Chapter 1

Introduction

My dissertion contains the main results of my research activity carried out during the time of my PhD studies at the University of Szeged. It includes three papers of mine which have been published. These works are related to the following two different fields of mathematics:

1- complete polynomial vector fields

2- geodesics on warped product manifolds.

In accordance with our papers, the dissertation is divided into three main chapters:

1) In **Chapter 2** we shall describe complete polynomial vector fields on a finite-dimensional simplex $S := (x_1+x_2+\cdots+x_n = 1)$ with an application to differential equations in genetical dynamic systems,
2) **Chapter 3** deals with the complete polynomial vector fields on the Euclidean unit ball $B := (x_1^2 + \ldots + x_n^2 < 1)$,
3) **Chapter 4** is devoted to the geometry of the central symmetric warped product structures on $\mathbb{R}_0^N \times \mathbb{R}$.

In **Chapter 2** we are going to describe the complete polynomial vector fields and their fixed points in a finite-dimensional simplex. We apply the results to differential equations of genetical evolution models.

There are several well-known models in literature [4], [5], [2] on the time evolution of a closed population consisting of $\mathbb{N}$ different species - with the whole population at time $t \geq 0$ as the solution of a system of ordinary differential equations $\frac{d}{dt}v_k(t) = F_k(v_1(t), v_2(t), \ldots, r_N(t))$ $(k = 1, 2, \ldots, N)$ where the functions F_k are some polynomials of at most 3-rd. degree. During a seminar on such models one has raised the problem what are the strange consequences of the assumption that the evolution has no starting point in time, in particular what can be stated on non-changing distribution in that case. In this chapter we provide the complete algebraic description of all polynomial vector fields (with arbitrary degrees), $V(x) = (F_1(x), F_2(x), \ldots, F_N(x))$ on $\mathbb{R}^N$ which give rise to solutions for the evolution equation defined for all time parameters $t \in \mathbb{R}$, and satisfying the natural rate conditions $r_1(t)$, $r_2(t)$, $r_3(t)$, $\ldots$, $r_N(t) \geq 0$; $\sum_{k=1}^N r_k(t) = 1$ whenever $r_1(0)$, $r_2(0)$, $\ldots$, $r_N(0) \geq 0$ and $\sum\limits_{k=1}^{N} r_k(0) = 1$. On the basis of the explicit formulas obtained we describe the structure of the set of zeros for such vector fields which corresponded to the non-changing distribution.

In **Chapter 3** we are going to describe the complete polynomial vector fields in the unit ball $B := (x_1^2 + x_2^2 + \cdots + x_N^2 < 1)$ of $\mathbb{R}^N$. This work originates from a nice parametric formula due to L.L. Stachó [3] for the complete real polynomial vector fields on the unit disc $\mathbb{K}$ of the space $\mathbb{C}$ of complex numbers. He has shown that a real polynomial vector field $p : \mathbb{C} \to \mathbb{C}$ is complete in $\mathbb{K}$ iff p is a finite real linear combination formed by the functions iz, $\gamma\overline{z}^m - \overline{z}z^{m+2}$, $(z \in \mathbb{C},\ m = 0, 1, \ldots)$ and $(1 - |z|^2)Q$ where Q is any real polynomial from $\mathbb{C}$ to $\mathbb{C}$. Our result in this chapter establishes that $p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a complete polynomial vector field in the unit ball B if and only if $p(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$ for some polynomials $Q, R : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. This theorem not only generalizes the result of [3] on $\mathbb{K}$, but it even simplifies it by showing that the complete polynomial vector fields on the unit disc of $\mathbb{C}$ have the form $[ip(z)z + q(z)(1 - |z|^2)]$ where $p, q : \mathbb{C} \to \mathbb{R}$ are any real polynomials.

In **Chapter 4** we shall study the geometry of the central symmetric warped product manifold structures on $\mathbb{R}_0^N \times \mathbb{R}^1$ where $\mathbb{R}_0^N = \mathbb{R}^N \backslash \{0\}$, which correspond to the potential functions $a\|x\|$, $a \geq 0$, and equipped with the Riemannian scalar product $\langle \cdot, \cdot \rangle$ defined by the following properties:

i) the projection onto $\mathbb{R}^N$ along $\mathbb{R}^1$ of this Riemannian scalar $\langle \cdot, \cdot \rangle$ is canonical Euclidean,

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}^N$ with respect to $\langle\cdot,\cdot\rangle$,

iii) the projection onto $\mathbb{R}^1$ along $\mathbb{R}^N$ of $\langle\cdot,\cdot\rangle$ at $(a,\alpha)\in\mathbb{R}_0^N\times\mathbb{R}^1$ is the canonical one multiplied by $U(|a|^2)$, where $U:\mathbb{R}_+\to\mathbb{R}_+$ is smooth.

Notice that these properties determine uniquely the scalar product of the vectors (X,ξ), $Y(\eta)\in T_{(a,\alpha)}(\mathbb{R}_0^N\times\mathbb{R}^1)$ and it can be written in the form

$$\langle (X,\xi)\cdot(Y,\eta)\rangle = \langle X,Y\rangle + U(|a|^2)\cdot\xi\cdot\eta.$$

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Chapter 2

Complete polynomial vector fields in a simplex

Throughout the whole work $\mathrm{I\!R}^N := \{(\xi_1, \ldots, \xi_N) : \xi_1, \ldots, \xi_N \in \mathrm{I\!R}\}$ denotes the vector space of all real N-tuples. We reserve the notations $x_1, \ldots, x_N$ for the standard coordinate functions $x_k : (\xi_1, \ldots, \xi_N) \to \xi_k$ on $\mathrm{I\!R}^N$. Also we reserve the notation S for the unit simplex

$$S := \big(x_1 + \cdots + x_N = 1, \quad x_1, \ldots, x_N \geq 0\big) =$$
$$= \{p \in \mathrm{I\!R}^N : \ x_1(p) + \cdots + x_N(p) = 1, \quad x_1(p), \ldots, x_N(p) \geq 0\} \ .$$

Recall [6] that by a vector field on S we simply mean a function $S \to \mathrm{I\!R}^N$. A function $\varphi : S \to \mathrm{I\!R}$ is said to be polynomial if it is the restriction of some polynomial of the linear coordinate functions $x_1, \ldots, , x_N$: for some finite system of coefficients $\alpha_{k_1 \ldots k_N} \in \mathrm{I\!R}$ with $k_1, \ldots, k_N \in \{0, 1, \ldots\}$) we can write $\varphi(p) = \sum_{k_1, \ldots, k_N} \alpha_{k_1 \ldots k_N} x_1^{k_1} \cdots x_N^{k_N}$ $(p \in S)$. In accordance with this terminology, a vector field V on S is a polynomial vector field if its components $V_k := x_k \circ V$ (that is $V(p) = (V_1(p), \ldots, V_N(p))$ for $p \in S$) are polynomial functions. It is elementary that given two polynomials $P_m = P_m(x_1, \ldots, x_N) : \mathrm{I\!R}^N \to \mathrm{I\!R}$ $(m = 1, 2)$, their restrictions to S coincide if and only if the difference $P_1 - P_2$ vanishes on the affine subspace $A_S := \big(x_1 + \cdots + x_N = 1\big)$ generated by S. We shall see later that a polynomial $P = P(x_1, \ldots, x_N)$ vanishes on the affine subspace $M := \big(\gamma_1 x_1 + \cdots + \gamma_N x_N = \delta\big)$ iff $P = (\gamma_1 x_1 + \cdots + \gamma_N x_N - \delta) Q(x_1, \ldots, x_N)$ for some polynomial Q. Thus polynomial vector fields on S admit several polynomial extensions to $\mathrm{I\!R}^N$ but any two such extensions differ only by a vector field of the form $(x_1 + \cdots + x_N - 1)W$.

Definition. A locally Lipschitzian (e.g. polynomial) vector field $V : \mathrm{I\!R}^N \to \mathrm{I\!R}^N$ is said to be *complete* in a (non-empty) subset $K \subset \mathrm{I\!R}^N$ if for any point $p \in K$ there is a (necessarily unique) curve $C_p : \mathrm{I\!R} \to K$ such that $C_p(0) = p$ and $\frac{d}{dt} C_p(t) = V(C_p(t))$ $(t \in \mathrm{I\!R})$.

Our purpose will be to describe the complete polynomial vector fields on the simplex S and we apply the results to differential equations of genetical evolutions models.

Well-known genetical models [4], [5], [2] of the time evolution of a closed population consisting of N different species describe the rates $r_1(t), r_2(t), \ldots, r_N(t)$ of the respective species within the whole population at time $t \geq 0$ as the solution of a system of ordinary differential equations $dr_k(t)/dt = F_k(r_1(t), \ldots, r_N(t), \ (k = 1, \ldots, N)$ where the functions F_k are some polynomials of degree at most 3. During a seminar on such models held at the Bolyai Institute one has asked the following question. What are the consequences of the assumption that the evolution has no starting point in time, in particular what can be stated on the non-changing distribution in that case.

In this chapter we provide the complete algebraic discription of all polynomial vector fields (with arbitrary degrees) $v(x) = (F_1(x), F_2(x), \ldots, F_N(x))$ on $\mathbb{R}^N$, which give rise to solutions for all the evolution equation defined for all time parameters $t \in \mathbb{R}$ and satisfying the natural rate conditions $v_1(t), \ldots, v_N(t) \geq 0$, $\sum_k^N r_k(t) = 1$, whenever $v_1(0), \ldots, v_N(0) \geq 0$ and $\sum_{k=1}^N r_k(0) = 1$. Observe that the vector fields v described above are complete polynomial vector fields of third degree in the simplex. On the basis of the explicit formulas obtained, we describe the structure of the set of zeros for such vector fields (which correspond to the non-changing distributions).

Our main results are as follows.

2.2. Theorem. *A polynomial vector field $V : S \to \mathbb{R}^N$ is complete in S if and only if with the vector fields*

$$Z_k := x_k \sum_{j=1}^{N} x_j (e_j - e_k) \qquad (k = 1, \ldots, N)$$

where e_j *is the standard unit vector* $e_j := (0,\ldots,0,\overbrace{1}^{j},0,\ldots,0)$, *we have*

$$V = \sum_{k=1}^{N} P_k(x_1,\ldots,x_N) Z_k$$

for some polynomial functions $P_1,\ldots,P_N : \mathbb{R}^N \to \mathbb{R}$.

2.3. Theorem. *Given a complete polynomial vector field* V *of* S, *there are polynomials* $\delta_1,\ldots,\delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ *of degree less than that of* V *such that the vector field*

$$\widetilde{V} := \sum_{k=1}^{N-1} x_k \Big[\delta_k(x_1,\ldots,x_{N-1}) - \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1,\ldots,x_{N-1})\Big] e_k +$$
$$+ (x_1+\cdots+x_{N-1}-1) \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1,\ldots,x_{N-1}) e_N$$

coincides with V *on* S. *The points of the zeros of* V *inside the facial subsimplices* $S_K := S \cap (x_1,\ldots,x_K > 0 = x_{K+1} = \cdots = x_N)$ $(K=1,\ldots,N)$ *can be described as*

$$(*)\qquad \begin{aligned} S_N \cap (V=0) &= S \cap \bigcup_{k=1}^{N-1} \big(\delta_k(x_1,\ldots,x_{N-1}) = 0\big), \\ S_K \cap (V=0) &= S_K \cap \big(\delta_1(x_1,\ldots,x_{N-1}) = \cdots = \\ &\qquad = \delta_K(x_1,\ldots,x_{N-1})\big) \qquad (K<N). \end{aligned}$$

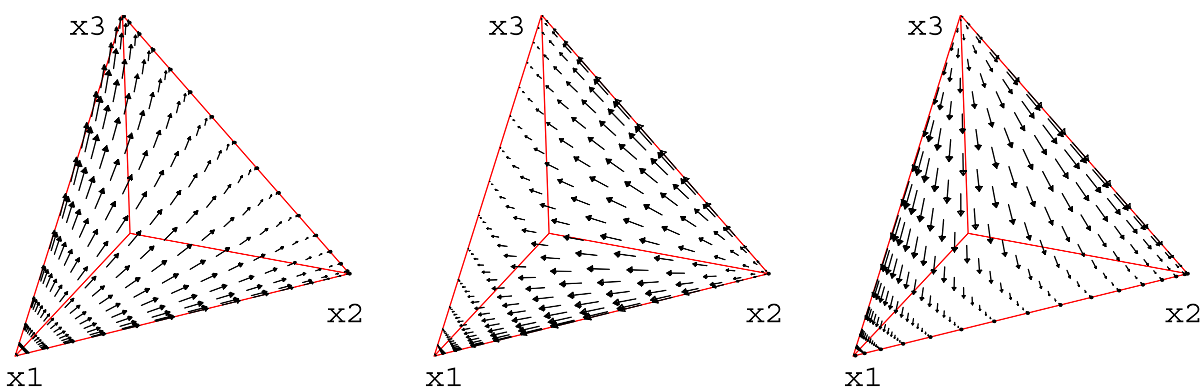

Fig1. The fundamental vector fields Z_1, Z_2, Z_3 in the case $N=3$.

Finally we turn back to our motivativation, the genetical time evolution equation for the distribution of species within a closed population. Namely in [7] we have the system

$$(V) \qquad \begin{aligned} \frac{d}{dt}x_k = & \Big(\sum_{i=1}^{N} g(i)x_i - g(k)\Big)x_k + \\ & + \sum_{i,j=1}^{N} w(i,j)x_i x_j \Big[\sum_{\ell=1}^{N} M(i,j,\ell)\varepsilon(i,j,\ell,k) - x_k\Big] \end{aligned}$$

for describing the behaviour of the rates $x_1(t),\ldots,x_N(t)$ at time t of the N species of the population. Here the terms $g(k), M(i,j,\ell)$ and $\varepsilon(i,j,\ell,k)$ are non-negative constants with $\sum_{\ell=1}^{N} M(i,j,\ell) = \sum_{k=1}^{N} \varepsilon(i,j,\ell,k) = 1$. Observe that this can be written as

$$\frac{d}{dt}x = \sum_{k=1}^{N} g(k)Z_k + W$$

with the vector fields

$$\begin{aligned} Z_k &:= x_k \sum_{j=1}^{N} x_j(e_j - e_k), \\ W &:= \sum_{i,j,k=1}^{N} w(i,j)x_i x_j \Big[\sum_{\ell=1}^{N} M(i,j,\ell)\varepsilon(i,j,\ell,k) - x_k\Big] e_k, \end{aligned}$$

respectively. As a consequence of Theorems 2.1 and 2.2 we obtain the following.

