DECIDABILITY IN ALGEBRA

Ph.D. dissertation

by

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The three chapters of my dissertation are based on the papers [18, 19] and [20], respectively. The first paper is not related to the main topic of the dissertation—decidability problems—but gives a complete description of the simple algebras in the variety of semilattices expanded by an abelian group of automorphisms. In the second paper we study the decidability of the near-unanimity problem, posed ten years ago in [5], and prove a partial version of it to be undecidable. In the last, unpublished paper we show that the original problem, contrary to expectations, is decidable. As a consequence, we obtain the decidability of the natural duality problem for finitely generated, congruence distributive quasi-varieties.

We assume basic knowledge of universal algebra and direct the reader to either [2] or [23] for reference. Although the study of the near-unanimity problem stems from that of natural dualities (see [4, 5, 6]), the reader is not required to know this theory. For easier reference, we kept the original numbering of definitions and theorems of the dissertation.

F-semilattices

One of the primary goals of universal algebraic investigations is the full description of broad classes of algebras. According to a theorem of G. Birkhoff, in equational classes of algebras, such as in the varieties of groups, rings and lattices, every algebra can be expressed as a subdirect product of subdirectly irreducible members of the class. Therefore, these subdirectly irreducible algebras can be considered as the building blocks of varieties. The description of subdirectly irreducible algebras is particularly important because the study of many algebraic properties can be reduced to that of subdirectly irreducible algebras.

This description is trivial in the variety of semilattices because only the two-element semilattice is subdirectly irreducible. The situation is not this simple in other varieties, e.g., every subdirectly irreducible algebra in the variety generated by tournaments is a tournament, but not every algebra is [21]. There are (residually large) varieties, such as the variety generated by the quaternion group, where the subdirectly irreducible algebras form a proper class, and their full description is practically beyond hope. Therefore, in many cases we restrict ourselves to the study of simple algebras, i.e., subdirectly irreducible algebras that have only trivial congruences. Even this problem is extremely difficult in general, as witnessed by the classification of finite simple groups.

Algebras with a commuting semilattice operation, i.e., satisfying the identity

$$f(x_1 \wedge y_1, \dots, x_n \wedge y_1) \approx f(x_1, \dots, x_n) \wedge f(y_1, \dots, y_n)$$

for all basic operations f, have been studied in various forms. In many respects these algebras behave similarly to modules. For example, it is proved in [15] that if a locally finite variety of type-set $\{\mathbf{5}\}$ satisfies a term-condition similar to the term-condition for abelian algebras, then it has a semilattice term that commutes with all other term operations.

Within the class of modes—that is, idempotent algebras whose basic operations commute with each other—those having a semilattice term operation play an important role (see [28, 29]); these algebras are called semilattice modes. The structure of locally finite varieties of semilattice modes is described in [14].

An interesting class of algebras with a commuting semilattice operation arises if we add automorphisms, as basic operations, to a semilattice. This is a special case of the construction studied in [3]. In general, one can expand any variety $\mathcal{V}$ by a fixed monoid $\mathbf{F}$ of endomorphisms in a natural way. The expanded variety is the variety of $\mathcal{V}$-algebras $\mathbf{A}$ equipped with new unary basic operations, acting as endomorphisms on $\mathbf{A}$. We study only the following special case.

Definition 1.1. An algebra $\mathbf{S} = \langle S; \wedge, F\rangle$ with a binary operation $\wedge$ and a set F of unary operations is an $\mathbf{F}$-*semilattice*, if $\mathbf{F} = \langle F; \cdot, ^{-1}, \mathrm{id}\rangle$ is a group and $\mathbf{S}$ satisfies the following identities:

(1) the operation $\wedge$ is a semilattice operation,

(2) $\mathrm{id}(x) \approx x$,

(3) $f(g(x)) \approx (f \cdot g)(x)$ for all $f, g \in F$, and

(4) $f(x \wedge y) \approx f(x) \wedge f(y)$ for all $f \in F$.

