Asymptotic Bernstein type inequalities

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To the memory of my father

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Contents

List of Figures

Chapter 1

Preliminaries

1.1 Introduction

The subject of this dissertation is a generalization of Bernstein's inequality [11].

The complex Bernstein's inequality is (see [19, Corollary 1.3 p. 98] or [16, Corollary 5.1.6 p. 233] or [41, Theorem 1.2.3 p. 531])

$$\left|p_n'(z_0)\right| \leq n||p_n||_D \ , \tag{1.1}$$

where p_n is an arbitrary complex polynomial of degree n, $||p_n||_D$ denotes its supremum norm over the unit disk $D = \{z \in \mathbf{C} : |z| \leq 1\}$ and $|z_0| = 1$.

With standard substitutions one obtains the following inequality on $I = [-1, 1]$

$$\left|p_n'(t)\right| \leq n\frac{1}{\sqrt{1-t^2}}||p_n||_I \tag{1.2}$$

where p_n is an algebraic polynomial of degree n, $||p_n||_I$ is the supremum norm of p_n over $I = [-1, 1]$ and $t \in I$ (see [19, Corollary 1.2 p. 98] or [16, Theorem 5.1.7 p. 233] or [41, Theorem 1.2.5 p. 532] or [45, formula (158) p. 128]).

This inequality has recently been extended to the following inequality by Baran [5] and Totik [56]. Let $K \subset \mathbf{R}$ be a compact set and ν_K be its equilibrium measure (see [51] or [52] or the following section). In the interior of K, $\mathrm{Int}(K)$, the measure ν_K is absolutely continuous with respect to Lebesgue measure and write $\omega_K(t)dt = d\nu_K(t)$. Then

$$\left|p_n'(t)\right| \leq n\pi\omega_K(t)||p_n||_K \tag{1.3}$$

where p_n is an algebraic polynomial with degree n and $||p_n||_K$ is the supremum norm of p_n over K and $t \in \mathrm{Int}(K)$.

There is a vast literature for various generalizations of Bernstein's and Markov's inequality. It is more or less hopeless to find all the papers and articles which is related to these inequalities. Some surveys have appeared recently, see e.g. [54] which details the different approaches to these inequalities and also their history. Our aim here is to find the analogue of Bernstein's inequality for sets bounded by Jordan curves. This dissertation is based on the papers [42], [44] and [43]. The main result of this work is Theorem 9. To prove it we shall have to use several tools and results from potential theory, and to generalize Hilbert's lemniscate theorem.

The basic idea is to exhaust compact sets with lemniscates. See Hilbert's lemniscate theorem, Theorem 6. This exhaustion is useful in potential theoretic calculations, see Lemma 4 and also useful in supremum norm estimates. Recently, such "exhaustion" has appeared in different forms and different situations. The "inscribed ellipse method" is one of such examples. See for example Sarantopoulos' article [53] or the more recent Milev-Révész paper [40]. Another example is the "Padova" points by Len Bos and Shayne Waldron which is unpublished yet. And, from our point of view, the most important example is Totik's paper [56].

1.2 Notions and tools from potential theory

We extensively use potential theory. For a detailed introduction and also as a reference book, we refer to Ransford's [51] or Saff-Totik's [52] book.

Let us briefly recall some important notions and theorems. On the complex plane, the logarithmic kernel $-\log|x-y|$ plays an important role which also has an interesting physical interpretation: if we have one electron at x and we want to move another electron from y_1 to y_2, then we do $\log|x-y_1| - \log|x-y_2|$ work.

Let μ be a positive Borel measure on $\mathbf{C}$ with compact support and with total mass 1. We call the following convolution the potential of μ

$$U^\mu(x) := \int \log \frac{1}{|x-y|} d\mu(y)$$

where this function may attain $+\infty$ as well. Actually for such μ's, the set where the potential attains plus infinity is not large. $U^\mu(x)$ is a superharmonic function and is harmonic outside $\operatorname{supp}\mu$. We define the logarithmic energy of the measure μ with the double integral

$$I(\mu) := \int\int \log \frac{1}{|x-y|} d\mu(y) d\mu(x)$$

Again, this quantity may be $+\infty$. For a compact set K let

$$\mathcal{M}(K) := \{\mu : \mu \text{ Borel measure, } \mu \geq 0, \\ \operatorname{supp}\mu \text{ is compact, } ||\mu|| = 1, \operatorname{supp}\mu \subset K\},$$

which is the set of probabilty measures supported on K. We define the energy of the compact set K as

$$I(K) := \inf\{I(\mu) : \mu \in \mathcal{M}(K)\}.$$

With it, we set

$$\operatorname{cap}(K) := \exp\big(-I(K)\big)$$

which is called the logarithmic capacity of K. If $I(K) = \infty$, then K is of/has zero capacity and is "small" in potential theory. For example, a compact set consisting of only countably many points is of zero capacity. We say that a propetry holds *quasi-everywhere* (q.e.) if it holds everywhere except for a zero capacity set.

The following theorem (see [51, Theorem 3.3.2 p. 58 and Theorem 3.7.6 p. 75] or [52, Theorem I.1.3 (b) p. 27]) is based on a simple compactness argument. For every compact set K with positive capacity there exists a unique measure ν_K with minimal energy:

$$I(\nu_K) = I(K) = \inf\{I(\mu) : \mu \in \mathcal{M}(K)\}$$

and $\operatorname{supp}\nu_K \subset \partial_e K$ where $\partial_e K$ is the exterior boundary of K, that is, the boundary of the unbounded component of $\mathbf{C} \setminus K$. Furthermore, Frostman's theorem describes the potential of ν_K:

a) $U^{\nu_K}(x) \leq I(K)$ for every $x \in \mathbf{C}$

b) $U^{\nu_K}(x) = I(K)$ for quasi every $x \in K$.

We call this unique measure ν_K the equilibrium measure of K. We also use the notion of Green's function. Let K be a compact set with positive capacity. Then Green's function of the complement of K with pole at infinity is defined as

$$g_K(z) = g(K, z) := I(K) - U^{\nu_K}(z).$$

Green's function has the properties

- $g_K(z) \geq 0$,
- $g_K(z)$ is harmonic outside K,

- $g_K(z)$ is subharmonic on $\mathbf{C}$,
- $g_K(z)$ has logarithmic growth near infinity, more precisely, $\lim_{|z|\to\infty}(g_K(z)-\log|z|) = I(K)$
- $g_K(z)$ is zero on K quasi everywhere and everywhere on $\mathrm{Int}(K)$.

Equivalently, these properties can be used as the defining properties of Green's function.
Examples.

1. $K = D = \{z \in \mathbf{C} : |z| \le 1\}$. Then Green's function is $g_D(z) = \log|z|$ and $\mathrm{cap}(D) = 1$.
2. $K = [-1, 1]$. Then Green's function is $g_{[-1,1]}(z) = \log|z + \sqrt{z^2 - 1}|$ and $\mathrm{cap}([-1, 1]) = 1/4$.
3. $K = \{z \in \mathbf{C} : |r(z)| = 1\}$ where r is a complex polynomial of degree m, that is, K is a so called lemniscate. Then Green's function is $g_K(z) = \frac{1}{m}\log|r(z)|$ and $\mathrm{cap}(K) = a^{-1/m}$ where a is the leading coefficient of r.
4. $K = \{z \in \mathbf{C} : |r(z)| \le 1\}$, where, again, r is a complex polynomial of degree m with leading coefficient a, then the same holds as in 3.
5. If K is a compact set such that $\mathbf{C}_\infty \setminus K$ is connected, and f is a conformal map from $\mathbf{C}_\infty \setminus K$ onto $\mathbf{C}_\infty \setminus K_1$ for some compact K_1 and $f(\infty) = \infty$, then $g_K(z) = g_{K_1}\big(f(z)\big)$.

The relation between Green's function and equilibrium measure has a "converse": If K is a compact set such that ∂K is a union of finitely many $C^{1+\delta}$ smooth curves ($\delta > 0$), then the equilibrium measure is absolutely continuous with respect to arc length measure, furthermore,

$$\frac{d\nu_K(z)}{ds} = \frac{1}{2\pi}\frac{\partial}{\partial \mathbf{n}_z} g_K(z)$$

where ds denotes the arc-length measure on ∂K and $\partial/\partial\mathbf{n}_z$ denotes differentation at z in the direction of the outer normal $\mathbf{n}_z$ (see [52], p. 92. Theorem II.1.5 and p. 211. Theorem IV.2.2) .

In some cases, the equilibrium measure is known explicitly, for example, if $K = D$ is the unit disk, then $d\nu_K = 1/2\pi ds$, where ds denotes the arc length measure on the unit circle. If $K = [-1, +1]$, then $d\nu_K = \frac{1}{\pi\sqrt{1-t^2}}dt$, where dt is the Lebesgue measure. $\nu_{[-1,+1]}$ is called the arcsine distribution.

Chapter 2

Asymptotic Bernstein type inequality on lemniscates

2.1 Notations, some remarks

Denote the unit disk by D, $D = \{z \in \mathbf{C} : |z| \le 1\}$.

Definition 1. *The set $L \subset \mathbf{C}$ is a* lemniscate *if for some complex polynomial r, $L = r^{-1}[\partial D]$, that is, $z \in L \Leftrightarrow |r(z)| = 1$. The set $r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \le 1\}$ is called the* interior of the lemniscate L.

Note that the interior of a lemniscate is not the topological interior of the lemniscate (which is actually $\{z \in \mathbf{C} : |r(z)| < 1\}$).

A lemniscate is a system of finitely many closed Jordan curves. They are not necessarily simple curves, so we distinguish their points. If $z \in L = r^{-1}[\partial D]$ is a point from the lemniscate L with $r'(z) \ne 0$, then we say z is a simple point (of the lemniscate L). In other words, z is not a critical point of r. It is also equivalent to the fact that L is a simple curve near z_0 (does not cross itself). Moreover, if $r'(z) \ne 0$, then $L = r^{-1}[\partial D]$ is a smooth (actually, analytic) curve near z.

In this chapter we assume that K is the interior of a lemniscate.

If ∂K is differentiable at $z_0 \in \partial K$, then the normal vector (with norm 1) at z_0 pointing outward is denoted by $\mathbf{n}_{z_0}$. We will usually consider $\mathbf{n}_{z_0}$ as a vector and as a complex number simoultaneously. So ∂K near z_0 can be parametrized in the form $z_0 + i\mathbf{n}_{z_0}t + o(|t|)$ for small real values of t.

To generalize the classical Bernstein's inequality (1.1), let us rephrase it in a different way first. We can write Bernstein's inequality for the disk $\{z \in \mathbf{C} : |z| \le \rho\}$ ($\rho > 0$) with the simple substitution $w = \rho z$, hence

$$\left|P_n'(w)\right| \le n\frac{1}{\rho}||P_n||$$

where P_n is arbitrary complex polynomial of degree n, $||P_n||$ is the supremum norm of P_n over the disk $\{z \in \mathbf{C} : |z| \leq \rho\}$ and $|w| = \rho$.

The Green's function of the complement of the disk $\{z \in \mathbf{C} : |z| \leq \rho\}$ is $g(z) = \log|z/\rho|$ which follows immediately from the defining properties. Its derivative at z_0 ($|z_0| = \rho$) with respect to the normal vector $\mathbf{n}_{z_0}$ pointing outward is

$$\frac{\partial}{\partial \mathbf{n}_{z_0}} g(z_0) = \frac{\partial}{\partial \mathbf{n}_{z_0}} \log|z_0/\rho| = \frac{1}{\rho} \ ,$$

which implicitly appears on the right hand side of (1.1).

So the classical Bernstein's inequality (1.1) for any disk $\{z \in \mathbf{C} : |z| \leq \rho\}$ can be written in the following form

$$|P_n'(z_0)| \leq n||P_n|| \frac{\partial}{\partial \mathbf{n}_{z_0}} g(z_0) \qquad (|z_0| = \rho).$$

Now we state

Theorem 2. *Let $K \subset \mathbf{C}$ be the interior of a lemniscate of some polynomial r, that is, $K = r^{-1}[D]$ and let $z_0 \in \partial K$ be fixed. Assume that z_0 is a simple point of ∂K. Denote the Green's function of the unbounded component of $\mathbf{C}_\infty \setminus K$ by $g_K(z)$. Then, for every polynomial P_n with $\deg P_n = n$ we have*

$$|P_n'(z_0)| \leq \big(1 + o(1)\big) \cdot \ n \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \cdot ||P_n||_K \ , \tag{2.1}$$

where the term $o(1)$ is to be understood as $n \to \infty$ and depends only on K and z_0 and is independent of P_n.

The result is sharp in the following two senses.

Theorem 3. *i) For a given fixed n, the factor $1 + o(1)$ can be arbitrarily large, if we choose the set K and the polynomial P_n appropriately.*

ii) For every interior of a lemniscate K there exists a sequence of nonzero polynomials $\{P_n\}$ with degrees tending to infinity such that

$$\big|P_n'(z_0)\big| = n||P_n||_K \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0)$$

where $\deg P_n = n$, $z_0 \in \partial K$ and z_0 is a simple point of ∂K.

In other words, the $1+o(1)$ factor cannot be left out if we choose the compact set and the polynomial suitably, and the constant (the factor $\frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0)$ on the right hand side) cannot be replaced by anything smaller. The proof of this latter Theorem will be given at the end of this chapter.

Brief outline of the proof (of Theorem 2) is as follows. First we prove the statement when P_n is a polynomial of r, that is, there exists a polynomial p such that $P_n = p(r)$. In this case the Bernstein type inequality (2.1) simply follows from the fact that $g_K(z) = \frac{1}{\deg r} \log |r(z)|$. Then we prove the inequality for polynomials that are not (necessarily) polynomials of r. This will be achieved by summing up the P_n on different branches of $r^{-1}[\partial D]$ so that the sum will be a polynomial of r.

For sake of convenience, the notations may change, but this will be explicitly mentioned.

2.2 The proof of Theorem 2 when P_n is a polynomial of r

In this section we prove the Bernstein type inequality (2.1) provided there exists a polynomial p such that $P_n = p(r)$. For simpler notation, we write P for P_n, $P = P_n$. The degree of $P_n = P$ is denoted by n and let $N_r = \deg r, N_p = \deg p$ ($N_p = \deg P / \deg r = n/N_r$).

