictly monotone nonlinearity is considered. Motivated by papers [11, 17], the more general equation

$$\dot{x}(t) = -\mu x(t) + af(x(t)) + bf(x(t-1)) + I$$

is investigated, where  $\mu > 0$ ,  $a \in \mathbb{R}$ ,  $b > 0$ ,  $I \in \mathbb{R}$  and  $f$  is the piecewise linear Hopfield activation function

$$f : \mathbb{R} \ni x \mapsto \frac{1}{2} (|x+1| - |x-1|) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1. \end{cases}$$

Based on numerical studies, Györi and Hartung [11] conjectured that for  $b > 0$ , all solutions tend to an equilibrium as  $t \rightarrow \infty$ . Theorem 6.3.1 and Theorem 6.3.2 analyze

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the truth of the conjecture for those choices of parameters that were not covered in [11]. In particular, it is shown that although most of the solutions are convergent, there exists a slowly oscillatory periodic orbit for certain choices of parameters. In the course of the proof, one has to overcome the difficulty that the solution operator is neither injective nor differentiable everywhere. The key step is to project the unstable set of the unstable equilibrium to the two-dimensional plane together with its boundary. A Poincaré return map is defined on the plane, and its fixed point yields the initial segment of the periodic solution. The analysis uses the generalizations of results in [32] for the discrete Lyapunov functional counting sign changes.

The dissertation is based on two papers of the author and on one paper with co-author Tibor Krisztin. These publications are the following:

- Krisztin, T., Vas, G., Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback, submitted to *Journal of Dynamics and Differential Equations*.
- Vas, G., Asymptotic constancy and periodicity for a single neuron model with delay, *Nonlinear Anal.* **71** (2009), no. 5-6, 2268–2277.
- Vas, G., Infinite number of stable periodic solutions for an equation with negative feedback, *E. J. Qualitative Theory of Diff. Equ.*, **18** (2011), 1–20.



## 8 Összefoglalás

A disszertáció témáját az

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \quad (1.1)$$

alakú skaláris funkcionál-differenciálegyenletek képezik  $\mu > 0$  paraméter és különböző nemlineáris  $f$  visszacsatolási függvények esetén. Folytonosan differenciálható és nem-sima, monoton növvő és monoton csökkenő nemlinearitásokat is tekintünk. Célunk a periodikus pályák létezésének kimutatása és a globális attraktor lehető legrészletesebb leírása  $f$ -re tett széles körű feltételek mellett.

Ilyen egyenletek mesterséges neuronhálózatok tanulmányozásánál fordulnak elő. A disszertáció bevezetője felsorol néhány, e munkát motiváló példát. Wu részletesebb leírást ad az alkalmazásokról [50]-ben.

A 2. fejezet áttekintést nyújt a dolgozat elméleti háttéréről. Miután ismertetjük a terület alapvető fogalmait, a 2.2. szakasz összefoglalja a monoton nemlinearitásokra vonatkozó eddigi legfontosabb eredményeket. Krisztin, Walther és Wu jellemezte a megoldások aszimptotikus viselkedését pozitív visszacsatolás esetén, azaz amikor  $f$  folytonos és  $xf(x) > 0$  minden nullától különböző valós  $x$ -re [20, 22, 23, 25, 26, 27]. Negatív visszacsatolás esetén ( $f$  folytonos és  $xf(x) < 0$  minden  $x \in \mathbb{R} \setminus \{0\}$ -ra) elsősorban Walther és Yebdri [45]-[49] munkáit érdemes kiemelni. E disszertáció az ő eredményeikre, illetve Győri és Hartung [11] publikációjára épül. A 2.3. szakasz a később használt legfontosabb analitikai eszközöket taglalja. Mallet-Paret és Sell [32]-ben bevezettek egy diszkrét Ljapunov-függvényt, amely a  $C$  állapotter elemeinek előjelváltásait számlálja. Habár eredményeik többsége közvetlenül nem alkalmazható ebben a dolgozatban, tételeik egyszerű általánosításai hatékony eszköznek bizonyulnak a globális dinamika megértésében. A Poincaré-féle visszatérési leképezések is lényeges szerepet játszanak, mivel fixpontjaik adják a periodikus megoldások kezdeti szegmenseit. A Floquet-elmélet értelmében a Poincaré-leképezés fixpontban vett deriváltjának spektruma határozza meg a társított periodikus pálya stabilitását. Alkalmazzuk Lani-Wayda [28] munkáját is, amely igazolja, hogy a visszacsatolási függvények kis perturbációi megőrzik a periodikus pályákat.

