ABSOLUTE CONVERGENCE OF DOUBLE TRIGONOMETRIC FOURIER SERIES AND WALSH-FOURIER SERIES

SUMMARY OF THE PhD THESES

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Our theses are based on the classical results of Bernstein and Zygmund giving sufficient conditions for the absolute convergence of the Fourier series of a complex valued function with period 2π. Namely, if the function $f(x)$ satisfies either a Lipschitz condition of order α where $\alpha > 1/2$, or is of bounded variation and satisfies a Lipschitz condition of order α where $\alpha > 0$, then its Fourier series converges absolutely (see for example in [19]).

Many generalizations of these theorems were proved, for example by Szász [12], Salem [9] and the latest one by Gogoladze and Meskhia [4]. In the first part of our theses we extend these results from single to double Fourier series.

In the second part of our theses we study the absolute convergence of double Walsh-Fourier series. Inspired by the results of F. Móricz [5] on the absolute convergence of single Walsh-Fourier series, we give sufficient conditions for the absolute convergence of double Walsh-Fourier series in terms of (either global or local) dyadic moduli of continuity and bounded s-fluctuation of f.

New results on double Fourier series

Let $f = f(x, y)$ be a complex-valued function periodic with period 2π in each variable. We recall that if $f \in L^1(\mathbb{T}^2)$, where $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$, then the double Fourier series of f is given by

$$f(x, y) \sim \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \hat{f}(m, n)e^{i(mx+ny)}, \quad (x, y) \in \mathbb{T}^2,$$

where the Fourier coefficients $\hat{f}(m, n)$ are defined by

$$\hat{f}(m, n) := \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f(x, y)e^{-i(mx+ny)}dxdy, \quad (m, n) \in \mathbb{Z}^2.$$

We will introduce the notion of moduli of continuity for functions of two variables. To this end, we use the notation

(1)
$$\Delta_{1,1}(f; x, y; h_1, h_2) :=$$
$$f(x + h_1, y + h_2) - f(x, y + h_2) - f(x + h_1, y) + f(x, y)$$

where $(x, y) \in \mathbb{T}^2$ and $h_1, h_2 > 0$. The *integral modulus of continuity* (in the norm of L^p) of a function $f \in L^p(\mathbb{T}^2)$ for some $1 \le p < \infty$ is defined by

$$\omega(f; \delta_1, \delta_2)_p := \sup\Bigg\{ \left(\frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} |\Delta_{1,1} f; x, y; h_1, h_2)|^p dxdy \right)^{1/p},$$
$$0 < h_1 \le \delta_1 \quad \text{and} \quad 0 < h_2 \le \delta_2 \Bigg\}, \ \delta_1, \delta_2 > 0.$$

In the case when $f \in C(\mathbb{T}^2)$, the *modulus of continuity* of f is defined analogously:

$$\omega(f; \delta_1, \delta_2) := \sup\{|\Delta_{1,1}(f; x, y; h_1, h_2)| :$$

$$(x,y)\in\mathbb{T}^2,\ 0<h_1\le\delta_1 \text{ and } 0<h_2\le\delta_2\},\ \delta_1,\delta_2>0.$$

Analogously to the definition for a single sequence $\{\gamma_m\}$ to be in $\mathfrak{A}_\alpha$ (see [4]), we say that a double sequence $\gamma=\{\gamma_{mn}:(m,n)\in\mathbb{N}_+^2\}$ of nonnegative numbers belongs to the class $\mathfrak{A}_\alpha$ for some $\alpha\ge 1$ if the inequality

$$(2)\qquad \left(\sum_{m\in D_\mu}\sum_{n\in D_\nu}\gamma_{mn}^\alpha\right)^{1/\alpha}\le \kappa\, 2^{(\mu+\nu)(1-\alpha)/\alpha}\sum_{m\in D_{\mu-1}}\sum_{n\in D_{\nu-1}}\gamma_{mn}$$

is satisfied for all $\mu,\nu\ge 0$, where

$$(3)\qquad D_0:=\{1\},\quad D_\mu:=\{2^{\mu-1}+1,2^{\mu-1}+2,\dots,2^\mu\},\quad \mu\in\mathbb{N}_+,$$

and we agree to put

$$(4)\qquad D_{-1}:=D_0=\{1\}.$$

For convenience in writing, we agree to put

$$(5)\qquad \gamma_{-m,n}=\gamma_{m,-n}=\gamma_{-m,-n}:=\gamma_{mn},\quad (m,n)\in\mathbb{N}_+^2.$$

For more details see [4] and [13].

