Width-type graph parameters

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1 Introduction

The thesis deals with a cornerstone of a modern subject — Graph Minor Theorem (GMT) — called path-width, and the possible generalizations of it.

A graph H is a *minor* of a graph G, $G \succeq H$ in notation, if H can be obtained from a subgraph of G by contracting edges.

A *graph parameter* is a graph property which is expressed by natural numbers. A graph parameter π is called *minor-monotone* if $G \succeq H$ implies $\pi(G) \geq \pi(H)$.

A *path-decomposition* of a graph G is a pair (P, W), where P is a path and $W = (W_p : p \in V(P))$ is a family of subsets of $V(G)$, satisfying

(1) $\bigcup_{p \in V(P)} W_p = V(G)$, and every edge of G has both ends in some W_p, and

(2) if $p, p', p'' \in V(P)$ and p' lies on the path from p to p'', then $W_p \cap W_{p''} \subseteq W_{p'}$.

(the sets W_i are usually called bags)

The *width* of a path-decomposition is $\max(|W_p| - 1 : p \in V(P))$, and the *path-width* of G ($pw(G)$ in notation) is the minimum width over all path-decompositions of G.

This graph parameter is minor-monotone and related to a cops-and-robber game on graphs.

Remark 1.1 $pw(K_n) = n - 1$.

Trees can have arbitrarily large path-width. To see this, let T_k denote the symmetric, ternary tree of height k. More exactly T_k has one specified vertex r of degree 3, all other vertices (except the leaves) have degree 4, and all leaves have distance k from r.

Lemma 1.2 $pw(T_k) \geq k$.

There is another thing which can make path-width big. Namely a big grid-minor. This fact somehow means that the graph is 'highly' connected.

Definition 1.3 *Consider the graph on* $\{1, \ldots, n\}^2$ *with the edge set*

$$\{(i,j)(i',j') : |i - i'| + |j - j'| = 1\}.$$

This graph is called the $n \times n$ *grid and denoted* J_n. *(Clearly the adjacency graph of an* $n \times n$ *chess-board.)*

Lemma 1.4 *The* $n \times n$ *grid has path-width* n.

The historical overview of the subject can be found in Chapter 1. We mention that any minor-closed class of graphs can be characterized by a finite list of so-called excluded minors. This fact is a consequence of the GMT proved by Robertson and Seymour. The complete proof of the theorem itself is very long and difficult. A key concept of the proof is a minor-monotone graph parameter.

This measures the tree-likeness of the graph in some sense, thus it is called tree-width.

The main chapters of the thesis are arranged as follows:

Chapter 3: Cops-and-robber games
3.1 Cops-and-robber games on graphs
3.2 Cops-and-robber games on directed graphs
3.3 Monotonicity for directed graphs
3.4 Directed path-width
3.5 Blockages for directed graphs

Chapter 4: Characterization of graphs with path-width two
4.1 Basics
4.2 $PW2$-safe operations
4.3 Non-reducible graphs characterization theorem
4.4 Path-width of the non-reducible graphs
4.5 Partial tracks
4.6 The structure of graphs with path-width two
4.7 Recognition of graphs with path-width at most two

Chapter 5: New minor-monotone graph parameters
5.1 Arc-width of graphs
5.2 Arc-width of the complete bipartite graph
5.3 Arc-width of non-connected graphs; the mM parameter
5.4 Excluded minor theorems for mM

2 Cops-and-robber games

In Chapter 3, we first present the known results and methods related to cops-and-robber games on graphs. The importance of these games is that the arising minor-monotone graph parameters are equivalent to tree- resp. path-width. Moreover some proofs previously requiring many technical ideas became essentially simpler. E.g. the previously mentioned Lemma 1.4 is an example.