2.4. Theorem. *Let $N \geq 3$. Then the time evolution of the population can be retrospected up to any time $t \leq 0$ starting with any distribution $(x(0),\ldots,x_N(0)) \in S$ if and only if the term W vanishes on S, that is if simply $d/dt\ x = \sum_{k=1}^{N} g(k)Z_k(x_1,\ldots,x_N)$. In this case the set of the stable distributions has the form*

$$\bigcup_{\gamma\in\{g(1),\ldots,g(N)\}} S \cap (x_m = 0 \text{ for } m \notin J_\gamma) \qquad \text{where } J_\gamma := \{m:\ g(m) = \gamma\}\ .$$

2.5. Corollary. *If* $g(1),\dots,g(N)\ge 0$ *and the vector field* (V) *is complete in* S *then*

$$\frac{d}{dt}\sum_{k=1}^{N} g(k)x_k(t)\ge 0$$

for any solution $t\mapsto x(t)\in S$ *of the evolution equation* $dx/dt=V(x)$*.*

Proof of Theorem 2.1

As in the previous section, we keep fixed the notations $e_1,\dots,e_N$, $x_1,\dots,x_N$, S for the standard unit vectors, coordinate functionals and unit simplex in $\mathbb{R}^N$, and $V:\mathbb{R}^N\to\mathbb{R}^N$ is an arbitrarily fixed polynomial vector field. We write $\langle u,v\rangle := \sum_{k=1}^N x_k(u)x_k(v)$ for the usual scalar product in $\mathbb{R}^N$.

According to [9, (2,2)], V is complete in S if and only if

$$V(p)\in T_p(S):=\{v\in\mathbb{R}^N:\ \exists\, c:\mathbb{R}\to S\ ,\quad c(0)=p\ ,\quad \frac{d}{dt}\Big|_{t=0}c(t)=v\}$$

for all $p\in S$. By writing

$$\overline{e}:=\frac{1}{N}\sum_{k=1}^{N}e_k,\quad u_k:=e_k-\overline{e},\quad S_k:=S\cap(x_k=0)\qquad(k=1,\dots,N)$$

for the center, the vectors connecting the vertices with the center and the maximal faces of S, it is elementary that

$$\begin{aligned}
&T_p(S)=\big\{v:\ \langle v,\overline{e}\rangle=0\big\}\quad \text{if } p\in S\setminus \textstyle\bigcup_{k=1}^N S_k),\\
&T_p(S)=\big\{v:\ \langle v,\overline{e}\rangle=\langle v,u_k\rangle=0\ \ (k\in K_p)\big\}\\
&\qquad\qquad\qquad\text{if } p\in \textstyle\bigcup_{k=1}^N S_k \text{ and } K_p:=\{k:\ p\in S_k\}
\end{aligned}$$

for any non-empty subset K of $\{1,\dots,N\}$. Since the vector field V is polynomial by assumption, it follows that

V is complete in S $\iff$

$$\langle V(p),\overline{e}\rangle=0\quad(p\in S)\quad\text{and}\quad\langle V(p),u_m\rangle=0\quad(p\in S_m,\ m=1,\dots,N).$$

Let us write

$$L_S:=(x_1+\cdots+x_N=1),\quad L_{S_m}:=L_S\cap(x_m=0)\quad(m=1,\dots,N)$$

for the hyperplane supporting S, and for the affine submanifolds generated by the faces S_m, respectively. Since $e_k = u_k + \overline{e}$ and since polynomials vanishing on a convex set vanish also on its supporting affine submanifold, equivalently we can say

V is complete in $S \iff$

$\langle V(p), \overline{e}\rangle = 0$ for $p \in L_S$ and $\langle V(p), e_m\rangle = 0$ for $p \in L_{S_m}$ $\quad (m = 1, \ldots, N)$.

If $P_1, \ldots, P_N : \mathbb{R}^N \to \mathbb{R}$ are polynomial functions then, with the vector fields $Z_k := x_k \sum_{j=1}^N x_j(e_j - e_k)$ $(k = 1, \ldots, N)$, we have

$$\begin{aligned}
&\left\langle \sum_{k=1}^N P_k(p) Z_k(p), \overline{e} \right\rangle = \\
&= \sum_{k=1}^N P_k(p) \langle Z_k(p), \overline{e}\rangle = \\
&= \sum_{k=1}^N P_k(p) \left\langle Z_k(p), \frac{1}{N} \sum_{\ell=1}^N e_\ell \right\rangle = \\
&= \frac{1}{N} \sum_{k=1}^N P_k(p) \sum_{j,\ell=1}^N \langle x_k(p) x_j(p)(e_j - e_k), e_\ell\rangle = \\
&= \frac{1}{N} \sum_{k=1}^N P_k(p) x_k(p) \sum_{j:\ j \neq k} \sum_{\ell = j,k} x_j(p) \langle e_j - e_k, e_\ell\rangle = \\
&= \frac{1}{N} \sum_{k=1}^N P_k(p) x_k(p) \sum_{j:\ j \neq k} x_j(p) \sum_{\ell = j,k} \langle e_j - e_k, e_\ell\rangle = \\
&= \frac{1}{N} \sum_{k=1}^N P_k(p) x_k(p) \sum_{j:\ j \neq k} x_j(p) [1 - 1] = 0
\end{aligned}$$

for any point $p \in \mathbb{R}^N$ (not only for $p \in S$). On the other hand, if $p \in S_m$

then $x_m(p)=0$ and

$$\begin{aligned}
&\left\langle \sum_{k=1}^N P_k(p) Z_k(p), e_m \right\rangle = \\
&= \sum_{k=1}^N P_k(p) \left\langle Z_k(p), e_m \right\rangle = \\
&= \sum_{k=1}^N P_k(p) x_k(p) \sum_{j:\ j\neq k} x_j(p) \langle e_j - e_k, e_m\rangle = \\
&= \sum_{k:\ k\neq m} P_k(p) x_k(p) \sum_{j:\ j\neq k,m} x_j(p) \langle e_j - e_k, e_m\rangle = 0.
\end{aligned}$$

This means that the vector fields of the form $V := \sum_{k=1}^N P_k Z_k$ with arbitrary polynomials $P_1,\dots,P_N$ are complete in S, moreover $\langle V(p), \overline{e}\rangle = 0$ for all $p\in \mathrm{I\!R}^N$.

To prove the remaining part of the theorem, we need the following lemma.

2.6. Lemma. *If $P:\mathrm{I\!R}^N\to\mathrm{I\!R}$ is a polynomial function and $0\neq\phi:\mathrm{I\!R}^K\to\mathrm{I\!R}$ is an affine function* such that $P(q)=0$ for the points q of the hyperplane $\{q\in\mathrm{I\!R}^N:\ \phi(q)=0\}$ then ϕ is a divisor of P in the sense that $P=\phi Q$ with some (unique) polynomial $Q:\mathrm{I\!R}^N\to\mathrm{I\!R}$.*

Proof. Trivially, any two hyperplanes are affine images of each other. In particular there is a one-to-one affine (i.e linear + constant) mapping $A:\mathrm{I\!R}^N\leftrightarrow\mathrm{I\!R}^N$ such that $\{q\in\mathrm{I\!R}^N:\ \phi(p)=0\} = A\big(\{q\in\mathrm{I\!R}^N:\ x_1(q)=0\}\big)$. Then $R:=P\circ A$ is a polynomial function such that $R(q)=0$ for the points of the hyperplane $\{q\in\mathrm{I\!R}^N:\ x_1(q)=0\}$. We can write $R=\sum_{k_1,\dots,k_N=0}^d \alpha_{k_1,\dots,k_N} x_1^{k_1}\cdots x_N^{k_N}$ with a suitable finite family of coefficients $\alpha_{k_1,\dots,k_N}$. By the Taylor formula, $\alpha_{k_1,\dots,k_N} = \frac{\partial^{k_1+\cdots+k_N}}{\partial x_1^{k_1}\cdots\partial x_N^{k_N}}\Big|_{x_1=\cdots=x_N=0} R$. It follows that $\alpha_{k_1,\dots,k_N}=0$ for $k_1>0$, since R vanishes for $x_1=0$. This means that $R=x_1R_0$ with the polynomial $R_0:=\sum_{k_1=1}^d\sum_{k_2,\dots,k_N=0}^d x_1^{k_1-1}x_2^{k_2}\cdots x_N^{k_N}$. By the same argument

* That is ϕ is the sum of a linear functional with a constant.

applied for the polynomial function ϕ of degree $d = 1$ in place of R, we see that $\phi \circ A = \alpha x_1$ for some constant (polynomial of degree 0) $\alpha \neq 0$. That is $\phi = \alpha x_1 \circ A^{-1}$. Therefore

$$P = R \circ A^{-1} = [x_1 R_0] \circ A^{-1} = (x_1 \circ A^{-1})(R_0 \circ A^{-1}) = \phi \cdot (\frac{1}{\alpha} R_0 \circ A^{-1}).$$

Since the inverse of an affine mapping is affine as well, the function $Q := \frac{1}{\alpha} R_0 \circ A^{-1}$ is a polynomial which suits the statement of the lemma. $\square$

2.7. Corollary. *A polynomial vector field $\widetilde{V} : \mathbb{R}^N \to \mathbb{R}^N$ coincides with V on S iff it has the form $\widetilde{V} = V + (x_1 + \cdots + x_N - 1)W$ with some polynomial vector field $W : \mathbb{R}^N \to \mathbb{R}^N$.*

Proof. Observe that $\widetilde{D}$ and V coincide on S iff they coincide on the hyperplane L_S supporting S. We can write $\widetilde{V} = \sum_{k=1}^N \widetilde{P}_k e_k$ resp. $V = \sum_{k=1}^N P_k e_k$ with some scalar valued polynomials $\widetilde{P}_k$ resp. P_k and, by the lemma, we have $\widetilde{P}_k - P_k = 0$ on L_S iff $\widetilde{P}_k - P_k = (x_1 + \cdots + x_N - 1)Q_k$ with some polynomial $Q_k : \mathbb{R}^N \to \mathbb{R}$ $(k = 1, \ldots, N)$, that is if $\widetilde{V} - V = (x_1 + \cdots + x_N - 1)W$ with the vector field $W := \sum_{k=1}^N Q_k e_k$. $\square$

Instead of the generic polynomial vector field V complete in S, it is more convenient to study another $\widetilde{V}$ coinciding with V on S but having additional properties. As in the proof of the previous corollary, we decompose V as $V = \sum_k P_k e_k$. Recall that $V(p) \in T_p(S) \subset \{v : \langle v, \overline{e} \rangle = 0\}$ for the points $p \in S$. In terms of the component functions P_k, this means that $\frac{1}{N} \sum_{k=1}^N P_k = 0$ that is $P_N = -\sum_{k:\ k \neq N} P_k$ on S. On the other hand, $x_1 + \cdots + x_N = 0$ that is $x_N = -\sum_{k:\ k \neq N}$ on S. Introduce the vector field

$$\widetilde{V} := \sum_{k=1}^N \widetilde{P}_k e_k$$

where

$$\begin{aligned}\widetilde{P_k} &:= \pi_k(x_1,\dots,x_{N-1}) := \\ &:= P_k(x_1,x_2,\dots,x_{N-1},1-x_1-\dots-x_{N-1}) \quad (k<N),\\ \widetilde{P}_N &:= \pi_N(x_1,\dots,x_{N-1}) := -\sum_{k=1}^{N-1}\widetilde{P}_k = -\sum_{k=1}^{N-1}\pi_k(x_1,\dots,x_{N-1}).\end{aligned}$$

By its construction, $\widetilde{V}$ coincides with V on S, it is a polynomial of the same degree as V but only in the variables $x_1,\dots,x_{N-1}$ and it has the property $\sum_{k=1}^{N}\widetilde{P}_k = 0$ on the whole space $\mathbb{R}^N$. The relations $\widetilde{V}(p) = V(p) \in T_p(S) \subset \{v: \ \langle v, e_k\rangle = 0\}$ for $p \in S_k$ $(k = 1,\dots,N)$ mean

$$\begin{aligned}&\widetilde{P}_k(p) = \langle\widetilde{V}(p), e_k\rangle = 0\\ &\qquad \text{for} \quad p \in S_k = (x_k = 0,\ x_1+\dots+x_N = 1,\ x_1,\dots,x_N \geq 0).\end{aligned}$$

In terms of the polynomials π_k of $N-1$ variables this can be stated as

$$(**)\qquad \begin{aligned}&\pi_k(\xi_1,\dots,\xi_{N-1}) = 0 \quad \text{whenever} \ \ \xi_k = 0 \qquad (k=1,\dots,N-1) \quad \text{and}\\ &-\sum_{k=1}^{N-1}\pi_k(\xi_1,\dots,\xi_{N-1})\big[= \pi_N(\xi_1,\dots,\xi_{N-1})\big] = 0\\ &\ \text{whenever} \ \ \xi_1+\dots+\xi_{N-1} = 1.\end{aligned}$$

By the lemma (applied with $N-1$ instead of N), the first $N-1$ equations are equivalent to

$$\pi_k(\xi_1,\dots,\xi_{N-1}) = \xi_k\varrho_k(\xi_1,\dots,\xi_{N-1}) \qquad (k=1,\dots,N-1)$$

with some polynomials $\varrho_k : \mathbb{R}^{N-1} \to \mathbb{R}$ with degree less than the degree of π_k and V. Also by the lemma (with $N-1$ instead of N), the last equation can be interpreted as

$$\begin{aligned}-\sum_{k=1}^{N-1}\pi_k(\xi_1,\dots,\xi_{N-1}) &= \pi_N(\xi_1,\dots,\xi_{N-1}) =\\ &= [1-(\xi_1+\dots+\xi_{N-1})]\varrho_N(\xi_1,\dots,\xi_{N-1})\end{aligned}$$

with some polynomial $\varrho_N : \mathbb{R}^{N-1} \to \mathbb{R}$ of degree less than that of V. Thus

$$
\begin{aligned}
-\sum_{k=1}^{N-1} \xi_k \varrho_k(\xi_1, \dots, \xi_{N-1}) &= [1 - (\xi_1 + \dots + \xi_{N-1})]\varrho_N(\xi_1, \dots, \xi_{N-1}), \\
\sum_{k=1}^{N-1} \xi_k[\varrho_N - \varrho_k](\xi_1, \dots, \xi_{N-1}) &= \varrho_N(\xi_1, \dots, \xi_{N-1}).
\end{aligned}
$$

By introducing the polynomials $\delta_k := \varrho_k - \varrho_N$ $(k = 1, \dots, N-1)$ of $N-1$ variables, we can reformulate the relationships $(**)$ as

$$
\begin{aligned}
\pi_k &= \xi_k \varrho_k = \xi_k(\delta_k + \varrho_N) \qquad (k = 1, \dots, N-1), \\
\pi_N &= (1 - \xi_1 - \dots - \xi_N)\varrho_N, \\
\varrho_N &= -\xi_1\delta_1 - \dots - \xi_{N-1}\delta_{N-1}
\end{aligned}
$$

which is the same as

$$
(***) \quad
\begin{aligned}
&\pi_k(\xi_1, \dots, \xi_{N-1}) = \\
&\qquad = \xi_k \Big[\delta_k(\xi_1, \dots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \dots, \xi_{N-1})\Big] \quad \text{for } k \neq N, \\
&\pi_N = (\xi_1 + \dots + \xi_N - 1) \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \dots, \xi_{N-1})
\end{aligned}
$$

where $\delta_1, \dots, \delta_{N-1}$ are arbitrary polynomials of the variables $\xi_1, \dots, \xi_{N-1}$.