The following lemma will help us.

Lemma 4. *Let $K := r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \leq 1\}$. Denote the Green's function of the unbounded component of the complement of K by g_K. If $z_0 \in \partial K$ and $r'(z_0) \neq 0$, then*

$$\frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) = \frac{1}{N_r} |r'(z_0)| \ . \tag{2.2}$$

Proof. First, Green's function of $\mathbf{C}_\infty \setminus K$ is $g_K(w) = \frac{1}{N_r} \log |r(w)|$ which immediately follows from the defining properties of Green's function. Second, the following computation holds. If $\chi \in \mathbf{C}$, $|\chi| = 1$ and f holomorphic, then for the directional derivative $\frac{\partial}{\partial \chi} \log |f(z)|$ of $\log |f(z)|$ we have

$$\begin{aligned} \frac{\partial}{\partial \chi} \log |f(z)| = \lim_{t \to 0\ t>0} \frac{\log |f(z + t \cdot \chi)| - \log |f(z)|}{t} = \\ \lim_{t \to 0\ t>0} \frac{\mathrm{Re}\ \big(\log f(z + t \cdot \chi)\big) - \mathrm{Re}\ \big(\log f(z)\big)}{t} = \\ \mathrm{Re} \lim_{t \to 0\ t>0} \frac{\log f(z + t \cdot \chi) - \log f(z)}{t \cdot \chi} \chi = \mathrm{Re}\ \left(\frac{f'(z)}{f(z)} \cdot \chi \right) \ . \end{aligned} \tag{2.3}$$

Applying this with $f = r$, we obtain

$$\frac{\partial}{\partial \mathbf{n}_{z_0}} \frac{1}{N_r} \log |r(z_0)| = \frac{1}{N_r} |r'(z_0)| \ \cdot \ \mathrm{Re}\ \left(\frac{\arg r'(z_0)}{r(z_0)} \mathbf{n}_{z_0} \right) \ .$$

Since $r'(z_0) \neq 0$, r behaves in a small neighbourhood of z_0 like a $\mathbf{R}^2 \to \mathbf{R}^2$ linear mapping which preserves angles. Denote the unit length tangent vector to ∂K at z_0 by $\mathbf{v}_{z_0}$; the actual direction of $\mathbf{v}_{z_0}$ is not important. Since $\mathbf{n}_{z_0}$ is perpendicular to $\mathbf{v}_{z_0}$, that is, to ∂K at z_0, $r'(z_0)\mathbf{n}_{z_0}$ is perpendicular to $r'(z_0)\mathbf{v}_{z_0}$, that is, to ∂D at $r(z_0)$. Furthermore, if $t > 0$ is small enough, then $z_0 + t\mathbf{n}_{z_0} \notin K$, that is, $|r(z_0 + t\mathbf{n}_{z_0})| > 1$. So, from the $r(z_0 + t\mathbf{n}_{z_0}) = r(z_0) + r'(z_0) \cdot t\mathbf{n}_{z_0} + o(|t|)$ representation it follows that $r'(z_0) \cdot \mathbf{n}_{z_0}$ is an outward normal vector to ∂D at $r(z_0)$. Therefore, the direction of $r'(z_0)\mathbf{n}_{z_0}$ coincides with the direction of $r(z_0)$, and this means that $\big(\arg r'(z_0)\big)\mathbf{n}_{z_0} = r(z_0)$. Substituting this into the previous formula, we obtain that $\mathrm{Re}\,(...) = 1$, that is

$$\frac{\partial}{\partial \mathbf{n}_{z_0}} \frac{1}{N_r} \log |r(z_0)| = \frac{1}{N_r} |r'(z_0)| \ .$$

□

Using $P = p(r)$ with $w = r(z)$ and $w_0 = r(z_0)$ ($z_0 \in \partial K$) we can write

$$|P'(z_0)| = |p'\big(r(z_0)\big) \cdot r'(z_0)| = |p'(w_0)| \cdot |r'(z_0)| \ .$$

But $p(w)$ already acts on the unit circle so Bernstein's inequality (1.1) can be applied. Clearly $||p(r(.))||_K = ||p(.)||_D$ holds, furthemore $\deg p = n/N_r$, so (2.2) yields

$$|P'(z_0)| \leq (\deg p) \cdot ||p||_D \cdot |r'(z_0)| = n \cdot ||P||_K \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \ ,$$

which is the inequality (2.1) without the $(1 + o(1))$ here.

2.3 The proof of Theorem 2 for arbitrary polynomials

We will use the already introduced notations: $\deg P = n$, $\deg r = N_r$, $r(z) = w$, $K = r^{-1}[D]$.

Let $z^{(0)} := z$ and $z^{(1)}, \ldots, z^{(N_r - 1)}$ denote those points (counting multiplicity) for which $r(z^{(0)}) = r(z^{(1)}) = \ldots = r(z^{(N_r-1)})$. From the assumption in the theorem, we have $z_0 \neq z_0^{(j)}$ $j = 1, 2, \ldots, N_r$.

Now we construct the weight that we will use when summing up P on different branches of $r^{-1}[\partial D] = \partial K$.

Lemma 5. *For arbitrary $\varepsilon > 0$ and any fixed $z_0 \in \partial K$ where z_0 is simple point ($r'(z_0) \neq 0$) there exists a polynomial $Q(.) = Q(z_0; .) = Q(\varepsilon, z_0; .)$ satisfying the following properties*

$$Q(z_0) = 1 \ , \tag{2.4}$$

$$Q(z_0^{(1)}) = Q'(z_0^{(1)}) = \ldots = Q(z_0^{(N_r-1)}) = Q'(z_0^{(N_r-1)}) = 0 \quad \textit{and} \tag{2.5}$$

$$\sum_{j=0}^{N_r-1} |Q(z^{(j)})| \leq 1 + \varepsilon \quad \textit{for all } z \in \partial K \ . \tag{2.6}$$

Proof. The proof consists of two steps. In the first step we construct a preliminary polynomial, and in the second step we use this preliminary polynomial to construct Q. Since z_0 is fixed, we assume in this proof that $r(z_0) = 1$.

First step.
Consider the following polynomial

$$q_1(m, z_0; z) = q_1(z) := \Big(\frac{1 + r(z)}{2}\Big)^m \ ,$$

where m is a positive integer parameter which we will choose later. Since $r(z_0) = 1$, $q_1(z_0) = 1$. Moreover, $|q_1(z)| < 1$ for all $z \in K$ except for finitely many points (namely, $z_0^{(0)}, z_0^{(1)}, \ldots, z_0^{(N_r-1)}$).

Second step.

Let $q_2(z_0; z) = q_2(z)$ be the polynomial with the lowest possible degree such that

$$q_2(z_0; z_0) = 1, \quad {q_2}'(z_0; z_0) = -1,$$
$$q_2(z_0, z_0^{(1)}) = {q_2}'(z_0; z_0^{(1)}) = \ldots = q_2(z_0, z_0^{(N_r-1)}) = {q_2}'(z_0; z_0^{(N_r-1)}) = 0 \ .$$

The z_0 is not a critical point of r, but the other $z_0^{(j)}$s ($j \neq 0$) may be, so $\deg q_2 \leq 2 \cdot N_r$.

Let $Q = Q(z_0; .)$ be the following polynomial

$$Q(z) = Q(m, z_0; z) := q_1(m, z_0; z) \cdot q_2(z_0; z) \ ,$$

where m will be chosen later. Property (2.5) for Q immediately follows since $q_2(z_0; z_0^{(j)}) = {q_2}'(z_0; z_0^{(j)}) = 0$ at every $z_0^{(j)}$, $z_0^{(j)} \neq z_0$. Property (2.4) is also true because of $q_1(z_0) = q_2(z_0) = 1$.

Now we verify (2.6) for all $z \in \partial K$. The family $\{|q_2(z_0; z)| : z_0 \in \partial K\}$ is uniformly bounded if $z \in K$ and let $M < \infty$ be an upper bound where M is independent of ε and m. Moreover, the derivatives (with respect to z) $\{|{q_2}'(z_0; z)|\}$ are bounded too and let M_1 denote its upper bound. So the functions $q_2(z_0; z)$ $z_0 \in \partial K$ are uniformly equicontinuous on K. That is, there exists $\delta = \delta(\varepsilon)$ which is independent of z_0 and we can reindex the solutions $z^{(0)}, z^{(1)}, \ldots, z^{(N_r)}$ of the equation $r(t) = r(z)$ so that if $z \in \partial K$ and $w = r(z)$ then we have the following assertion:

$$\text{if } |w-1| < \delta, \text{ then } \left|q_2(z_0; z_0^{(j)}) - q_2(z_0; z^{(j)})\right| < \frac{\varepsilon}{N_r} \text{ for all } j.$$

Using this and the definition of q_2, we get the following estimate

$$\sum_{j=0}^{N_r-1} |Q(z_0; z^{(j)})| = \left|q_1(m, z_0; z) \cdot q_2(z_0; z)\right| + \sum_{j=1}^{N_r-1} |Q(z_0; z^{(j)})| \le$$
$$\le 1(1 + \frac{\varepsilon}{N_r}) + (N_r - 1)\frac{\varepsilon}{N_r} = 1 + \varepsilon \ .$$

So if z is such that $|w-1| < \delta$ (where $w = r(z)$), then $\sum |Q| \le 1 + \varepsilon$. Note that we used here that $z_0 \ne z_0^{(j)}$, $j = 1, 2, \ldots, N_r$.

On the other hand, let $z \in \partial K$ be such that $|w - 1| \ge \delta$. Then let us choose m so that

$$\left|q_1(m, z_0; z)\right| < \frac{\varepsilon}{3N_r M} \text{ for all } z \text{ with } |r(z) - 1| > \delta,\ z \in K \ . \tag{2.7}$$

The m depends on ε but is independent of z_0. Then, in this case

$$\sum_{j=0}^{N_r-1} |Q(z_0; z^{(j)})| \le \sum_{j=0}^{N_r-1} \frac{\varepsilon}{3N_r M} \cdot 3M = \varepsilon < 1 + \varepsilon \ .$$

So (2.6) holds.

□

For an arbitrary polynomial $P = P_n$ define

$$P^*(z) := \sum_{j=0}^{N_r-1} P(z^{(j)}) \cdot Q(z_0; z^{(j)}) \ . \tag{2.8}$$

This P^* is symmetric in $z^{(0)}, z^{(1)}, \ldots, z^{(N_r-1)}$ so it is a polynomial in their elementary symmetric polynomials. Consider the equation $r(t) - r(z) = 0$ where z is a parameter and t is the variable. The solutions are $z^{(0)}, z^{(1)}, \ldots, z^{(N_r-1)}$, hence an elementary symmetric polynomial is constant ($\pm$ ratios of coefficients of r) if its degree is smaller than N_r and is a linear polynomial of $r(z)$ if its degree is N_r.

This shows that P^* is a polynomial of $r(z)$, $P^* = p(r)$ where p is a suitable polynomial.

Differentiate P^* at z_0 :

$$\big(P^*\big)'(z_0) = P'(z_0)\, Q(z_0; z_0) + P(z_0)\, Q'(z_0; z_0) + \\ + \sum_{j=1}^{N_r-1} \big(P(z_0^{(j)})\big)'\, Q(z_0; z_0^{(j)}) + \sum_{j=1}^{N_r-1} P(z_0^{(j)})\, Q'(z_0; z_0^{(j)}) \ .$$

For all j with $z_0^{(j)} \neq z_0$ we have $Q(z_0; z_0^{(j)}) = Q'(z_0; z_0^{(j)}) = 0$. Since $Q(z_0; z_0) = 1$ it follows that

$$\big(P^*\big)'(z_0) = P'(z_0) + P(z_0) Q'(z_0; z_0) \ .$$

We estimate the second term as follows

$$\big|P(z_0) Q'(z_0; z_0)\big| \le ||P||_K \cdot \big(m|r'(z_0)| + M_1\big) \ ,$$

where $M_1 = M_1(K)$ and also $m = m(\varepsilon, K, z_0)$ is independent of $\deg P = n$ so this estimate can be written

$$\big|P(z_0) Q'(z_0; z_0)\big| = o(1) n \, \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \, ||P||_K \ ,$$

where $o(1)$ tends to zero as n tends to infinity and is independent of P (but depends on m, z_0, K).

On the other hand, the supremum norm of P^* on K can be estimated as follows (using $r(K) = D$)

$$||P^*(z)||_K = \Big|\Big| \sum_{j=0}^{N_r-1} P(z_0^{(j)}) \cdot Q(z_0; z^{(j)}) \Big|\Big|_D \le \\ \le ||P||_K \sup_{z \in K} \sum_{j=0}^{N_r-1} |Q(z_0; z^{(j)})| \le ||P||_K (1 + \varepsilon) \ .$$

The P^* is a polynomial of r, so we can use the Bernstein type inequality of the previous section. We get

$$\Big|\big(P^*\big)'(z_0)\Big| \le \deg P^* \cdot ||P^*||_K \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \ .$$

We know that $\big(P^*\big)'(z_0) = P'(z_0) + o(1) n \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) ||P||_K$, and $||P^*||_K \le (1 + \varepsilon) \cdot ||P||_K$ and $\deg P^* = \deg P + \deg Q$ and $\deg Q = m \cdot \deg r + \deg q_2 \le$

$mN_r + 2N_r$. The $\varepsilon > 0$ is fixed, so is $m = m(\varepsilon)$. So $\deg Q \le m(\varepsilon) \cdot N_r + 2 \cdot N_r$, which is a fixed value too. So we have

$$\left|P'(z_0)\right| \le n \cdot \left(1 + \frac{mN_r}{n} + \frac{2N_r}{n} + o(1)\right) \cdot (1+\varepsilon) \cdot ||P||_K \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) =$$
$$= (1+o(1)) \cdot n \cdot ||P||_K \cdot \frac{\partial}{\partial \mathbf{n}_{z_0}} g_K(z_0) \ .$$

It is easy to verify that the $o(1)$ error term depends on z_0, because the degree of $Q(z_0; z)$ depends on z_0.