Definition 2.1 *Let G be a graph. There is a robber standing on a vertex of G. There are k cops willing to capture the robber. The robber can run at any time to another vertex along edges with great speed. The movement of the cops is only possible by helicopter, but they can fly to an arbitrary vertex. Let the robber be invisible for the cops. (We can think that the surface is covered by forests.) So the cops cannot see the robber from the helicopter. The meaning of the great speed is as follows. When the robber see a helicopter approaching a vertex, he can still decide to run somewhere. However the robber cannot run through a vertex which is occupied by a cop. The cops can only capture the robber if they occupy all neighbors of the vertex where the robber is standing, and then with one extra cop they capture the robber.*

If there is a winning strategy for k cops, we say that 'there is a capture with k cops', or 'k cops can search the graph'.

The goal is to decide how many cops are necessary to capture the robber. This minimum is denoted by $\overline{cn}(G)$. (cn stands for cop number, overline for the invisible case.)

Remark 2.2 *A graph G is called a caterpillar iff it is a path with pendant edges attached to some of its vertices.*

An easy excluded minor theorem is the following.

Lemma 2.3 *Two cops can capture an invisible robber in a connected graph G iff G is a caterpillar. Equivalently iff* $G \not\succeq K_3, Y_1$, *where* Y_1 *is the graph on Figure 1.*

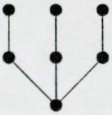

Figure 1:

One side of the relation between path-width and the above mentioned cops-and-robber game is indicated in the next lemma.

Lemma 2.4 *If G has path-width at most* $k-1$, *then k cops can capture an invisible robber in G. In notation:* $pw(G) \le k-1 (\Leftrightarrow pw^*(G) \le k) \Rightarrow \overline{cn}(G) \le k$

⋆

In Chapter 3, we generalize many of the relevant definitions and results on graphs to directed graphs.

There are several possible ways to define cops-and-robber games on directed graphs. For us the most natural one is the following:

Definition 2.5 *[2] Let a directed graph D be given. The robber can run along the directed edges in the indicated direction. The cops move by helicopters. Assume in this version, that the robber is invisible. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by* $\overline{cn^*}(D)$. *(cn stands for cop number, overline for the invisible case, * indicates that the robber's move is not restricted like in Definition 2.9.)*

First we mention some basic properties of this new parameter.

Lemma 2.6 *One cop is enough to capture the invisible robber in D, iff D has no directed circuit as a subgraph, hence D is acyclic.*

Lemma 2.7 *(a) If* $A \subset V(D)$, $|A| = k$ *is a set of vertices s.t.* $V \setminus A$ *has no directed circuit, then* $\overline{cn^*}(D) \le k+1$

(b) Let D be a directed circuit with at least two vertices. Then $\overline{cn^*}(D) = 2$

Remark 2.8 *The game when the robber is invisible can be similarly defined. The number of necessary cops to capture the robber is denoted by* $cn(G)$ *in that case.*

Let us consider another version of cops-and-robber games appearing in [14].

Definition 2.9 *Let a directed graph D be given. The cops are either standing on a vertex or in a helicopter (temporarily removed from the game). The robber stands on a vertex of D, and can at any time with great speed run to another vertex in the same strong component of $D \setminus Z$, where Z is the set of vertices occupied by the cops. In other words, the robber can only move from a to b, if there is also a cop-free directed path from b to a. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by $cn(D)$ if the robber is visible, and $\overline{cn}(D)$ if the robber is invisible.*

In the aforementioned [14] only the visible case was considered. There are some trivial connections between the so far defined parameters, which we indicate on the next figure.

Remark 2.10

$$\begin{array}{ccc} cn^*(D) & \leq & \overline{cn^*}(D) \\ \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} \\ cn(D) & \leq & \overline{cn}(D) \end{array}$$

⋆

We also simplify the search of the cops in a game to monotone search. This game is the directed counterpart of the game described in [6].

