DAVID BENKO

THE EQUILIBRIUM MEASURE AND THE SAFF CONJECTURE

(Az Egyensúlyi Mérték és Saff Sejtése)

THESIS

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Advisor: Vilmos Totik, Ph.D.

1 The Approximation Problem

In [28], V. Totik settled the Saff conjecture, which was a basic long standing conjecture in the theory of approximation by weighted polynomials. Let $w : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that $\lim_{x \to \pm\infty} x w(x) = 0$. Let us define q by the $w(x) = \exp(-q(x))$ equation, and call $q(x)$ to be an *external field*. The problem of Saff was the following: what $f(x)$ functions can we uniformly approximated by weighted polynomials of the form $w(x)^n P_n(x)$.

This approximation problem appears in several applications. It must be emphasized that the exponent of the weight changes with n, so this is a different (and in some sense more difficult) type of approximation than what is usually called weighted approximation. In fact, the polynomial P_n must balance exponential oscillations in w^n.

Clearly, the function which we approximate must be continuous. We shall see that exactly those continuous functions can be approximated by $w^n P_n$ which vanishes on a particular closed set Z_w. So the problem reduces to finding Z_w.

Interestingly, a measure called the *equilibrium measure associated with w* plays an important role in the problem. We will denote the support of this measure by S_w, this is a compact set.

Let us assume that $q(x)$ is a convex function. Saff conjectured that in this case a continuous $f(x)$ can be uniformly approximated by weighted polynomials if and only if it vanishes outside of the support S_w. In other worlds, Z_w is the complement of the interior of S_w.

Originally this statement was believed to be true without the convexity restriction until a counterexample was given by Totik in [29]. In [29] he also proved the conjecture under an additional smoothness condition on q.

We remark that convex external fields have the useful property that S_w is an interval.

Instead of the whole real line we may define $w(x)$ on a closed subset Σ of the real line. Suppose now that $\Sigma = [0, +\infty)$. There is a theorem which says that if $xq'(x)$ is increasing on Σ then S_w is an interval. This is again a useful statement, since there are external fields which are not convex but $xq'(x)$ is increasing.

Totik raised the following question. Assume that $xq'(x)$ is increasing on Σ and let $[a, b]$ denote the support S_w. Can we uniformly approximate continuous functions vanishing outside (a, b) by weighted polynomials $w(x)^n P_n(x)$? This was the original problem which has lead to this dissertation.

The positive answer to this question was given in [3]. There the approximation problem was solved for so called "weak convex" function, which is a large class of functions which contains both the convex functions as well as those when $xq'(x)$ is increasing. As an example, consider the external field $q(x) := x^\lambda$, where $\lambda \in (0, 1)$. Obiously q is not convex on $[0, +\infty)$ but $xq'(x)$ is increasing there. So the S_w will be an interval $[a, b]$ (which interval may even be found in terms of λ). And continuous functions vanishing outside (a, b) can be approximated by weighted polynomials $w(x)^n P_n(x)$.

In this dissertation we extend even further the validity of the above results (see Chapter 4). We define "weak convex" functions with basepoints A and B, and consider external field which belongs to this class. This class is larger than those in [3], nevertheless we will still use the "weak convex" denomination. We will also prove that "weak convex" external fields generate an equilibrium measure whose support is an interval (see Chapter 2). In particular, the support will be an interval when $\exp(q(x))$ is a convex function.

In Chapter 3 we will also consider the equilibrium problem on the unit circle.

We give conditions which guarantee that the support of the equilibrium measure will be an arc of the circle.

2 Logarithmic Potential Theory

Let $\Sigma \subset \mathbb{R}$ be a closed set. A weight function w on Σ is said to be admissible, if it satisfies the following three conditions:

(i) w is upper semi-continuous,

(ii) $\{x \in \Sigma : w(x) > 0\}$ has positive capacity,

(iii) if Σ is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in \Sigma$.

In the approximation problem will always assume in the theorems that w is continuous, and that Σ is regular with respect to the Dirichlet problem in $\mathbf{C}\backslash\Sigma$. Condition (iii) can be relaxed, but that would make the proofs more complicated. (For related results when (iii) is not assumed see [5], [6] and [25].)