2.4 Sharpness of the results

Proof of Theorem 3 i). Let $r(z) = z^l - 1$ and $P(z) = z$ and $K = r^{-1}[\partial D]$. Then, the equilibrium measure of K is absolutely continuous with respect to arc length (denote its density by ω) and the length of lemniscate K is at least $2l$. (Furthermore, $z \in K \Rightarrow |z| \le 2$, so $||P||_K \le 2$ for every l.) That is, if $l \to \infty$, then there will exist a $z \in K$ such that $\partial/\partial \mathbf{n}_{z_0} g_K(z) = \pi\omega(z) \le 1/(2l)$ and

$$1 \cdot 2 \cdot \frac{1}{2l}$$

which is larger than the right hand side (of (2.1) for P on K) and is much smaller than $1 = |P'(z)|$. □

Proof of Theorem 3 ii). If K is a lemniscate, that is, $K = r^{-1}[\partial D]$ for some complex polynomial r, then let $P(z) := \big(r(z)\big)^m$ where m is an arbitrary positive integer. Then, $\deg P = m \deg r$ and if $z \in K$, then $P'(z) = m\big(r(z)\big)^{m-1} \cdot r'(z)$. So the left hand side (of (2.1) for P at z) is

$$m\left|\big(r(z)\big)^{m-1} \cdot r'(z)\right| = m\left|r'(z)\right|$$

while the right hand side (of (2.1) for P at z) is (see Lemma (4))

$$m \deg r \, ||\big(r(.)\big)^m||_{r^{-1}[\partial D]} \frac{1}{\deg r} \left|r'(z)\right| = m\left|r'(z)\right|$$

which is the same. □

Chapter 3

Sharpening of Hilbert's lemniscate theorem

3.1 Preliminaries to Theorem 9

As in the preceeding chapter, a lemniscate σ is a level curve of a polynomial, i.e. $\sigma = \{z : |T_N(z)| = c\}$ for some polynomial T_N and some constant c (which may always be assumed to be 1). Hilbert's lemniscate theorem claims the following (see [51, Theorem 5.5.8, p. 158] or [52, p. 79]).

Theorem 6. *If K is a compact set on the plane and U is a neighborhood of K then there is a lemniscate σ that separates K and $\mathbf{C} \setminus U$, i.e. it lies within U but encloses K. In other words, $K \subset \operatorname{Int} \sigma$ and $\overline{\operatorname{Int} \sigma} \subset U$.*

An equivalent formulation is the following. Let γ_j, Γ_j, $j = 1, \ldots, m$ be Jordan curves (i.e. homeomorphic images of the unit circle), γ_j lying interior to Γ_j and the Γ_j's lying exterior to each other, and set $\gamma^* = \cup_j \gamma_j$, $\Gamma^* = \cup_j \Gamma_j$. Then there is a lemniscate σ contained in the interior of Γ^* that contains γ^* in its interior, i.e. σ separates γ^* and Γ^* in the sense that it separates each γ_j from the corresponding Γ_j.

In this and the following few sections we shall extend this lemniscate theorem to the case when γ^* can touch Γ^* at finitely many points. It will follow that at the touching points the normal derivative of Green's function for the (unbounded component of the) complement of σ can be as close to the normal derivative of Green's function for the complement of Γ^* as we wish. This fact will be applied to derive the analogue of Bernstein's inequality for polynomials on Γ^* with asymptotically sharp constants.

Let γ^* and Γ^* be twice continuously differentiable in a neighborhood of P and touching each other at P. We say that they $\mathcal{K}$-*touch* each other if

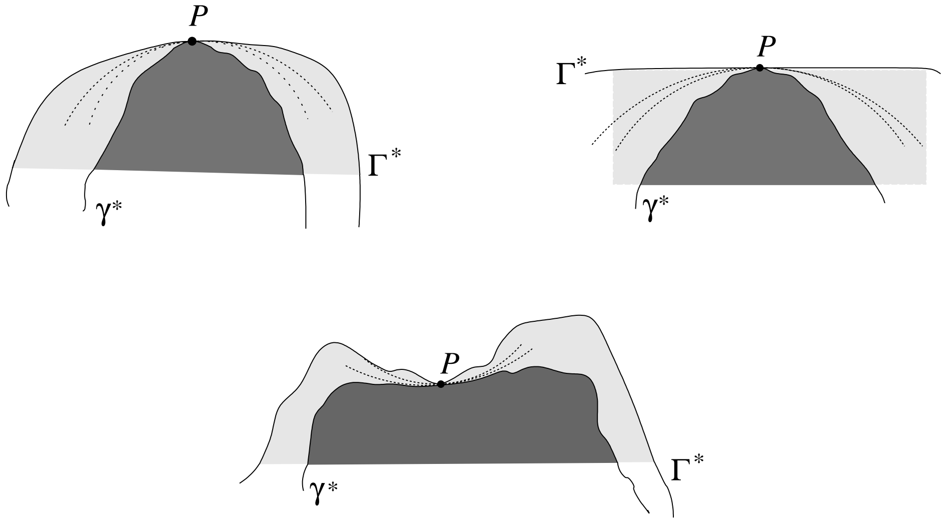

Figure 3.1: Various possibilities for the separating circles

their (signed) curvature at P is different (signed curvature is seen from the outside of Γ^*). Equivalently we can say that in a neighborhood of P the two curves are separated by two circles one of them lying in the interior of the other one. See Figure 3.1 for the various possibilities for these circles.

One of our main theorems is

Theorem 7. *Let $\gamma^* = \cup_{j=1}^m \gamma_j$ and $\Gamma^* = \cup_{j=1}^m \Gamma_j$ be as above, and let γ^* $\mathcal{K}$-touch Γ^* in finitely many points $P_1, \ldots, P_k$ in a neighborhood of which both curves are twice continuously differentiable. Then there is a lemniscate σ that separates γ^* and Γ^* and $\mathcal{K}$-touches both γ^* and Γ^* at each P_j.*

Furthermore, σ lies strictly in between γ^* and Γ^* except for the points $P_1, \ldots, P_k$, and has precisely one connected component in between each γ_j and Γ_j, $j = 1, \ldots, m$, and these m components are Jordan curves.

It should be noted that since in Theorem 7 the lemniscate σ is strictly in between γ^* and Γ^* except for the points $P_1, \ldots, P_k$ and in these points it has curvature bigger then the corresponding curvature of Γ^*, this σ can play the role of the inner curve γ^*, and this way we can get an exhaustion of the domain enclosed by Γ^* by interiors of lemniscates $\sigma_0, \sigma_1, \ldots$ touching in more and more points, as is depicted in Figure 3.2.

Let K be the closed domain enclosed by Γ^* and K_0 the closed domain enclosed by γ^*. Denote by $g(K, z)$ Green's function of $\mathbf{C}_\infty \setminus K$ with pole at infinity. Finally, let L be the closed domain enclosed by σ. We shall need

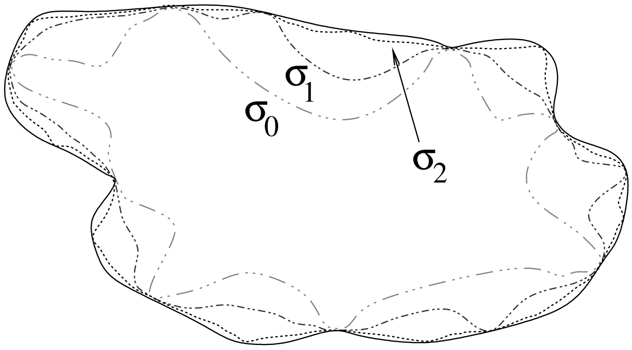

Figure 3.2: Exhaustion by interiors of lemniscates touching in more and more points

Theorem 8. *Let Γ^*, γ^* and $P_1, \ldots, P_k \in \Gamma^*$ be as in Theorem 7. Then for every $\varepsilon > 0$ there is a lemniscate σ as in Theorem 7 such that for each P_j we have*

$$\frac{\partial g(L, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(K, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.1}$$

where $\partial(\cdot)/\partial\mathbf{n}$ denotes (outward) normal derivative.

In a similar manner, for every $\varepsilon > 0$ there is a lemniscate σ as in Theorem 7 such that for each P_j we have

$$\frac{\partial g(K_0, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(L, P_j)}{\partial \mathbf{n}} + \varepsilon. \tag{3.2}$$

Note that

$$\frac{\partial g(K, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(L, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(K_0, P_j)}{\partial \mathbf{n}},$$

because $K_0 \subset L \subset K$.

As an application of these results we prove the following Bernstein'stype inequality with asymptotically best constant for derivatives of polynomials. By approximating a compact set K from the inside by touching lemniscates we deduce from Theorem 2 a general Bernstein type inequality. We want to do that for more general sets than those bounded by finitely many Jordan curves, so we make the following definition. We say that the compact set K is *Jordan fat*, if the boundary of every connected component of its interior $\mathrm{Int}(K)$ is a Jordan curve and K is the closure of its interior: $K = \overline{\mathrm{Int}(K)}$. In particular, every component of its interior is a simply connected domain, but K may have infinitely many connected components or it may have cut points on the boundary.

Now we can state

Theorem 9. *Let K be a Jordan fat compact set on the plane with connected complement. Let z_0 be a point on the boundary of K and let us suppose that this boundary is a twice continuously differentiable Jordan arc in a neighborhood of z_0. Then*

$$|P_n'(z_0)| \le n(1+o(1))\frac{\partial g(K,z_0)}{\partial \mathbf{n}}\|P_n\|_K, \tag{3.3}$$

where the $o(1)$ tends to 0 uniformly in the polynomials P_n of degree at most n as $n \to \infty$.

Recall that a Jordan arc is a homeomorphic image of the interval $(0,1)$.

Theorem 9 is sharp regarding the constant $\partial g(K,z_0)/\partial \mathbf{n}$:

Theorem 10. *Let K and z_0 be as in Theorem 9. Then for every n there is a polynomial P_n of degree at most n such that*

$$|P_n'(z_0)| > n(1-o(1))\frac{\partial g(K,z_0)}{\partial \mathbf{n}}\|P_n\|_K. \tag{3.4}$$

Actually, this theorem is true for any compact K for which z_0 belongs to the boundary of the interior of K, and in a neighborhood of z_0 the boundary ∂K is a twice differentiable Jordan arc. This follows from the proof below by first approximating K from the outside by a compact set which is bounded by finitely many Jordan curves and which coincides with K in a neighborhood of z_0.

The proofs of the theorems use some basic tools from potential theory, for which see for example [51], [59] or [52]. Placing lemniscates in between touching curves will be done by the Brouwer fixed point theorem and by a local version of Blaschke's rolling theorem ([12, Ch. 4., Section 24., subsection II.]) claiming that out of two touching curves the one with a larger curvature stays inside the other one.

We also remark that having just a single touch point is conceptually simpler (and a simpler translation-rotation technique would work) than having finitely many points. In order to facilitate the general discussion, when dealing with a single touching point in Section 3.2 we shall follow the more involved approach that will lead to the general case of finitely many touching points.

The outline of the paper is as follows. In the next section we shall prove the analogue of Theorem 7 for $k = 1$ (i.e. when there is only one touch point), but for Green lines (i.e. level lines of some Green's functions) instead of lemniscates. Then in Section 3.3 we extend this to any k points still for

Green lines. Section 3.4 contains the completion of the proof by showing that the Green lines in Section 3.3 can be replaced by lemniscates. Section 3.5 contains the proof of Theorem 8, while in Section 3.6 we present some lemmas that are frequently used in the proofs. Finally, in Section 3.7 we give short proofs for Theorems 9 and 10.

3.2 Green lines touching in one point

In this section we prove Theorem 7 for a single touching point ($k = 1$) and for Green lines (level lines of Green's functions) instead of lemniscates. Thus, let γ_j, Γ_j, $j = 1, \ldots, m$ be Jordan curves, γ_j lying interior to Γ_j and the Γ_j's lying exterior to each other, $\gamma^* = \cup_j \gamma_j$, $\Gamma^* = \cup_j \Gamma_j$, and denote by K the closed domain enclosed by Γ^* and by K_0 the closed domain enclosed by γ^*. Some of the topological properties are easier to see for smooth curves, and by suitable approximation we may assume that the curves γ_j and Γ_j are twice continuously differentiable (apply e.g. conformal mapping of the interior of the curves in question on the unit disk and make use of [50, Theorem 3.6]), though we shall not explicitly use this assumption (except the twice differentiability around touching points).

For simpler notation Γ will mean any one (but fixed) of the curves Γ_j, and then γ is the corresponding inner curve γ_j.

The proof is fairly technical, therefore first we present an outline:

- First we remove a small part of the closed inner domain K_0 around the point P, the rest will be denoted by K_1.
- The removed part will be replaced by a rotated and shifted copy $T^{\theta,\delta}(S)$ of a lens shaped region S for which the bounding circular arcs have curvature lying in between the curvatures of Γ and γ at the point P.
- The Green line will be for some small τ the τ-level curve of Green's function $g(K_1 \cup T^{\theta,\delta}(S), z)$ of $\mathbf{C}_\infty \setminus (K_1 \cup T^{\theta,\delta}(S))$ with pole at infinity.
- To analyze these τ-level lines close to the boundary of $T^{\theta,\delta}(S)$ we use the reflection principle to continue the Green's functions $g(K_1 \cup T^{\theta,\delta}(S), z)$ over the circular arc $\partial T^{\theta,\delta}(S)$, and complete these continued harmonic functions to analytic functions. This way the τ-level line of $g(K_1 \cup T^{\theta,\delta}(S), z)$ coincides with the image of a line segment under the inverse of these analytic functions, and simple analytic properties can be used for the analysis (Lemma 11).

- We shall use the Brouwer fixed point theorem to prove that for appropriate (and small) rotation (by angle θ) and shift (by δ), the τ-level line will pass through the point P and will have the same tangent line there as Γ (and γ).

- For small τ this τ-level line will lie very close to $K_1 \cup T^{\theta,\delta}(S)$, hence it will separate each γ_j from Γ_j, and along the boundary of $T^{\theta,\delta}(S)$ it will have curvature very close to that of $\partial T^{\theta,\delta}(S)$, which is the same as the curvature of ∂S.