Summarizing the arguments, we have obtained the following result.

2.8. Proposition. *Let $V = \sum_{k=1}^{N} P_k e_k$ be a vector field where $P_1, \dots, P_N : \mathbb{R}^N \to \mathbb{R}$ are polynomials of the coordinate functions $x_1, \dots, x_N$. Then V is complete in the simplex $S := (x_1 + \dots + x_N = 1, x_1, \dots, x_N \geq 0)$ if and only if there exist polynomials $\delta_1, \dots, \delta_{N-1}$ of $N-1$ variables and degree less than that of V such that the vector field $\widetilde{V} := \sum_{k=1}^{N} \pi_k(x_1, \dots, x_{N-1})e_k$, where the polynomials π_k are given by $(***)$ in terms of $\delta_1, \dots, \delta_{N-1}$, coincides with V on the hyperplane $L_S := (x_1 + \dots + x_N = 1)$.*

On the basis of the proposition we can finish the proof of Theorem 2.1 as follows. Let V be a polynomial vector field complete in S. By the proposition, we can find a vector field $\widetilde{V}$ of the form (*) coinciding with V on S such that $\delta_1, \ldots, \delta_{N-1} : \mathbb{R}^{N-1} \to \mathbb{R}$ are polynomials. It suffices to show that the vector field

$$\widehat{V} := -\sum_{k=1}^{N-1} \delta(x_1, \ldots, x_{N-1}) Z_k(x_1, \ldots, x_N) =$$
$$= \sum_{k=1}^{N-1} \delta(x_1, \ldots, x_{N-1}) \sum_{\ell=1}^{N} x_k x_\ell (e_k - e_\ell)$$

coincides with $\widetilde{V}$ on S. Consider any point $p \in S$ and let $\xi_k := x_k(p)$ $(k = 1, \ldots, N)$. Since $\xi_N = 1 - \xi_1 - \cdots - \xi_{N-1}$, it is straightforward to check that indeed

$$\widetilde{V}(p) - \widehat{V}(p) = \sum_{k=1}^{N-1} \xi_k \Big[\delta_k(\xi_1, \ldots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1}) \Big] e_k +$$
$$+ (\xi_1 + \cdots + \xi_{N-1} - 1) \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1}) e_N +$$
$$+ \sum_{k=1}^{N-1} \delta(\xi_1, \ldots, \xi_{N-1}) \sum_{\ell=1}^{N} \xi_k \xi_\ell (e_k - e_\ell) = 0 \,. \square$$

Proof of Theorem 2.2

According to Proposition 3.3, we can take a vector field $\widetilde{V}$ of the form $(*)$ coinciding with V on S where $\delta_1, \ldots, \delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ are polynomials of degree less than that of V. Consider a point $p := (\xi_1, \ldots, \xi_N) \in S$. Necessarily $\xi_N = 1 - \xi_1 - \cdots - \xi_{N-1} \geq 0$ and $\xi_1, \ldots, \xi_{N-1} \geq 0$. We have $V(p) = 0$ iff

$$\xi_k \Big[\delta_k(\xi_1, \ldots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1})\Big] = 0 \qquad (k = 1, \ldots, N-1).$$

Assume these equations hold with $\xi_1, \ldots, \xi_N > 0$, that is $p \in S_N$. Then

$$\delta_1(\xi_1, \ldots, \xi_{N-1}) = \cdots = \delta_{N-1}(\xi_1, \ldots, \xi_{N-1}) = \\ = \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1}).$$

However, by writing δ for the common value of the $\delta_k(\xi_1, \ldots, \xi_{N-1})$, we have $\delta = \sum_{\ell=1}^{N-1} \xi_\ell \delta$, that is $\xi_N \delta = \big(1 - \sum_{\ell=1}^{N-1} \xi_\ell\big)\delta = 0$ and $\delta = 0$.

Assume finally that $K < N$ and $\xi_1, \ldots, \xi_K > 0 = \xi_{K+1} = \cdots = \xi_N$, that is $p \in S_K$. Then $V(p) = 0$ iff

$$\delta_k(\xi_1, \ldots, \xi_K, 0, \ldots, 0) = \\ = \sum_{\ell=1}^{K} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_K, 0, \ldots, 0) \qquad (k = 1, \ldots, K).$$

Again the $\delta_k(\xi_1, \ldots, \xi_K, 0, \ldots, 0)$ assume a common value δ. However, in this case $\sum_{\ell=1}^{K} \xi_\ell = 1$ and hence δ may be arbitrary for $V(p) = 0$. $\square$

Proof of Theorem 2.3

First we check that $\sum_{k=1}^{N} \big(\sum_{i=1}^{N} g(i) x_i - g(k)\big) x_k e_k = \sum_{k=1}^{N} g(k) Z_k$ on S. Indeed, given any index m, from the fact that $\sum_{i=1}^{N} x_i = 1$ on S, it follows

$$
\begin{aligned}
&\Big\langle \sum_{k=1}^{N} g(k) Z_k \ , \ e_m \Big\rangle = \\
&= \sum_{k=1}^{N} g(k) \langle Z_k, e_m \rangle = \\
&= \sum_{k=1}^{N} g(k) \big\langle x_k \sum_{i=1}^{N} x_i (e_i - e_k), e_m \big\rangle = \\
&= \sum_{i,k=1}^{N} g(k) x_k x_i \langle e_i, e_m \rangle - \sum_{i,k=1}^{N} g(k) x_k x_i \langle e_k, e_m \rangle = \\
&= \sum_{k=1}^{N} g(k) x_k x_m - g(m) x_m \sum_{i=1}^{N} x_i = \\
&= \big(\sum_{i=1}^{N} g(i) x_i - g(m) \big) x_m = \\
&= \Big\langle \sum_{k=1}^{N} \big(\sum_{i=1}^{N} g(i) x_i - g(k) \big) x_k e_k \ , \ e_m \Big\rangle \ .
\end{aligned}
$$

Since, in general, (real-)linear combinations of complete vector fields are complete vector fields (see e.g. [1]), and since the Z_k are complete in S, the field $V = \sum_{k=1}^{N} g(k) Z_k - W$ is complete in S iff W is complete in S. As we have seen, the polynomial vector field W is complete in S iff $\big\langle W, \sum_{k=1}^{N} e_k \big\rangle = 0$ and $\langle W(x_1, \ldots, x_N), e_k \rangle = 0$ whenever $x_k = 0$ for some index k and $\sum_{i=1}^{N} x_i = 1$. It is well known that, by its construction, $\big\langle V(x_1, \ldots, x_N), \sum_{i=1}^{N} e_i \big\rangle = 0$ and hence $\big\langle W(x_1, \ldots, x_N), \sum_{i=1}^{N} e_i \big\rangle = 0$ if $\sum_{i=1}^{N} x_i$ even in the case if V is not complete in S. Fix any index k. By the definition $W := \sum_{m=1}^{N} \sum_{i,j=1}^{N} w(i,j) x_i x_j \big[\sum_{\ell=1}^{N} M(i,j,\ell) \varepsilon(i,j,\ell,m) - x_m \big] e_m$ we have $\langle W(p), e_k \rangle = 0$ for all points $p := (\xi_1, \ldots, \xi_N)$ with $\xi_k = 0$ and $\sum_{i=1}^{N} \xi_i = 1$ if and only if

$$
\sum_{\substack{i,j=1 \\ i,j \neq k}}^{N} w(i,j) \xi_i \xi_j \sum_{\ell=1}^{N} M(i,j,\ell) \varepsilon(i,j,\ell,k) = 0 \qquad \text{if } \textstyle\sum_{\substack{i=1 \\ i \neq k}}^{N} \xi_i = 1.
$$

By elementary properties of bilinear forms, this latter relation holds iff

$$w(i,j)\sum_{\ell=1}^{N} M(i,j,\ell)\varepsilon(i,j,\ell,k) = 0 \qquad \text{if } i,j \neq k.$$

Since $N \geq 3$, the field W has this property for all indices $k = 1,\ldots,N$ iff all these terms vanish and hence $W = 0$. Thus V is complete in S iff $W = 0$ that is $V = \sum_{k=1}^{N} g(k)Z_k$ on S. In this case, the equation $V(\xi_1,\ldots,\xi_N) = 0$ with $(\xi_1,\ldots,\xi_N) \in S$ means

$$\xi_k\big(\sum_{i=1}^{N} g(i)\xi_i - g(k)\big) = 0 \qquad (k = 1,\ldots,N)$$

along with the conditions $\xi_1+\cdots+\xi_N = 1$ and $\xi_1,\ldots,\xi_N \geq 0$. Consider a point $(\xi_1,\ldots,\xi_N) \in S$ and write $J := \{j : \ \xi_j > 0\}$. Then $\xi_k = 0$ for $k \notin J$ and hence $\sum_{j\in J}\xi_j = 1$ and $V(\xi_1,\ldots,\xi_N) = 0$ iff $g(k) = \sum_{i=1}^{N}\xi_i g(i) = \sum_{j\in J}\xi_j g(j)$ for the indices $k \in J$. By writing $\gamma := \sum_{j\in J}\xi_j g(j)$ for the common value of the $g(k)$ with $(k \in J)$, we see that $V(\xi_1,\ldots,\xi_N) = 0$ for any $\xi_1,\ldots,\xi_N) \in S \cap \bigcap_{j\in J}(x_j > 0) \cap \bigcap_{i\notin J}(x_i = 0)$. This completes the proof. $\square$

Chapter 3

Complete polynomial vector fields of the Euclidean ball

In this chapter we will describe the complete polynomial vector fields in the unit ball of a finite dimensional inner product space which we identify with $\mathbb{R}^N$.

Our work arises from an idea of a nice result of L.L. Stachó [4] in 2001 where he characterized the complete real polynomial vector fields in the (two-dimensional) unit disc $\mathbb{K}$ of the complex plane $\mathbb{C}$. We will show that our result not only generalizes the results of [4] on $\mathbb{K}$, but it even simplifies them.

3.1. Definition. Given any subset K in $\mathbb{R}^N$ the set of real n tuples and a mapping $v : \mathbb{R}^N \to \mathbb{R}^N$, we say that v is a *complete vector field* in K if for every point $k_0 \in K$ there exists a curve $x : \mathbb{R} \to K$ such that $x(0) = k_0$ and $\frac{dx(t)}{dt} = v(x(t))$ for all $t \in \mathbb{R}$.

In Chapter 2 we represented complete polynomial vector fields on a simplex as polynomial combinations of some finite family of complete vector fields of third degree. This idea motivates the formulation of our main result in this section.

First let us reformulate Stachó's theorem [4] in terms of polynomial combinations instead of linear combinations asserting (in complex notations, when identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner) that a polynomial vector field $v : \mathbb{C} \to \mathbb{C}$ is complete in the unit disc $\mathbb{K}$ if and only if it is a finite $\mathbb{R}$-linear combination of the vector fields from the family

$$\mathcal{F} := \big\{ iz,\ \mu\overline{z}^n - \overline{\mu}z^{n+2},\ (1-|z|^2)Q : \\ n = 0,1,\ldots;\ \mu = 1,i;\ Q \in \mathrm{Pol}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \big\}\ .$$

Actually we have the simpler form for the real linear span (the family of all finite linear combinations) of $\mathcal{F}$ as

$$\begin{aligned}\mathrm{Span}_{\mathbb{R}}\mathcal{F} = \big\{ & P \cdot iz + Q(1-|z|^2) : \\ & P \in \mathrm{Pol}(\mathbb{C}, \mathbb{R}),\ Q \in \mathrm{Pol}(\mathbb{C}, \mathbb{C}) \big\} .\end{aligned}$$

3.2. Remark. Recall that a mapping $v : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a polynomial vector field if $v(x) = (p_1(x), \dots, p_N(x))$; $x \in \mathbb{R}^N$ for some polynomials $p_1, \dots, p_n : \mathbb{R}^N \to \mathbb{R}$, of N variables (that is each p_i is a finite linear combination of functions of the form $x_1^{m_1} \dots x_N^{m_N}$ with non-negative integers m_j where $x_j : (\xi_1, \xi_2, \dots, \xi_N) \mapsto \xi_j$ denotes the j-th canonical coordinate function of $\mathbb{R}^N$).

3.3. Definition. By writing $\langle (\xi_1, \xi_2, \dots, \xi_N), (\eta_1, \eta_2, \dots, \eta_N) \rangle := \sum_{i=1}^N \xi_i \eta_i$ for the inner product in $\mathbb{R}^N$, it is easy to see that a polynomial (or even smooth) vector field is complete in the ball $B := (\langle x, x \rangle < 1)$ if and only if it is complete in the sphere $S := (\langle x, x \rangle = 1)$. Furthermore, v is complete in S if and only if it is orthogonal to the radius vector on S, i.e. if $\langle v(x), x \rangle = 0$ for $x \in S$.

We know from Chapter 2 that if $F : \mathbb{R}^N \to \mathbb{R}$ is a polynomial and $p : \mathbb{R}^N \to \mathbb{R}$ be any polynomial such that $p(M) = 0$ and $M \subset R^N$, then there is a polynomial $q : \mathbb{R}^N \to \mathbb{R}$ such that $P = q \cdot F$, when $F(x) = \phi_1(x) \cdot \phi_2(x) \cdots \phi_N(x)$, and the ϕ_i are linearly independent affine functions. Now we will prove the case when $f(x) = 1 - \langle x, x \rangle$ which is important to formulate our main result.

3.4. Lemma. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a polynomial such that $f(x) = 0$ for $x \in S$ where $S := (\langle x, x \rangle = 1)$. Then there exists a polynomial $Q : \mathbb{R}^N \to \mathbb{R}$, such that $f(x) = (1 - \langle x, x \rangle) Q(x)$.*

Proof. Let $g : B \to \mathbb{R}$ be the function on the unit ball $B := (\langle x, x \rangle < 1)$, defined by $g(x) = \frac{f(x)}{(1 - \langle x, x \rangle)}$.