We define q by

$$w(x) =: \exp(-q(x)),$$

so $q : \Sigma \to (-\infty, \infty]$ is a lower semi-continuous function.

Let $\mathcal{M}(\Sigma)$ be the collection of all positive unit Borel measures with compact support in Σ. We define the *logarithmic potential* of $\mu \in \mathcal{M}(\Sigma)$ as

$$U^{\mu}(x) := \int \log \frac{1}{|x - t|} d\mu(t),$$

and the *weighted energy integral* as

$$I_w(\mu) := - \int \int \log(|x - y|w(x)w(y)) d\mu(x) d\mu(y).$$

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We say that a property holds quasi everywher (q.e.) if it holds everywhere except a set with capacity zero. We will need the following basic theorem ([24], Theorem I.1.3):

Theorem 1 *Let w be an admissible (not necessarily continuous) weight on the closed set Σ and let $V_w := \inf\{I_w(\mu) : \mu \in \mathcal{M}(\Sigma)\}$. Then*

(a) V_w is finite,

(b) there exists a unique element $\mu_w \in \mathcal{M}(\Sigma)$ such that $I_w(\mu_w) = V_w$,

(c) setting $F_w := V_w - \int q d\mu_w$, the inequality

$$U^{\mu_w}(x) + q(x) \geq F_w$$

holds quasi-everywhere on Σ,

(d) the inequality

$$U^{\mu_w}(x) + q(x) \leq F_w \tag{1}$$

holds for all $x \in S_w := \operatorname{supp}(\mu_w)$.

Remark According to our definition, every measure in $\mathcal{M}(\Sigma)$ has compact support, so the support S_w is a compact set.

The measure μ_w is called *the equilibrium* or *extremal measure* associated with w.

Notation 2 *When we say that a property holds* inside G *- where G is a subset of $\mathbb{R}$ - we mean that the property is satisfied on every compact subset of G.*

3   Totik's Results

We are considering uniform approximation of continuous functions on Σ by weighted polynomials of the form $w^n P_n$, where $deg P_n \leq n$. Theorem 3 is a Stone-Weierstrass type theorem for this kind of approximation (see [13], or see [24], Theorem VI.1.1).

Theorem 3 *There exists a closed set $Z = Z(w) \subset \Sigma$, such that a continuous function f on Σ is the uniform limit of weighted polynomials $w^n P_n$, $n = 1, 2, ...$, if and only if f vanishes on Z.*

Thus the problem of what functions can be approximated is equivalent to determining what points lie in $Z(w)$. This latter problem is intimately related to the density of μ_w. The support $S_w := \text{supp}(\mu_w)$ plays a special role (see [29], or [13], or see [24], Theorem VI.1.2):

Theorem 4 *The complement of S_w belongs to Z.*

The definition of functions with smooth integrals was introduced by Totik in [28]. This was a crucial definition in solving the Saff conjecture.

Definitions 5 *We say that a function $f(x)$ has smooth integral on $R \subset \mathbb{R}$, if $f(x)$ is non-negative a.e. on R and*

$$\int_I f = (1 + o(1)) \int_J f \tag{2}$$

where I, $J \subset R$ are any two adjacent intervals, both of which has length $0 < \epsilon$, and $\epsilon \to 0$. The $o(1)$ term depends on ϵ and not on I and J.

Clearly, all continuous functions which have a positive lower bound have smooth integral. But $\log(1/|t|)$ also has smooth integral on $[-1/2, 1/2]$.

The next three theorems are due to Totik, [28].

Theorem 6 *Let us suppose that $[a, b]$ is a subset of the support S_w, and the extremal measure has a density v on $[a, b]$ that has a positive lower bound and smooth integral there. Then $(a, b) \cap Z(w) = \emptyset$.*

Theorem 7 *Let us suppose that (a, b) is a subset of the support S_w, and that q is convex on (a, b). Then μ_w has a density v in (a, b) which has a positive lower bound and smooth integral inside (a, b).*

From these two theorems follows

Theorem 8 *Let us suppose that (a, b) is a subset of the support S_w, and that q is convex on (a, b). Then $(a, b) \cap Z(w) = \emptyset$. In particular, every function that vanishes outside (a, b) can be uniformly approximated by weighted polynomials of the form $w^n P_n$.*

Notice that Theorem 8 is a local result; it works for any part of the extremal support where q is convex.