Note that every semilattice can be considered as an $\mathbf{F}$-semilattice in a trivial way: every unary operation of F acts as the identity function. A much more interesting example of an $\mathbf{F}$-semilattice is the following.

Definition 1.2. For a group $\mathbf{F} = \langle F; \cdot, ^{-1}, \mathrm{id}\rangle$ let $\mathbf{P}(F) = \langle P(F); \wedge, F\rangle$ be the $\mathbf{F}$-semilattice which is defined on the set $P(F)$ of all subsets of F by setting

(1) $A \wedge B = A \cap B$ for all $A, B \subseteq F$, and

(2) $f(A) = A \cdot f^{-1}$ for all $f \in F$ and $A \subseteq F$.

Our first important statement reduces the study of subdirectly irreducible $\mathbf{F}$-semilattices to that of the subalgebras of $\mathbf{P}(F)$.

Lemma 1.6. *If $\mathbf{S}$ is a subdirectly irreducible $\mathbf{F}$-semilattice, then $\mathbf{S}$ is isomorphic to a subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$. The algebra $\mathbf{U}$ can be selected so that it has a unique element $M \subseteq F$ with the following properties:*

(1) $\mathrm{id} \in M$ *and* $M \cdot M = M$,

(2) $A = M \cdot A$ *for all* $A \in U$, *and*

(3) $M = \bigcap\{\, A \in U \mid \mathrm{id} \in A \,\}$.

In [12] J. Ježek has described all subdirectly irreducible $\langle \mathbb{Z}; + \rangle$-semilattices. Using our lemma above, we can easily describe the finite subdirectly irreducible $\mathbf{F}$-semilattices (Proposition 1.4), and all subdirectly irreducible $\mathbf{F}$-semilattices when $\mathbf{F}$ is locally finite (Corollary 1.7). In these special cases the subdirectly irreducible subalgebra $\mathbf{U}$ of $\mathbf{P}(F)$ contains the empty set and some subgroup M of $\mathbf{F}$. These elements also play an important role in the following class of simple $\mathbf{F}$-semilattices.

Definition 1.8. If $\mathbf{F}$ is a fixed group and M is a subgroup of $\mathbf{F}$, then let $\mathbf{S}_M$ denote the subalgebra of $\mathbf{P}(F)$, the elements of which are the empty set and the right cosets of M.

Thus the empty set is the least element in $\mathbf{S}_M$, and the right cosets of M are the atoms. The set F of unary operations of $\mathbf{S}_M$ acts as a transitive permutation group on the set of atoms. It is not hard to show that the algebras $\mathbf{S}_M$ are exactly those simple subalgebras of $\mathbf{P}(F)$ that have a least element and some atoms.

In [11] J. Ježek has described all simple algebras in the variety of semilattices expanded by two commuting automorphisms, that is, in the variety of $\langle \mathbb{Z} \times \mathbb{Z}; + \rangle$-semilattices. We generalize this result to arbitrary commutative groups, which is our main result in this chapter.

Definition 1.13. Let $\mathbf{F}$ be a fixed commutative group. Then for every non-constant homomorphism β from $\mathbf{F}$ to the additive group $\langle \mathbb{R}; + \rangle$ of the real numbers let us define an $\mathbf{F}$-semilattice $\mathbf{R}_\beta = \langle \mathbb{R}; \min, F \rangle$ as follows:

(1) $\min(a, b)$ is taken with respect to the natural order of $\mathbb{R}$, and

(2) $f(a) = a - \beta(f)$ for all $f \in F$ and $a, b \in \mathbb{R}$.

Definition 1.16. A homomorphism $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ is called *dense* if for each real number $\varepsilon > 0$ there exists an element $f \in F$ such that $0 < \beta(f) \leq \varepsilon$.