The function g is analytic, since it is the quotient of two polynomials. Thus $g(x) = \sum_{k=0}^{\infty} g_k(x)$ where g_k are k-homogeneous polynomials on $\mathbb{R}^N$. We have $f(\pm e) = 0$ $\langle e, e\rangle = 1$, where e is the unit vector. So given $e \in \mathbb{R}^N$ with $\langle e, e\rangle = 1$, there exists a polynomial $P_e : \mathbb{R} \to \mathbb{R}$, of degree $\leq$ deg $f - 2$ such that $(1 - t^2)P_e(t) = f(te)$. If follows that, for every fixed unit vector $e \in \mathbb{R}^N$, $g(te) = \frac{f(te)}{(1-t^2)} = P_e(t) = \sum_{k=0}^{\deg f-2} \alpha_k(e)t^k$, with suitable constants $\alpha_0(e), \ldots, \alpha_{\deg f-2}(e) \in \mathbb{R}$. Hence we deduce that $g_k(te) = 0$ for $k > \deg f - 2$ and for all $t \in \mathbb{R}$ and unit vectors e. Then $g = \sum^{\deg} f - 2_{k=0} g_k$ is a polynomial. This completes the proof.

Lemmas with such a character seem to be very important in the theory of complete polynomial vector fields of domains defined by polynomial inequalities. In the complex case, due to the algebraic closedness of the field $\mathbb{C}$, there are similar results but the proofs cannot be imitated in the real case, even in the case of a ball.

3.5. Theorem. *Let $P : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial mapping. Then P is a complete polynomial vector field in the sphere $S := (\langle x, x\rangle = 1)$ if and only if*

$$[P(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)]$$

for some polynomial mappings $R, Q : \mathbb{R}^N \to \mathbb{R}^N$.

Proof. Suppose that $P(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$ where R, $Q : \mathbb{R}^N \to \mathbb{R}^N$ are polynomials. Then

$$\langle R(x) - \langle R(x), x\rangle x, x\rangle = \langle R(x), x\rangle - \langle R(x), x\rangle\langle x, x\rangle = 0$$

on S. Since $\langle x, x\rangle = 1$ for $x \in S$ and also $(1 - \langle x, x\rangle)Q(x) = 0$ on S. Theorefore $\langle P(x), x\rangle = 0$ for $x \in S$ that P is tangent to S.

Conversely, suppose that the polynomial vector field $P : \mathbb{R}^N \to \mathbb{R}^N$, is complete in S. Let $\tilde{P}(x) = P(x) - \langle P(x), x\rangle x$. Since P is tangent to S, we have $P(x) \perp x$ (i.e. $\langle P(x), x\rangle = 0$. This implies that

$P(x) = \tilde{P}(x)$ or $P(x) - \tilde{P}(x) = 0$ on S. By Lemma 3.4, $P(x) - \tilde{P}(x) = (1 - \langle x, x\rangle)Q(x)$ for some $Q \in P$ the $(\mathrm{I\!R}^N, \mathrm{I\!R}^N)$. Theorefore

$$P(x) = \tilde{P}(x) + (1 - \langle x, x\rangle)Q(x) = P(x) - \langle P(x), x\rangle x + (1 - \langle x, x\rangle)Q(x).$$

This completes the proof. $\square$

3.6. Corollary. *Let $V_k : x \to e_k - \langle e_k, x\rangle x$ where $k = 1, 2, 3, \ldots, N$. Then every complete polynomial vector field on the sphere $S := \big(\sum\limits_{i=1}^{N} x_i^2 = 1\big)$ coincides with some vector field of the form $V(x) = \sum_{k=1}^{N} p_k(x)V_k(x)$ when restricted to S where $p_1, \ldots, p_n : \mathrm{I\!R}^n \to \mathrm{I\!R}$ are appropriate polynomials.*

Proof. Let $x = \sum x_i e_i$ be fixed, and let $\alpha_x : \mathrm{I\!R}^N \to \mathrm{I\!R}^N$ be the linear mapping $y \mapsto y - \langle x, y\rangle x$. Consider the operation $\beta_x : y \to (1 - \langle x, x\rangle)y^+\langle x, y\rangle x$ where y^+ stands for the adjoint of y. Observe that

$$\begin{aligned}
\alpha_x(\beta_x(y)) &= \beta_x(y) - \langle x, \beta_x(y)\rangle x = \\
&= (1 - \langle x, x\rangle)y + \langle x, y\rangle x - \langle x, (1 - 2x, x\rangle)y + \langle x, y\rangle x\rangle x = \\
&= (1 - \langle x, x\rangle)y + \langle x, y\rangle x - \langle x, y\rangle x + \langle x, x\rangle\langle x, y\rangle x - \\
&\quad - \langle x, y\rangle\langle x, x\rangle x = (1 - \langle x, x\rangle)y.
\end{aligned}$$

Therefore $V_k(x) = \alpha_x(e_k)$ and $Q^X(x) = \beta_x(Q, (x)) = \sum_{k=1}^{N} q_k^* e_k$ and

$$\alpha_x(Q^*(x)) = \alpha_x(\beta_x(Q(x)) = (1 - \langle x, x\rangle)Q(x),$$
$$\alpha_x(\sum_{k=1}^{N} q_k^* e_k) = \sum_{k=1}^{N} q_k^* \alpha_x(e_k) = \sum q_k^*(x)V_k(x).$$

Then, by writing $v_k(x) := \langle V(x), e_k\rangle$ $(k = 1, \ldots, N)$ for the component functions of the vector field V, we have $V(x) := \sum v_k(x)e_k$ and

$$\begin{aligned}
R(x) - \langle x, R(x)\rangle x = \alpha_x(R(x)) &= \alpha_x(\sum_{k=1}^{N} v_k(x)e_k) = \\
&= \sum v_k(x)\alpha(e_k) = \sum_{k=1}^{N} v_k(x)V_k(x)
\end{aligned}$$

By this we get

$$(1 - \langle x, \rangle) Q(x) = \sum q_k^*(x) v_k(x).$$

Thus with the scalar valued polynomials $p_k(x) := v_k(x) + q_k(x)$ we have

$$\begin{aligned} V(x) &= \sum_{k=1}^{N} v_k(x) V_k(x) + \sum_{k=1}^{N} q_k^*(x) V_k(x) = \\ &= \sum_{k=1}^{N} (v_k(x) + q_k(x)) v_k(x) = \\ &= \sum_{k=1}^{N} p_k(x) V_k(x) \end{aligned}$$

This completes the proof. $\square$

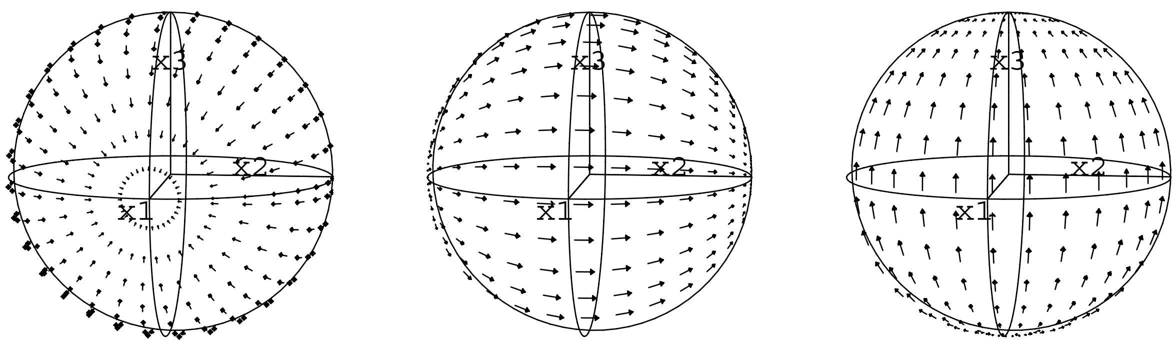

Figure 2. The vector fields $V_k : x \mapsto e_k - \langle e_k, x \rangle x$, ($k{=}1,2,3$) on S in $N{=}3$ dimensions.

3.7. Corollary *The complete polynomial vector fields on S are exactly the resrictrictions of the vector fields of the form*

$$\widetilde{V} : x \mapsto x A(x)$$

where A is any polynomial mapping $\mathbb{R}^N \to \mathrm{Mat}^{(-)}(N, \mathbb{R})$ into the space af all antisymmetric $N \times N$-matrices.

Proof. Given any polynomial mapping $A : \mathrm{I\!R}^N \to \mathrm{Mat}^{(-)}(N, \mathrm{I\!R})$, we have

$$\langle xA(x), x\rangle = \langle x, xA(x)^{\mathrm{T}}\rangle = \langle x, x(-A(x))\rangle =$$
$$= -\langle xA(x), x \ .$$

Thus necessarily $\langle xA(x), x\rangle = 0$ that is $xA(x) \perp x$ on the whole $\mathrm{I\!R}^N$. In particular $eA(e) \perp e$ for all unit vectors e which means that $x \mapsto xA(x)$ is a complete polynomial vector field of second degree in the unit sphere $S = (x_1^2 + \cdots + x_N^2 = 1)$.

Conversely, let V be any complete polynomial vector field on S. We know that $V(x) = \sum_{k=1}^N v_k(x) V_k(x)$ for some scalar-valued polynomials $v_k : \mathrm{I\!R}^N \to \mathrm{I\!R}$ with the fundamental vector fields

$$V_k(x) = e_k - \langle e_k, x\rangle x = e_k - x_k \sum_{i=1}^N x_i e_i \ .$$

Since the function $1 - (x_1^2 + \cdots + x_N^2)$ vanishes on S, the vector field

$$\widetilde{V}(x) := V(x) - [1 - (x_1^2 + \cdots + x_N^2)] \sum_{k=1}^N v_k(x) e_k$$

coincides with V on the sphere S. However, with the standard matrices E_{ik} with 1 at the (i,k)-entry and 0 elsewhere, we can write

$$\begin{aligned}
\widetilde{V}(x) &= \sum_{k=1}^N v_k(x) \Big[e_k - x_k \sum_{i=1}^N x_i e_i \Big] - [1 - (x_1^2 + \cdots + x_N^2)] \sum_{k=1}^N v_k(x) e_k = \\
&= \sum_{k=1}^N v_k(x) \sum_{i=1}^N x_i [x_i e_k - x_k e_i] = \\
&= \sum_{k=1}^N v_k(x) \sum_{i=1}^N x_i x [E_{ik} - E_{ki}] = \\
&= x \sum_{1 \le i < k \le N} (x_i v_k(x) - x_k v_i(x)) [E_{ik} - E_{ki}] = \\
&= xA(x)
\end{aligned}$$

where $A(x) := \sum_{1 \le i < k \le N} (x_i v_k(x) - x_k v_i(x)) [E_{ik} - E_{ki}]$ is a polynomial mapping from $\mathrm{I\!R}^N$ into $\mathrm{Mat}^{(-)}(N, \mathrm{I\!R})$. $\square$

3.8. Remark. There is an interesting link between the complete polynomial vector fields of the unit simplex $P := (x_1+\cdots+x_N,\ x_1,\ldots,x_N \geq 0)$ and those of the sphere $S := (x_1^2+\cdots+x_N^2 = 1)$. Namely, the mapping

$$T : (x_1,\ldots,x_N) \mapsto (x_1^2,\ldots,x_N^2)$$

maps the positive part $S_+ := S \cap (x_1,\ldots,x_N \geq 0)$ of the sphere onto P in a one-to-one manner. Given any smooth complete vector field $W : P \to \mathrm{I\!R}^N$ $(W(x) = (w_1(x),\ldots,w_N(x)))$ of the simplex P, its pull-back to S_+ is

$$T^{\#}V : S_+ \ni (x_1,\ldots,x_N) \mapsto \frac{d}{d\tau}\bigg|_{\tau=0} T^{-1}\big(T(x)+\tau W(T(x))\big) =$$
$$= \frac{d}{d\tau}\bigg|_{\tau=0}\big([x_1^2+\tau w_1(x_1^2,\ldots,x_N^2)]^{1/2},\ldots,[x_N^2+\tau w_N(x_1^2,\ldots,x_N^2)]^{1/2}\big) =$$
$$= \frac{1}{2}\big(x_1^{-1}w_1(x_1^2,\ldots,x_N^2),\ldots,x_N^{-1}w_N(x_1^2,\ldots,x_N^2)\big) \ .$$

In particular the operation $T^{\#}$ establishes the following relationship between the fundamental complete polynomial vector fields $Z_k(x) := x_k\sum_{i=1}^N x_i(e_i-e_k)$ of P of P and $V_k(x) := e_k - \langle e_k, x\rangle x$ of S, respectively:

$$T^{\#}Z_k(x) = \tfrac{-1}{2}x_kV_k(x) \qquad (k=1,\ldots,N) \ .$$

Therefore all complete polynomial vector fields of P are pulled back to complete polynomial vector fields of S_+. Namely we have

$$T^{\#}\left(\textstyle\sum_{k=1}^N p_k(x)Z_k(x)\right) = -\sum_{k=1}^N \tfrac{1}{2}x_kp_k(x_1^2,\ldots,x_N^2)V_k(x).$$

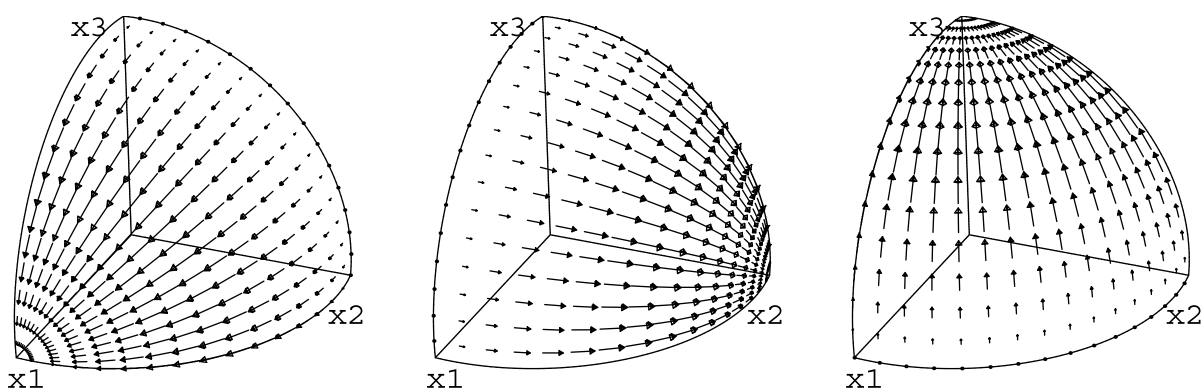

Figure 3. The vector fields $x_kV_k(x)$ in $N=3$ dimensions.