Theorem 1. *Suppose $f\in L^p(\mathbb{T}^2)$ for some $1<p\le 2$. If*

$$\gamma=\{\gamma_{mn}\}\in\mathfrak{A}_{p/(p-rp+r)}\ \textit{for some}\ r\in(0,q),$$

where $1/p+1/q=1$, then

$$(6)\qquad \sum(\gamma;f)_r:=\sum_{|m|\ge1}\sum_{|n|\ge1}\gamma_{mn}|\hat f(m,n)|^r\le$$

$$\le\kappa\, C\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}2^{-(\mu+\nu)r/q}\,\Gamma_{\mu-1,\nu-1}\,\omega^r\left(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)_p,$$

where κ is from (2) corresponding to $\alpha:=p/(p-rp+r)$,

$$(7)\qquad \Gamma_{\mu\nu}:=\sum_{m\in D_\mu}\sum_{n\in D_\nu}\gamma_{mn}\ \textit{for}\ \mu,\nu\ge -1,$$

with the agreement (4) that

$$(8)\qquad \Gamma_{-1,\nu}:=\Gamma_{0\nu},\ \Gamma_{\mu,-1}:=\Gamma_{\mu0}\ \textit{for}\ \mu,\nu\ge0,\ \textit{and}\ \Gamma_{-1,-1}:=\Gamma_{00}=\{\gamma_{11}\}.$$

Corollary 1. *Under the conditions of Theorem* 1, *we have*

$$\sum (\gamma; f)_r \le \kappa\, C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{-r/q}\, \gamma_{mn}\, \omega^r \left(f; \frac{\pi}{m}, \frac{\pi}{n}\right)_p . \tag{9}$$

We recall that a function $f \in C(\mathbb{T}^2)$ is said to belong to the *Lipschitz class* $\operatorname{Lip}(\alpha_1, \alpha_2)$ for some $\alpha_1, \alpha_2 > 0$ if

$$\omega(f; \delta_1, \delta_2) = O(\delta_1^{\alpha_1} \delta_2^{\alpha_2}). \tag{10}$$

It is worth formulating Theorem 1 in the particular case when $f \in \operatorname{Lip}(\alpha_1, \alpha_2)$ and $\gamma_{mn} \equiv 1$ or $\gamma \equiv m^{\beta_1} n^{\beta_2}$ and $r = 1$.

Corollary 2. *Suppose* $f \in \operatorname{Lip}(\alpha_1, \alpha_2)$ *for some* $\alpha_1, \alpha_2 > 0$ *and* $1 < p \le 2$. *If*

$$\frac{q}{1 + q \min\{\alpha_1, \alpha_2\}} < r < q,$$

then

$$\sum_{|m| \ge 1} \sum_{|n| \ge 1} |\hat{f}(m, n)|^r < \infty. \tag{11}$$

Corollary 3. *Suppose* $f \in \operatorname{Lip}(\alpha_1, \alpha_2)$ *for some* $\alpha_1, \alpha_2 > 0$ *and* $1 < p \le 2$. *If* $\beta_1, \beta_2 \in \mathbb{R}$ *are such that*

$$\beta_j < \alpha_j - \frac{1}{p}, \quad j = 1, 2,$$

then

$$\sum_{|m| \ge 1} \sum_{|m| \ge 1} m^{\beta_1} n^{\beta_2} |\hat{f}(m, n)| < \infty. \tag{12}$$

Now, Theorem 1 and Corollary 1 were proved in [7] in the case when $\lambda_{mn} \equiv 1$, $p = 2$ and $r = 1$. In particular, in this case the double series on the right-hand side of (6) is convergent if

$$f \in \operatorname{Lip}(\alpha_1, \alpha_2) \quad \text{for some} \quad \alpha_1, \alpha_2 > 1/2.$$

It is worth observing that Theorem 1 and Corollary 1 remain valid if the modulus of continuity is replaced by the modulus of smoothness in them.