If the homomorphism β in Definition 1.13 is not dense, then the range of β is isomorphic to $\langle \mathbb{Z}; + \rangle$. We will consider this case separately:

Definition 1.18. Let $\mathbf{F}$ be a fixed commutative group. Then for every surjective homomorphism α from $\mathbf{F}$ onto the additive group $\langle \mathbb{Z}; + \rangle$ of the integers let $\mathbf{Z}_\alpha = \langle \mathbb{Z}; \min, F \rangle$ be the $\mathbf{F}$-semilattice defined as follows:

(1) $\min(a, b)$ is taken with respect to the natural order of $\mathbb{Z}$, and

(2) $f(a) = a - \alpha(f)$ for all $f \in F$ and $a, b \in \mathbb{Z}$.

Theorem 1.21. *If* $\mathbf{F}$ *is a commutative group, then every simple* $\mathbf{F}$*-semilattice is isomorphic to one of the following algebras:*

(1) $\mathbf{S}_M$, *where* M *is a subgroup of* $\mathbf{F}$,

(2) $\mathbf{Z}_\alpha$, *where* $\alpha\colon \mathbf{F} \to \langle \mathbb{Z}; + \rangle$ *is a surjective group homomorphism, and*

(3) *the subalgebras of* $\mathbf{R}_\beta$, *where* $\beta\colon \mathbf{F} \to \langle \mathbb{R}; + \rangle$ *is a dense group homomorphism.*

Furthermore, these simple $\mathbf{F}$*-semilattices are pairwise nonisomorphic, except for the case when* β_1, β_2 *are dense homomorphisms,* $\mathbf{S}_1, \mathbf{S}_2$ *are subalgebras of* $\mathbf{R}_{\beta_1}, \mathbf{R}_{\beta_2}$ *respectively, and there exist real numbers* $t > 0$ *and* d *such that* $\beta_2 = t\beta_1$ *and* $S_2 = tS_1 + d$.

We conclude this chapter by noting that there exists a simple $\mathbf{F}$-semilattice in the nonabelian case that has a least element but no atom and its semilattice order is not linear.

Duality theory and the near-unanimity problem

General duality theory is capable of describing various well-known dualities—for example Pontryagin's, Stone's and Priestley's—between a category $\mathcal{A}$ of algebras with homomorphisms and a category $\mathcal{X}$ of topological structures with continuous structure preserving maps (see [6]). In these cases the class $\mathcal{A}$ is a quasi-variety generated by a single algebra $\mathbf{P} \in \mathcal{A}$, and $\mathcal{X}$ is the class of closed substructures of powers of an object $\underset{\sim}{\mathbf{P}} \in \mathcal{X}$ having the same underlying set as $\mathbf{P}$. Without getting into the details, we note that the points of the dual $\underset{\sim}{\mathbf{A}} \in \mathcal{X}$ of an algebra $\mathbf{A} \in \mathcal{A}$ are the homomorphisms $\varphi\colon \mathbf{A} \to \mathbf{P}$; while the elements of the dual $\mathbf{X} \in \mathcal{A}$ of a topological structure $\underset{\sim}{\mathbf{X}} \in \mathcal{X}$ are the $f : \underset{\sim}{\mathbf{X}} \to \underset{\sim}{\mathbf{P}}$ continuous structure preserving maps.

Example. *For the Pontryagin duality,* $\mathcal{A}$ *is the class of abelian groups,* $\mathbf{P} = \langle P; \cdot, ^{-1}, 1 \rangle$ *is the circle group on the set* $P = \{\, z \in \mathbb{C} : |z| = 1 \,\}$ *of complex numbers with multiplication,* $\mathcal{X}$ *is the category of compact topological abelian groups, and* $\underset{\sim}{\mathbf{P}} = \langle P; \cdot, ^{-1}, 1, \tau \rangle$ *where* τ *is the restriction of the natural topology of the complex plane to* P.

Example. *For the Stone duality,* $\mathcal{A}$ *is the category of Boolean algebras,* $\mathbf{P} = \langle \{0, 1\}; \wedge, \vee, ', 0, 1 \rangle$ *is the two-element Boolean algebra,* $\mathcal{X}$ *is the category of totally disconnected Hausdorff spaces, and* $\underset{\sim}{\mathbf{P}} = \langle \{0, 1\}; \tau \rangle$ *where* τ *is the discrete topology. It is easy to see that the ultra filters of a Boolean algebra* $\mathbf{A} \in \mathcal{A}$ *correspond to the homomorphisms of* $\mathbf{A}$ *onto* $\mathbf{P}$.