Chapter 4

Geodesics on a central symmetric warped product manifold

Let $U : \mathbb{R}_+ \to \mathbb{R}_+$ be a given smooth function. We consider the manifold $\mathbb{R}_0^n \times \mathbb{R}^1$, where $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$ is equipped with a Riemannian scalar product $\langle \cdot, \cdot \rangle$ satisfying the following conditions:

i) The projection onto $\mathbb{R}_0^n$ along $\mathbb{R}^1$ of the Riemannian scalar product $\langle \cdot, \cdot \rangle$ is the canonical Euclidean one.

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}_0^n$ with respect to $\langle \cdot, \cdot \rangle$.

iii) The projection onto $\mathbb{R}^1$ along $\mathbb{R}_0^n$ of $\langle \cdot, \cdot \rangle$ at $(a, p) \in \mathbb{R}_0^n \times \mathbb{R}^1$ is the canonical one multiplied by the function U.

These properties determine uniquely the scalar product of the tangent vectors (X, ξ), $(Y, \eta) \in T_{(a,\beta)}(\mathbb{R}_0^n \times \mathbb{R}^1)$ and it can written in the form

$$g_{(a,\beta)}((X, \xi), (Y, \eta)) = \langle X, Y \rangle + \xi \cdot \eta \cdot \cup(|a|^2). \tag{1}$$

where $\langle X, Y \rangle = \sum_{i=1}^{n} X_i \cdot Y_i$. For the sake of simplicity we shall write

$$\langle (X, \xi), (Y, \eta) \rangle_* = g_{(a,\beta)}((X, \xi)(Y, \eta)).$$

This simplification will not lead to any confusion since we know every time which point the tangent vector belongs to. We will regard β in (a, β) like the $(n+1)$-th coordinate.

One of our basic results is formulated in the following theorem.

4.1. Theorem. *The Levi-Civita connection of the Riemannian metric* (1) *introduced above has the following Christoffel symbols*

$$\Gamma_{i,j}^k(a,\beta) = \begin{cases} 0 & \textit{if} \quad i,j,k \le n \\ 0 & \textit{if} \quad i,j \le n, \;\; k = n+1 \\ 0 & \textit{if} \quad i,k \le n, \;\; j = n+1 \\ 0 & \textit{if} \quad j,k \le n, \;\; i = n+1 \\ -\partial_k(U(z))/2 & \textit{if} \quad k \le n, \;\; i,j = n+1 \\ \partial_i(U(z))/2U(z) & \textit{if} \quad j,k = n+1 \\ \partial_j(U(z))/2U(z) & \textit{if} \quad i,k = n+1 \\ 0 & \textit{if} \quad i,j,k = n+1 \end{cases},$$

where $1 \leq i, j, k \leq n+1$, $z = \langle a, \beta \rangle$ *and* ∂_s *is the derivative with respect to the* s*-th coordinate.*

Proof. The Levi-Civita connection is torsion free thus we have $\Gamma^k_{i,j} = \Gamma^k_{j,i}$ (the symmetry of Christoffel symbols). The other defining equation for this connection is

$$(T, \tau)[g_{(a,\beta)}((X, \xi), (Y, \eta))] =$$
$$= \langle \nabla_{(T,\tau)}(X, \xi), (Y, \eta)\rangle_* + \langle (X, \xi), \nabla_{(T,\tau)}(Y, \eta)\rangle_*$$

where (T, τ) is a tangent vector. Let $\{E_i\}_{i=1}^n$ be an orthonormal base in the tangent space, and

$$\partial_i = \begin{cases} (E(i,0) & \text{if} \quad 1 \leq i \leq n \\ \quad (0,1) & \text{if} \quad i = n+1. \end{cases}$$

We obtain that

$$\partial_i g_{(a,\beta)}(\partial_j, \partial_k) = \sum_{s=1}^{n+1} (\Gamma^S_{i,j}((a,\beta)) \cdot \langle \partial_S, \partial_k \rangle_* + \Gamma^S_{i,k}((a,\beta)) \cdot \langle \partial_j, \partial_S \rangle_*),$$

where $1 \leq i,\ k \leq n+1$. The simple way in which this system can be solved is presented in the pattern below.

In the first column we have written the delimited cases according to values of indices which we were just investigating. In the second column are the equations corresponding to the indices in the first column. The solutions of the respective equations can be found in the third column. The solutions of the equations in the third and fourth rows are obtained as a solution of a system of linear (for the Γ-s) equations.

CASE	EQUATION	SOLUTION
$j = k = n+1$	$\partial_i U(Z) = 2 \cdot \Gamma^{n+1}_{i,n+1} \cdot U(Z)$	$\Gamma^{n+1}_{i,n+1} = \frac{\partial_i U(Z)}{2U(Z)}$
$i = k = n+1, j \leq n$	$\Gamma^{n+1}_{j,n+1} \cdot U(Z) = -\Gamma^j_{n+1,n+1}$	$\Gamma^j_{n+1,n+1} = \frac{\partial_i U(Z)}{2}$
$k = n+1, i, j \leq n$	$\Gamma^{n+1}_{i,j} \cdot U(Z) = -\Gamma^j_{i,n+1}$	$\Gamma^{n+1}_{i,j} = 0$
$i = n+1, j, k \leq n$	$\Gamma^k_{n+1,j} = -\Gamma^j_{n+1,k}$	$\Gamma^j_{i,n+1} = 0$
$i, j, k \leq n$	$\Gamma^k_{i,j} = -\Gamma^j_{i,k}$	$\Gamma^i_{k,j} = 0$

These solutions show the statements in the theorem. $\square$

4.2. Corollary. *The system of differential equations of the geodesics is*

$$\dot{\beta} = h/U(Z)$$

$$\ddot{a}_j = a_j h^2 U'(Z)/U^2(Z) \quad 1 \leq j \leq n$$

where h is a suitable constant, $Z = \langle a, \beta \rangle$, and $(a(s), \beta(s))$ is the geodesic whose coordinates are $\{a_j\}_{j=1}^n$ and β.

Proof. The general differential equation for geodesics is

$$\ddot{x}_j + \sum_{s,i=1}^{n+1} \dot{x}_s \cdot \dot{x}_1 \cdot \Gamma^j_{s,i}(x) = 0,$$

where $x(s) = (x_1(s), \ldots, x_{n+1}(s))$ is geodesic. In our case we get the following

$$\ddot{\beta} + 2\dot{\beta} \sum_{i=1}^{n} \dot{a}_1 \cdot \frac{\partial_i(U(Z))}{2U(Z)} = 0,$$

$$\ddot{a}_j + (\dot{\beta})^2 \cdot \frac{-\partial_i(U(Z))}{2U(Z)} = 0 \quad (1 \leq j \leq n),$$

where $(a(a), \beta(s)) = (a_1(s), a_2(s), \ldots, a_n(s), \beta(s))$ is a geodesic. Since

$$\sum_{i=1}^{n} \dot{a}_1 \cdot \frac{\partial_i(U(Z))}{2U(Z)} = \frac{d}{ds}(U(Z))/U(Z),$$

we obtain from the first equation that

$$U(Z) \cdot \dot{\beta} + \frac{d}{ds}(U(Z)) \cdot \dot{\beta} = 0,$$

which implies the existence of a constant h satisfying

$$\dot{\beta} U(Z) = h.$$

This and a simple calculation give from our second differential equation that

$$\ddot{a} - a_j \cdot \frac{h^2 U'(Z)}{U^2(Z)} = 0 \quad (1 \leq j \leq n).$$

Notation:

The function $U(Z)$ is called the deformation function of the metric. The corresponding potential function of the mechanical system is $v(a_1, a_2) = \frac{-1}{2U(Z)}$ where $z = a_1^2 + a_2^2$. There are two interesting cases where the trajectories are closed.

i) $v(a_1, a_2) = k(a_1^2 + a_2^2)$;

ii) $v(a_1, a_2) = \frac{-k}{\sqrt{a_1^2+a_2^2}}$.

According to [1, p.42] we deal with first one where $U(Z) = m/z$ where m is constant. The second case with $U(Z) = -k\sqrt{z}$ is called Kepler case. Our Riemannian manifold described by (1) is a special case of the so-called warped product Riemannian space, with warping function $x \to \|x\|^{-2} : \mathbb{R}_0^n \to \mathbb{R}^1$. Consider an arbitrary warping function $\psi : \mathbb{R}_0^n \to \mathbb{R}$ it is easy to see that the orthogonal projection onto $\mathbb{R}_0^n$ of geodesics of the warped product manifold $\mathbb{R}_0^n X_\psi \mathbb{R}$ are exactly the trajectories of the mechanical systems on $\mathbb{R}_0^n$ with potential function $c\psi(x)^{-1}$ with arbitrary constant. The geometry of the trajectories of mechanical systems with Central symmetric potential function is presented in Arnold's book Chapter 2 page 41. In particular there has been proved that all bounded trajectories of mechanical system with central symmetric potential function are closed if and only if the potential has the form $-k\|x\|^{-1}$, $(k \geq 0)$ or $a\|x\|^2$, $(a \geq 0)$. We will study the geometry of central symmetric warped product manifold which corresponds to potential function $a\|x\|^2$, $(a \geq 0)$ first and also geometry of Kepler $-k\|x\|^{-1}$ $(k \geq 0)$,

4.3. Theorem *Let $(\mathbf{x}(s), \xi(s))$ be a geodesic in $\mathbb{R}_0^n \times \mathbb{R}^1$ with respect to the Riemannian metric (1). We denote its initial values at $s = 0$ by $\mathbf{x}(0) = \mathbf{x}_0$, $\xi(0) = \xi_0$, $\dot{\mathbf{x}}(0) = \mathbf{t}_0$, $\dot{\xi}(0) = \tau_0$. Then one has the following possibilities:*

a) *If $\tau_0 = 0$ then the geodesic $(\mathbf{x}(s), \xi(s))$ is contained in the line $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$, $\xi(s) = \xi_0$; this geodesic is complete except in* *the case if* $\xi_0 = 0$ *and the vectors* $\mathbf{t}_0$ *and* $\mathbf{x}_0$ *are collinear.*

b) *If* $\tau_0 > 0$ *then the projection of the geodesic onto* $\mathbb{R}_0^n$ *is an ellipse with centre* $\mathbf{0}$*. Its equation has the shape*

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

The corresponding geodesic is complete except in the case if the vectors $\mathbf{t}_0$ *and* $\mathbf{x}_0$ *are collinear and the projected ellipse is degenerated to a segment with the midpoint* $\mathbf{0}$*.*

c) *If* $\tau_0 < 0$ *then the projection of the geodesic onto* $\mathbb{R}_0^n$ *is a hyperbola with center* $\mathbf{0}$*. Its equation has the shape*

$$\mathbf{x}(s) = \cos h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

If the vectors $\mathbf{t}_0$ *and* $\mathbf{x}_0$ *are collinear then the projected hyperbola is degenerated to a half line. The corresponding geodesic is complete.*

Proof. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis in the vector space $\mathbb{R}^n$ satisfying $\mathbf{x}_0 = r\mathbf{e}_1$, $\mathbf{t}_0 = \cos\gamma\mathbf{e}_1 + \sin\gamma\mathbf{e}_2$ and let $\mathbf{e}_0$ be a unit vector of $\mathbb{R}^1$. In the corresponding coordinate system $\{x_0, x_1, \dots, x_n\}$, defined by $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ and $x_0 = \xi$, from Theorems 4.1 and 4.2 the Riemannian metric tensor g_{ij} in Theorem 4.1 and Corollary 4.2 has the following components:

$$g_{\lambda\mu} = \delta_{\lambda\mu}, \quad g_{\lambda 0} = g_{0\lambda} = 0, \quad g_{00} = \|\mathbf{x}\|^{-2}, \quad (\lambda, \mu = 1, \dots, n)$$

at the point $(\mathbf{x}, \xi)$. An easy calculation gives that the non-vanishing coefficients Γ^i_{jk} in the equation $\ddot{x}^i + \sum \Gamma^i_{jk}\dot{x}^k = 0$, $(i, j, k = 0, \dots, n)$ of geodesics can be expressed by

$$\Gamma^\lambda_{00} = \|\mathbf{x}\|^{-3}\frac{\partial\|\mathbf{x}\|}{\partial x_\lambda}, \quad \Gamma^0_{\mu 0} = \Gamma^0_{0\mu} = -\|\mathbf{x}\|^{-1}\frac{\partial\|\mathbf{x}\|}{\partial x_\mu}.$$

It follows that the equation of a geodesic $(\mathbf{x}(s), \xi(s))$ is of the form

$$\ddot{x}^\lambda(s) + \|\mathbf{x}(s)\|^{-4}x^\lambda(s)\dot{\xi}^2(s) = 0, \quad (\lambda = 1, \dots, n),$$

$$\ddot{\xi}(s) - 2\|\mathbf{x}(s)\|^{-2}\sum_{\mu=1}^{n} x^{\mu}(s)\dot{x}^{\mu}(s)\dot{\xi}(s) = 0,$$

where the dot denotes the derivation $\frac{d}{ds}$. These equations can be written in the form

$$\ddot{\mathbf{x}}(s) = \|\mathbf{x}(s)\|^{-4}\dot{\mathbf{x}}(s)\dot{\xi}^2(s), \quad \ddot{\xi}(s) = 2\|\mathbf{x}(s)\|^{-1}\frac{d\|\mathbf{x}(s)\|}{ds}\dot{\xi}(s).$$

The last equation is equivalent to the expression

$$\dot{\xi}(s) = c\|\mathbf{x}(s)\|^2$$

with an arbitrary $c =$ constant. Substituting this into the preceding equations one has $\ddot{\mathbf{x}}(s) + c\mathbf{x}(s) = \mathbf{0}$.
If $\dot{\xi}(0) = \tau_0 = 0$ then $c = 0$, the function $\xi(s) = \xi_0$ is constant and the vector valued function $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$ is linear. It means that the corresponding geodesic is a line. If the initial values satisfy $\xi_0 = 0$ and the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear then the geodesic should contain the origin $(\mathbf{0}, 0$ which does not belong to the manifold. Hence in this case the corresponding geodesic is non-complete.
Now, we assume $\tau_0 \neq 0$. In this case we have $\tau_0 = c\|\mathbf{x}(0)\|^2 = c\|\mathbf{x}_0\|^2$ and

$$\ddot{\mathbf{x}}(s) + \frac{\tau_0}{\|\mathbf{x}_0\|^2}\mathbf{x}(s) = \mathbf{0}.$$

If $\tau_0 > 0$ then the general solution of this equation has the following form

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{a} + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors satisfying $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{t}_0 = \sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}\mathbf{b}$. Clearly, if the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly independent then the solution curve is an ellipse with centre $\mathbf{0}$ which is contained in the 2-dimensional subspace W of $\mathbb{R}^n$ spanned by the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$. Hence the corresponding geodesic is complete.
If the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly dependent then the solution ellipse with centre $\mathbf{0}$ is degenerated to a segment containing $\mathbf{0}$. But the

origin $\mathbf{0}$ does not belong to our manifold and hence the corresponding geodesic is non-complete.