Next, we recall the notion of bounded s-variation (in the sense of Vitali) for functions of two variables, where $s \in \mathbb{R}_+$. We consider an arbitrary partition $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ of the closed square $\overline{\mathbb{T}}^2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$\mathcal{P}_1 : -\pi = x_0 < x_1 < x_2 < \cdots < x_m = \pi,$$

$$\mathcal{P}_2 : -\pi = y_0 < y_1 < y_2 < \cdots < y_n = \pi.$$

A function $f = f(x, y)$ periodic in each variable with period 2π is said to be of *bounded s-variation*, in symbols: $f \in BV_s(\overline{\mathbb{T}}^2)$, if

(13)
$$V_s(f) := \sup \sum_{k=1}^{m} \sum_{\ell=1}^{n} |f(x_k, y_\ell) - f(x_{k-1}, y_\ell) -$$
$$- f(x_k, y_{\ell-1}) + f(x_{k-1}, y_{\ell-1})|^s < \infty,$$

where the supremum is extended for all partitions $\mathcal{P}$ of $\overline{\mathbb{T}}^2$. In the case when $s = 1$, see this definition for example in [3].

If a function $f \in BV_s(R)$ is such that the marginal functions $f(\cdot, c)$ and $f(a, \cdot)$ are of bounded variation over the intervals $[a, b]$ and $[c, d]$, respectively, then f is said to be of bounded variation over R in the sense of Hardy and Krause, in symbols: $f \in BV_H^s(R)$. (See also [3].)

Theorem 2. *Suppose* $f \in C(\mathbb{T}^2) \cap BV_s(\overline{\mathbb{T}}^2)$ *for some* $s \in (0, 2)$*. If*

$$\gamma = \{\gamma_{mn}\} \in \mathfrak{A}_{2/(2-r)} \textit{ for some } r \in (0, 2),$$

then

(14)
$$\sum(\gamma; f)_r := \sum_{|m|\geq 1} \sum_{|n|\geq 1} \gamma_{mn} |\hat{f}(m, n)|^r \leq$$
$$\leq \kappa\, C\, V_s^{r/2}(f) \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{-(\mu+\nu)r}\, \Gamma_{\mu-1,\nu-1}\, \omega^{(2-s)r/2} \left(f; \frac{\pi}{2^\mu}, \frac{\pi}{2^\nu}\right),$$

where κ *is from* (2) *corresponding to* $\alpha := 2/(2 - r)$*,* $V_s(f)$ *is defined in* (13)*, and* $\Gamma_{\mu\nu}$ *is defined in* (7) *and* (8)*.*

Corollary 4. *Under the conditions of Theorem* 2*, we have*

(15)
$$\sum(\gamma; f)_r \leq \kappa\, C\, V_s^{r/2}(f) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{-r}\, \gamma_{mn}\, \omega^{(2-s)r/2} \left(f; \frac{\pi}{m}, \frac{\pi}{n}\right).$$

We formulate Theorem 2 in the particular case when the function $f \in BV_s(\overline{\mathbb{T}}^2) \cap \mathrm{Lip}(\alpha_1, \alpha_2)$ and $\gamma \equiv 1$ or $r = 1$ and $\gamma_{mn} = m^{\beta_1} n^{\beta_2}$, where $\beta_1, \beta_2 \in \mathbb{R}$.

Corollary 5. *Suppose* $f \in \mathrm{Lip}(\alpha_1, \alpha_2) \cap BV_s(\overline{\mathbb{T}}^2)$ *for some* $\alpha_1, \alpha_2 > 0$ *and* $0 < s < 2$*. If*

$$r > \frac{1}{1 + (1 - s/2) \min\{\alpha_1, \alpha_2\}},$$

then (11) *is satisfied.*

Corollary 6. *Suppose* $f \in \mathrm{Lip}(\alpha_1, \alpha_2) \cap BV_s(\overline{\mathbb{T}}^2)$ *for some* $\alpha_1, \alpha_2 > 0$ *and* $0 < s < 2$. *If*

$$\beta_j < (1 - s/2)\alpha_j, \quad j = 1, 2,$$

then (12) *is satisfied.*

Theorem 2 and Corollary 4 were proved in [6] in the case when $\gamma_{mn} \equiv 1,\ s = 1$ and $r = 1$ and in [14] in the case when $\gamma_{mn} \equiv 1$ and $s = 1$. More generally, in this particular case the double series on the right-hand side of (15) is convergent if

$$\omega\left(f; \frac{\pi}{m}, \frac{\pi}{n}\right) = O\left(\left(\log \frac{\pi}{m}\right)^{\beta_1} \left(\log \frac{\pi}{n}\right)^{\beta_2}\right) \quad \text{for some} \quad \beta_1, \beta_2 > \frac{2}{2-s}.$$