Definition 2.11 *A mixed-search in a directed graph D is a sequence of pairs*

$$(A_0, Z_0), \ldots, (A_n, Z_n)$$

(intuitively Z_i is the set of vertices occupied by the cops immediately before the (i+1)st step, and A_i is the set of clear edges) such that

(I) $0 \leq i \leq n$, $A_i \subseteq E(D)$, $Z_i \subseteq V(D)$,

(II) $0 \leq i \leq n$, any vertex which is a head of an edge in $E(D) \setminus A_i$ and tail of an edge in A_i is in Z_i,

(III) $A_0 = \emptyset$, $A_n = E(D)$,

(IV) (List of possible moves) for $1 \leq i \leq n$, either

(a) (placing new cops) $Z_i \supseteq Z_{i-1}$, and $A_i = A_{i-1}$, or

(b) (removing cops) $Z_i \subseteq Z_{i-1}$, and A_i is the set of edges e, s.t. every directed path containing an edge of $E(D) \setminus A_{i-1}$ before e in order, has an internal vertex in Z_i, and $A_i \subseteq A_{i-1}$, or

(c) (node searching e) $Z_i = Z_{i-1}$ and $A_i \subseteq A_{i-1} \cup \{e\}$ for some edge $e \in E(D) \setminus A_{i-1}$ with both ends in Z_{i-1}, or

(d) (sliding) $Z_i = (Z_{i-1} \setminus \{u\}) \cup \{v\}$ for some $u \in Z_{i-1}$ and $v \in V(D) \setminus Z_{i-1}$ and $e = (v, u) \in E(D)$, s.t. every other in-edge to u belongs to A_i, and $A_i = A_{i-1} \cup \{e\}$, or

(e) (clearing an edge with one cop) $Z_i = Z_{i-1}$, *and* $A_i = A_{i-1} \cup \{e\}$ *for some edge* $e = (u, v) \in E(D) \setminus A_{i-1}$ *with head* v *in* Z_{i-1} *and every (possibly* 0*) edge with head* u *in* A_{i-1}.

If $|Z_i| \leq k$ *for* $1 \leq i \leq n$, *then* $\overline{cn}_m(D) \leq k$ *in notation.*

Definition 2.12 *A mixed-search of* D *is called monotone, if every edge of* D *is cleared exactly once. This is the same as saying that the cleared edges form a monotone increasing set.*

Lemma 2.13 *[2] If there is a mixed-search of* D *with at most* k *cops, then there is a monotone mixed-search of* D *with at most* k *cops.*

⋆

The notion of directed path-width came up in a joint work of Bruce Reed, Paul Seymour and Robin Thomas:

Definition 2.14 *One can define a directed path-decomposition (dpd) as a sequence* $W_1, W_2, \ldots, W_k$ *such that*

(i) the union of W_i *is* $V(D)$, *and*

(ii) if $i < j < k$, *then* $W_i \cap W_k$ *is a subset of* W_j, *and*

(iii) an edge either has both endpoints in the same W_i *or has its head in* W_i *and tail in* W_j, *where* $i \leq j$.

The width of a dpd is the maximum size of a W_i *minus one. (The* W_i*'s are called bags again.) The directed path-width (dpw) of a digraph* D *is the minimum width over all possible dpd's.*

$$dpw = \min_{W_i \text{ is a dpd}} \left(\max_{1 \leq i \leq k} (|W_i| - 1) \right)$$

We can call dpw a generalization of path-width to directed graphs, more precisely the following is true:

Lemma 2.15 *Let* G *be a graph, and let* D *be the graph obtained from* G *by replacing every edge by two directed edges in opposite directions. Then the path-width of* G *is equal to the directed path-width of* D.

We deduce an equivalence theorem, which includes directed path-width. An equivalence showing that dpw corresponds to $\overline{cn}^*$, so Definition 2.5 and 2.14 lead to the same thing.

Theorem 2.16 *[2] For* $k \geq 1$ *the following are equivalent:*

(i) $dpw(D) \leq k - 1$,

(ii) $\overline{cn}^*(D) \leq k$,

(iii) there is a monotone capture of an invisible robber in D with at most k cops.