If $\tau_0 < 0$ then the general solution of this equation has the following form

$$\mathbf{x}(s) = \cos h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{a} + \sin h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors satisfying $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{t}_0 = \sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}\mathbf{b}$. If $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly independent then the solution curve is a connected component of a hyperbola with centre $\mathbf{0}$ which is contained in the subspace W spanned by the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$. Hence the corresponding geodesic is complete.
If the vectors $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly dependent then the solution hyperbola with centre $\mathbf{0}$ is degenerated to a repeated half line fully contained in the manifold. Hence the corresponding geodesic is complete. $\square$

Now we deal with the second case to give the geometry of Kepler Motions.

In this case the determining function of the metric is $U(z) = c\sqrt{z}$. We have the following description of the geodesics.

4.4. Theorem. *Let $(a(s), \alpha(s))$ be a geodesic in $\mathrm{IR}_0^n \times \mathrm{IR}^1$ with respect to the Riemannian metric (1). We denote its initial values at $s = 0$ by $a_0 = a(0)$, $\alpha_0 = \alpha(0)$, $T = \dot{a}(0)$, $\tau = \dot{\alpha}(0)$. Let E_1, $E_2 \in \mathrm{IR}_0^n$ be orthogonal unit vectors in W which are spanned by a_0 and T. Choose E_1, E_2 satisfying the following*

$$a_0 = a_1 \cdot E_1, \quad T = T_1 \cdot E_1 + T_2 \cdot E_2.$$

If $T_2 \neq 0$ we get the following description of geodesics:

The geodesics do not leave the space spanned by W and IR^1. Furthermore, if we denote the projection of T to IR^1 along IR_0^n by T_3, there are three possibilities:

- *i) if $|(T,\tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 < 0$, then the projection of the geodesic onto W is an ellipse,* *ii) if* $|T,\tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 = 0$, *then the projection of the geodesic onto* W *is a parabola,*

iii) if $|T,\tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 > 0$, *then the projection of the geodesic onto* W *is a hyperbola.*

The equation of the projected geodesic in polar-coordinate is

$$P(\gamma) = \frac{2 \cdot |a_1^0|^3 \cdot T_2^2}{-c \cdot T_3^2 \cdot |a_1^0|^3 + v \cdot \cos(\varphi - \omega)},$$

where

$$v = \operatorname{sgn}(c) = \cdot \sqrt{4 \cdot T_1^2 \cdot T_2^2 \cdot |a_1^0|^4 + (2 \cdot T_2^2 \cdot |a_1^0|^2 + c \cdot T_3^2 \cdot |a_1^0|^3)^2},$$

$$\omega = \arcsin\Big(\frac{2 \cdot T_1 \cdot a_1^0 \cdot \operatorname{sgn}(c)}{U}\Big)$$

and $p = |a|$, $\cos\varphi = \langle a, E_1\rangle/|a|$.

Proof. Let $\{E_1, \ldots, E_n\}$ be an orthonormal base in $\mathbb{R}^n$ such that

$$a_0 = a_1^0 \cdot E_1, \quad T = t_1 \cdot E_1 + T_2 \cdot E_2$$

and E_{n+1} be a unit vector in $\mathbb{R}^1$. From Corollary 4.2 we get the following differential equation for the geoedesic $(a(s), \alpha(s))$:

(5) $$\ddot{a}_j - a_j \cdot \frac{h^2}{2 \cdot c \cdot |a_1|^3} = 0 \quad (1 \le j \le n)$$

(6) $$\dot{\alpha} = \frac{h}{c \cdot |a_1|},$$

where $h = \tau \cdot c \cdot |a_0|$. In our coordinate system $a_j(0) = \dot{a}_j(0)$ for $3 \le j \le n$ hence by the Picard-Lindelöf theorem we conclude, that $a_j(0) \equiv 0$. So it is enough to investigate the case if $n = 2$.

Let $p(s) = |a(s)|$ and take the polar coordinate system in $\mathbb{R}_0^2$, i.e.

$$a_1(s) = p(s) \cdot \cos(\varphi(s)), \quad a_2(s) = p(s) \cdot \sin(\varphi(s)).$$

¿From the differential equations (5) we have

(5′) $$2 \cdot c \cdot p^2 \cdot (\ddot{p} \cdot \cos\varphi - 2 \cdot \dot{p} \cdot \dot{\varphi} \cdot \sin\varphi - p \cdot \varphi^2 \cdot \cos\varphi - p \cdot \ddot{\varphi} \cdot \sin\varphi) = h^2 \cdot \cos\varphi,$$

$$(5'') \qquad 2\cdot c\cdot p^2\cdot(\ddot{p}\sin\varphi - 2\cdot\dot{p}\cdot\dot{\varphi}\cdot\cos\varphi - p\cdot\varphi^2\cdot\sin\varphi - p\cdot\ddot{\varphi}\cdot\cos\varphi) = h^2\cdot\sin\varphi.$$

Take the linear combination of these equations by $(\sin\varphi, -\cos\varphi)$ and $(\cos\varphi, \sin\varphi)$ to obtain the following ones:

$$2\cdot\dot{p}\cdot\varphi + p\cdot\ddot{\varphi} = 0$$

$$2\cdot c\cdot p^2\cdot(\ddot{p} - p\cdot\varphi^2) = h^2.$$

After multiplying the first one by p, a simple integration gives

$$(7) \qquad p^2\cdot\dot{\varphi} = q,$$

where q is a suitable constant. On substituting this into the second equation and dividing it by p^2, multiplying it by $\dot{p}$ we can integrate it, that yields

$$(8) \qquad c\cdot\dot{p}^2 + c\cdot\frac{q^2}{p^2} + \frac{h^2}{p} = q',$$

where q' is constant.

Using the equations (7), (8) it is easy to get, that

$$\frac{dp}{d\varphi} = \pm\frac{p}{q}\cdot\sqrt{\frac{p^2\cdot q' - p\cdot h^2 - c\cdot q^2}{c}}.$$

On subsituting $p = 1/\rho$ into this equation it appears in the following integrable form

$$\mp 1 = \frac{q\cdot\frac{d\rho}{d\varphi}}{\frac{q'}{c} - \frac{h^2}{c}\cdot\rho - q^2\cdot\rho^2},$$

hence

$$(9) \qquad p(\varphi) = \frac{-2\cdot a_1\cdot q^2}{h^2 - \cos(k\mp\varphi)\cdot\sqrt{4\cdot a_1\cdot q^2\cdot q' + h^4}},$$

where k is constant given by the integration of the previous equation.

It must be noted here, that q is zero if and only if $\dot{\varphi}\cdot p^2 \equiv 0 \Longleftrightarrow \dot{\varphi}\equiv 0$. i.e. φ is constant. Hence the geodesic is a straight line passing through the origin and the equation (9) is not true. Easy calculation shows, that $\dot{\varphi} = T_2/|a_0|$ and $\varphi(0) = 0$. Thus $q = 0$ if and only if $T_2 = 0$, i.e. the projection of geodesic onto $\mathbb{R}^2_0$ along $\mathbb{R}^1$ is straight line if and only if its starting speed T is parallel to a_0. In this case (8) shows that $\dot{p} = \sqrt{q'/c - h^2/cp}$. This equation together with (6) gives

$$\frac{d\varphi}{dp} = \frac{\dot{a}}{\dot{p}} = \frac{h}{\sqrt{c \cdot p^2 \cdot q' - c \cdot p \cdot h^2}}.$$

There are three possibilities now:

i) if $c \cdot q' > 0$, then

$$\varphi(p) = q'' + \frac{h}{\sqrt{c \cdot q'}} \cdot \ln\Big(p - \frac{h^2}{2 \cdot q'} + \sqrt{p^2 - h^2 \cdot p/q'}\Big),$$

ii) if $c \cdot q' < 0$, then

$$\varphi(p) = q'' + \frac{h}{\sqrt{-c \cdot q'}} \cdot \arcsin\Big(\frac{4 \cdot p \cdot q'^2 - 2 \cdot q' \cdot h^2}{h^4}\Big),$$

iii) if $c \cdot q' = 0 \iff q' = 0$. Since $c \cdot \dot{p}^2 = -h^2/p$, c has to be negative, and so $dp/d\varphi = \sqrt{-c \cdot p}$, which leads to

$$p(\varphi) = (\alpha \cdot \sqrt{-c} + q'')^2/4.$$

The most interesting case is the second one, where p is bounded and the geodesic vibrates in the interval $\left(\frac{h^2}{2q'} - \frac{h^4}{4q'^2}, \frac{h^2}{2q'} + \frac{h^4}{4q'^2}\right)$. We will not deal with these cases further.

From the border conditions one can show by a straightforward but tedious caclulation that

(10) $$p(\varphi) = \frac{|a_0| \cdot T_2^2}{u - v \cdot \cos(\varphi - \infty)},$$

where

$u = -\tau^2 \cdot |a_0| \cdot \frac{c}{2}$,
$v = -\mathrm{sgn}(c) \cdot \sqrt{T_1^2 \cdot T_2^2 + (\frac{2}{2} + \tau^2 \cdot |a_0| \cdot \frac{c}{2})^2}$,
$\omega = \arcsin\big(\frac{T_1 \cdot T_2}{-\mathrm{sgn}(c) \cdot v}\big)$.

It is well know, that this equation defines conic sections. To determine its shape we have to investigate its eccentricity

$$\varepsilon = \frac{v}{u} = \frac{\sqrt{T_1^2 \cdot T_2^2 + (T_2^2 + \tau^2 \cdot |a_1| \cdot \frac{c}{2})^2}}{\tau^2 \cdot |a_1| \cdot \frac{c}{2}}. \tag{11}$$

A quick calculation shows, that

$$\varepsilon^2 = 1 + \frac{4 \cdot T_2^2}{c^2 \cdot a_1^2 \cdot \tau^2} \cdot (T_1^2 + T_2^2 + c \cdot |a_1| \cdot T_3^2),$$

which implies in an easy way, that ε more than, less than or equal to 1 according to the sign of $|(T, \tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_0| \cdot T_3^2$, which was to be proved. □

4.5. Corollary. *If $c > 0$, then all the projections of geodesics are hyperbolas which have two asymptotic straight lines through the origin with the direction $\omega - \arccos(1/\varepsilon)$ and $\omega + \arccos(1/\varepsilon)$. The nearest point of these asymptotic lines to the origin is $(\omega, |a_0| \cdot T_2^2/(u - v))$. Thus the origin is not contained inside the hyperbola.*

Proof. If $c > 0$, then $|(T, \tau)|^2 > 0$ and so $\varepsilon > 1$. Thus the equation of the projection of geodesics is

$$p(\varphi) = \frac{|a_0| \cdot T_2^2}{(-v) \cdot \cos(\varphi - \omega) - (-u)},$$

where $-v, -u > 0$ and $\varepsilon = \frac{-v}{-u} > 1$. It is clear that $p(\varphi)$ is minimal if $\cos(\varphi - \omega)$ is maximal. This proves the second statement of the Corollary.

On the other hand the denominator can not be zero, and $p(\varphi)$ tends to infinite if φ tends to $\omega - \arccos(u/v)$ or $\omega + \arccos(u/v)$. This completes the proof.

4.6. Corollary. *The projection of a geodesic is a circle if and only if $c < 0$, T is perpendicular to a_0 and $|T|^2 + |(T, \tau)|_*^2 = 0$. The radius of this circle is $2 \cdot |T|^2/(-c \cdot \tau^2)$. Its center is the origin.*

Proof. The projection is a circle if and only if $\nu = 0$. Since $c \neq 0$, this gives

$$T_1 \cdot T_2 = 0 \quad \text{and} \quad T_2^2 + \frac{c}{2} \cdot \tau^2 \cdot |a_0| = 0.$$

4.7. Corollary. *If the projection of a geodesic is an ellipse, and for its eccentricity $\varepsilon \neq 0$, then its long axis has direction ω and length $\frac{2 \cdot |a_0| \cdot T_2^2 \cdot u}{u^2 - v^2}$. It has two focal points: the origin and $\left(\omega, \frac{2 \cdot |a_0| \cdot T_2^2 \cdot u}{u^2 - v^2}\right)$. Its short axis has length $2 \cdot |a_0| \cdot T_2^2$.*

Proof. It is clear that the nearest and the most far point of the projection are on the long axis. We can get these points and their distances from the origin, when $\cos(\varphi - \omega) = \pm 1$. The length of the long axis is the sum of their distances. The difference of these distances gives the distance of second focal points from the origin. If x is the half of short axis, the Pythagoras theorem gives

$$\frac{|a_0| \cdot T_2^2 \cdot u}{u^2 - v^2} = \left(\frac{|a_0|^2 \cdot T_2^4 \cdot v^2}{(u^2 - v^2)^2} + x^2 \right)^{\frac{1}{2}}.$$

The solution of this equation completes the proof. $\square$

4.8. Corollary. *If the projection of the geodesic is a parabola, then it is open in direction ω. Its nearest point is $(\omega + \pi, -T_2^2/(c \cdot \tau^2))$ and its focal point is the origin.*

Proof. This corollary can be easily obtained on substituting $v = \varepsilon \cdot u = u$ into (10). $\square$

4.9. Corollary. *If the projection of the geodesic is a hyperbola and $c < 0$, then its focal point is the origin. It has two asymptotic straight lines with direction*

$$\omega + \arccos(1/\varepsilon)(1/\varepsilon) \quad \text{and} \quad \omega - \arccos(1/\varepsilon).$$

Proof. The proof of our first corollary shows the way in which we can get this one. $\square$

4.10. Theorem. *If $\tau > (<) 0$ then α is strictly increasing (decreasing) and it depends on $p = |a|$ according to the following differential equation*

$$\frac{d\alpha}{dp} = \frac{\operatorname{sgn}\sin(\varphi-\omega)\cdot|a_0|\cdot T_2}{\sqrt{p^2(v^2-u^2)+2|a_0|T_2^2\cdot u\cdot p-|a_0|^2\cdot T_2^4}}, \tag{12}$$

where we have used the notations of our first theorem.