A 3. és 4. fejezet pozitív visszacsatolás esetén vizsgálja az (1.1) egyenletet. Ezekben a fejezetekben egy olyan folytonosan differenciálható, szigorúan monoton növvő nemlinearitást tekintünk, amelyre a  $\xi \mapsto -\mu\xi + f(\xi)$  függvénynek öt egymást követő

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

zérushelye van, a  $\xi_{-2}$ ,  $\xi_0$ ,  $\xi_2$  zérushelyek által meghatározott egyensúlyi helyzetek stabilak, és a  $\xi_{-1}$ ,  $\xi_1$  zérushelyek által definiált  $\hat{\xi}_{-1}$ ,  $\hat{\xi}_1$  egyensúlyi helyzetek instabilak. Mivel  $f$  monoton, ezért a  $C = C([-1, 0], \mathbb{R})$  állapotter

$$C_{-2,0} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq 0 \text{ minden } s \in [-1, 0] \text{-re}\},$$

$$C_{0,2} = \{\varphi \in C : 0 \leq \varphi(s) \leq \xi_2 \text{ minden } s \in [-1, 0] \text{-re}\}$$

részalmazai pozitívan invariánsak. Jelölje  $\mathcal{A}_{-2,0}$  és  $\mathcal{A}_{0,2}$  a szemidinamikai rendszer  $C_{-2,0}$ -ra és  $C_{0,2}$ -re vett megszorításainak globális attraktorait (ezek léteznek). Krisztin, Walther és Wu eredményei szerint az  $\mathcal{A}_{-2,0}$  és  $\mathcal{A}_{0,2}$  halmazoknak orsó-szerű struktúrájuk van. A kérdés, hogy a szemidinamikai rendszer

$$C_{-2,2} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq \xi_2 \text{ minden } s \in [-1, 0] \text{ esetén}\}$$

halmazra vett megszorításának  $\mathcal{A}$  globális attraktora előáll-e  $\mathcal{A}_{-2,0}$  és  $\mathcal{A}_{0,2}$  uniójaként, már [26]-ban felmerült.

A 3. fejezetben olvasható 3.1.1. tétel igazolja, hogy  $\mathcal{A}$  szerkezete összetettebb is lehet: megadunk egy olyan folytonosan differenciálható, szigorúan monoton növekvő  $f$  nemlineáris függvényt, amelyre az  $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$  halmazban van pontosan 2 periodikus pálya,  $\mathcal{O}_p$  és  $\mathcal{O}_q$ . A periodikus pályákat definiáló  $p$  és  $q$  megoldások ún. LSOP megoldások: nagy az amplitúdójuk abban az értelemben, hogy  $p(\mathbb{R}) \supsetneq (\xi_{-1}, \xi_1)$  és  $q(\mathbb{R}) \supsetneq (\xi_{-1}, \xi_1)$ , valamint lassan oszcillálnak abban az értelemben, hogy minden 1 hosszú intervallumon egy vagy két előjelváltásuk van. Ilyen periodikus megoldások nem keletkeznek lokális bifurkáció révén, így létezésük igazolása kihívást jelentő feladat. A 3.1.1. tételben szereplő  $f$  függvény "közel" van az

$$f^{K,0}(x) = \begin{cases} 0 & \text{ha } |x| \leq 1, \\ K \operatorname{sgn}(x) & \text{ha } |x| > 1 \end{cases}$$

lépcsős függvényhez, ahol  $K > 0$  paraméter. A bizonyítás első lépéseként explicit periodikus megoldásokat adunk meg abban az esetben, amikor  $\mu = 1$  és  $f = f^{K,0}$ . Ez a probléma véges dimenziós, így kezelhető. Majd az implicitfüggvény-tétel és Poincaré-leképezések perturbációinak segítségével megmutatjuk, hogy az (1.1) egyenletnek pontosan két LSOP pályája van, ha  $\mu = 1$  és  $f$  függvény az  $f^{7,0}$ -hoz közeli nemlinearitás.

