Graph Parameters

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1. The interval number of graphs

The interval number of a graph G, denoted by $i(G)$, is the least natural number t such that G is the intersection graph of sets, each of which is the union of at most t intervals. Here we settle a conjecture of Griggs and West about bounding $i(G)$ in terms of e, that is the number of edges in G. Namely, it will be shown that $i(G) \leq \lceil 1/2\sqrt{e} \rceil + 1$. It is also observed that the edge bound induces $i(G) \leq \sqrt{3/2\gamma(G)} + o(1)$, where $\gamma(G)$ is the genus of G.

In the first chapter we are concerned with representing graphs as special intersection graphs. That is, we assign a set to each vertex of G so that v is adjacent to w if and only if the common part of the assigned sets is not empty. A *t-interval representation* is such an assignment, in which each set consists of at most t closed intervals. The *interval number* of G, denoted by $i(G)$, is the least integer t for which a t-representation of G exists. We need one more definition: a representation is *displayed* if each set of the representation has an open interval, which is disjoint from the other sets. Such an interval is called *displayed segment*.

Several interesting aspects of the interval number had been investigated before, here we wish to pursue only one of these in depth. More than one and a half decade ago Griggs and West conjectured that $i(G) \leq$

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$\lfloor 1/2\sqrt{e} \rfloor + 1$ for any graph G, where e is the number of edges in G. They also showed that this would be the best possible because the bound is attained for the complete bipartite graphs $K_{2m,2m}$, where m is a positive integer. In fact their conjecture can be restated as $i(G) \leq \lceil 1/2\sqrt{e} \rceil + 1$, since the two forms coincide for the previous graphs. In a series of papers it was proved that $i(G) \leq \lceil c\sqrt{e} \rceil + 1$ for $c = 1$, $c = \sqrt{2}/2$ and $c = 2/3$, see [21], [33] and [24]. The main goal of this paper is to conclude this process by showing that the right constant is really $c = 1/2$.

We also mention some connected problems, which are worth further study.

Theorem 1. *Every graph with e edges has a displayed interval representation with at most $\lceil 1/2\sqrt{e} \rceil + 1$ intervals for each vertex.*

Most known bounds on $i(G)$ are grossly excessive when G has more than half of the possible edges. A plausible remedy is to develop bounds on $i(G)$ that are monotone decreasing in G. In Chapter 2, we bound $i(G)$ in terms of $e(\bar{G})$, the number of edges in the complement of G. We prove that $i(G) \leq \lceil \frac{1}{2}\sqrt{e(\bar{G})} \rceil + O(n/\log n)$.

Conjecture 2. *[10] $i(G) \leq \frac{1}{2}\lceil \sqrt{e(\bar{G})} \rceil + 1$.*

This bound would be sharp, since it is attained for the complete bi-

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partite graph $K_{2m,2m}$ and for the graph obtained by adding an edge from each vertex of $K_{2m,2m}$ to each vertex of a clique of order t. These are the only extremal graphs known so far.

Here we take a step toward settling Conjecture 2. As a byproduct, we also obtain upper bounds in terms of $\Delta(\bar{G})$.

Theorem 3. $i(G) \leq \lceil \frac{1}{2}\sqrt{e(\bar{G})} \rceil + (3/2 + o(1))n/\log n$.

Theorem 4. $i(G) \leq \lceil (\Delta(\bar{G}) + 1)/2 \rceil + 2^{n/(\Delta(\bar{G})+1)}$.

Theorem 5. $i(G) \leq \Delta(\bar{G}) + \frac{1}{2}\chi(\bar{G}) + 1$, *where χ denotes the chromatic number.*

Corollary 6. $i(G) \leq \frac{3}{2}(\Delta(\bar{G}) + 1)$.

Perhaps every bounds in terms of density has an analogue in terms of the complement.

Conjecture 7. $i(G) \leq \lceil \frac{1}{2}(\Delta(\bar{G}) + 1) \rceil$ *and* $i(G) \leq \lceil \rho(\bar{G}) \rceil + 1$.