4 Main Results

Here are some agreements we will follow.

Notation 9 *Unless otherwise noted, all intervals in the dissertation are arbitrary intervals (that is, they may be open, half open, closed, bounded or unbounded). The notation $I = [(a, b)]$ will mean that I is an interval (of any type) with endpoints a and b. The notation $\mathrm{int}(I)$ is used for the interior points of I. Absolute continuity inside I means that the function is absolutely continuous on any compact set which is lying in I.*

When we consider a function which is increasing on $[a, b]$, it is assumed that it is increasing only almost everywhere. This we will define more precisely in the dissertation. In this sense we can say that if q is convex thenn q' is increasing (although q' exists only a.e.).

Now we are ready to define "weak convexity". We remark that the convexity of q implies the convexity of $\exp(q)$, so the class of weak convex functions contains the convex functions.

Definition 10 *We say that a function $q : D \to \overline{\mathbb{R}}$ ($D \subset \mathbb{R}$) is weak convex on an interval $I = [(a, b)] \subset D$ ($a, b \in \overline{\mathbb{R}}$) with basepoints $A, B \in \overline{\mathbb{R}}$, $A < B$, if the following properties hold:*

(i) $I \subset [A, B]$,

(ii) q is absolutely continuous inside (a, b) (so q' exists a.e. in I),

(iii) if $a \in I$, then

$$\liminf_{x \to a+0} q(x) = q(a),$$

and if $b \in I$, then

$$\liminf_{x \to b-0} q(x) = q(b),$$

(iv) I can be written as the disjoint union of finitely many intervals $I_1, ..., I_n$ such that for any interval I_k ($1 \leq k \leq n$)):

$$\exp(q(x)) \quad \text{is convex on } I_k, \quad \text{or} \tag{3}$$

$$(x - A)(B - x)q'(x) + x \quad \text{is increasing on } I_k. \tag{4}$$

If $-\infty < A$ and $B = +\infty$ then (4) should be replaced by:

$$(x - A)q'(x) \quad \text{is increasing on} \quad I_k. \tag{5}$$

If $-\infty = A$ and $B < +\infty$ then (4) should be replaced by:

$$(B - x)q'(x) \quad \text{is increasing on} \quad I_k. \tag{6}$$

If both $-\infty = A$ and $B = +\infty$ then (4) should be ignored, thus in this case $\exp(q(x))$ must be convex on the whole I.

(v) if x_0 is any endpoint of any I_k but different of a and b then:

$$\limsup_{x \to x_0^-} q'(x) \leq \liminf_{x \to x_0^+} q'(x), \tag{7}$$

We will simply just say that q is weak convex on an interval I (without mentioning the basepoints), if q is weak convex on the interval $I = [(a, b)] \subset D$ $(a, b \in \overline{\mathbb{R}})$ with basepoints a and b.

The following theorem is our *main theorem.*

Theorem 11 *Let w be a continuous admissible weight on $\mathbb{R}$. Suppose that q is weak convex on $[A, B]$ with finite basepoints A, B satisfying $S_w \subset (A, B)$. Then $Z(w) = (\text{int } S_w)^c$. Thus a continuous function $f(x)$ can be uniformly approximated by weighted polynomials $w^n P_n$ if and only if $f(x)$ vanishes outside S_w.*

We will also give several conditions on q which guarantees that the support of the equilibrium measure is an interval. Our most general theorem which guarantees that the support is an interval is the following.

Theorem 12 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and suppose that q is weak convex on the interval $I \subset \Sigma$ with basepoints $A, B \in \overline{\mathbb{R}}$ satisfying $S_w \subset [A, B]$. Then $S_w \cap I$ is an interval.*

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In the next theorem we give an integral representation for the density of the equilibrium measure. For the sake of simple notations we state the next theorem for the interval $[-1, 1]$, instead of $[A, B]$.