Example. *For the Priestley duality, $\mathcal{A}$ is the category of bounded distributive lattices, $\mathbf{P} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ is the two-element bounded distributive lattice, $\mathcal{X}$ is the category of totally order-disconnected spaces, and $\underset{\sim}{\mathbf{P}} = \langle \{0,1\}; \leq, \tau\rangle$ where τ is the discrete topology. It is easy to see that the prime filters of a distributed lattice $\mathbf{A} \in \mathcal{A}$ correspond to the homomorphisms of $\mathbf{A}$ to $\mathbf{P}$.*

We say that an algebra $\mathbf{P}$ *admits a natural duality*, if there exists a topological structure $\underset{\sim}{\mathbf{P}}$ defined on P such that the quasi-variety generated by $\mathbf{P}$ is dually represented, as defined by duality theory, by the category $\mathcal{X}$ of closed substructures of powers of $\underset{\sim}{\mathbf{P}}$. Therefore, to leverage the power of duality, it is natural to ask which finite algebras admit a natural duality. Is this characterization possible? Is it decidable of a finite algebra $\mathbf{P}$ whether it admits a natural duality? This second question is known as the *natural duality problem.* Currently, we do not know the answer to this problem, but many expect it to be undecidable.

The natural duality problem was partially reduced to a pure algebraic problem in the following way. We call a term t of an algebra $\mathbf{P}$ a *near-unanimity term* if it satisfies the following identities:

$$t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y) \approx x.$$

An algebra is congruence join-semi-distributive if its congruence lattice satisfies the quasi-identity

$$x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y.$$

B. A. Davey and H. Werner proved in [6] that in the presence of a near-unanimity term of $\mathbf{P}$, the quasi-variety $\mathcal{A}$ generated by $\mathbf{P}$ admits a natural duality. The converse was proved in [5] under the assumption that $\mathcal{A}$ is congruence join-semi-distributive:

Theorem (B. A. Davey, L. Heindorf and R. McKenzie [5])**.** *Let $\mathbf{P}$ be a finite non-trivial algebra and let $\mathcal{A}$ be the quasivariety generated by $\mathbf{P}$. The following are equivalent:*

(1) $\mathbf{P}$ *has a near-unanimity term;*

(2) $\mathbf{P}$ *admits a natural duality, and every algebra in $\mathcal{A}$ is congruence distributive; and*

(3) $\mathbf{P}$ *admits a natural duality, and every finite algebra in $\mathcal{A}$ is congruence join-semi-distributive.*

This theorem, known as the *near-unanimity obstacle theorem*, motivates the *near-unanimity problem*, the problem of deciding whether a finite algebra has a near-unanimity term. Clearly, if the arity of the near-unanimity term of **P** is known, then finding the near-unanimity term is easy by simply calculating the free algebra in $\mathcal{A}$ generated by the appropriate number of elements. The difficulty lies in the fact that we do not even have an upper bound for the arity of a possible near-unanimity term.

Near-unanimity term operations come up naturally in the study of algebras. For example, all lattices have a ternary near-unanimity term

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

From E. L. Post's classification [26] we know that almost all clones on a two element set contain a near-unanimity operation. It is also well known that an algebra having a near-unanimity term lies in a congruence distributive variety, and has a finite base of identities provided it is of finite signature (see [30]).

It is easy to decide whether the quasi-variety $\mathcal{A}$ generated by a finite algebra **P** is congruence distributive because it is enough to search for Jónsson terms among the finitely many ternary terms of **P**. Therefore, by the near-unanimity obstacle theorem, if the near-unanimity problem were undecidable, then the natural duality problem would also be undecidable.

The undecidability of a partial near-unanimity term

In an attempt to prove the undecidability of the near-unanimity problem the following approach was taken by R. McKenzie.