⋆

The notion of a blockage was introduced by Bienstock et al. in [7] as obstructions in graphs for having small pw. We generalize this notion to digraphs. The main task here is to define the concepts in the appropriate way for directed graphs. Then the same proof techniques can be used.

Definition 2.17 *Let $k \geq 0$ be an integer. A blockage (in D, of order k) is a set $\mathcal{B}$ s.t.*

(i) each $X \in \mathcal{B}$ is a subset of V with $\alpha(X) \leq k$,

(ii) if $X \in \mathcal{B}$ and $Y \subseteq X$ and $\alpha(Y) \leq k$, then $Y \in \mathcal{B}$,

(iii) if X_1 and X_2 are complementary and $|X_1 \cap X_2| \leq k$, then $\mathcal{B}$ contains exactly one of X_1, X_2.

We call these the blockage axioms.

We prove the non-existence of a blockage of order k if the directed path-width was less than k. In the undirected case the two claims are equivalent. We conjecture that the reverse implication does probably not hold for directed graphs.

Lemma 2.18 *[2]* (i) *implies* (ii).

(i) $dpw(D) \leq k-1$,

(ii) *there is no blockage of order k in D.*

⋆

Still in Chapter 3 we consider two classes of digraphs, which seems to be interesting regarding cops-and-robber games. We give explicitly the exact values of their parameters. Moreover we state a conjecture saying that one of the classes is extremal in some sense.

Definition 2.19 *[14] For $k = 1, 2, \ldots$ let J_k be the union of k directed circuits $C_1, C_2, \ldots, C_k$ of length $2k$, and $2k$ directed paths $P_1, P_2, \ldots, P_k$ of length k resp. $Q_1, Q_2, \ldots, Q_k$ of length k. Here for $i = 1, 2, \ldots k$ C_i has vertex set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,k}, v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ (in order), P_i has vertex set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,k}\}$ (in order), and Q_i has vertex set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ (in order). Thus J_k has a planar drawing, where the circuits are concentric, the P's are disjoint paths linking C_1 to C_k, and the Q's are disjoint paths linking C_k to C_1. (See Figure 2.)*

Lemma 2.20 $cn(J_k) = \overline{cn}(J_k) = k$

Lemma 2.21 $cn^*(J_k) = \overline{cn^*}(J_k) = k + 1$

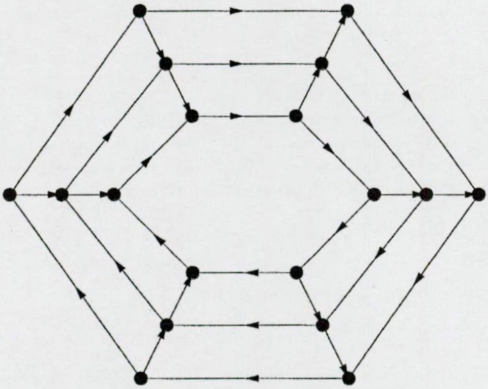

Figure 2: The 'candidate' directed grid; case $k = 3$

Definition 2.22 *For* $k = 1, 2, \ldots$ *let* I_k^s *be the union of* k *directed circuits* $C_1, C_2, \ldots, C_k$ *of length* s, *and* s *copies of the complete undirected graph on* k *vertices,* $K_k^1, K_k^2, \ldots, K_k^s$, *where* C_i *has vertex set* $\{u_{i,1}, u_{i,2}, \ldots, u_{i,s}\}$, *and* K_k^j *has vertex set* $\{u_{1,j}, u_{2,j}, \ldots, u_{k,j}\}$.