Theorem 13 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ such that $\min S_w = -1$, $\max S_w = 1$. Suppose that q is absolutely continuous inside $(-1, 1)$ and q' is bounded on $[-1, 1]$. Assume further that $\liminf_{x \to -1+0} q(x) = q(-1)$, $\liminf_{x \to 1-0} q(x) = q(1)$ and with some constants $-1 \leq u \leq v \leq 1$ we have*

$$q'(x) \leq \frac{-1}{1 - x} \quad on \quad (-1, u),$$

$$\frac{-1}{1 - x} \leq q'(x) \leq \frac{1}{x + 1} \quad on \quad (u, v),$$

$$\frac{1}{x + 1} \leq q'(x) \quad on \quad (v, 1),$$

$$(1 - x)^2 q'(x) - x \quad is\ increasing\ on \quad (-1, u),$$

$$(1 - x^2) q'(x) + x \quad is\ increasing\ on \quad (u, v),$$

$$(x + 1)^2 q'(x) - x \quad is\ increasing\ on \quad (v, 1).$$

Then $S_w = [-1, 1]$ and the density of μ_w is $d\mu_w(t) = v(t)dt$ a.e. $t \in [-1, 1]$, where we define $v(t)$ for a.e. $t \in [-1, 1]$ as follows:

For $t \in (-1, u)$, let

$$v(t) := \frac{1}{\pi^2 \sqrt{1 - t^2}} \left[\frac{t + 1}{1 - t} \int_{-1}^{1} \frac{(1 - s)^2 q'(s) - s - [(1 - t)^2 q'(t) - t]}{(s - t)\sqrt{1 - s^2}} ds \right.$$

$$\left. + \frac{2}{1 - t}(\pi + \int_{-1}^{1} \frac{(1 - s)q'(s)}{\sqrt{1 - s^2}} ds) \right] \tag{8}$$

For $t \in (u, v)$, let

$$v(t) := \frac{1}{\pi^2 \sqrt{1 - t^2}} \int_{-1}^{1} \frac{(1 - s^2)q'(s) + s - [(1 - t^2)q'(t) + t]}{(s - t)\sqrt{1 - s^2}} ds. \qquad (9)$$

For $t \in (v, 1)$, let

$$v(t) := \frac{1}{\pi^2 \sqrt{1 - t^2}} \left[\frac{1 - t}{t + 1} \int_{-1}^{1} \frac{(s + 1)^2 q'(s) - s - [(t + 1)^2 q'(t) - t]}{(s - t)\sqrt{1 - s^2}} ds \right.$$

$$\left. + \frac{2}{t + 1} (\pi - \int_{-1}^{1} \frac{(s + 1)q'(s)}{\sqrt{1 - s^2}} ds) \right] \qquad (10)$$

In Chapter 3 we will consider the equilibrium problem on the unit circle C. Using the unit circle we also reveal the connections between different conditions on the real line, and we give a new condition not covered by the weak convexity definition at Definition 10.

Let now $\alpha, \beta \in \mathbb{R}$ be two angles, $|\beta - \alpha| < 2\pi$. We define $\widehat{[\alpha, \beta]}$ to be the arc $[e^{i\alpha}, e^{i\beta}] \subset C$, where we go from $e^{i\alpha}$ to $e^{i\beta}$ in a counterclockwise direction. We define $\widehat{[\alpha, \alpha + 2\pi]}$ to be the full circle C.