Definition 2.1. Let **A** be a fixed finite algebra, $t(x_1, \dots, x_n)$ be a term of **A**, and $S \subseteq A$. We say that t is a *partial near-unanimity term on* S if

$$t(y, x, \dots, x) = t(x, y, x, \dots, x) = \dots = t(x, \dots, x, y) = x$$

for all $x, y \in S$.

Clearly, a term of **A** is a near-unanimity term if and only if it is a partial near-unanimity term of the two-generated free algebra in the variety generated by **A** on the set $\{x, y\}$ of generators. Thus it is natural to study the decidability of the partial near-unanimity problem on some fixed subset of a finite algebra. It is proved in [22] that the existence of a partial near-unanimity term on a fixed two-element subset is undecidable. In Chapter 2 we extend this result to a subset excluding two fixed elements:

Theorem 2.2. *There exists no algorithm that can decide of a finite algebra* **A** *and two fixed elements* $r, w \in A$ *if* **A** *has a partial near-unanimity term on the set* $A \setminus \{r, w\}$.

This theorem does not seem to be significant after learning the decidability of the near-unanimity problem. Nevertheless, the methods used in the proof are interesting on their own and may be useful for the study of other decidability problems.

In the proof of this theorem we employ Minsky machines, which are equivalent to Turing machines (see [24, 25]). The "hardware" of a Minsky machine consists of a pair of *registers* that can contain arbitrary natural numbers. The "software" is a finite set of *states* containing an *initial* and a *halting state* together with a list of *commands*. There are two types of commands: the first instructs the machine to increase the value stored in one of the registers by one, and then to go to another state. The second command first checks the value stored in one of the registers; if it is zero, then the machine goes to one state; otherwise the value stored in the register is decremented by one and the machine goes to another state. The *computation* of the machine is a possibly infinite sequence of states together with the values of the registers at each step.

Since the halting problem for Minsky machines is undecidable, it is enough to construct (by an effective algorithm) for each Minsky machine $\mathcal{M}$ an algebra $\mathbf{A}(\mathcal{M})$ with two special elements $r, w \in A(\mathcal{M})$ such that $\mathbf{A}(\mathcal{M})$ has a partial near-unanimity term on the set $A(\mathcal{M}) \setminus \{r, w\}$ if and only if $\mathcal{M}$ halts.

In the construction the universe and the set of basic operations of $\mathbf{A}(\mathcal{M})$ depend on the set of states and commands of $\mathcal{M}$, respectively. Our goal is to encode the halting computation of $\mathcal{M}$ into a partial near-unanimity term. The key step is to show that given a partial near-unanimity term t on $A(\mathcal{M}) \setminus \{r, w\}$ one can reconstruct the halting computation of $\mathcal{M}$ from the term tree of t. The definition of the basic operations forces the shape of the tree to be almost "linear" with the basic operations encoding a sequence of commands of $\mathcal{M}$. With the proper definition of the basic operations, for example making their range pairwise disjoint, we can easily ensure that the sequence of states is correct except possibly those steps where the next state depends whether the content of a register is zero or not. We solve this difficulty by encoding whether the contents of registers are zero at each step together with the states. We cannot encode the actual values of the registers, which can be arbitrary large natural numbers, because $\mathbf{A}(\mathcal{M})$ must be finite. Our final task is to verify whether the sequence of states together with these special markings for zero values correspond to the halting computation of $\mathcal{M}$. We achieve this by forcing an appropriate matching of the variables of t via the known value of t at near-unanimous evaluations.

The element $w \in A(\mathcal{M})$ has (essentially) the *absorbing property*: for all basic operations $f(x_1, \dots, x_n)$ and elements $x_1, \dots, x_n \in A(\mathcal{M})$ the implication

$$w \in \{x_1, \dots, x_n\} \implies f(x_1, \dots, x_n) = w$$

holds. We use w to indicate that either the shape of t is incorrect, or the sequence of encoded states does not correspond to that of the halting computation. If some local inconsistency is detected, then one of the basic operations in the term tree returns w at an appropriate near-unanimous evaluation $(x, \dots, x, y, x \dots, x)$. Then the element w propagates to the root of t by the absorbing property, thus $t(x, \dots, x, y, x, \dots, x) = w$, which is a contradiction.