- As a consequence, in the neighborhood of P we are working in, the curvature of the τ-level line will lie in between the curvatures of γ and Γ and at the same time it touches both of these curves at P. Hence, by a variant of Blaschke's rolling theorem (given in Lemma 13) the level line will lie in between these two curves in a smaller neighborhood.

- Elsewhere the τ-level line follows closely the boundary of $K_1 \cup T^{\theta,\delta}(S)$, hence it lies outside γ^* but inside Γ^*.

To carry out this project we have found it convenient to separate γ and Γ in a neighborhood of P by two circles $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ specified in the second paragraph below. In what follows we may encounter compact sets $\mathcal{L}$ (like $K_1 \cup S$ or $K_1 \cup T^{\theta,\delta}(S)$ below) which may have unconnected component, in which case Green's function $g(\mathcal{L}, \cdot)$ will mean Green's function of the unbounded component of $\mathbf{C}_\infty \setminus \mathcal{L}$ with pole at infinity.

If σ is a curve which is twice continuously differentiable in a neighborhood of $P \in \sigma$, then let $\kappa(\sigma, P)$ denote the curvature of σ at P and $\varphi(\sigma, P)$ the tangent direction angle (i.e. the angle with the positive half of the real axis of the tangent line to σ at P), which we consider modulo π.

In this section we shall assume that γ^* and Γ^* $\mathcal{K}$-touch each other at a single point P, and they are twice continuously differentiable in a neighborhood of P. Thus, γ^* lies strictly within Γ^* except for the point P, where we have $\kappa(\gamma^*, P) > \kappa(\Gamma^*, P)$ (note that $\mathcal{K}$-touching means that $\kappa(\gamma^*, P) \neq \kappa(\Gamma^*, P)$, and $\kappa(\Gamma^*, P) > \kappa(\gamma^*, P)$ is impossible because γ^* lies inside Γ^*, see e.g. Lemma 13).

Let $\Delta_r(P) = \{\zeta \,|\, |z - P| < r\}$ denote the open disk of radius r about P, and for simpler notation in this section we shall write Δ_r for $\Delta_r(P)$.

Let $P \in \Gamma$ (recall that Γ is one of the Γ_j's), and with $\kappa(\Gamma, P) < 1/R_1 < 1/R_0 < \kappa(\gamma, P)$ consider the circles $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ of radii R_0 and R_1, respectively, that touch Γ at P in appropriate sense, see Figure 3.3 (if the curvature $\kappa(\Gamma, P)$ is negative, then we set $|\kappa(\gamma, P)| < 1/R_0 < 1/R_1 < |\kappa(\Gamma, P)|$, see

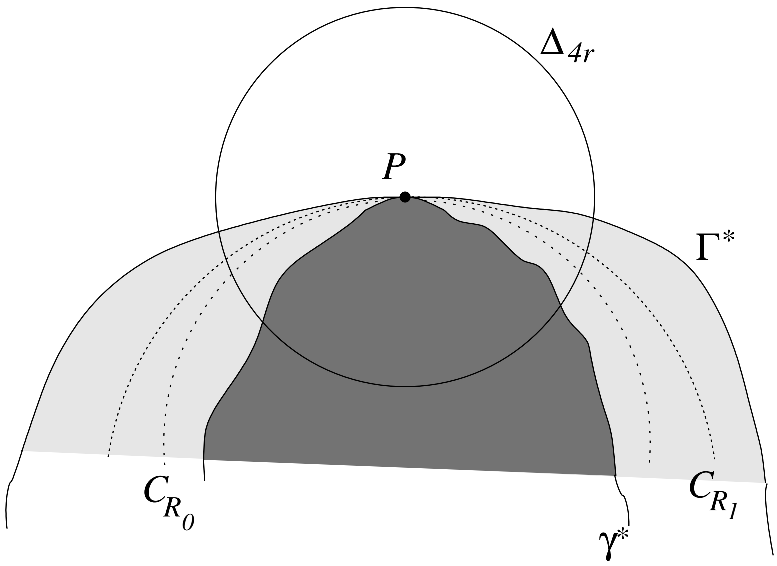

Figure 3.3: The choice of $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$

Figure 3.4). Then in a neighborhood of P these circles lie inside Γ and outside γ (see e.g. Lemma 13), and in this section our aim is to show that there is a Green line (the level curve of a Green's function) that separates γ_j and Γ_j, $j = 1, \ldots, m$, and in a neighborhood of P it also lies in between $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$. We shall only deal with the nonnegative curvature case, the argument is similar when the curvature at P is negative.

In what follows $\mathcal{C}_R$ always means a circle touching Γ at P in the appropriate sense and $\mathcal{D}_R$ denotes the closed disk that it encloses.

Choose a number $R_0 < R < R_1$. Then (see Figure 3.3) there is a small $r < R_0/8$ such that $\kappa(\Gamma, z) < 1/R_1$ and $\kappa(\gamma, z) > 1/R_0$ for $z \in \Delta_{4r}$, $\mathcal{D}_{R_1} \cap \Delta_{4r}$ lies inside Γ and $\mathcal{D}_{R_0}$ contains the part of K_0 that lies in Δ_{4r}. We may also suppose $r < 1$ so small that Δ_{8r} intersects only the curve Γ (which contains P) out of the curves $\Gamma_1, \ldots, \Gamma_m$. Let A and B be the two points on $\mathcal{C}_R$ lying of distance $4r$ from P, and let S be a closed lens shaped domain bounded by the (shorter) arc $\widetilde{AB}$ of the circle $\mathcal{C}_R$ and by its reflection onto the segment AB, see Figure 3.5. When the curvature of Γ at P is negative then we have to make a slight change in the definition of S: then let S be bounded by $\widetilde{AB}$ and by a curve lying inside Γ and going close to $\widetilde{AB}$, see Figure 3.6. We cut off a small part of K_0, namely if Q is the open half-plane with boundary line passing through the points A and B and containing P, then we cut off

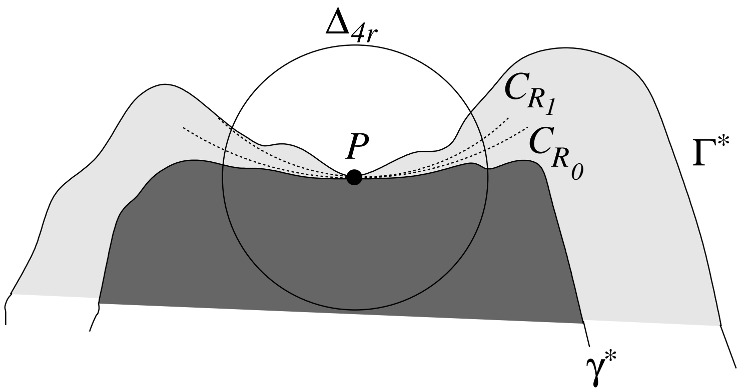

Figure 3.4: The choice of $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ for negative curvature

$\Delta_{4r} \cap K_0 \cap Q$, and set

$$K_1 = K_0 \setminus \Big(\Delta_{4r} \cap K_0 \cap Q \Big), \tag{3.5}$$

for the remainder, see Figure 3.5, where the darkest shaded region is the cut off part $\Delta_{4r} \cap K_0 \cap Q$. Then $K_0 \subset K_1 \cup S$, $K_1 \cup S$ has one connected component inside Γ and $K_1 \cup S$ lies inside Γ^* except for its point at P. It is possible however, that $K_1 \cup S$ (or the sets $K_1 \cup T^{\theta,\delta}(S)$ considered below) has unconnected complement. Let $T^{\theta,\delta}$ be the transformation that consists of a counterclockwise rotation about P by angle θ followed by a translation in the direction of PO by δ, where O is the center of $\mathcal{C}_R$ (this is the inner normal direction). With S we also consider the domain $T^{\theta,\delta}(S)$. We restrict θ, δ with some small but fixed numbers $0 \le \theta^*, \delta^* < r/8$ so that for $-\theta^* \le \theta \le \theta^*$ and $0 \le \delta \le \delta^*$ the circle $T^{\theta,\delta}(\mathcal{C}_R)$ hits $\partial\Delta_{r/2}$ in between the circles $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$, and $T^{\theta,\delta}(S) \setminus \Delta_{r/2}$ lies inside Γ. If θ^*, δ^* are sufficiently small, then we have $K_0 \setminus \Delta_{r/2} \subset K_1 \cup T^{\theta,\delta}(S)$.

Note also that the system of curves

$$\Sigma = \Big(\mathcal{C}_{R_1} \cap \overline{\Delta_{r/2}} \Big) \cup \big(\Gamma^* \setminus \Delta_{r/2} \big) \cup \big((\partial\Delta_{r/2} \cap K) \setminus \mathcal{D}_{R_1} \big) \tag{3.6}$$

(see Figure 3.7) lies within Γ^* and outside γ^* (except for the point P). We shall put a Green line outside γ^* that lies within Σ and also in between $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ in $\Delta_{r/2}$.

The Green's functions $g(K_1 \cup T^{\theta,\delta}(S), z)$, $|\theta| \le \theta^*$, $0 \le \delta \le \delta^*$ of $\mathbf{C}_\infty \setminus (K_1 \cup T^{\theta,\delta}(S))$ (or of their unbounded component if these sets are not connected) are uniformly bounded on compact subsets of the plane. $g(K_1 \cup T^{\theta,\delta}(S), z)$ vanishes on $T^{\theta,\delta}(\widetilde{AB})$, hence by the reflection principle we can reflect it on

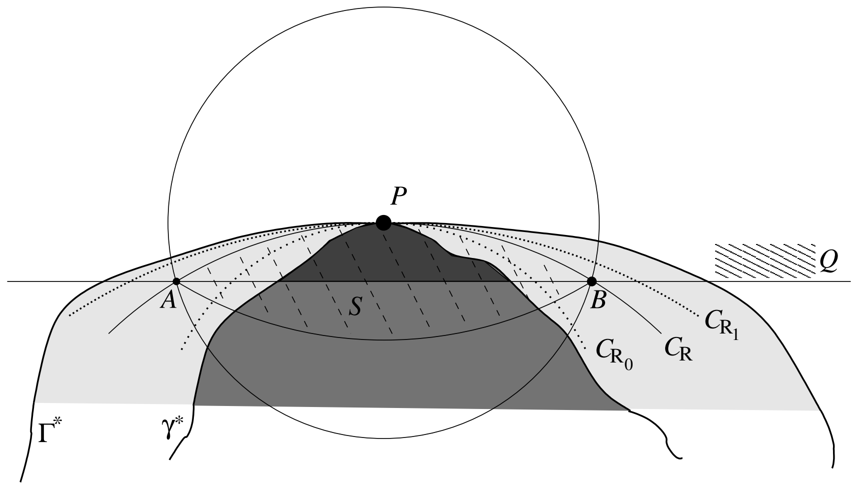

Figure 3.5: The darkest shaded region is cut off from K_0 and the NE-SW striped lens shaped region is added

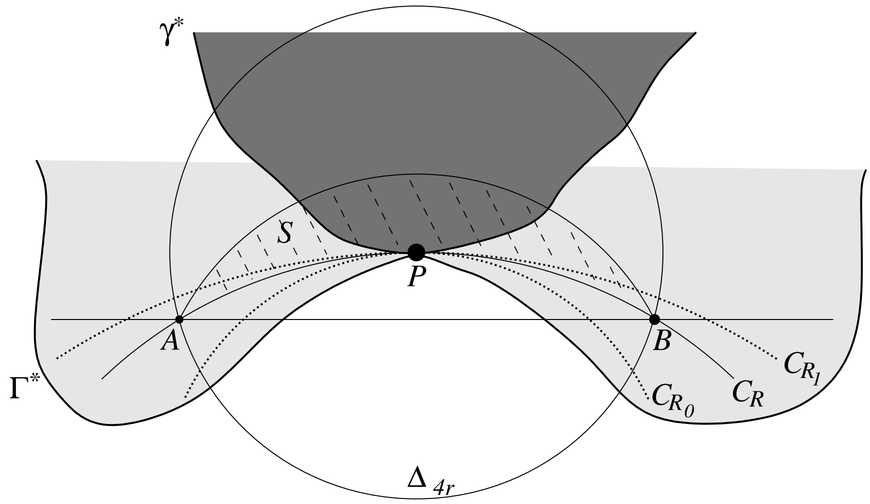

Figure 3.6: Selection of S for negative curvature

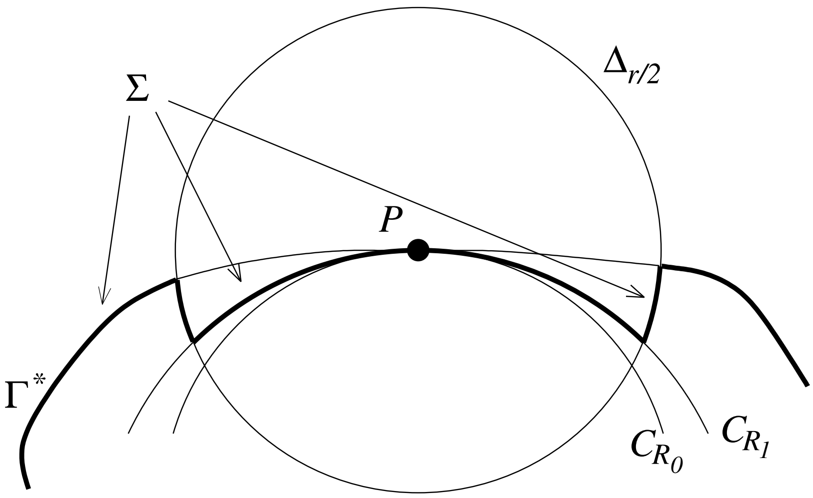

Figure 3.7: The curve Σ

the circular arc $T^{\theta,\delta}(\widetilde{AB})$, and let this extended function be denoted by $g^E(K_1 \cup T^{\theta,\delta}(S), z)$. Thus, these $g^E(K_1 \cup T^{\theta,\delta}(S), z)$ are uniformly bounded harmonic functions on Δ_{3r}, and let $h_{\theta,\delta}$ be their analytic completion in Δ_{3r} such that the imaginary part vanishes at P: Im $h_{\theta,\delta}(P) = 0$. Then $h_{\theta,\rho}$ and their first and second derivatives are uniformly bounded in Δ_{2r}.