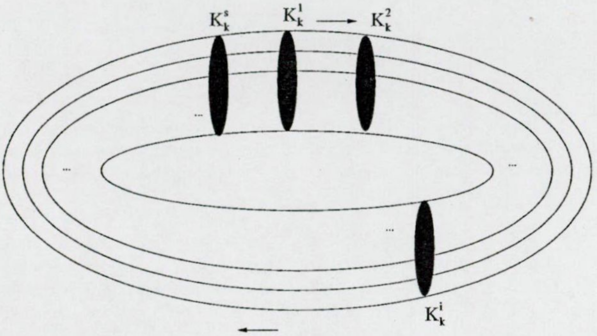

Figure 3:

Lemma 2.23 *Assume that* $s \gg k$ *($s > 2k$ say).*

(i) $cn^*(I_k^s) = \overline{cn^*}(I_k^s) = 2k$

(ii) $cn(I_k^s) = \overline{cn}(I_k^s) = k + 1$

The above Lemmas prove that the different cop-parameters behave different. However we conjecture that Lemma 2.23 is best possible in some sense.

Conjecture 2.24 $cn^*(D) \leq 2(cn(D) - 1)$ *and* $\overline{cn^*}(D) \leq 2(\overline{cn}(D) - 1)$.

3 Characterization of graphs with path-width two

Chapter 4 uses an equivalent definition of path-width. There are intervals of the real line assigned to the vertices of a graph. If two vertices are adjacent, then the corresponding intervals must intersect. (Not necessarily vice versa.) The width of a point is the number of intervals containing it. The width of such an interval-representation is the maximum width of the points. (Hence the maximum number of pairwise intersecting intervals.) The width of a graph is the minimum width of its interval-representations. This parameter is minor-monotone, and essentially equivalent to path-width. But this language allows us to prove some of the results in Chapter 4.

The excluded minor characterization of the graphs with path-width two was one of the goals of this thesis. This was considered as a difficult question, and only computer-aided proof existed before. The thesis achieves this result by other methods. Introducing certain operations, it considers the minimal graphs respect to an ordering finer than the minor relation. We had to prove that the introduced operations preserve path-width. A prototype of such a lemma is the following:

Operation 4: Instead of a formal description, let Figure 4 define operation O_4. (The attachments are drawn as full circles.)

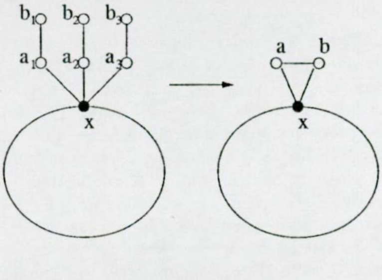

Figure 4: O_4 reduction

Lemma 3.1 *O_4 is a PW2-safe reduction, i.e.* $pw(G) \leq 2$ *iff* $pw(O_4(G)) \leq 2$.

There are 6 new path-width preserving operations introduced other than edge-deletion and edge-contraction.

⋆

Instead of the 110 excluded minors, we describe the class of path-width two graphs with 10 excluded non-reducible graphs.

Theorem 3.2 *[3] The following statements are equivalent:*

(i) G has path width at most two,

(ii) G is not reducible to any of the graphs listed on Figure 5.

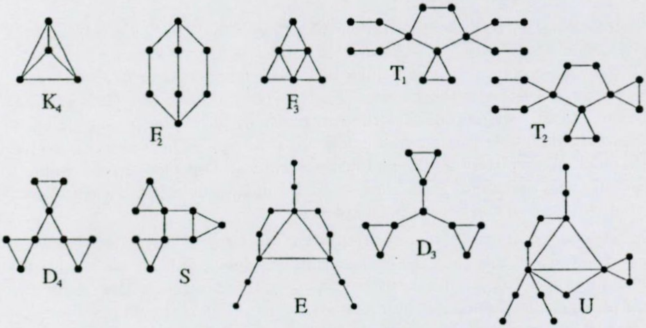

Figure 5: Non-reducible graphs

In Section 4.5 we describe a wide class of graphs with path-width at most two.