We will prove the following two theorems:

Theorem 14 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor \gamma, \delta \rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Assume that q is absolutely continuous inside I and*

$$\liminf_{\substack{x \to y \\ x \in I}} q(x) = q(y) \qquad (11)$$

whenever y is an endpoint of I with $y \in I$. Let e^{ic} be any point which is not an interior point of $\widehat{I}$. Let $\widehat{[\alpha_1, \beta_1]}, ..., \widehat{[\alpha_k, \beta_k]}$ be $k \geq 0$ arcs of C. Here, for all $1 \leq i \leq k$, $0 < \beta_i - \alpha_i \leq 2\pi$ and $(S_w \cup \widehat{I}) \subset \widehat{[\alpha_i, \beta_i]}$. Suppose further that I

can be written as a disjoint union of $n \geq 1$ intervals $I_1, ..., I_n$ and for any fixed $1 \leq j \leq n$, either

$$e^{q(\theta)} \left[2 \sin \left(\frac{\theta - c}{2} \right) q'(\theta) - \cos \left(\frac{\theta - c}{2} \right) \right] \mathrm{sgn} \left(\sin \left(\frac{\theta - c}{2} \right) \right) \qquad (2.2)$$

is increasing on I_j or for some $1 \leq i \leq k$:

$$\sin \left(\frac{\theta - \alpha_i}{2} \right) \sin \left(\frac{\beta_i - \theta}{2} \right) q'(\theta) + \frac{1}{4} \sin \left(\theta - \frac{\alpha_i + \beta_i}{2} \right) \qquad (2.3)$$

is increasing on I_j. Finally we assume that

$$\limsup_{\theta \to \theta_0^-} q'(\theta) \leq \liminf_{\theta \to \theta_0^+} q'(\theta),$$

whenever θ_0 is an endpoint of I_j $(1 \leq j \leq n)$ but not an endpoint of I. Then $S_w \cap \widehat{I}$ is an arc of C.

Here sgn denotes the signum function.

Example: The following example illustrates the theorem.

Let $q(\theta) = \cos(5\theta) \sin(3\theta)$ defined on $\Sigma = [2.9, 3.18] \cup [3.95, 4]$. (We may define w to be zero outside Σ so that w is defined on C.) We claim that both $S_w \cap \widehat{[2.9, 3.18]}$ and $S_w \cap \widehat{[3.95, 4]}$ are arcs of C. (One of them may be an empty set.)

Take $\alpha_1 = 2.9, \beta_1 = 4$ and $\alpha_2 = 3.95, \beta_2 = 3.18 + 2\pi$.

One can verify that (2.2) is satisfied on $[2.9, 3.17]$ but not on the whole $[2.9, 3.18]$. (At (2.2) c can be chosen to be any number such that e^{ic} is not an interior point of $\widehat{[2.9, 3.18]}$.) Also, using α_1 and β_1 we see that (2.3) is not satisfied on the whole $[2.9, 3.18]$. However (2.3) is satisfied on the subinterval $[3.17, 3.18]$. So the combination of the (2.2) and (2.3) conditions implies that

$S_w \cap \widehat{[2.9, 3.18]}$ is an arc. Note that (2.3) is satisfied using α_2 and β_2 on the whole $[3.95, 4]$. So $S_w \cap \widehat{[3.95, 4]}$ is an arc.

Theorem 15 *For given $k \in \mathbb{N}^+$ let*

$$\Sigma := \cup_{i=1}^k [A_i, B_i] \subset \bar{\mathbb{R}}, \quad where \tag{12}$$

$$-\infty < A_1 \le B_1 < A_2 \le B_2 < ... < A_k \le B_k < +\infty.$$

Let $W = \exp(-Q)$ be a weight on Σ, $I \subset \Sigma$ be an interval and assume that Q is absolutely continuous inside I and

$$\liminf_{\substack{X \to Y \\ X \in I}} Q(X) = Q(Y) \tag{13}$$

whenever Y is an endpoint of I with $Y \in I$. Assume further that I can be written as a disjoint union of intervals $I_1, ..., I_n$ such that for any fixed $1 \le j \le n$ either

$$e^{Q(X)} \quad is \ convex \ on \ I_j, \tag{14}$$

or for some $1 \le i \le k - 1$

$$(X - B_i)(A_{i+1} - X)Q'(X) + X \quad is \ decreasing \ on \ I_j, \tag{15}$$

or

$$(X - A_1)(B_k - X)Q'(X) + X \quad is \ increasing \ on \ I_j. \tag{16}$$

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Finally we assume that

$$\limsup_{X \to X_0^-} Q'(X) \leq \liminf_{X \to X_0^+} Q'(X),$$

whenever X_0 is an endpoint of I_j ($1 \leq j \leq n$) but not an endpoint of I. Then $S_W \cap I$ is an interval.