Next we claim that

$$\frac{\partial g(K_1 \cup T^{\theta,\delta}(S), z)}{\partial \mathbf{n}} \geq c_0 \tag{3.7}$$

with some $c_0 > 0$ independent of θ, δ, where the partial derivative is taken in the direction of the normal to $T^{\theta,\delta}(\widetilde{AB})$, and the inequality is claimed for $z \in T^{\theta,\delta}(\widetilde{AB}) \cap \Delta_{2r}$. To this end let $g_w(K_1 \cup T^{\theta,\delta}(S), z)$ denote Green's function of $\mathbf{C}_\infty \setminus (K_1 \cup T^{\theta,\delta}(S))$ with pole at w. Then the normal derivative in question is $\partial g_\infty(K_1 \cup T^{\theta,\delta}(S), z)/\partial \mathbf{n}$. Let w_0 be the point on $\partial \Delta_r$ that is the farthest away from the arc $\widetilde{AB}$, see (see Figure 3.8). The function $\partial(g_w(K_1 \cup T^{\theta,\delta}(S), z))/\partial \mathbf{n}$ is non-negative and harmonic in w, hence Harnack's inequality gives with a $c_1 > 0$ independent of $|\theta| \leq \theta^*$, $0 \leq \delta \leq \delta^*$ and $z \in T^{\theta,\delta}(\widetilde{AB}) \cap \Delta_r$

$$\frac{\partial g_\infty(K_1 \cup T^{\theta,\delta}(S), z)}{\partial \mathbf{n}} \geq c_1 \frac{\partial g_{T^{\theta,\delta}(w_0)}(K_1 \cup T^{\theta,\delta}(S), z)}{\partial \mathbf{n}}. \tag{3.8}$$

But it is easy to see that the right hand side is uniformly bounded from below on $T^{\theta,\delta}(\widetilde{AB}) \cap \Delta_r$. In fact, just attach a domain $\mathcal{H}$ to $\widetilde{AB} \cap \Delta_{3r}$ with C^2 boundary in such a way that it contains w_0 and lies in $\mathbf{C} \setminus K$ (see Figure 3.8).

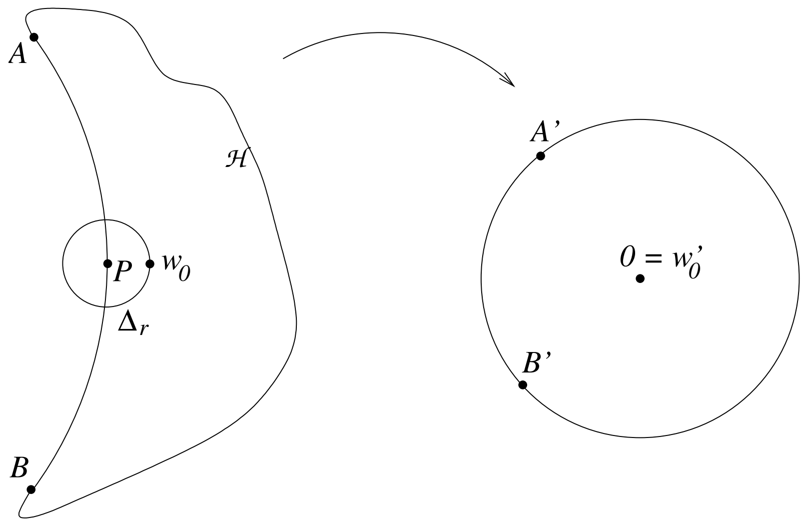

Figure 3.8: The choice of w_0 and $\mathcal{H}$

Then $g_{T^{\theta,\delta}(w_0)}(K_1 \cup T^{\theta,\delta}(S), z)$ at $z = T^{\theta,\delta}(\zeta)$ is bigger than Green's function g of $\mathcal{H}$ with pole at w_0 at ζ, hence the right hand side of (3.8) is at least as large as the appropriate normal derivative for g at $\zeta = (T^{\theta,\delta})^{-1}(z)$. But $\mathcal{H}$ can be conformally mapped into the unit disk so that w_0 is mapped into the origin, and this conformal map is C^1 up to the boundary of $\mathcal{H}$. Since Green's functions are conformal invariant, the lower boundedness of the right hand side of (3.8) is a consequence of the same result on the disk (in which case Green's function is just $\log 1/|z|$).

A consequence of (3.7) is that $|h'_{\theta,\delta}(z)| \geq c_0$ for $T^{\theta,\delta}(\widetilde{AB}) \cap \Delta_{2r}$ (recall that $h_{\theta,\delta}$ was the analytic completion of $g^E(K_1 \cup T^{\theta,\delta}(S), \cdot))$, hence, by the uniform boundedness of the second derivatives of $h_{\theta,\delta}$ in Δ_{2r}, it follows that there is a neighborhood U of $\widetilde{AB} \cap \Delta_{2r}$ such that in $T^{\theta,\delta}(U)$ we have $|h'_{\theta,\delta}(z)| \geq c_0/2$, and this is even true in a neighborhood of the closure of U. Let

$$\sigma(\tau, \theta, \delta) = \{z \,|\, g(K_1 \cup T^{\theta,\delta}(S), z) = \tau\}$$

be the τ-level line of Green's function $g(K_1 \cup T^{\theta,\delta}(S), \cdot)$. From the properties of $h_{\theta,\delta}$ and from Lemma 11 below it follows that there are constants C_0 and $\tau_0 > 0$ such that for $0 \leq \tau \leq \tau_0$ and $z \in \sigma(\tau, \theta, \delta) \cap T^{\theta,\delta}(U)$

1. we have for the distance from z to the boundary arc $T^{\theta,\delta}(\widetilde{AB})$

$$\frac{1}{C_0}\tau \leq \operatorname{dist}(z, T^{\theta,\delta}(\widetilde{AB})) \leq C_0\tau, \tag{3.9}$$

2. we have for the curvatures

$$|\kappa(\gamma(\tau,\theta,\delta),z)-\frac{1}{R}|\le C_0\tau \tag{3.10}$$

(recall that the curvature of $T^{\theta,\delta}(\widetilde{AB})$ is $1/R$), and

3. if z^* is the intersection point of $T^{\theta,\delta}(\widetilde{AB})$ with the segment $\overline{zT^{\theta,\delta}(O)}$ connecting z with the center $T^{\theta,\delta}(O)$ of the circle $T^{\theta,\delta}(\mathcal{C}_R)$, then we have for the tangent direction angles

$$|\varphi(\sigma(\tau,\theta,\delta),z)-\varphi(T^{\theta,\delta}(\widetilde{AB}),z^*)|\le C_0\tau. \tag{3.11}$$

The first and third estimate we shall only need around P, but the second one along $\sigma(\tau,\theta,\delta)\cap T^{\theta,\delta}(U)$.

Let $\sigma_{\theta,\delta}$ be the level line of $g(K_1\cup T^{\theta,\delta}(S)),\cdot)$ passing through the point P, and let $\varphi_0=\varphi(\Gamma,P)$ be the tangent direction angle to the bounding curve Γ at P. The distance from P to $K_1\cup T^{\theta,\delta}(S)$ is the same as to $T^{\theta,\delta}(S)$, and this is $\ge\delta\cos\theta>\delta/2$ (recall that $T^{\theta,\delta}$ consists of a rotation by angle θ and by a shift δ, and this latter one moves $T^{\theta,0}(S)$ away from P by δ). Therefore, (3.9) implies for $\theta\in[-\theta^*,\theta^*]$ and $\delta=2C_0\tau$ that the point P lies outside the τ level line (i.e. $\sigma(\tau,\theta,\delta)$ lies inside $\sigma_{\theta,2C_0\tau}$):

$$g(K_1\cup T^{\theta,2C_0\tau}(S),P)>\tau. \tag{3.12}$$

Next, if $0<\tau<\tau_0$ is sufficiently small, then for $0\le\delta\le 2C_0\tau$ (3.11) gives

$$\varphi(\sigma_{\theta^*,\delta},P)-\varphi_0>\frac{\theta^*}{2} \tag{3.13}$$

and

$$\varphi(\sigma_{-\theta^*,\delta},P)-\varphi_0<-\frac{\theta^*}{2}. \tag{3.14}$$

Now fix $0<\tau<\tau_0$ so small that all these are satisfied, as well as the inequalities

$$C_0\tau<(1/R_0-1/R)/2,\qquad C_0\tau<(1/R-1/R_1)/2$$

and

$$\tau<\inf\left\{g(K_1\cup T^{\theta,\delta}(S),z)\,\Big|\,-\theta^*\le\theta\le\theta^*,\ 0\le\delta\le\delta^*,\ \ z\in\Sigma\setminus\Delta_{r/2}\right\}, \tag{3.15}$$

where Σ is the curve defined in (3.6). Since all points of $\Sigma\setminus\Delta_{r/2}$ lie outside every $K_1\cup T^{\theta,\delta}(S)$, $-\theta^*\le\theta\le\theta^*$, $0\le\delta\le\delta^*$ (by the choice of θ^*,δ^*), the latter infimum is positive, hence such a choice of τ is possible.

On the set $[-\theta^*, \theta^*] \times [0, 2C_0\tau]$ consider the functions

$$f(\theta, \delta) = g(K_1 \cup T^{\theta,\delta}(S), P) - \tau, \tag{3.16}$$

and

$$\Phi(\theta, \delta) = \varphi(\sigma_{\theta,\delta}, P) - \varphi_0. \tag{3.17}$$

These are continuous functions of (θ, δ), and their behavior on the boundary is as follows:

$$f(\theta, 2C_0\tau) > 0 \qquad \text{by (3.12)}, \tag{3.18}$$

$$f(\theta, 0) < 0 \qquad \text{because} \quad g(K_1 \cup T^{\theta,0}(S), P) = 0, \tag{3.19}$$

$$\Phi(\theta^*, \delta) > 0 \qquad \text{by (3.13)}$$

and

$$\Phi(-\theta^*, \delta) < 0 \qquad \text{by (3.14)}.$$

Therefore Lemma 12 can be applied to the function $F(\theta, \delta) = (\Phi(\theta, \delta), f(\theta, \delta))$ on the box $[-\theta^*, \theta^*] \times [0, 2C_0\tau]$ to conclude that there is a $\theta \in [-\theta^*, \theta^*]$ and a $0 \leq \delta \leq 2C_0\tau$ such that $f(\theta, \delta) = \Phi(\theta, \delta) = 0$. In other words, for this θ and δ the τ-level line $\sigma(\tau, \theta, \delta)$ of the Green's function of $K_1 \cup T^{\theta,\delta}(S)$ passes through the point P and at P it has the same tangent line as Γ.

This is true for all sufficiently small $\tau > 0$. We claim that for small τ this level line $\delta(\tau, \theta, \delta)$ separates each γ_j from Γ_j, it consists of m components and it lies in between $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ in $\Delta_{r/2}$. For the latter one consider that by (3.10) and the choice of τ we have $1/R_1 < \kappa(\sigma(\tau, \theta, \delta), z) < 1/R_0$ for all $z \in \sigma(\tau, \theta, \delta) \cap T^{\theta,\delta}(U)$, hence for all $z \in \Delta_r \cap \sigma(\tau, \theta, \delta)$, and so we may apply Lemma 13 to conclude that in $\Delta_{r/2}$ the level line $\sigma(\tau, \theta, \delta)$ lies in between $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$, for it has the same starting point P and the same tangent line at P as these latter circles. The curve (3.6) encloses the Green line $\sigma(\tau, \theta, \delta)$ because of what we have just proved and because of (3.15). Finally, $\sigma(\tau, \theta, \delta)$ lies outside $K_1 \cup T^{\theta,\delta}(S)$, and since this set has exactly one component in each Γ_j, we get that for small $\tau > 0$ the level curve $\sigma(\tau, \theta, \delta)$ has exactly m components and it separates each γ_j from Γ_j. That this level curve $\sigma(\tau, \theta, \delta)$ consists of precisely m Jordan curves follows from the fact that each bounded component of the complement of $\sigma(\tau, \theta, \delta)$ must contain a point of $K_1 \cup T^{\theta,\delta}(S)$, so there are precisely m such bounded components, one-one containing γ_j, $j = 1, \ldots, m$ (in other words, $\sigma(\tau, \theta, \delta)$ cannot intersect itself).

3.3 Green lines touching in finitely many points

In this section we extend the construction in Section 3.2 to k touching points.

Let γ_j, Γ_j, $j = 1, \ldots, m$ be the given Jordan curves, γ_j lying interior to Γ_j and the Γ_j's lying exterior to each other, and γ^* touching Γ^* in the finitely many points $P_1, \ldots, P_k$, where we assume the curves to be twice continuously differentiable. For each j let there be given two touching circles $\mathcal{C}_{R_{j,0}}$ and $\mathcal{C}_{R_{j,1}}$ with

$$\kappa(\gamma^*, P_j) > 1/R_{j,0} > 1/R_{j,1} > \kappa(\Gamma^*, P_j)$$

(with appropriate modification for negative curvatures). We want to prove that there is a Green line separating γ^* and Γ^* which also goes in between $\mathcal{C}_{R_{j,0}}$ and $\mathcal{C}_{R_{j,1}}$ in a neighborhood of each P_j. We follow the proof from Section 3.2, just do what was done there simultaneously around each P_j. We follow the notations there, but let us agree that the relevant objects from Section 3.2 for a point P_j (instead of the point P of Section 3.2) will be denoted by affixing the subscript j. In particular, we fix radii $R_{j,0} < R_j < R_{j,1}$ and consider touching circles $\mathcal{C}_{R_j}$ etc. The radius r can be chosen to be common for all P_j, and then let A_j and B_j be the two points on $\mathcal{C}_{R_j}$ lying of distance $4r$ from P_j, and let S_j be the closed lens shaped domain bounded by the arc $\widetilde{A_jB_j}$ and by its reflection onto A_jB_j (with obvious modifications for the negative curvature case), see Figure 3.9. We cut off a small part from K_0 as in (3.5) (with modifications for the negative curvature case) and set

$$K_1 = K_0 \setminus \Big(\bigcup_{j=1}^{k}(\Delta_{4r}(P_j) \cap K_0 \cap Q_j)\Big).$$

For each $j = 1, \ldots, k$ we consider the transformation $T_j^{\theta_j,\delta_j}$ that consists of a rotation about P_j by angle θ_j followed by a translation in the direction of P_jO_j by δ_j, where O_j is the center of $\mathcal{C}_{R_j}$ (inward normal direction at P_j). With each S_j we also consider the domains $T_j^{\theta_j,\delta_j}(S_j)$, where $-\theta^* \le \theta_j \le \theta^*$ and $0 \le \delta_j \le \delta^*$ with some small positive numbers θ^*, δ^*. Thus, in this case we rotate and translate each S_j independently of each other, and we have $2k$ parameters $\theta_1, \ldots, \theta_k$, $\delta_1, \ldots, \delta_k$. We set $(\boldsymbol{\theta}, \boldsymbol{\delta}) = (\theta_1, \ldots, \theta_k, \delta_1, \ldots, \delta_k)$.