The following characterization is proved in [1]: a function f is absolutely continuous on $R := [a,b] \times [c,d]$ (see, for example, [1]) if and only if there exist functions $g \in AC([a,b])$, $h \in AC([c,d])$ and $\phi \in L^1(R)$ such that

$$f(x,y) = g(x) + h(y) + \int_a^x \int_c^y \phi(u,v)\,du\,dv, \quad x \in [a,b],\ y \in [c,d]. \tag{16}$$

Now, in the special case when $\gamma_{mn} \equiv 1$ and $r = 1$ Theorem 2 yields the following

Theorem 3. *If* $f \in AC(\overline{\mathbb{T}}^2)$ *and* $f_{x,y} \in L^p(\mathbb{T}^2)$ *for some* $p > 1$, *then*

$$\sum_{|m| \geq 1} \sum_{|n| \geq 1} |\hat{f}(m,n)| \leq C_p V_2^{1/2}(f) \|f_{x,y}\|_{L^p}^{1/2}. \tag{17}$$

Combining Theorems 1, 2 and 3 and the theorems for single Fourier series (see [4]), we can easily find sufficient conditions for the convergence of the double series

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma_{mn} |\hat{f}(m,n)|^r. \tag{18}$$

As an illustration, it is worth to present some simple corollaries in the particular case when $\gamma_{mn} \equiv 1$ and $r = 1$, these conditions ensure the absolute convergence of the double Fourier series of f.

We note that the majority of the results were extended from double to multiple trigonometric Fourier series (see [6] and [7] for details).

New results on the absolute convergence of double Fourier series

To the convergence of the series in (18) in the special case when $\gamma_{mn} \equiv 1$ and $r = 1$ let

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\hat{f}(m,n)| =$$

$$\sum_{|m|\ge 1}\sum_{|n|\ge 1}|\hat f(m,n)| + \sum_{n\in\mathbb{Z}}|\hat f(0,n)| + \sum_{m\in\mathbb{Z}}|\hat f(m,0)| - |\hat f(0,0)|.$$

In the special case when $n=0$ or $m=0$, we may write that

$$\hat f(m,0) = \hat f_1(m), \quad \text{where} \quad f_1(x) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)dy,\ x\in\mathbb{T}; \tag{19}$$

and

$$\hat f(0,n) = \hat f_2(n), \quad \text{where} \quad f_2(x) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)dx,\ y\in\mathbb{T}. \tag{20}$$

Combining the famous theorems of Bernstein and Zygmund for the absolute convergence of single Fourier series (see for example in [2], [15], [17], [18]) with Theorem 1, 2 and 3 yield the following corollaries.

Corollary 7. *If a function $f:\mathbb{T}^2\to\mathbb{R}$ is such that $f\in \mathrm{Lip}(\alpha_1,\alpha_2)$, $f_1\in\mathrm{Lip}(\alpha_3)$ and $f_2\in\mathrm{Lip}(\alpha_4)$ for some $\alpha_j>1/2$, $j=1,2,3,4$; where f_1 and f_2 are defined in* (19) *and* (20)*, then the double Fourier series of f converges absolutely.*

Corollary 8. *If a funtion $f:\mathbb{T}^2\to\mathbb{R}$ is such that $f\in\mathrm{Lip}(\alpha_1,\alpha_2)\cap BV_H^s(\overline{\mathbb{T}}^2)$, $f_1\in\mathrm{Lip}(\alpha_3)\cap BV_{s_1}(\overline{\mathbb{T}})$ and $f_2\in\mathrm{Lip}(\alpha_4)\cap BV_{s_2}(\overline{\mathbb{T}})$ for some $\alpha_j>0$, $j=1,2,3,4$ and $0<s,s_1,s_2<2$; where f_1 and f_2 are defined in* (19) *and* (20)*, then the double Fourier series of f converges absolutely.*

Corollary 9. *If $f\in AC(\overline{\mathbb{T}}^2)$, $f_{x,y}\in L^p(\mathbb{T}^2)$, $g'\in L^{p_1}(\mathbb{T})$ and $h'\in L^{p_2}(\mathbb{T})$ for some $p,p_1,p_2>1$, where the functions g and h occur in* (16)*, then the double Fourier series of f converges absolutely.*