An improvement of this method might be possible to the subset $A(\mathcal{M}) \setminus \{w\}$, which could be formulated, analogously to the results in [13], as the undecidability of the near-unanimity problem for partial algebras:

Problem 1. *Given a finite partial algebra, decide whether it has a term that is defined on all near-unanimous evaluations and satisfies the near-unanimity identities.*

The decidability of a near-unanimity term

In the last chapter of the dissertation we prove the decidability of the near-unanimity problem, a rather surprising development after the negative partial results. We state this theorem in the language of clones:

Theorem 3.17. *Given a finite set A and a finite set $\mathcal{F}$ of operations on A, it is decidable whether the clone generated by $\mathcal{F}$ contains a near-unanimity operation.*

Instead of working with operations and their composition, we introduce an equivalence relation on the set of operations in such a way that

(1) the near-unanimity operations form an equivalence class of the relation,

(2) a new notion of composition can be introduced on the equivalence classes, and

(3) it is possible to algorithmically compute the closure of equivalence classes under this new notion of composition.

Based on these requirements, our next definition might not be so surprising. We will need the following notations. Let ω and ω^+ be the set of all finite and countable cardinals, respectively. Let $\mathcal{O}_A$ be the set of all operations on the set A, and for $n \in \omega$ let $\mathcal{O}_A^{(n)} = A^{A^n}$, that is, the set of all n-ary operations on A. Given an operation $f \in \mathcal{O}_A$, we consider those binary operations—called polymers—with their multiplicities that arise as $f(x, \dots, x, y, x, \dots, x)$ where the lone y is at a fixed coordinate:

Definition 3.1. For $f \in \mathcal{O}_A^{(n)}$ and $i \in \omega$, the ith *polymer* of f is $f|_i \in \mathcal{O}_A^{(2)}$ defined as

$$f|_i(x, y) = \begin{cases} f(x, \dots, x, \overbrace{y}^{i\text{th}}, x, \dots, x) & \text{if } 0 \le i < n, \\ f(x, \dots, x) & \text{if } i \ge n, \end{cases}$$

where y occurs at the ith coordinate of f in the first case. The collection of polymers of f together with their multiplicities is the *characteristic function* of f, which is formally defined as the map $\chi_f : \mathcal{O}_A^{(2)} \to \omega^+$ where

$$\chi_f(b) = |\{ i \in \omega : f|_i = b \}| .$$

Clearly, near-unanimity operations are characterized by their polymers; namely all of them must be equal to x. Therefore, the characteristic functions of near-unanimity operations are the same and equal to

$$\chi_{\mathrm{nu}}(b) = \begin{cases} \omega & \text{if } b(x,y) \approx x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{X}_A$ be the set of characteristic functions of operations on $\mathbf{A}$. Now the kernel of the operator

$$\mathsf{X}\colon \mathcal{O}_A \to \mathcal{X}_A, \qquad \mathsf{X}\colon f \mapsto \chi_f$$

satisfies our condition (1) stated above.

We do not give the technical definition of the composition operator $\mathsf{C}_{\mathcal{F}}$ as the following shall be sufficient. We distinguish the "outer" set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations from the "inner" objects on which we apply the members of $\mathcal{F}$. For $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$ the set $\mathsf{C}_{\mathcal{F}}(\mathcal{G})$ contains all operations t of the form

$$t(y_1, \dots, y_k) = f\big(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})\big),$$

where $f \in \mathcal{F}$ and $g \in \mathcal{G}$ are m and n-ary operations, respectively, and $\{x_{11}, \dots, x_{mn}\} \subseteq \{y_1, \dots, y_k\}$. We employ the same operator symbol $\mathsf{C}_{\mathcal{F}}$ for characteristic functions, thus $\mathsf{C}_{\mathcal{F}}(\mathcal{U}) \subseteq \mathcal{X}_A$ for every $\mathcal{U} \subseteq \mathcal{X}_A$. The connection between the two composition operators, the real meaning of condition (2), is expressed by the next lemma.