Now copy the proof from Section 3.2 word for word with the set

$$K_1 \cup (\cup_{j=1}^{k} T_j^{\theta_j,\delta_j}(S_j)).$$

If θ^*, δ^* are sufficiently small then no change is needed in the proof, and for small $\tau > 0$ the analogues of (3.13)–(3.19) hold for each point P_j instead of P for the functions

$$f_j(\boldsymbol{\theta}, \boldsymbol{\delta}) = g\left(K_1 \cup (\cup_{j=1}^{k} T_j^{\theta_j,\delta_j}(S_j)), P_j\right) - \tau,$$

and

$$\Phi_j(\boldsymbol{\theta}, \boldsymbol{\delta}) = \varphi(\sigma_{j,\boldsymbol{\theta},\boldsymbol{\delta}}, P) - \varphi_{j,0},$$

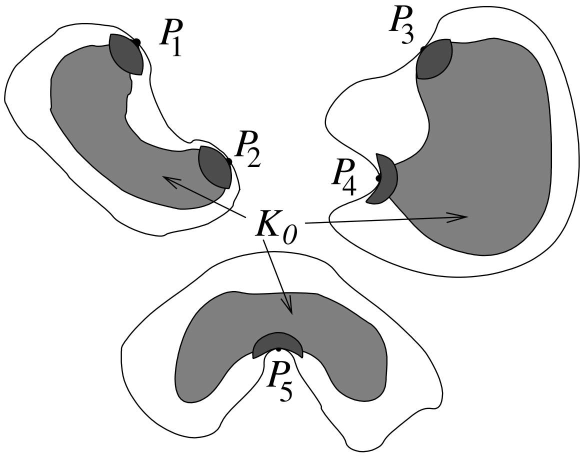

Figure 3.9: The choice of the S_j's

where $\sigma_{j,\boldsymbol{\theta},\boldsymbol{\delta}}$ is the level curve of $g\big(K_1 \cup (\cup_{j=1}^k T^{\theta,\delta}{}_j(S_j)), \cdot\big)$ passing through the point P_j, and $\varphi_{j,0} = \varphi(\Gamma^*, P_j)$ is the tangent direction angle to Γ^* at P_j. All these for $\tau > 0$ sufficiently small. Now an application of Lemma 12 to the function

$$F(\boldsymbol{\theta},\boldsymbol{\delta}) = \Big(\Phi_1(\boldsymbol{\theta},\boldsymbol{\delta}), \cdots, \Phi_k(\boldsymbol{\theta},\boldsymbol{\delta}), f_1(\boldsymbol{\theta},\boldsymbol{\delta}), \cdots, f_k(\boldsymbol{\theta},\boldsymbol{\delta}),\Big)$$

on the box $[-\theta^*, \theta^*]^k \times [0, 2C_0\tau]^k$ gives $\theta_1, \ldots, \theta_k \in [-\theta^*, \theta^*]$ and $\delta_1, \ldots, \delta_k \in [0, 2C_0\tau]$ such that the τ-level line $\sigma = \sigma(\tau, \boldsymbol{\theta}, \boldsymbol{\delta})$ of Green's function $g\big(K_1 \cup (\cup_{j=1}^k T^{\theta,\delta}{}_j(S_j)), \cdot\big)$ of $\mathbf{C}_\infty \setminus \big(K_1 \cup (\cup_{j=1}^k T^{\theta,\delta}{}_j(S_j))\big)$ passes through each P_j and has the same tangent line there as Γ^*. Furthermore, in $\Delta_{r/2}(P_j)$ its curvature is close to the curvature of $\mathcal{C}_{R_j}$, and the same proof that was used at the end of Section 3.2 shows that σ lies in between γ^* and Γ^*, and also lies in between $\mathcal{C}_{R_{j,0}}$ and $\mathcal{C}_{R_{j,1}}$ in each $\Delta_{r/2}(P_j)$. Since $K_1 \cup (\cup_{j=1}^k T^{\theta,\delta}{}_j(S_j))$ has exactly one connected component inside every Γ_j, it also follows that the Green line σ consists of m connected components.

3.4 Completion of the proof of Theorem 7

In this section we shall replace the Green line $\sigma = \sigma(\tau, \boldsymbol{\theta}, \boldsymbol{\delta})$ from Section 3.3 by lemniscates.

The outline is the following.

- σ is the τ-level line of some Green's function, and first using the integral representation for Green's functions in terms of equilibrium measures and discretizing these equilibrium measures, we get polynomials T_N for which the e^τ-level curve lies very close to σ (and this approximation is getting better and better as $N \to \infty$).

- Next, for each $j = 1, \ldots, k$ we select (for some large M) N/M zeros of T_N lying close to P_j, and apply to all these N/M zeros a small rotation and dilation with center at P_j in such a way that these rotations and dilations are done independently of each other for different j's. Thus, in this step we introduce k rotation and k dilation parameters $\theta_1, \ldots, \theta_k$ and $(1+\rho_1), \ldots, (1+\rho_k)$.

- Using the Brouwer fixed point theorem we show that these rotation and dilation parameters can be selected in such a way that for the so modified polynomials T_N^* the e^τ-level curve σ^* passes through each point P_j and has the same tangent line there as Γ^*.

- By controlling the curvature of σ^* around each P_j and using that elsewhere σ^* is very close to σ and σ lies strictly in between γ^* and Γ^*, we can conclude that σ^* has similar properties as σ, in particular it separates each γ_j from the corresponding Γ_j.

We use the notations from Sections 3.2, 3.3, but denote the disks $\Delta_r(P_j)$ there by $\Delta_{r_0}(P_j)$. Recall that the Green line σ in question was the Green line associated with a set

$$K_2 = K_1 \bigcup \bigcup_{j=1}^{k} T^{\theta_j,\delta_j}(S_j),$$

where each S_j has a circular arc on its boundary in the neighborhood $\Delta_{r_0}(P_j)$, more precisely $\Delta_{r_0}(P_j) \cap \partial S_j$ is a circular arc of some fixed radius R_j, going closer to P_j than $r_0/4$. Recall also that this arc was lying on some circle $\mathcal{C}_{R_j}$ touching Γ^* at P_j and σ lies in between two touching circles $\mathcal{C}_{R_{j,0}}$ and $\mathcal{C}_{R_{j,1}}$ in the neighborhood $\Delta_{r_0}(P_j)$ of P_j, and lies strictly in between γ^* and Γ^* outside these neighborhoods, see Figure 3.10. We shall also need that in $\Delta_{r_0}(P_j)$ the curvature of σ satisfies an inequality

$$1/R_{j,1} + \varepsilon < \kappa(\sigma, z) < 1/R_{j,0} - \varepsilon \tag{3.20}$$

with some $\varepsilon > 0$ (this is how the construction went in Sections 3.2 and 3.3).

In Section 3.2 we also verified that the normal derivative to level lines of $g(K_2, z)$ is strictly positive in the given neighborhood $\Delta_{2r_0}(P_j)$ of each P_j.

Now choose a small $0 < r < r_0/2$ so that $\Delta_{4r}(P_j) \cap K_2 = \emptyset$ for all j, i.e. the disks $\Delta_{4r}(P_j)$ lie outside K_2 (see Figure 3.10).

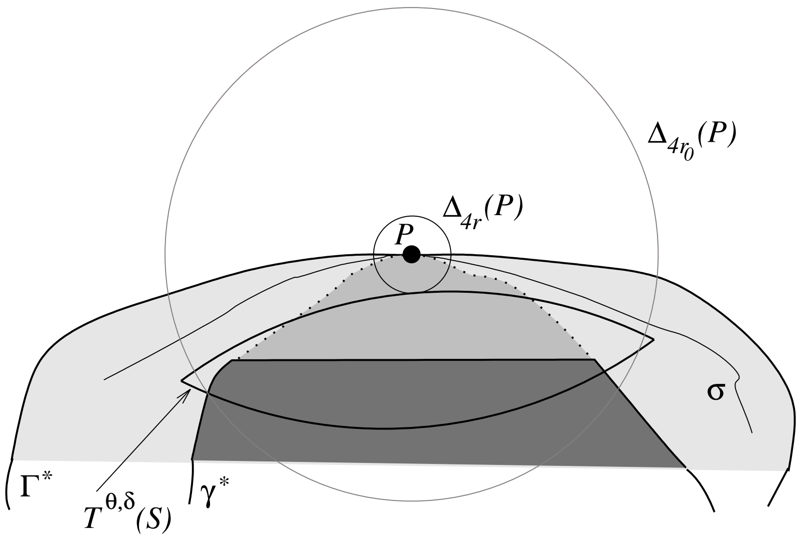

Figure 3.10: The form of K_2 and the position of σ

Let μ be the equilibrium measure of K_2, and $\mathrm{cap}(K_2)$ the logarithmic capacity of K_2 (see e.g [51, p. 107] or [52, (I.4.8)]). Then

$$g(K_2, z) = \int \log|z-t|d\mu(t) - \log \mathrm{cap}(K_2),$$

and locally this is the same as the real part of

$$h(z) = \int \log(z-t)d\mu(t) - \log \mathrm{cap}(K_2),$$

(with an appropriate local branch of log). What we have just mentioned on the normal derivative implies that $h'(z) \neq 0$ in any of the neighborhoods $\Delta_{2r}(P_j)$. Note also that the Green line σ is just the level line $\{\mathrm{Re}\, h(z) = \tau\}$.

For each N choose N points $\{x_s^{(N)}\}_{s=1}^N$ on the boundary of K_2 so that their asymptotic distribution is μ (i.e. if we put mass $1/N$ to each $x_s^{(N)}$, then the so obtained measures tend to μ in the weak* topology on measures as $N \to \infty$), and set

$$T_N(z) = \prod_{s=1}^{N}(z - x_s^{(N)}).$$

Then (note that all zeros of T_N lie in K_2) we have

$$\frac{1}{N}\log T_N(z) \to \int \log(z-t)d\mu(t), \tag{3.21}$$

and

$$\frac{1}{N}\log|T_N(z)| \to \int \log|z-t|d\mu(t) \tag{3.22}$$

locally uniformly in $\mathbf{C}\setminus K_2$ as $N\to\infty$. Thus,

$$\frac{1}{N}\log T_N(z) \to h(z) + \log \operatorname{cap}(K_2) \tag{3.23}$$

uniformly on each $\Delta_{2r}(P_j)$, $j=1,\dots,k$ (with some local branches of the logarithm). Then for sufficiently large N the absolute value of the derivative of $\frac{1}{N}\log T_N(z)$ stays above a fixed positive number on each $\Delta_r(P_j)$ (because the same is true of their limit in $\Delta_{2r}(P_j)$), and all derivatives $\frac{1}{N}\log T_N(z)$ tend to the appropriate derivative of h uniformly on each $\Delta_r(P_j)$. Hence it follows from Lemma 11 that for the level line $\sigma_{N,j}$ of

$$\frac{1}{N}\log|T_N(z)| = \operatorname{Re}\frac{1}{N}\log T_N(z)$$

that passes through the point P_j we have for all $j=1,\dots,k$

$$\varphi(\sigma_{N,j},P_j)\to\varphi(\sigma,P_j)=\varphi(\Gamma^*,P_j), \qquad N\to\infty,$$

where, as always, $\varphi(\sigma,P_j)$ is the tangent direction angle of σ at P_j taken modulo π. Thus, there is sequence $\{d_N\}$ tending to 0 such that

$$|\varphi(\sigma_{N,j},P_j)-\varphi(\sigma,P_j)| < d_N^2 \tag{3.24}$$

and

$$(\operatorname{cap}(K_2)e^{\tau})^N e^{-Nd_N^2} \le |T_N(P_j)| \le (\operatorname{cap}(K_2)e^{\tau})^N e^{Nd_N^2}. \tag{3.25}$$

Choose and fix a large number M, and consider only N's that are divisible by M. For each j let X_j be the set of the M closest zero of T_N to P_j. As we remarked at the beginning of this section the normal derivative $\partial g(K_2,z)/\partial\mathbf{n}$ is strictly positive for $z\in\Delta_{r_0}(P_j)\cap\partial K_2$, and this latter set is a circular arc of $\Delta_{r_0}(P_j)$. But this normal derivative is just the density of the equilibrium measure with respect to arc length ds, more precisely

$$\frac{d\mu(z)}{ds} = \frac{1}{2\pi}\frac{\partial g(K_2,z)}{\partial\mathbf{n}}$$

(see [52, Theorem I.1.5] and formula [52, (I.4.8)]), hence there is a fixed constant C such that $\mu(\Delta_{C/M}(P_j)) > 1/M$, and so for large N there are at least N/M zeros in each $\Delta_{C/M}(P_j)$. This implies $X_j\subset\Delta_{C/M}(P_j)$ for each $j=1,\dots,k$. In particular, if M is sufficiently large, then the sets X_j, $j=1,2,\dots,k$ are disjoint.