A 4. fejezet a megoldások szerkezetét vizsgálja a 3.1.1. tétel teljesülése esetén. A 4.1.1. tétel szerint  $f$  választható úgy, hogy a 3.1.1. tétel igaz legyen, és az  $\mathcal{A}$  globális attraktorra teljesüljön az

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q)$$

egyenlet, ahol  $\mathcal{W}^u(\mathcal{O}_p)$  és  $\mathcal{W}^u(\mathcal{O}_q)$  rendre az  $\mathcal{O}_p$  és  $\mathcal{O}_q$  LSOP pályák instabil halmazait jelölik. A 4. fejezet a  $\mathcal{W}^u(\mathcal{O}_p)$  és  $\mathcal{W}^u(\mathcal{O}_q)$  instabil halmazokat is leírja, többek között a  $\xi_{-1}$ ,  $0$ ,  $\xi_1$  körül vett diszkrét Ljapunov-függvények, a Poincaré–Bendixson-tétel, a Poincaré-leképezés fixpontjának instabil halmazával kapcsolatos ismereteink és elemi topológiai érvelések segítségével. A globális attraktor jellemzése azért kulcsfontosságú feladat, mivel ez a  $C$  fázistér azon részhalmaza, amely  $C_{-2,2}$  összes megoldásának aszimptotikus viselkedését meghatározza. Ilyen eredmények csupán a végtelen dimenziós dinamikai rendszerek egy szűk osztályára léteznek.

Az 5. fejezet áttér a negatív visszacsatolás esetére. Minden  $\mu > 0$ -hoz konstruálunk egy olyan lokálisan Lipschitz-folytonos  $f$  függvényt, amelyre  $xf(x) < 0$  teljesül minden  $0$ -tól különböző valós  $x$  esetén, és amelyre az (1.1) egyenletnek végtelen számú periodikus pályája van. A periodikus pályákat definiáló megoldások mindegyike lassan oszcillál  $0$  körül abban az értelemben, hogy a szomszédos előjelváltásaik távolsága nagyobb a késleltetésnél, azaz  $1$ -nél. Ha  $f$  folytonosan differenciálható, akkor a periodikus pályák stabilak és hiperbolikusak. Ebben a példában  $f$  "közel" van az  $f^*$  páratlan lépcsős függvényhez, ahol

$$f^*(x) = \begin{cases} 0 & \text{ha } x \in [0, 1], \\ Kr^n & \text{ha } n \geq 0 \text{ és } x \in (r^n, r^{n+1}], \end{cases}$$

és ahol  $K$ ,  $r$  nagy konstansok. Erre a tulajdonságra építve kontraktív Poincaré-leképezések végtelen sorozatát adjuk meg, amelyek fixpontjai a periodikus megoldások kezdeti szegmensei. A konstrukció egyszerű módosításával megadhatunk egy olyan lokálisan Lipschitz-folytonos  $f$  leképezést, amelyre létezik lassan oszcilláló periodikus pályák két irányban végtelen sorozata.

A 6. fejezetben egy nemsima és nem szigorúan monoton visszacsatolási függvényt tekintünk. A [11, 17] cikkek indíttatására az általánosabb

$$\dot{x}(t) = -\mu x(t) + af(x(t)) + bf(x(t-1)) + I$$

egyenletet tekintjük, ahol  $\mu > 0$ ,  $a \in \mathbb{R}$ ,  $b > 0$ ,  $I \in \mathbb{R}$  és  $f$  a szakaszonként lineáris

$$f : \mathbb{R} \ni x \mapsto \frac{1}{2} (|x+1| - |x-1|) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1 \end{cases}$$

Hopfield-féle aktiválási függvény. Györi és Hartung numerikus eredményeikre alapozva azt a sejtést fogalmazták meg, hogy  $b > 0$  esetén minden megoldás egyensúlyi helyzethez tart, ha  $t \rightarrow \infty$ . A 6.3.1. és a 6.3.3. tételek a sejtés igazságtartalmát vizsgálják azokra a

paraméterválasztásokra, amelyeket [11] nem fed le. Megmutatjuk, hogy bár a megoldások többsége konvergens, bizonyos paraméterválasztások esetén létezik lassan oszcilláló periodikus pálya. A bizonyítás során le kell küzdenünk azt a nehézséget, hogy a megoldásoperátor nem injektív és nem mindenhol differenciálható. A bizonyítás kulcslépéseként levetítjük az instabil egyensúlyi helyzet instabil halmazát a lezártjával együtt a 2 dimenziós síkra. Poincaré-féle visszatérési leképezést definiálunk a síkon, amelynek fixpontja adja a periodikus megoldás kezdeti szegmensét. Vizsgálódásunk során a diszkrét Ljapunov-függvényre vonatkozó, [32] -ben igazolt eredmények általánosításait használjuk.

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- Vas, G., Infinite number of stable periodic solutions for an equation with negative feedback, *E. J. Qualitative Theory of Diff. Equ.*, **18** (2011), 1–20.

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