Proof. On investigating (7) at the startpoint, we get $q = T_2 \cdot |a_0|$. From (10) we conclude

$$p\cdot v\cdot\sin(\varphi-\omega) = \operatorname{sgn}\sin(\varphi-\omega)\cdot\sqrt{p^2(v^2-u^2)+2|a_0|T_2^2\cdot u\cdot p-|a_0|^2\cdot T_2^4}.$$

Let $\alpha(t) = \alpha(p(\varphi(t)))$. The theorem will be implied by (6), (7), (10) in the following way:

$$\begin{aligned}
\tau|a_0| &= \frac{d\alpha}{dt}\cdot p = \frac{d\alpha}{dp}\cdot\frac{dp}{d\varphi}\cdot\frac{d\varphi}{dt}\cdot p\\
&= \frac{d\alpha}{dt}\cdot\frac{-p\cdot v\cdot\sin(\varphi-\omega)}{|a_0|\cdot T_2^2}\cdot p\cdot\frac{q}{p^2}\cdot p\\
&= \frac{d\alpha}{dp}\cdot\operatorname{sgn}\sin(\varphi-\omega)\cdot\frac{\sqrt{p^2(v^2-u^2)+2|a_0|T_2^2\cdot u\cdot p-|a_0|^2\cdot T_2^4}}{T_2}.
\end{aligned}$$

The monotonicity of α follows from (6) directly, since

$$\operatorname{sgn}\Big(\frac{h}{c\cdot|a|}\Big) = \operatorname{sgn}\tau. \square$$

4.11. Corollary. *If the projection of the geodesic is an ellipse, then*

$$p(\alpha) = \frac{c\cdot a_0^2\cdot\tau^2}{|T,\tau}|_*^2 - \frac{|a_0|\cdot v}{|(T,\tau)|_*^2}\cdot\sin\Big(\frac{\sqrt{|T|^2+\tau^2}}{\tau\cdot|a_0|\cdot\operatorname{sgn}(\sin(\varphi-\omega))} - \text{const}\Big),$$

where const is such a number, that $p(\alpha) = |a_0|$.

Proof. Since the projection is an ellipse, $v^2 - u^2 < 0$. We can rewrite (12) in the form

$$\frac{d\alpha}{dp} = \frac{T_2\cdot\tau\cdot|a_0|\cdot\operatorname{sgn}(\sin(\varphi-\omega))}{\sqrt{u^2-v^2}}\cdot\Big(\frac{|a_0|^2\cdot T_2^4\cdot v^2}{(u^2-v^2)^2} - \big(p - \frac{|a_0|\cdot T_2^2\cdot u}{u^2-v^2}\big)^2\Big)^{\frac{1}{2}}.$$

The integration of this formula implies the Corollary. □

4.12. Corollary. *If the projection of the geodesic is a parabola, then*

$$p(\alpha) = \frac{c \cdot \tau^2}{4} \cdot (\alpha_0 - \alpha) + |a_0|.$$

Proof. In this case, $v^2 = u^2$ thus (12) appears in the orm

$$\frac{d\alpha}{dp} = \frac{\operatorname{sgn} \sin(\varphi - \omega) \cdot |a_0| \cdot \tau}{\sqrt{2 \cdot |a_0| \cdot u \cdot p - |a_0|^2 \cdot T_2^2}}.$$

On integrating this equation we get the corollary. □

Proof. Since $v^2 > u^2$, we have from (12) that

$$\frac{d\alpha}{dp} = \frac{T_2 \cdot \tau \cdot |a_0| \cdot \operatorname{sgn}(\sin(\varphi - \omega))}{\sqrt{v^2 - u^2} \cdot \left(\frac{|a_0|^2 \cdot T_2^4 \cdot v^2}{(u^2 - v^2)^2} + \left(p + \frac{|a_0| \cdot T_2^2 \cdot u}{v^2 - u^2}\right)^2 \right)^{\frac{1}{2}}}.$$

The integration gives α like a function of p, from which the corollary follows. □

4.13. Remark. All the above give the result, that we would be able to write down the geodesics completely in the cylindrical coordinate system (p, φ, α) if we choose α for the parameter.

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List of publications

Nuri Muhammed Ben Yousif

[1] M.ben Y. Nuri, *Complete polynomial vector fields in simplexes with application to evolutionary dinamics,* Electronic Journal od Qualitative Theory of Differential Equations Szeged, to appear 2004.

[2] M.ben Y. Nuri, *Complete polynomial vector fields in Euclidean ball,* Publ. Math. Nyíregyháza, to appear 2004.

[3] M.ben Y. Nuri, *Geodesics on a central symmetric warped product manifold,* Publ. Math. Nyíregyháza, to appear 2004.

ABSTRACT

The dissertion contains the main results of three papers of mine concerning complete polynomial vector fields with an application to differential equations in genetical dynamic systems and geodesics on warped product manifolds, respectively.

In **Chapter 2** we describe the complete polynomial vector fields and their fixed points in a finite-dimensional simplex. As main result we get the following. A polynomial vector field $V : S \to \mathbb{R}^N$ is complete in $T := (x_1 + \cdots + x_N)$ if and only if with the vector fields

$$Z_k := x_k \sum_{j=1}^{N} x_j (e_j - e_k) \qquad (k = 1, \ldots, N)$$

where e_j is the standard unit vector $e_j := (0, \ldots, 0, \overbrace{\;1\;}^{j}, 0, \ldots, 0)$, we have

$$V = \sum_{k=1}^{N} P_k(x_1, \ldots, x_N) Z_k$$

for some polynomial functions $P_1, \ldots, P_N : \mathbb{R}^N \to \mathbb{R}$. Given a complete polynomial vector field V of T, there are polynomials $\delta_1, \ldots, \delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ of degree less than that of V such that the vector field

$$\begin{aligned}\widetilde{V} := \sum_{k=1}^{N-1} x_k \Big[\delta_k(x_1, \ldots, x_{N-1}) - \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1})\Big] e_k + \\ + (x_1 + \cdots + x_{N-1} - 1) \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1}) e_N\end{aligned}$$

coincides with V on T. The points of the zeros of V inside the facial subsimplices $S_K := S \cap (x_1, \ldots, x_K > 0 = x_{K+1} = \cdots = x_N)$ $(K = 1, \ldots, N)$ can be described as

$$\begin{aligned}S_N \cap (V = 0) &= S \cap \bigcup_{k=1}^{N-1} \big(\delta_k(x_1, \ldots, x_{N-1}) = 0\big), \\ S_K \cap (V = 0) &= S_K \cap \big(\delta_1(x_1, \ldots, x_{N-1}) = \cdots = \\ &\qquad = \delta_K(x_1, \ldots, x_{N-1})\big) \qquad (K < N).\end{aligned}$$

We apply the results to differential equations of genetical evolution models to answer the question what are the strange consequences of the assumption that the evolution has no starting point in time, in particular what can be stated on non-changing distributions of populations in that case. Several well-known models in literature on the time evolution of a closed population consisting of N different species - with the whole population at time $t \geq 0$ as the solution of a system of ordinary differential equations $\frac{d}{dt}v_k(t) = F_k(v_1(t), v_2(t), \ldots, v_N(t))$ $(k = 1, 2, \ldots, N)$ where the functions F_k are some polynomials of at most 3-rd. degree and we have $(v_1(t), v_2(t), \ldots, v_N(t)) \in T$ for all time $t \in \mathbb{R}$. As a consequence of our description, if $N \geq 3$ then the time evolution of the population can be retrospected up to any time $t \leq 0$ starting with any distribution $(x(0), \ldots, x_N(0)) \in S$ if and only if the term W vanishes on S, that is if simply $d/dt\ x = \sum_{k=1}^{N} g(k)Z_k(x_1, \ldots, x_N)$. In this case the set of the stable distributions has the form

$$\bigcup_{\gamma \in \{g(1), \ldots, g(N)\}} S \cap (x_m = 0 \text{ for } m \notin J_\gamma) \qquad \text{where } J_\gamma := \{m : \ g(m) = \gamma\} \ .$$

If $g(1), \ldots, g(N) \geq 0$ and the vector field (**) is complete in S then

$$\frac{d}{dt} \sum_{k=1}^{N} g(k) x_k(t) \geq 0$$

for any solution $t \mapsto x(t) \in S$ of the evolution equation $dx/dt = V(x)$.

In **Chapter 3** we describe the complete polynomial vector fields in the unit sphere $S := (x_1^2 + x_2^2 + \cdots + x_N^2 = 1)$ of $\mathbb{R}^N$. We get the following. With the vector fields $V_k : x \to e_k - \langle e_k, x \rangle x$ $(k = 1, \ldots, N)$, every complete polynomial vector field on S coincides with some vector field of the form $V(x) = \sum_{k=1}^{N} p_k(x) V_k(x)$ when restricted to S where $p_1, \ldots, p_n : \mathbb{R}^n \to \mathbb{R}$ are appropriate polynomials. Also the complete polynomial vector fields on S are exactly the resrictrictions of the vector fields of the form

$$\widetilde{V} : x \mapsto xA(x)$$

where A is any polynomial mapping $\mathbb{R}^N \to \mathrm{Mat}^{(-)}(N, \mathbb{R})$ into the space af all antisymmetric $N \times N$-matrices. This work originates from a nice parametric formula due to L.L. Stachó for the complete real polynomial vector fields on the unit disc $\mathbb{K}$ of the space $\mathbb{C}$ of complex numbers. He has shown that a real polynomial vector field $p : \mathbb{C} \to \mathbb{C}$ is complete in $\mathbb{K}$ iff p is a finite real linear combination formed by the functions $iz,\ \gamma\overline{z}^m - \overline{z}z^{m+2},\ (z \in \mathbb{C},\ m = 0, 1, \ldots)$ and $(1 - |z|^2)Q$ where Q is any real polynomial from $\mathbb{C}$ to $\mathbb{C}$. Our result in this chapter not only generalizes the 2-dimensional result of Stachó, but it even simplifies it by showing that the complete polynomial vector fields on the unit disc of $\mathbb{C}$ have the form $[ip(z)z + q(z)(1 - |z|^2)]$ where $p, q : \mathbb{C} \to \mathbb{R}$ are any real polynomials.

Notice that there is an interesting link between the complete polynomial vector fields of the unit simplex $P := (x_1+\cdots+x_N,\ x_1, \ldots, x_N \geq 0)$ and those of the sphere $S := (x_1^2 + \cdots + x_N^2 = 1)$. Namely, the mapping

$$T : (x_1, \ldots, x_N) \mapsto (x_1^2, \ldots, x_N^2)$$

maps the positive part $S_+ := S \cap (x_1, \ldots, x_N \geq 0)$ of the sphere onto P in a one-to-one manner. Given any smooth complete vector field $W : P \to \mathbb{R}^N$ $(W(x) = (w_1(x), \ldots, w_N(x)))$ of the simplex P, its pull-back to S_+ is

$$\begin{aligned} T^{\#}V : S_+ \ni (x_1, \ldots, x_N) \mapsto \frac{d}{d\tau}\Big|_{\tau=0} T^{-1}\big(T(x) + \tau W(T(x))\big) = \\ = \frac{d}{d\tau}\Big|_{\tau=0} \big([x_1^2 + \tau w_1(x_1^2, \ldots, x_N^2)]^{1/2}, \ldots, [x_N^2 + \tau w_N(x_1^2, \ldots, x_N^2)]^{1/2}\big) = \\ = \frac{1}{2}\big(x_1^{-1} w_1(x_1^2, \ldots, x_N^2), \ldots, x_N^{-1} w_N(x_1^2, \ldots, x_N^2)\big) \ . \end{aligned}$$

In particular the operation $T^{\#}$ establishes the following relationship between the fundamental complete polynomial vector fields $Z_k(x) := x_k \sum_{i=1}^N x_i(e_i - e_k)$ of P of P and $V_k(x) := e_k - \langle e_k, x\rangle x$ of S, respectively:

$$T^{\#} Z_k(x) = \tfrac{1}{2} x_k V_k(x) \qquad (k = 1, \ldots, N) \ .$$

Therefore all complete polynomial vector fields of P are pulled back to complete polynomial vector fields of S_+. Namely we have

$$T^{\#}\left(\sum_{k=1}^{N} p_k(x)Z_k(x)\right) = \sum_{k=1}^{N} \tfrac{1}{2}x_k p_k(x_1^2,\dots,x_N^2)V_k(x).$$

In **Chapter 4** we study the geometry of the central symmetric warped product manifold structures on $\mathbb{R}_0^N \times \mathbb{R}^1$ where $\mathbb{R}_0^N = \mathbb{R}^N \backslash \{0\}$, which correspond to the potential functions $a\|x\|$, $a \geq 0$, and equipped with the Riemannian scalar product $\langle \cdot,\cdot \rangle$ defined by the following properties:

i) the projection onto $\mathbb{R}^N$ along $\mathbb{R}^1$ of this Riemannian scalar $\langle \cdot,\cdot \rangle$ is canonical Euclidean,

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}^N$ with respect to $\langle \cdot,\cdot \rangle$,

iii) the projection onto $\mathbb{R}^1$ along $\mathbb{R}^N$ of $\langle \cdot,\cdot \rangle$ at $(a,\alpha) \in \mathbb{R}_0^N \times \mathbb{R}^1$ is the canonical one multiplied by $U(|a|^2)$, where $U : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth.

According to our basic result, the Levi-Civita connection of the natural Riemannian metric has the following Christoffel symbols

$$\Gamma_{i,j}^k(a,\beta) = \begin{cases} 0 & \text{if} \quad i,j,k \leq n \\ 0 & \text{if} \quad i,j \leq n, \;\; k = n+1 \\ 0 & \text{if} \quad i,k \leq n, \;\; j = n+1 \\ 0 & \text{if} \quad j,k \leq n, \;\; i = n+1 \\ -\partial_k(U(z))/2 & \text{if} \quad k \leq n, \;\; i,j = n+1 \\ \partial_i(U(z))/2U(z) & \text{if} \quad j,k = n+1 \\ \partial_j(U(z))/2U(z) & \text{if} \quad i,k = n+1 \\ 0 & \text{if} \quad i,j,k = n+1 \end{cases},$$

where $1 \leq i,j,k \leq n+1$, $z = \langle a,\beta \rangle$ and ∂_s is the derivative with respect to the s-th coordinate. The system of differential equation of the geodesics is

$$\dot{\beta} = h/U(Z)$$

$$\ddot{a}_j = a_j h^2 U'(Z)/U^2(Z) \quad 1 \leq j \leq n$$

where h is a suitable constant, $Z = \langle a,\beta \rangle$, and $(a(s),\beta(s))$ is the geodesic whose coordinates are $\{a_j\}_{j=1}^n$ and β. Hence we get the following description of geodesics. Given a geodesic $(\mathbf{x}(s), \xi(s))$ in $\mathrm{I\!R}_0^n \times \mathrm{I\!R}^1$ with respect to the natural Riemannian metric with initial values $\mathbf{x}(0) = \mathbf{x}_0$, $\xi(0) = \xi_0$, $\dot{\mathbf{x}}(0) = \mathbf{t}_0$, $\dot{\xi}(0) = \tau_0$ at $s = 0$, we have the following possibilities:

a) If $\tau_0 = 0$ then the geodesic $(\mathbf{x}(s),\, \xi(s))$ is contained in the line $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$, $\xi(s) = \xi_0$; this geodesic is complete except in the case if $\xi_0 = 0$ and the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear.

b) If $\tau_0 > 0$ then the projection of the geodesic onto $\mathrm{I\!R}_0^n$ is an ellipse with centre $\mathbf{0}$. Its equation has the shape

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

The corresponding geodesic is complete except in the case if the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear and the projected ellipse is degenerated to a segment with the midpoint $\mathbf{0}$.

c) If $\tau_0 < 0$ then the projection of the geodesic onto $\mathrm{I\!R}_0^n$ is a hyperbola with center $\mathbf{0}$. Its equation has the shape

$$\mathbf{x}(s) = \cos h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

If the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear then the projected hyperbola is degenerated to a half line. The corresponding geodesic is complete.