Consider the transformations $\mathfrak{T}_j^{\theta_j,\rho_j}$, $j = 1,\ldots,k$, where

$$\mathfrak{T}_j^{\theta_j,\delta_j} z = P_j + e^{i\theta_j}(1+\rho_j)(z - P_j)$$

is a rotation about P_j with angle θ_j followed by a dilation with factor $(1+\rho_j)$, and let $T_N^*(z)$ be the polynomial obtained by replacing each zero x of T_N in X_j by a corresponding zero $\mathfrak{T}_j^{\theta_j,\rho_j}x$ in $\mathfrak{T}_j^{\theta_j,\rho_j}X_j$ (and do this for all $j = 1,\ldots,k$). Let

$$X^* = \left(X \setminus (\cup_j X_j)\right) \bigcup \left(\cup_j \mathfrak{T}_j^{\theta_j,\delta_j} X_j\right) = \{x_s^{*(N)}\}_{s=1}^N$$

be the zero set of T_N^*. We restrict θ_j, ρ_j to lie in the interval $[-d_N, d_N]$.

We think of the transformation $x \to \mathfrak{T}_j^{\theta_j,\rho_j}x$ as moving the zero x. Note first of all that no zero in X is moved by more than $2Cd_N/M$, and all the N/M points in X_j get farther away from P_j by a factor $(1+\rho_j)$ (or closer by this factor if $\rho_j < 0$). Hence, if d is the minimum distance between the points P_j, then for any $j_0 = 1,\ldots,k$ if $\rho_{j_0} = -d_N$ then

$$\frac{|T_N^*(P_{j_0})|}{|T_N(P_{j_0})|} \le (1-d_N)^{N/M}\left(1 + \frac{2Cd_N/M}{d/2}\right)^{(k-1)N/M} < e^{-d_N N/4M}$$

provided M is so large that $2C(k-1)/M(d/2) < 1/8$. In a similar manner, if $\rho_{j_0} = d_N$ then

$$\frac{|T_N^*(P_{j_0})|}{|T_N(P_{j_0})|} \ge (1+d_N)^{N/M}\left(1 - \frac{2Cd_N/M}{d/2}\right)^{(k-1)N/M} > e^{d_N N/4M}.$$

Combining these with (3.25) we can see that for the functions

$$f_j(\rho_1,\ldots,\rho_n,\theta_1,\ldots,\theta_n) = \frac{1}{N}\log|T_N^*(P_j)| - \log\operatorname{cap}(K_2) - \tau \tag{3.26}$$

we have

$$\operatorname{sign} f_{j_0}(\rho_1,\ldots,\rho_n,\theta_1,\ldots,\theta_n) = \pm 1 \tag{3.27}$$

if $\rho_{j_0} = \pm d_N$ and N is sufficiently large.

Next we consider the change of the tangent direction angle to the lemniscates when we go from T_N to T_N^*. By Lemma 11 the tangent direction angle $\varphi(\sigma_{N,j}, P_j)$ on the left hand side of (3.24) equals (mod π)

$$\frac{\pi}{2} - \arg\frac{1}{N}(\log T_N(z))'\big|_{z=P_j} = \frac{\pi}{2} + \frac{1}{N}\sum_{s=1}^N \arg(x_s^{(N)} - P_j).$$

Subtract this from the corresponding expression for T_N^*, the result is

$$\Phi_j(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) = \frac{1}{N} \sum_{l=1}^{k} \sum_{x \in X_l} \left(\arg(\mathcal{T}_l^{\theta_l, \rho_l} x - P_j) - \arg(x - P_j) \right),$$

and this quantity is the difference between the tangent direction angles at P_j to the level lines of $|T_N^*|$ resp. $|T_N|$ going through the point P_j. Here for $x \in X_j$ the change of the argument is

$$\arg(T_j^{\theta_j, \rho_j} x - P_j) - \arg(x - P_j) = \theta_j,$$

while for all other $l \neq j$ and $x \in X_l$ this change is at most

$$|\arg(\mathcal{T}_l^{\theta_l, \rho_l} x - P_j) - \arg(x - P_j)| \leq \frac{2Cd_N/M}{d/2}$$

because the distance between x and $\mathcal{T}_l^{\theta_l, \rho_l} x$ is at most $2Cd_N/M$, and the distance from P_j to x is at least $d/2$ (d was the minimum distance between the points P_j). Therefore, if for a particular $j = j_0$ we have $\theta_{j_0} = -d_N$, then

$$\Phi_{j_0}(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) \leq \frac{1}{N} \left(-d_N \frac{N}{M} + (k-1) \frac{N}{M} \frac{4Cd_N}{Md} \right) < -\frac{d_N}{2M} \tag{3.28}$$

if M is large, and similarly for $\theta_{j_0} = d_N$ we have

$$\Phi_{j_0}(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) \geq \left(d_N \frac{N}{M} - (k-1) \frac{N}{M} \frac{4Cd_N}{Md} \right) > \frac{d_N}{2M}. \tag{3.29}$$

This and (3.24) give that for $\theta_{j_0} = \pm d_N$ the sign of

$$\widehat{\Phi}_j(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) = \varphi(\sigma_{N,j}, P_j) - \varphi(\sigma, P_j) + \Phi_j(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) \tag{3.30}$$

for $j = j_0$ is ± 1 for all large N. Therefore we can applying Lemma 12 to the function

$$\begin{aligned} F(\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m) &= F(\boldsymbol{\rho}, \boldsymbol{\theta}) \\ &= \left(f_1(\boldsymbol{\rho}, \boldsymbol{\theta}), \cdots, f_k(\boldsymbol{\rho}, \boldsymbol{\theta}), \widehat{\Phi}_1(\boldsymbol{\rho}, \boldsymbol{\theta}), \cdots, \widehat{\Phi}_k(\boldsymbol{\rho}, \boldsymbol{\theta}) \right) \end{aligned}$$

with the f_j's from (3.26), to conclude that for all large N there are values

$$\rho_1, \ldots, \rho_m, \theta_1, \ldots, \theta_m \in [-d_N, d_N]$$

such that the lemniscate

$$\sigma_N^* = \{z \,\big|\, |T_N^*(z)| = (\mathrm{cap}(K_2)e^\tau)^N\}$$

passes through each P_j and has the same tangent line at P_j as the lemniscate σ, i.e. as Γ.

From what we have said it also follows that the distribution of the sets X_N^* is again the equilibrium distribution μ, and for any compact set in $\mathbf{C}\setminus K_2$ there are no points from X_N^* in that compact set for large N. Therefore, all the asymptotic formulae that we have verified for T_N hold also for T_N^*. In particular, the analogue of (3.23) is true:

$$\frac{1}{N}\log T_N^*(z) \to h(z) + \log \mathrm{cap}(K_2) \tag{3.31}$$

uniformly on each $\Delta_{2r}(P_j)$. Now this, Lemma 11 and (3.20) imply that for large N the curvature of σ_N^* lies strictly in between $1/R_{j,0}$ and $1/R_{j,1}$ in each $\Delta_{2r}(P_j)$, $j = 1, \ldots, k$. Therefore, we can apply Lemma 13 to conclude that in $\Delta_r(P_j)$ the lemniscate σ_N^* goes in between $\mathcal{C}_{R_j}$ and Γ.

Finally, the function

$$\frac{1}{N}\log\frac{|T_N^*(z)|}{\mathrm{cap}(K_2)^N}$$

converges to the Green's function $g(K_2, z)$ of $\mathbf{C}_\infty \setminus K_2$ uniformly on compact subsets of $\mathbf{C} \setminus K_2$. Therefore, if $\tau_1 < \tau < \tau_2$, then for large N the lemniscate σ_N^* lies in between the Green's lines

$$\{g(K_2, z) = \tau_1\} \qquad \text{and} \qquad \{g(K_2, z) = \tau_1\}.$$

For τ_1 and τ_2 sufficiently close to τ, away from the P_j's, more precisely outside $\cup_j \Delta_r(P_j)$ these level lines lie in between γ^* and Γ^* (because $\sigma = \{g(K_2, z) = \tau\}$ lies strictly in between these curves there), and this shows that the lemniscate σ_N^* lies in between γ^* and Γ^*, and, as we have just seen, in $\Delta_r(P_j)$ it also lies in between $\mathcal{C}_{R_j}$ and Γ for each j.

The same argument easily implies that the lemniscate σ_N^* has the same number of components as σ, i.e. it has precisely one component in between each γ_j and Γ_j, and this completes the proof.

3.5 Normal derivative of Green's function

We shall only prove the inequality in (3.1), the proof of (3.2) is completely analogous.

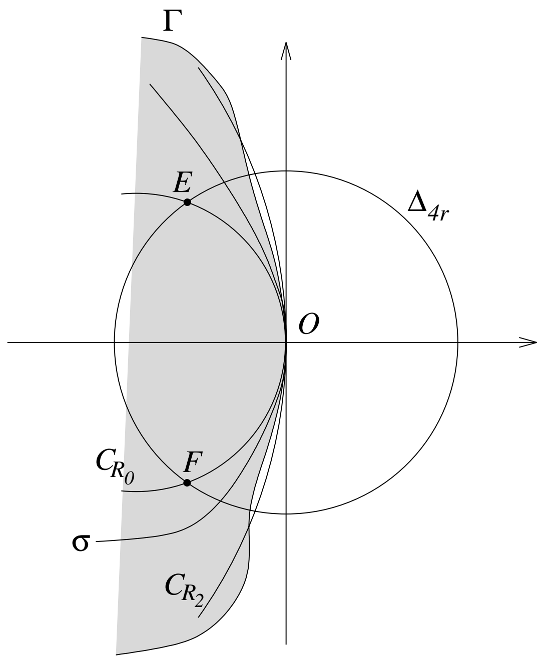

Figure 3.11: The choice of $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_2}$

Let $\sigma = \sigma(\tau, \boldsymbol{\theta}, \boldsymbol{\delta})$ be the lemniscate constructed in the preceding section, and let L be the closed region inside it. Recall also that K is the closed domain enclosed by the Γ^*, i.e. the union of the domains enclosed by the Γ_j's, $j = 1, \ldots, m$.

We pick any touching point $P = P_1, \ldots, P_k$, and work with this single P as in Section 3.2, and use the notations from there. We show that if $1/R_0$ is sufficiently close to $\kappa(\Gamma, P)$ and γ_j are sufficiently close Γ_j for all $j = 0, 1, \ldots, m$, then the normal derivative $\partial g(L, P)/\partial \mathbf{n}$ is close to $\partial g(K, P)/\partial \mathbf{n}$. Note that since L lies inside K, we necessarily have $g(L, z) \geq g(K, z)$, and hence

$$\frac{\partial g(L, P)}{\partial \mathbf{n}} \geq \frac{\partial g(K, P)}{\partial \mathbf{n}}.$$

We shall only consider the case when the curvature of Γ^* at P (seen from the outside of Γ^*) is nonnegative – the case of negative curvature can be similarly handled. As in Section 3.2, Γ is the Γ_j that contains the point P.

We shall use the notation from the previous section, but will only use the fact that L contains K_0 (and $K_0 \subset K$, i.e. the choice of the γ_j's is at our disposal at this moment), and σ runs in between $\mathcal{C}_{R_0}$ and Γ in a neighborhood of P.

For simpler notation we may assume $P = 0$, that the tangent line to Γ at 0 is the imaginary axis and $\mathcal{C}_{R_0}$ lies to the left of this axis, see Figure 3.11.

First we consider the case when the curvature $\kappa(\Gamma, 0)$ of Γ at 0 is positive. Let $\varepsilon > 0$ and choose $R_0 < 1/\kappa(\Gamma, 0) < 1/R_2$ so that $1/R_0 - 1/R_2 < \varepsilon$. We may assume $r < R_0/8$ so small that in $\Delta_{4r} = \Delta_{4r}(P) = \Delta_{4r}(0)$ the lemniscate σ runs in between $\mathcal{C}_{R_0}$ and Γ, and Γ runs strictly in between $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_2}$ except for the point 0, where all these curves touch each other. Let $\widetilde{EF}$ be the arc $\mathcal{C}_{R_0} \cap \overline{\Delta_{4r}}$. Then this is part of L, hence $g(L, z) \leq g(\widetilde{EF}, z)$. Since $\widetilde{EF}$ has diameter bigger than r and smaller than $8r$, if we consider $\widetilde{EF}/\mathrm{diam}(\widetilde{EF})$, then this is an arc of diameter 1 and of curvature ≥ 1, hence its Green's function is C^1-smooth inside $\widetilde{EF}$, i.e. there is an absolute constant $C_0 \geq 1$ such that for any z we have $g(\widetilde{EF}, z) \leq C_0 \mathrm{dist}(z, \widetilde{EF} \cap \Delta_{2r})/r$. It is easy to verify that if $z \in \mathcal{C}_{R_2} \cap \Delta_{2r}$ then

$$\mathrm{dist}(z, \widetilde{EF} \cap \Delta_{2r}) \leq \frac{|z|^2}{R_0} - \frac{|z|^2}{R_2} \leq \varepsilon |z|^2. \tag{3.32}$$

Therefore, for $z \in \mathcal{C}_{R_2} \cap \Delta_{2r}$

$$g(L, z) \leq C_0 \frac{\varepsilon}{r} |z|^2. \tag{3.33}$$

Next observe that as γ^* approaches Γ^*, the domain L approaches K from the inside, therefore $\mathrm{cap}(L)$ tends to $\mathrm{cap}(K)$ where cap denotes logarithmic capacity. Now the function $g(L, z) - g(K, z)$ is nonnegative and harmonic in $\mathbf{C}_\infty \setminus K$, and takes the value $\log(\mathrm{cap}(K)/\mathrm{cap}(L))$ at infinity, therefore it tends to 0 at infinity if γ^* approaches Γ^*. ¿From Harnack's inequality we can infer that in this case $g(L, z) - g(K, z)$ tends to 0 uniformly on compact subsets of $\mathbf{C}_\infty \setminus K$. Therefore, if we start from inner curves γ^* that lie sufficiently close to Γ^*, we can achieve that

$$g(L, z) - g(K, z) \leq \varepsilon r \tag{3.34}$$

for all $z \in \partial\Delta_{2r} \setminus \mathcal{D}_{R_2}$, where $\mathcal{D}_{R_2}$ is the disk enclosed by $\mathcal{C}_{R_2}$.

Consider the domain

$$G = \Delta_{2r} \setminus \mathcal{D}_{R_2}. \tag{3.35}$$

(3.33) can be applied on its boundary that lies on $\mathcal{C}_{R_2}$, while (3.34) can be applied on the part of its boundary that lies on $\partial\Delta_{2r}$. In particular,

$$g(L, z) - g(K, z) \leq 4C_0 \varepsilon r \tag{3.36}$$

on the whole boundary, and hence also on the whole G. We show that these are sufficient to conclude that the normal derivative $\partial(g(L, z) - g(K, z)/\partial\mathbf{n}$ is small at 0.