Lemma 3.6. $\mathsf{X}\mathsf{C}_{\mathcal{F}}(\mathcal{G}) = \mathsf{C}_{\mathcal{F}}\mathsf{X}(\mathcal{G})$ *for all* $\mathcal{F}, \mathcal{G} \subseteq \mathcal{O}_A$.

Up to this point, we showed that the clone $\langle \mathcal{F} \rangle$ generated by $\mathcal{F} \subseteq \mathcal{O}_A$ contains a near-unanimity operation if and only if the characteristic function χ_{nu} can be obtained from the characteristic function χ_{id} of the unary projection by finitely many applications of the composition operator $\mathsf{C}_{\mathcal{F}}$. However, we are still far from establishing requirement (3), our ultimate goal.

Suppose that the sets A and $\mathcal{F} \subseteq \mathcal{O}_A$ are finite, and that the clone $\langle \mathcal{F} \rangle$ contains a near-unanimity operation. Then, using a theorem of L. Lovász on the chromatic number of Kneser graphs [17], we can show that $\langle \mathcal{F} \rangle$ must contain an operation g of bounded arity (dependent only on $|A|$) that satisfies a set of technical identities similar to that of near-unanimity operations. We can effectively find g since its arity is bounded.

Recall that a partially ordered set is called *well-ordered*, if it has no infinite anti-chains and satisfies the descending chain condition, i.e., contains no strictly decreasing infinite sequence of elements. Using the properties of g, we introduce a well-ordered partial order on a special subset of $\mathcal{X}_A$. By applying the composition operator $\mathsf{C}_{\mathcal{F}}$ to an order filter of characteristic functions, we get another order filter whose minimal elements can be effectively computed from that of the original filter. If we apply the composition operator iteratively, we get an increasing sequence of order filters under inclusion. However, a well-ordered partially ordered set cannot have a strictly increasing infinite chain of order filters, therefore this process must terminate in finitely many steps. This proves that the closure of characteristic functions under the composition operator can be effectively calculated.

As an immediate consequence of the decidability of the near-unanimity problem and the near-unanimity obstacle theorem from [5], we also obtain the decidability of the natural duality problem for finite algebras in a congruence join-semi-distributive variety.

Since there are only finitely many algebras on a fixed n-element set whose basic operations are at most r-ary, by the decidability of the near-unanimity problem, there exists a recursive function $N(n, r)$ that puts an upper limit on the minimum arity of a near-unanimity term operation for those algebras that have one. Consequently, given an algebra $\mathbf{P}$ whose operations are at most r-ary, one can decide the near-unanimity problem by simply constructing all at most $N(|P|, r)$-ary terms and checking if one of them yields a near-unanimity operation. If no such is found, then $\mathbf{P}$ has no near-unanimity term operation. We know that such recursive function $N(n, r)$ exists, but currently we do not have a formula for one.

A very interesting group of open problems is related to the *constraint satisfaction problem*, which we do not define here and refer the reader to [8] for details. It is proved in [10] that if a set Γ of relations on a set admits a compatible near-unanimity operation, then the corresponding constraint satisfaction problem CSP(Γ) is solvable in polynomial time. Therefore, it is natural to consider the *near-unanimity problem for relations*:

Problem 2. *Given a finite set Γ of relations on a set, decide whether there exists a near-unanimity operation that is compatible with each member of Γ.*

Currently we are unable to solve this problem, even in the light of our result. We know that if a clone has a near-unanimity operation, then both the clone and its dual relational clone are finitely generated (see [30]). Inspired by this fact, we ask the following:

Problem 3. *Given a finite set of operations and a finite set of relations on the same underlying set, decide if the functional and relational clones they generate are duals of each other.*

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