Example: Let $\Sigma := [-2, -1] \cup [1, 2]$, and let $Q(X) = \log(X + 1)$, $X \in [1, 2]$, and on $[-2, -1]$ let $Q(X)$ an arbitrary lower semi-continous function. Then $S_W \cap [1, 2]$ is an interval. Indeed, $(X - (-1))(1 - X)Q'(X) + X$ is constant 1 ($X \in [1, 2]$), i.e., it is a decreasing function.

5 Some Lemmas

The following lemma is crucial in the proof of Theorem 12:

Lemma 16 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and let $I \subset \Sigma$ be an interval. Let μ_w be the equilibrium measure associated with w. Assume that q is absolutely continuous inside I and that*

$$\liminf_{\substack{x \to z \\ x \in I}} q(x) = q(z), \quad \text{if } z \text{ is an endpoint of } I \text{ with } z \in I. \tag{17}$$

If for some function $f : \text{int}(I) \to \mathbb{R}^+$, the function $f(x)\frac{d}{dx}[U^{\mu_w}(x) + q(x)]$ is strictly increasing on $\text{int}(I)$, then $S_w \cap I$ is an interval.

As we already have mentioned, an important special case of Theorem 12 is:

Theorem 17 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and suppose that $\exp(q)$ is convex on the interval $I \subset \Sigma$ and satisfies condition (17). Then $S_w \cap I$ is an interval.*

Another special case is the following theorem:

Theorem 18 *Let $w = \exp(-q)$ be an admissible weight on $\mathbb{R}$ and let $I \subset \Sigma$ be an interval. Suppose that q is absolutely continuous inside I and satisfies condition (17). Suppose that with some finite constants $A < B$ we have $I \subset [A, B]$, $S_w \subset [A, B]$ and $(x - A)(B - x)q'(x) + x$ is increasing on the interval I. Then $S_w \cap I$ is an interval.*

When is the "$\exp(q)$ is convex" condition weaker than the "$(x - A)(B - x)q'(x) + x$ is increasing" condition? The answer is given in the following proposition. For simplicity let $A := -1$ and $B := 1$.

Proposition 19 *Let $J \subset [-1, 1]$ be an open interval and q be a function defined on J which is absolute continuous inside J.*

a) If

$$q'(x) \in \left[\frac{-1}{1 - x}, \frac{1}{x + 1} \right], \quad x \in J,$$

and $\exp(q(x))$ is convex on J, then $(1 - x^2)q'(x) + x$ is increasing on J.

b) If

$$q'(x) \geq \frac{1}{x + 1}, \quad x \in J, \quad (\text{ or } q'(x) \leq \frac{-1}{1 - x}, \quad x \in J),$$

and $(1 - x^2)q'(x) + x$ is increasing on J, then $\exp(q)$ is convex on J.

We also remark that if q'' exists, then (3) holds if and only if

$$0 \leq (q'(x))^2 + q''(x), \quad x \in I_k,$$

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(4) holds if and only if

$$0 \leq \frac{1}{(x-A)(B-x)} + \left(\frac{1}{x-A} - \frac{1}{B-x}\right)q'(x) + q''(x), \quad x \in I_k, \tag{18}$$

finally, (5) holds if and only if

$$0 \leq \frac{1}{x-A}q'(x) + q''(x), \quad x \in I_k.$$

Remark At Theorem 14 the choice of c is not important. We also remark that if $\widehat{I}$ is the full circle, then one should check only condition (2.2) and ignore (2.3) which is a stronger assumption. A simple corollary of Theorem 14 is:

Corollary 20 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I_1 := (\gamma_1, \gamma_1 + 2\pi)$ and $I_2 := (\gamma_2, \gamma_2 + 2\pi)$ where $e^{i\gamma_1} \neq e^{i\gamma_2}$. Assume that (2.2) is increasing on I_1 where $c := \gamma_1$, and (2.2) is increasing on I_2 where $c := \gamma_2$. Then $S_w = C$.*