Definition 3.3 *[4] A graph G is called track graph (or shortly a track) iff it can be represented in the following way. Let P and Q be two vertex disjoint paths. Their vertex sets are $V(P) = \{ p_1, p_2, \dots, p_l \}$ and $V(Q) = \{ q_1, q_2, \dots, q_k \}$ (the indices reflect the order of vertices along the path). The graph G contains a disjoint copy of P and Q, and some connections between them. We allow two types of connections. First we can have edges connecting a vertex of P to a vertex of Q. The $p_i q_j$ edge is called ij-chord. Second we allow paths of length two connecting P and Q. We call these paths long chords. A long chord has three nodes, a p_i, a middle node m and a q_j. In this case we say that our long chord has type ij, it is a long ij-chord. We assume that for different long chords the middle nodes are different. We assume that if ij and $i'j'$ occur as types of chords or long chords, then $(i - i')(j - j') \geq 0$, i.e. the chords and long chords are not crossing. So G is a track graph if its vertex set is the disjoint union of $V(P)$, $V(Q)$ and M, and its edge set is the disjoint union of $E(P)$, $E(Q)$ and the edges of non-crossing chords, long chords (the last two types are called middle edges).*

Remark 3.4 *A track graph (and hence any subgraph of a track graph) has path-width at most two.*

Definition 3.5 *G is called a partial track graph iff it is a subgraph of a track graph.*

The second characterization is with excluded minors.

Theorem 3.6 *[4] The following claims are equivalent:*

(i) G is a partial track;

(ii) G has path-width at most two;

(iii) G has no minor listed in the Appendix.

We describe first the excluded minors for 2-connected partial tracks. The list of the forbidden graphs is on Figure 6. $F_1 \simeq K_4$.

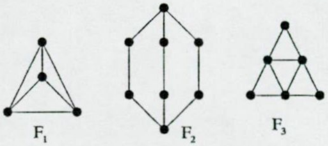

Figure 6: 2-connected forbidden minors

Lemma 3.7 *[4] The following statements are equivalent:*

(a) G is a 2-connected track graph;

(b) G is a 2-connected partial track graph;

(c) G is 2-connected and has no F_1, F_2 or F_3 as a minor.

This first step is easy to show. The rest of the proof of Theorem 3.6 is quite long and tedious, but without computers.

At the end of Chapter 4 we also deduce a linear-time recognition algorithm.

4 Arc-width

In Chapter 5 another graph-representation turns up. Here the vertices correspond to the arcs of a base circle. The width of a point on the base circle is the number of arcs containing it. The width of an arc-representation is the maximum of the width of the points. The arc-width of a graph is then the minimum width of such an arc-representation.

Lemma 4.1 *Arc-width is minor-monotone.*

By tradition pw is one less than the maximum width in an optimal interval representation. But the maximum width is the natural parameter for us. By this reason let us use the following notation: $pw^*(G) = pw(G) + 1$.

Arc-width is a natural modification of path-width. There is a quantitative connection, not just a formal one. Interesting and important is that in magnitude the arc-width of a given graph is between the path-width and its half. The first equality holds e.g. for trees. The complete graphs realize the other end.

Lemma 4.2 *[5]* $\left\lceil \frac{1}{2}(pw^*(G)+1) \right\rceil \leq aw(G) \leq pw^*(G)$

Lemma 4.3 *[5]* $aw(T) = pw^*(T)$ *where* T *is a tree. (The same statement holds for any graph which has no cycle.)*

Lemma 4.4 *[5]* $aw(K_n) = \left[\frac{n}{2}\right] + 1$

One of the curiosities of Chapter 5 is the exact determination of the arc-width of the complete bipartite graph. The proof of this is not easy at all. Also here it is necessary to give — in some sense — good constructions.

Lemma 4.5 *[5]* $aw(K_{s,s}) = s$.

Finally we present some of the possible excluded minor theorems. Actually not only for arc-width, but also for a finer measure of the arc-representations. Important is that also non-connected graphs can be excluded minors. Among the obstructions for a certain class, unexpectedly the Kuratowski graphs turn up.

The results of Chapter 4 and 5 are partially joint with the supervisor.

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