KIVONAT

A disszertáció három publikációm eredményeit tartalmazza, ezek egyrészt a teljes polinomiális vektormezőkről, illetve azok genetikai dinamikus rendszerekbeli alkalmazásairól szólnak, valamint torzított szorzatsokaságok geodetikusaival foglalkoznak.

A **2. fejezetben** leírjuk a végesdimenziós szimplexek teljes polinomiális vektormezőit és ezek fixpontjait. A fő eredményünk a következő. A $V : S \to \mathbb{R}^N$ polinomiális vektormező akkor és csak akkor teljes a $T := (x_1 + \cdots + x_n)$ halmazban, ha

$$V = \sum_{k=1}^{N} P_k(x_1, \ldots, x_N) Z_k$$

alakban előállíthatók, ahol $P_1, \ldots, P_N : \mathbb{R}^N \to \mathbb{R}$ polinomfüggvények,

$$Z_k := x_k \sum_{j=1}^{N} x_j (e_j - e_k) \qquad (k = 1, \ldots, N)$$

és $e_j = (0, \ldots, 0, \overbrace{1}^{j}, 0, \ldots, 0)$ szokásos egységvektorok. Adott T-nek V teljes polinomiális vektormezője, léteznek V-nél alacsonyabb fokú $\delta_1, \ldots, \delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ polinomok oly módon, hogy a

$$\begin{aligned} \widetilde{V} := \sum_{k=1}^{N-1} x_k \Big[\delta_k(x_1, \ldots, x_{N-1}) - \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1}) \Big] e_k + \\ + (x_1 + \cdots + x_{N-1} - 1) \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1}) e_N \end{aligned}$$

vektormezők egybeesnek V-vel T-n. V zérushelyei a lapok $S_K := S \cap (x_1, \ldots, x_K > 0 = x_{K+1} = \cdots = x_N)$ $(K = 1, \ldots, N)$ részszimplexein belül leírhatóak az alábbi alakban:

$$\begin{aligned} S_N \cap (V = 0) &= S \cap \bigcup_{k=1}^{N-1} \big(\delta_k(x_1, \ldots, x_{N-1}) = 0 \big), \\ S_K \cap (V = 0) &= S_K \cap \big(\delta_1(x_1, \ldots, x_{N-1}) = \cdots = \\ &\qquad = \delta_K(x_1, \ldots, x_{N-1}) \big) \qquad (K < N). \end{aligned} \tag{$*$}$$

Az eredményeinket felhasználjuk a genetikus evolúciós modellek differenciálegyenleteinek területén. Megválaszoljuk azt a kérdést, hogy milyen különleges következényei vannak annak a ténynek, hogy az evolúciónak nincs időbeli kezdőpontja. Részletesebben, hogy mit lehet mondani ebben az esetben a populáció eloszlásának változatlanságáról. Az irodalomban több jól ismert modell létezik az N különböző fajból álló zárt populáció időbeli alakulásának leírására, ahol a $t \geq 0$ időpillanatban a teljes populációt a $\frac{d}{dt}v_k(t) = F_k(v_1(t), v_2(t), \ldots, r_N(t))$ $(k = 1, 2, \ldots, N)$ közönséges differenciálegyenlet megoldásfüggvénye írja le. Itt az F_k függvények legfeljebb harmadfokú polinomok, a $(v_1(t), \ldots, v_N(t)) \in T$ vektorok pedig T-beliek minden $t \in \mathbb{R}$ időpontban. A leírásunk következménye, hogy az $N \geq 3$ esetben a populáció idő szerinti alakulása akkor és csak akkor vezethető vissza bármely $t \leq 0$ időpontra bármely $(x_1(0), \ldots, x_N(0)) \in S$ eloszlás esetén, ha a W tag eltűnik S-en, azaz, egyszerűen ha $d/dt\ x = \sum_{k=1}^{N} g(k)Z_k(x_1, \ldots, x_N)$. Ebben az esetben a stabil eloszlások halmaza

$$\bigcup_{\gamma \in \{g(1), \ldots, g(N)\}} S \cap (x_m = 0 \text{ for } m \notin J_\gamma) \qquad \text{where } J_\gamma := \{m : \ g(m) = \gamma\} \ .$$

alakú.

Ha $g(1), \ldots, g(N) \geq 0$ és a $(**)$ vektormező teljes S-ben, akkor

$$\frac{d}{dt} \sum_{k=1}^{N} g(k)x_k(t) \geq 0$$

fennáll a $dx/dt = V(x)$ evolúciós egyenlet minden $t \mapsto x(t) \in S$ megoldására.

A **3. fejezetben** az $\mathbb{R}^N$-beli $S := (x_1^2 + x_2^2 + \cdots + x_N^2 = 1)$ egységgömb teljes polinomiális vektormezőit írjuk le. A következő eredményt kapjuk. A minden S-beli teljes polinomiális vektormező megegyezik egy $V(x) = \sum_{k=1}^{N} p_k(x)V_k(x)$ alakú vektormező S-re vett megszorításával, ahol $V_k : x \to e_k - \langle e_k, x\rangle x$ $(k = 1, \ldots, N)$ vektormező és $p_1, \ldots, p_n : \mathbb{R}^n \to \mathbb{R}$ megfelelő polinomok. Más szóval, az S-beli teljes polinomiális vektormezők pontosan a $\tilde{V} : x \mapsto xA(x)$ alakú vektormezők megszorításai, ahol A tetszőleges polinomiális $\mathbb{R}^N \to Mat^{(-)}(N, \mathbb{R})$ leképezés az $N \times N$-es antiszimmetrikus mátrixok terébe. Ez a munka Stachó Lászlónak a komplex számok $\mathbb{K}$ egységkörén értelmezett teljes valós polinomiális vektormezőket megadó elegáns paraméteres formulájából ered. Stachó megmutatta, hogy a valós polinomiális $p : \mathbb{C} \to \mathbb{C}$ vektormező akkor és csak akkor teljes K-n ha p az alábbi függvények véges valós lineáris kombinációja: iz, $\gamma \bar{z}^m - \bar{z} z^{m+2}$, $(z \in \mathbb{C}, m = 0, 1, \ldots)$ valamint $(1 - |z|^2)Q$, ahol $Q : \mathbb{C} \to \mathbb{C}$ tetszőleges valós polinom. Ezen fejezetünk eredménye nem csak Stachó tételét általánosítja, de le is egyszerűsíti azt, amikor megmutatja, hogy $\mathbb{C}$ egység körlemezének teljes polinomiális vektormezői $[ip(z)z + q(z)(1 - |z|^2)]$ alakúak, ahol $p, q : \mathbb{C} \to \mathbb{R}$ tetszőleges valós polinomok.

Megjegyezzük, hogy érdekes kapcsolat áll fenn a $P := (x_1 + \cdots + x_N,\ x_1, \ldots, x_N \geq 0)$ egységszimplex, valamint az $S := (x_1^2 + \cdots + x_N^2 = 1)$ gömb teljes polinomiális vektormezői között. Nevezetesen, a

$$T : (x_1, \ldots, x_N) \mapsto (x_1^2, \ldots, x_N^2)$$

leképezés a gömb $S_+ := S \cap (x_1, \ldots, x_N \geq 0)$ pozitív részét 1-1 értelmű módon megfelelteti a P-vel. Adott a P szimplex tetszőleges $W : P \to \mathbb{R}^N$ $(W(x) = (w_1(x), \ldots, w_N(x)))$ sima teljes vektormezője, az S_+-ra vett pullback-je

$$\begin{aligned} T^{\#}V : S_+ \ni (x_1, \ldots, x_N) \mapsto & \left.\frac{d}{d\tau}\right|_{\tau=0} T^{-1}\big(T(x) + \tau W(T(x))\big) = \\ = \left.\frac{d}{d\tau}\right|_{\tau=0} & \big([x_1^2 + \tau w_1(x_1^2, \ldots, x_N^2)]^{1/2}, \ldots, [x_N^2 + \tau w_N(x_1^2, \ldots, x_N^2)]^{1/2}\big) = \\ = \frac{1}{2} & \big(x_1^{-1} w_1(x_1^2, \ldots, x_N^2), \ldots, x_N^{-1} w_N(x_1^2, \ldots, x_N^2)\big) \ . \end{aligned}$$

Speciálisan, a $T^{\#}$ művelet az alábbi kapcsolatot létesíti a P-beli $Z_k(x) := x_k \sum_{i=1}^N x_i(e_i - e_k)$ és az S-beli $V_k(x) := e_k - \langle e_k, x \rangle x$ vektormezők között:

$$T^{\#} Z_k(x) = \tfrac{1}{2} x_k V_k(x) \qquad (k = 1, \ldots, N) \ .$$

Tehát P bármely teljes polinomiális vektormezője visszahúzható S_+ egy teljes polinomiális vektormezőjére. Nevezetesen,

$$T^{\#}\left(\sum_{k=1}^{N} p_k(x)Z_k(x)\right) = \sum_{k=1}^{N} \tfrac{1}{2}x_k p_k(x_1^2,\ldots,x_N^2)V_k(x).$$

A **4. fejezetben** az $\mathbb{R}_0^N \times \mathbb{R}^1$-on értelmezett középpontosan szimmetrikus torzított szorzatsokaságok geometriáját tanulmányozzuk, ahol $\mathbb{R}_0^N = \mathbb{R}^N \setminus \{0\}$, amely az $a\|x\|$, $a \geq 0$ potenciálfüggvényeknek felel meg, és amelyen az alábbi módon bevezethetjük a $\langle .,. \rangle$ Riemann-féle skalárszorzatot:

i) ezen skalárszorzat $\mathbb{R}^1$ mentén $\mathbb{R}^N$-re vett projekciója kanonikusan euklideszi,

ii) $\mathbb{R}^1$ merőleges $\mathbb{R}^N$-re $\langle .,. \rangle$ szerint,

iii) $\langle .,. \rangle$ $(a,\alpha) \in \mathbb{R}_0^N \times \mathbb{R}^1$ pontbeli, $\mathbb{R}^N$ mentén $\mathbb{R}^1$-re vett projekciója a kanonikus skalárszorzat $U(|\alpha|^2)$-szerese, ahol $U : \mathbb{R}_+ \to \mathbb{R}_+$ sima függvény.

A fő eredményünk szerint, a természetes Riemann-metrika Levi-Civita konnexiójának Christoffel-szimbólumai az alábbiak:

$$\Gamma_{i,j}^k(a,\beta) = \begin{cases} 0 & \text{if} \quad i,j,k \leq n \\ 0 & \text{if} \quad i,j \leq n,\ k = n+1 \\ 0 & \text{if} \quad i,k \leq n,\ j = n+1 \\ 0 & \text{if} \quad j,k \leq n,\ i = n+1 \\ -\partial_k(U(z))/2 & \text{if} \quad k \leq n,\ i,j = n+1 \\ \partial_i(U(z))/2U(z) & \text{if} \quad j,k = n+1 \\ \partial_j(U(z))/2U(z) & \text{if} \quad i,k = n+1 \\ 0 & \text{if} \quad i,j,k = n+1 \end{cases},$$

ahol $1 \leq i,j,k \leq n+1$, $z = \langle \alpha, \beta \rangle$ és ∂_s az s-dik koordináta szerinti derivált. A geodetikus differenciálegyenlet-rendszere

$$\dot{\beta} = h/U(Z)$$

$$\ddot{a}_j = a_j h^2 U'(Z)/U^2(Z) \quad 1 \leq j \leq n$$

ahol h megfelelő konstans, $Z = (\alpha, \beta)$, és $(\alpha(s), \beta(s))$ az a geodetikus, melynek koordinátái $\{a_j\}_{j=1}^n$ és β. Tehát a geodetikusok következő leírását kapjuk. Adott az természetes Riemann-metrika szerinti $\mathbb{R}_0^N \times \mathbb{R}^1$-beli $(\mathbf{x}(s), \xi(s))$ geodetikus az $\mathbf{x}(0) = \mathbf{x}_0$, $\xi(0) = \xi_0$, $\dot{\mathbf{x}}(0) = \mathbf{t}_0$, $\dot{\xi}(0) = \tau_0$ $s = 0$-beli kezdeti értékekkel. Ekkor a következő lehetőségeink vannak:

a) Ha $\tau_0 = 0$, akkor az $(\mathbf{x}(s), \xi(s))$ geodetikust tartalmazza az $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$, $\xi(s) = \xi_0$ egyenes. Ez a geodetikus teljes, kivéve ha $\xi_0 = 0$ és a $\mathbf{t}_0$, $\mathbf{x}_0$ vektorok kollineárisak.

b) Ha $\tau_0 > 0$, akkor a geodetikus $\mathbb{R}_0^N$-re vett vetülete $\mathbf{0}$ középpontú ellipszis, melynek egyenlete

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

alakú. A megfelelő geodetikus teljes, kivéve azt az esetet, amikor a $\mathbf{t}_0$, $\mathbf{x}_0$ vektorok kollineárisak és a vetületi ellipszis $\mathbf{0}$ középpontú szakasszá fajul el.

c) Ha $\tau_0 < 0$, akkor a geodetikus $\mathbb{R}_0^N$-re vett vetülete $\mathbf{0}$ középpontú hiperbola. Ennek egyenlete

$$\mathbf{x}(s) = \cos h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

alakú. Ha a $\mathbf{t}_0$, $\mathbf{x}_0$ vektorok kollineárisak, akkor a vetületi hiperbola félegyenessé fajul. A megfelelő geodetikus teljes.