A special case of Theorem 14 is:

Lemma 21 *Let $w(z) = \exp(-q(z))$, $|z| = 1$ be a weight on C and let $I = \lfloor \gamma, \delta \rceil$ be an interval with $0 < \delta - \gamma \leq 2\pi$. Suppose q is absolutely continuous inside I and satisfies (11). Let e^{ic} be any point which is not an interior point of $\widehat{I}$. If*

$$e^{q(\theta)}\left[2\sin\left(\frac{\theta-c}{2}\right)q'(\theta) - \cos\left(\frac{\theta-c}{2}\right)\right]\mathrm{sgn}\left(\sin\left(\frac{\theta-c}{2}\right)\right) \tag{19}$$

is increasing on I, then $S_w \cap \widehat{I}$ is an arc of C.

We remark that if q is twice differentiable then condition (19) is equivalent to the condition

$$q'(\theta)^2 + q''(\theta) + \frac{1}{4} \geq 0, \quad \theta \in (\gamma, \delta)$$

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The following defintions are used in the proof of Theorem 11.

Let $x \in [-1, 1]$. Depending on the value of $c \in [-1, 1]$ the following integrals may be regular Lebesgue integrals or Cauchy principal value integrals.

$$v_c(x) := -PV \int_{-1}^{c} \frac{\sqrt{1-t^2}}{\pi^2 \sqrt{1-x^2}(t-x)} dt,$$

$$h_c(x) := PV \int_{c}^{1} \frac{\sqrt{1-t^2}}{\pi^2 \sqrt{1-x^2}(t-x)} dt.$$

For $0 < \iota$ and $a \in \mathbb{R}$ we define

$$a_\iota^+ := \max(a, \iota) \quad \text{and} \quad a_\iota^- := \max(-a, \iota).$$

In the proof of the approximation problem the following lemmas are crucial.

Lemma 22 *Let $-1 < a < b < 1$ and $0 < \iota$ be fixed. Then the family of functions $\mathcal{F}^+ := \{v_c(x)_\iota^+ : c \in [-1, 1]\}$ and $\mathcal{F}^- := \{v_c(x)_\iota^- : c \in [-1, 1]\}$ have uniformly smooth integrals on $[a, b]$.*

Lemma 23 *Let $F(x) = G(x) - H(x)$, where $F(x)$, $G(x)$, $H(x)$ are a.e. non-negative functions defined on an interval, $G(x)$ and $H(x)$ have smooth integrals and $H(x) \leq (1 - \eta)G(x)$ a.e. with some $\eta \in (0, 1)$. Then $F(x)$ has smooth integral.*

Lemma 24 *Let $N(x)$ be a right-continuous function on $[-1, 1]$ which is of bounded variation. Let $f(x) \in L^1([-1, 1])$ be non-negative. Then*

$$PV \int_{-1}^{1} \frac{f(t)N(t)}{t-x} dt = -N(1)f_1(x) + \int_{(-1,1]} f_t(x) dN(t), \quad a.e. \ x \in [-1, 1], \quad (20)$$

where the integral on the right hand side is a Lebesgue-Stieltjes integral and

$$f_c(x) := -PV \int_{-1}^{c} \frac{f(t)}{t-x} dt, \quad a.e. \ x \in [-1,1].$$

Definition 25 *We say that a function $g(x)$ has bounded variation almost everywhere inside a set $E \subset \mathbb{R}$ if for every compact set $F \subset E$ there exists $G \subset F$ such that $F \setminus G$ has measure zero and $g(x)$ has bounded variation on G.*

Lemma 26 *Let w be an admissible weight which is absolutely continuous inside $\mathbb{R}$. Let the interval $[a_0, b_0]$ be a subset of the support S_w. If $q'(x)$ has bounded variation a.e. inside (a_0, b_0) and the extremal measure has a density V on $[a_0, b_0]$ that has positive lower bound inside (a_0, b_0) then $(a_0, b_0) \cap Z(w) = \emptyset$.*

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