New results on double Walsh-Fourier series

Given a function $f:\mathbb{I}^2\to\mathbb{R}$, integrable in Lebesgue's sense on the unit square $\mathbb{I}^2=[0,1)\times[0,1)$, in symbols: $f\in L^1(\mathbb{I}^2)$, its *double Walsh-Fourier series* is defined by

$$f(x,y)\sim\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\hat f(m,n)w_m(x)w_n(y), \tag{21}$$

where the

$$\hat f(m,n) := \int_0^1\int_0^1 f(x,y)w_m(x)w_n(y)\,dx\,dy, \quad m,n\in\mathbb{N}, \tag{22}$$

are the *Walsh-Fourier coefficients* of f and $w_m(x)$ is the mth Walsh-function.

Denote by $C_W(\mathbb{I}^2)$ the collection of W-continuous functions on $\mathbb{I}^2$, where the dyadic topology on $\mathbb{I}^2$ is generated by *dyadic rectangles*

$$(23)\qquad I(k,m;l,n) := I(k,m) \times I(l,n)$$

$$= [k2^{-m},(k+1)2^{-m}) \times [l2^{-n},(l+1)2^{-n}),$$

$$0 \le k < 2^m,\ 0 \le l < 2^n \quad \text{and} \quad k,l,m,n \in \mathbb{N}.$$

The (global) *dyadic modulus of continuity* of a function $f \in C_W(\mathbb{I}^2)$ is defined by

$$\omega(f;\delta_1,\delta_2) := \sup\{|\Delta_{1,1}f(x,y;h_1,h_2)| : (x,y) \in \mathbb{I}^2,$$

$$0 \le h_j < \delta_j, j = 1,2\}, \quad 0 < \delta_j \le 1.$$

We recall that the difference operator $\Delta_{1,1}$ are defined in the usual way:

$$(24)\qquad \Delta_{1,1}f(x,y;h_1,h_2) := |f(x \dot{+} h_1, y \dot{+} h_2) - f(x, y \dot{+} h_2)$$

$$-f(x \dot{+} h_1, y) + f(x,y).$$

Furthermore, the *dyadic L^p-modulus of continuity* of a function $f \in L^p(\mathbb{I})$ for some $1 \le p < \infty$ is defined by

$$\omega(f;\delta_1,\delta_2)_p := \sup\left\{\left(\int_0^1\int_0^1 |\Delta_{1,1}f(x,y;h_1,h_2)|^p\,dx\,dy\right)^{1/p} :\right.$$

$$\left. 0 \le h_j < \delta_j,\ j = 1,2\right\}.$$

For $\alpha_1,\alpha_2 > 0$, the *dyadic Lipschitz class* $\mathrm{Lip}(\alpha_1,\alpha_2;W)$ is the collection of those functions $f \in C_W(\mathbb{I}^2)$ which satisfy the inequality

$$\omega(f;\delta_1,\delta_2) \le C\delta_1^{\alpha_1}\delta_2^{\alpha_2}, \quad 0 < \delta_1,\delta_2 \le 1,$$

where C is a constant which depends only on f. Analogously, for $\alpha_1,\alpha_2 > 0$ and $1 \le p < \infty$, we denote by $\mathrm{Lip}(\alpha_1,\alpha_2;L^p)$ the collection of functions $f \in L^p(\mathbb{I}^2)$ which satisfy the inequality

$$\omega(f;\delta_1,\delta_2)_p \le C\delta_1^{\alpha_1}\delta_2^{\alpha_2}, \quad 0 < \delta_1,\delta_2 \le 1.$$