In Chapter 3 we examine further properties of the interval number of graphs. Namely, we characterize some graphs, whose interval number is maximal possible for the degree bound, we determine up to a 1/2 factor the interval number of split graphs, and we claim that the main reason for the unusually high interval number is a "large" induced bipartite graph.

One may ask, what is the reason of high interval number in terms of induced subgraphs? For the random graph $G_{n,1/2}$ the equality $i(G_{n,1/2}) =$

$(1/2 + o(1))n/\log_2 n$ holds almost surely [29]. If G contains the complete bipartite graph $K_{k,k}$ as an induced subgraph, then $i(G) \geq \lceil (k+1)/2 \rceil$. The following result roughly states that the big induced complete bipartite graphs are responsible for the unusually high interval number.

Theorem 8. *Let k be a positive integer. If a graph G does not contain $K_{k,k}$ as an induced subgraph, then $i(G) \leq (1+o(1))n/\log_2 n$.*

We do not know matching lower bound here. Standard use of the probabilistic method shows the existence of a bipartite $K_{r,r}$-free graph G which has $2n$ vertices and $n^{2-2/r}$ edges. Applying the formula $i(G) \geq \lceil (e(G)+1)/v(G) \rceil$ to G, which is triangle-free of course, we get that $i(G) = \Omega(n^{1-2/r})$.

2. The game domination number

In Chapter 3 we define the game domination number. The game domination number of a (simple, undirected) graph is defined by a game related to the domination number, a well-known graph parameter.

A *dominating set* of a digraph $\vec{G}$ is a set S of vertices such that for every vertex $v \notin S$ there exists some $u \in S$ with $\vec{uv} \in E(\vec{G})$. The *domination number* $\gamma(\vec{G})$ of $\vec{G}$ is defined as the cardinality of the smallest dominating set.

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We define a "domination parameter" of an undirected graph G as the domination number of one of its orientations, determined by the following two player game. Players A and D orient the unoriented edges of the graph G alternately with D playing first, until all edges are oriented. Player D (frequently called the *Dominator*) is trying to minimize the domination number of the resulting digraph, while player A (*Avoider*) tries to maximize the domination number. This game gives a unique number depending only on G, if we suppose that both A and D play according to their optimal strategies. We call this number the *game domination number of G* and denote it by $\gamma_g(G)$.

We determine the game domination number for several classes of graphs and provide general inequalities relating it to other graph parameters.

Theorem 9. *For every graph $G = (V, E)$ with n vertices and minimum degree $\delta \geq 2$ and for every real number p between 0 and 1, $\gamma_g(G) \leq np + 2n(1-p)^\delta + 1 + n\delta p(1-p)^\delta$. Therefore, $\gamma_g(G) \leq (1 + o(1))\frac{n\ln(\delta+1)}{\delta+1}$, where the $o(1)$-term tends to zero as δ tends to infinity, and the above the estimate is tight, up to the $o(1)$ error term.*

3. A jump for graph properties

An extremal jump is a discrete step between measures guaranteed in certain situations. It has been known for some time that the density of

a graph jumps; recent work on hereditary graph properties has shown that properties with "large" or "small" speeds jump, but it was unknown whether there is a clean jump for properties with speed in a middle range. In Chapter 5, generalizations of the theorems of Dilworth, Ramsey, and Turán's are applied to answer this in the affirmative. In particular, we find a strict lower bound for the penultimate range of the speed hierarchy for hereditary properties of graphs.

Theorem 10. *If $|\mathcal{P}^n| \geq n^{(1+o(1))n}$, then $|\mathcal{P}^n| \geq B_n$ for all n, where B_n is the nth Bell number. Furthermore, equality is possible only if $\mathcal{P} = \mathcal{P}_{cl}$ or $\mathcal{P} = \mathcal{P}_{\overline{cl}}$.*

In the past, we have sought to give, in addition to bounds on the speed of properties, collections of minimal properties that "force" the speed to be in the range given. For the penultimate range, this type of result is only partially done.

4. Summary

In this dissertation we examine problems that have their roots in questions about properties and parameters of graphs. Each question develops into a deep theory about its subject.

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