For each dyadic rectangle

$$I(k,m;l,n) := I(k,m) \times I(l,n) =: I \times J,$$

where $0 \le k < 2^m$, $0 \le l < 2^n$; $k,l,m,n \in \mathbb{N}$ (see (23)), the *local dyadic modulus of continuity* of a function $f \in C_W(\mathbb{I}^2)$ is defined by

$$\omega(f; I \times J) := \sup\{|\Delta_{1,1}f(x,y;h_1,h_2)| : (x,y) \in I \times J,$$

$$0 \le h_1 < |I|,\ 0 \le h_2 < |J|\},$$

where $|I| = 2^{-m}$ and $|J| = 2^{-n}$ are the length of the intervals I and J, respectively. Morever, for each $1 \le p < \infty$, the *local dyadic L^p-modulus of continuity* of a function $f \in L^p(\mathbb{I}^2)$ is defined by

$$\omega(f; \delta_1, \delta_2)_p := \sup \left\{ \left(\frac{1}{|I| \cdot |J|} \int_I \int_J |\Delta_{1,1} f(x, y; h_1, h_2)|^p \, dx \, dy \right)^{1/p} : \right.$$

$$\left. 0 \le h_1 < |I|,\ 0 \le h_2 < |J| \right\}.$$

Finally, we say that a function $f : \mathbb{I}^2 \to \mathbb{R}$ is of *s-bounded fluctuation* for some $0 < s < \infty$, in symbols: $f \in BF_s(\mathbb{I}^2)$, if

$$Fl_s(f; \mathbb{I}^2) := \sup_{m \ge 1} \sup_{n \ge 1} \left(\sum_{k=0}^{2^m - 1} \sum_{l=0}^{2^n - 1} |\omega(f; I(k, m) \times I(l, n))|^s \right)^{1/s} < \infty; \tag{25}$$

Theorem 4. *Suppose $f \in L^p(\mathbb{I}^2)$ for some $1 < p \le 2$. If*

$$\{\gamma_{mn}\} \in \mathfrak{A}_{p/(p-rp+r)} \ \text{ for some } \ 0 < r < q, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{26}$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} |\hat{f}(m, n)|^r \tag{27}$$

$$\le 4^{-r} \kappa \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{-(\mu+\nu)r/q} \Gamma_{\mu-1, \nu-1} |\omega(f; 2^{-\mu}, 2^{-\nu})_p|^r,$$

where κ is from (2) *correspondings to $\alpha := p/(p - rp + r)$ and $\Gamma_{\mu,\nu}$ is defined in* (7) *and* (8)*.*

It is worth formulating Theorem 4 in the particular case when $f \in \mathrm{Lip}(\alpha_1, \alpha_2; W)_p$ and $\gamma_{mn} \equiv 1$ or $\gamma_{mn} = m^{\beta_1} n^{\beta_2}$ and $r = 1$.

Corollary 10. *Suppose $f \in \mathrm{Lip}(\alpha_1, \alpha_2; W)_p$ for some $\alpha_1, \alpha_2 > 0$ and $1 < p \le 2$. If*

$$\frac{q}{1 + q \min\{\alpha_1, \alpha_2\}} < r < q,$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\hat{f}(m, n)|^r < \infty. \tag{28}$$

Corollary 11. *Suppose $f \in \mathrm{Lip}(\alpha_1, \alpha_2; W)_p$ for some $\alpha_1, \alpha_2 > 0$ and $1 < p \le 2$. If $\beta_1, \beta_2 \in \mathbb{R}$ are such that*

$$\beta_j < \alpha_j - \frac{1}{p}, \quad j = 1, 2,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\beta_1} n^{\beta_2} |\hat{f}(m,n)| < \infty. \tag{29}$$

Theorem 5. *Suppose $f \in L^p(\mathbb{I}^2)$ for some $1 < p \leq 2$. If $\{\gamma_{mn} \geq 0\}$ satisfies condition* (26), *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn} |\hat{f}(m,n)|^r \tag{30}$$

$$\leq 4^{-r}\kappa \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} 2^{-(\mu+\nu)r}\Gamma_{\mu-1,\nu-1}\left(\sum_{k=0}^{2^\mu-1}\sum_{l=0}^{2^\nu-1} |\omega(f;I(k,\mu;l,\nu))_p|^p\right)^{1/p},$$

where κ is from (2) *corresponding to $\alpha := p/(p-rp+r)$, $\Gamma_{\mu,\nu}$ is defined in* (7) *and* (8), *$I(k,\mu;l,\nu)$ is defined in* (23).

Theorem 6. *Suppose $f \in C_W \cap BF_s(\mathbb{I}^2)$ for some $0 < s < 2$. If*

$$\{a_{mn} \geq 0\} \in \mathfrak{A}_{2/(2-r)} \quad \textit{for some} \quad 0 < r < 2, \tag{31}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn} |\hat{f}(m,n)|^r \tag{32}$$

$$\leq 4^{-r}\kappa |Fl_s(f;\mathbb{I}^2)|^{rs/2} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} 2^{-(\mu+\nu)r}\Gamma_{\mu-1,\nu-1}|\omega(f;2^{-\mu},2^{-\nu})|^{(1-s/2)r},$$

where κ is from (2) *corresponding to $\alpha := 2/(2-r)$, $Fl_s(f)$ is defined in* (25) *and $\Gamma_{\mu\nu}$ is defined in* (7).

Similarly to Theorem 4 we can formulate Theorem 6 in the particular case when $\gamma_{mn} \equiv 1$ or $r = 1$ and $\gamma_{mn} = m^{\beta_1} n^{\beta_2}$,

Corollary 12. *Suppose $f \in \mathrm{Lip}(\alpha_1,\alpha_2;W) \cap BF_s(\mathbb{I}^2)$ for some $\alpha_1, \alpha_2 > 0$ and $0 < s < 2$. If*

$$r > \frac{1}{1 + (1-s/2)\min\{\alpha_1,\alpha_2\}},$$

then (28) *is satisfied.*

Corollary 13. *Suppose $f \in \mathrm{Lip}(\alpha_1,\alpha_2;W) \cap BF_s(\mathbb{I}^2)$ for some $\alpha_1, \alpha_2 > 0$ and $0 < s < 2$. If*

$$\beta_j < (1-s/2)\alpha_j, \quad j = 1, 2,$$

then (29) *is satisfied.*

Combining Theorems 4 and 6 with the theorems for absolute convergence of single Walsh-Fourier series (see [10]), we can easily find sufficient conditions for the convergence of the double series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{mn}|\hat{f}(m,n)|^r. \tag{33}$$

As an illustration, it is worth to present some simple corollaries in the particular case when $\gamma_{mn} \equiv 1$ and $r = 1$, these conditions ensure the absolute convergence of the double Walsh-Fourier series of f.

Further references to this part are [8].

New results on the absolute convergence of double Walsh-Fourier series

To the convergence of the series in (33) in the special case when $\gamma_{mn} \equiv 1$ and $r = 1$ let

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\hat{f}(m,n)| =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\hat{f}(m,n)| + \sum_{n=0}^{\infty}|\hat{f}(0,n)| + \sum_{m=0}^{\infty}|\hat{f}(m,0)| - |\hat{f}(0,0)|.$$

In the special case when $n = 0$ or $m = 0$, we may write that

$$\hat{f}(m,0) = \hat{f}_1(m), \quad \text{where} \quad f_1(x) := \int_0^1 f(x,y)\,dy, \quad x \in \mathbb{I}; \tag{34}$$

and

$$\hat{f}(0,n) = \hat{f}_2(n), \quad \text{where} \quad f_2(y) := \int_0^1 f(x,y)\,dx, \quad y \in \mathbb{I}. \tag{35}$$

Combining the dyadic analogue of the famous theorems of Bernstein and Zygmund for the absolute convergence of single Walsh-Fourier series (see [10]) with Theorem 4 and 6 yield the following corollaries.

Corollary 14. *If a function $f : \mathbb{I}^2 \to \mathbb{R}$ is such that $f \in \mathrm{Lip}(\alpha_1, \alpha_2; W)$, $f_1 \in \mathrm{Lip}(\alpha_3; W)$ and $f_2 \in \mathrm{Lip}(\alpha_4; W)$ for some $\alpha_j > 1/2$, $j = 1, 2, 3, 4$; where f_1 and f_2 are defined in* (34) *and* (35), *then the Walsh-Fourier series of f converges absolutely.*

Corollary 15. *If a funtion $f : \mathbb{I}^2 \to \mathbb{R}$ is such that $f \in \mathrm{Lip}(\alpha_1, \alpha_2; W) \cap BF_s(\mathbb{I}^2)$, $f_1 \in \mathrm{Lip}(\alpha_3; W) \cap BF_{s_1}(\mathbb{I})$ and $f_2 \in \mathrm{Lip}(\alpha_4; W) \cap BF_{s_2}(\mathbb{I})$ for some $\alpha_j > 0$, $j = 1, 2, 3, 4$ and $0 < s, s_1, s_2 < 2$, where f_1 and f_2 are defined in* (34) *and* (35), *then the Walsh-Fourier series of f converges absolutely.*

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