The circle $\mathcal{C}_{R_2}$ is the one with radius R_2 and with center at $-R_2$, hence the Joukovskii transformation

$$\zeta = \frac{1}{2}\left(\frac{i(z+R_2)}{R_2} + \frac{R_2}{i(z+R_2)}\right)$$

maps the arc $\mathcal{C}_{R_2} \cap \Delta_{2r}$ into a segment around the origin, and simple calculation shows that the image of G contains the upper half of the disk Δ_{r/R_2}. By the mapping $w = \zeta R_2/r$ map this half disk onto the upper half of the unit disk. We have defined mappings $z \to \zeta \to w$, and set $h(w) = g(L,z) - g(K,z)$. For all w in the upper half of the unit disk $|w|/(|z|/r)$ lies in between two universal constants, therefore it follows from (3.33)–(3.36) that there is a universal constant C_1 such that $h(w) \le C_1\varepsilon r|w|^2$ for all $w \in [-1,1]$, and $h(w) \le C_1 r\varepsilon$ for $|w| = 1$, Im $w > 0$. Let $h_1(w)$ be the function that is harmonic and bounded on the upper half plane and takes the boundary value $C_1\varepsilon r|w|^2$ for $w \in [-1,1]$ and 0 on $\mathbf{R} \setminus [-1,1]$, and let $h_2(w)$ be the harmonic function in the upper half of the unit disk that takes boundary value $C_1 r\varepsilon$ for $|w| = 1$, Im $w > 0$ and 0 for $w \in [-1,1]$. Then $h(w) \le h_1(w) + h_2(w)$, and we can separately estimate $h_1(w)$ and $h_2(w)$ for w lying close to 0.

By the Poisson formula ([51, Theorem 4.3.13]) for the upper half plane we have for $w = x + iy$ lying close to 0

$$h_1(w) = \frac{1}{\pi}\int_{-1}^{1} \frac{y}{(u-x)^2 + y^2} C_1\varepsilon r u^2 du.$$

The integrand over the interval $|u - x| \le 2|w|$ is at most $y/((u-x)^2 + y^2)$ times $C_1\varepsilon r(3|w|)^2$, therefore

$$\int_{|u-x|\le 2|w|} \le 9C_1\varepsilon r|w|^2.$$

On the rest of $[-1,1]$ we have $|u - x| \ge |u|/2$, hence the integrand is at most $4C_1\varepsilon r y$, therefore

$$\int_{|u-x|\ge 2|w|} \le 4C_1\varepsilon r|w|.$$

These give $h_1(w) \le 13C_1\varepsilon r|w|$.

To estimate h_2 apply the Joukovskii transform $W = (w + 1/w)/2$, which maps the upper half of the unit disk onto the upper half plane $\mathbf{C}_+$, the image of the upper semi-circle being $[-1,1]$. Now $h_3(W) := h_2(w)/(C_1 r\varepsilon)$ is nothing else than the harmonic measure $\omega(W, [-1,1], \mathbf{C}_+)$ of the segment $[-1,1] \subset \partial\mathbf{C}_+$, which is $1/\pi$-times the angle that the interval $[-1,1]$ is seen

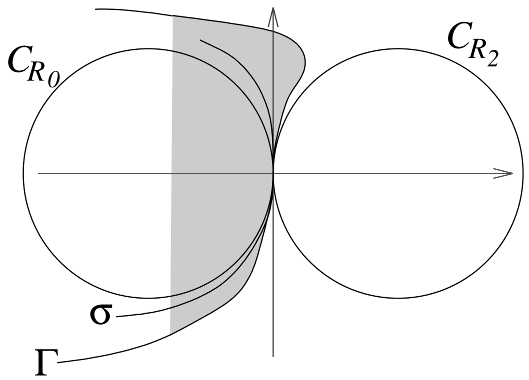

Figure 3.12: The choice of $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_2}$ in the zero curvature case

from W (see e.g. [51, Theorem 4.3.13] or [1, Example 3-1, p. 38]). For $|w| \le 1/4$ we have $1/4|w| \le |W| \le 1/|w|$, hence the angle in question is at most $2\arctan(1/|W|) \le 8|w|$. Thus, $h_2(w) \le 8C_1 r\varepsilon|w|$.

All in all we have obtained $h(w) \le 21C_1 r\varepsilon|w|$, if $|w| \le 1/4$, which gives for $g(L,z) - g(K,z) = h(w)$ the estimate $g(L,z) - g(K,z) \le C_2\varepsilon|z|$ with some universal constant C_2 for all z lying close to 0. This implies

$$\frac{\partial(g(L,z) - g(K,z))}{\partial \mathbf{n}}\Big|_{z=0} \le C_2\varepsilon,$$

and this is what we wanted to prove.

Finally let us consider the case when the curvature of σ at the origin is 0. Then let $\mathcal{C}_{R_2}$ be the reflection of $\mathcal{C}_{R_0}$ onto the origin, where $R_0 > 2/\varepsilon$, see Figure 3.12. In this case $\Delta_{4r} \cap \mathcal{D}_{R_0}$ lies in the exterior of K, and (3.32) is still true in the form

$$\operatorname{dist}(z, \widetilde{EF} \cap \Delta_{2r}) \le \frac{|z|^2}{R_0} + \frac{|z|^2}{R_2} \le \varepsilon|z|^2.$$

The rest of the argument is unchanged, if we set $G = \Delta_{2r} \cap \mathcal{D}_{R_2}$ and work with this G instead of the one defined above in (3.35).

3.6 Lemmas

Lemma 11. *Let h be an analytic function in a neighborhood of a point z_0, and suppose that $h'(z) \neq 0$ in that neighborhood. Then the tangent direction angle* *of the level curve* $\sigma = \{Re\ h(z) = Re\ h(z_0)\}$ *at* z_0 *is equal to* $\frac{\pi}{2} - \arg h'(z_0)$ $(\mathrm{mod}\ \pi)$ *and the curvature of* σ *at* z_0 *is given by*

$$|\,Re\,(h''(z_0)/h'(z_0)^2)||h'(z_0)|.$$

Furthermore, if $c \le |h'(z)| \le C$ *for* $|z - z_0| \le \rho$*, then for* $0 \le |\tau| < \rho c$ *the distance from* z_0 *to the level line* $\sigma(\tau) = \{Re\ h(z) = Re\ h(z_0) + \tau\}$ *satisfies the inequality*

$$\frac{|\tau|}{C} \le \operatorname{dist}\{z_0, \sigma(\tau)\} \le \frac{|\tau|}{c}. \tag{3.37}$$

Proof. . Without loss of generality we may assume $z_0 = 0$, $h(z_0) = 0$. Let $f(z) = (ih)^{-1}(z)$ be the inverse of $ih(z)$ in a small neighborhood of 0. Then the level curve $\{\mathrm{Re}\ h(z) = 0\}$ is the set of points that are mapped by ih into the real line, hence it is the same as the image of the real line under f. Let $[-a, a]$ be a small interval such that f exists on it. The direction of the tangent line for the curve $f(t)$, $t \in [-a, a]$ (which is part of σ) is $f'(t) = 1/ih'(f(t))$, hence the tangent direction angle of σ at 0 is $\arg 1/ih'(0) = \frac{\pi}{2} - \arg h'(0)$ $(\mathrm{mod}\ \pi)$.

As for the curvature, consider first the case when $f'(0) = 1$. The arc length element for the curve $f(t)$, $t \in [-a, a]$ is $ds = |f'(t)|dt$, the unit tangent vector is $f'(t)/|f'(t)|$, and the curvature is the absolute value of the derivative of the latter with respect to ds, i.e. it is

$$\left|\frac{d(f'(t)/|f'(t)|)}{dt}\frac{dt}{ds}\right| = \left|\frac{f''(t)|f'(t)| - f'(t)\frac{d|f'(t)|}{dt}}{|f'(t)|^2}\frac{1}{|f'(t)|}\right|. \tag{3.38}$$

At $t = 0$ we have $f'(0) = 1$, hence f has expansion about the origin $f(t) = t + ct^2 + \cdots$ with some c. Then

$$|f'(t)| = |1 + 2ct + \cdots| = \sqrt{1 + 2(c + \overline{c})t + \cdots} = 1 + (c + \overline{c})t + \cdots$$

for small real t, which gives

$$\frac{d|f'(t)|}{dt}\Big|_{t=0} = c + \overline{c} = 2\,\mathrm{Re}\ c = \mathrm{Re}\ f''(0).$$

Putting this into (3.38) and making use of $f'(0) = 1$ we obtain that the curvature to the curve $f(t)$ at $t = 0$ is $|\,\mathrm{Im}\ f''(0)|$.

If $f'(0) \ne 1$, then apply what we have got to the function $F(z) = f(z)/f'(0)$. Then the curvature for the curve $f(t)$ is $1/|f'(0)|$ times the curvature for the curve $F(t)$, hence it is equal to $|\,\mathrm{Im}\,(f''(0)/f'(0))|/|f'(0)|$. Finally, we can rewrite this back in terms of h using $f'(t) = 1/ih'(f(t))$, $f''(t) =$

$h''(f(t))/(h'(f(t))^3$ and we get the curvature in question is $|\,\mathrm{Re}\,(h''(0)/h'(0)^2)||h'(0)|$.

Finally, let us consider the distance of $\sigma(\tau)$ from z_0. We may assume $\tau > 0$. Let z_1 be the closest point of $\sigma(\tau)$ to z_0. If this point is outside the disk $|z - z_0| < \rho$, then the distance in question is $\geq \rho \geq \tau/c > \tau/C$. Otherwise

$$\tau = \mathrm{Re}\ h(z_1) - \mathrm{Re}\ h(z_0) = \mathrm{Re} \int_{z_0}^{z_1} h'(u)du \leq C|z_1 - z_0|,$$

and this proves the left inequality in (3.37). On the other hand, let us consider the vector field $\overline{h'(z)}/|h'(z)|$ where $\overline{z}$ means here the complex conjugate of z. Let $\chi(s)$ be the tangent curve to the vector field starting from z_0 with arc length parameter s. Then $d\chi(s)/ds = \overline{h'(\chi(s))}/|h'(\chi(s))|$ is the unit tangent vector to χ, and so for integration along the curve χ we have

$$\begin{aligned} h(\chi(\tau/c)) - h(z_0) &= \int_{z_0}^{\chi(\tau/c)} h'(z)dz = \int_0^{\tau/c} h'(\chi(s))\chi'(s)ds \\ &= \int_0^{\tau/c} |h'(\chi(s)|ds \geq (\tau/c)c = \tau, \end{aligned} \tag{3.39}$$

so one of the points on the curve $\chi(s)$, $0 \leq s \leq \tau/c$ must lie on the τ-level line $\sigma(\tau)$, and the distance of this point to z_0 is not bigger than the length of this curve, i.e. τ/c (which is smaller than ρ, so $\chi(s)$ for $0 \leq s \leq \tau/c$ stays within the disk $|z - z_0| < \rho$ and the lower estimate $|h'| \geq c$ in (3.39) holds by the assumption in the lemma). $\square$

Lemma 12. *Let $B = \prod_{j=1}^k [a_j^-, a_j^+]$ be a box in $\mathbf{R}^k$ and let $F : B \to \mathbf{R}^k$ be a continuous mapping in such a way that for any fixed $j_0 = 1, \ldots, k$ if $\mathbf{x} = (x_1, \ldots, x_k)$ is a point with $x_{j_0} = a_{j_0}^\pm$, then* $\mathrm{sign}F(\mathbf{x}) = \pm 1$ *(i.e. the function takes positive respectively negative values on opposite sides of the box). Then there is an $\mathbf{x} \in B$ such that $F(\mathbf{x}) = \mathbf{0}$, i.e. the origin is in the image set.*

Proof. . Without loss of generality we may assume $a_j^\pm = \pm 1$. Let us suppose to the contrary that the origin is not in the image set, and let $H(\mathbf{x})$ be the point where the half line emanating from the origin and going through $-F(\mathbf{x})$ intersects the boundary of the cube $B = [-1, 1]^k$. Then $\mathbf{x} \to H(\mathbf{x})$ is a continuous map of B into its boundary which does not have a fixed point (a fixed point could only be on the boundary of B, but a boundary point on face $x_j = \pm 1$ is mapped into a boundary point in the opposite half space $\mathrm{sign}x_j = \mp 1$). This however, contradicts the Brouwer fixed point theorem, and this contradiction proves the claim. $\square$

The next lemma is a local version of Blaschke's rolling theorem ([12, Ch. 4., Section 24., subsection II.]): suppose that two smooth convex curves G and g lie in the same side of a common tangent line and G has larger curvature at any point P than g at p, where the points $P \in G$ and $p \in g$ are such that the tangent line of G at P is parallel with the tangent line of g at p. Then G lies inside g.

Lemma 13. *Suppose that G, g are the curves $(t, F(t))$, and $(t, f(t))$, $t \in [0, a]$ respectively, where F, f are real valued twice continuously differentiable convex functions in $[0, a]$ such that the real line is their common tangent line at 0. If the curvature of G at any point $(t, F(t))$ is at least as large as the curvature of g at the point $(t, f(t))$, then $F(t) \geq f(t)$, i.e. G lies above g.*

In particular, let $R_0 < R_1$ and let $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$ be two circles of radii R_0 and R_1, respectively, with $\mathcal{C}_{R_0}$ lying inside $\mathcal{C}_{R_1}$, so that they touch each other at a point P and have common tangent line l there. Suppose that $r \leq R_0$ and a smooth curve γ lies on the same side of l as $\mathcal{C}_{R_1}$ and $\mathcal{C}_{R_2}$, and at all points of $\Delta_r \cap \gamma$ it is has curvature lying in between $1/R_1$ and $1/R_0$. Then in $\Delta_{r/2}$ the curve γ lies in between the two circles $\mathcal{C}_{R_0}$ and $\mathcal{C}_{R_1}$.

Proof. . It is sufficient to prove the first statement.

The normalized tangent vector to $(t, f(t))$ is

$$\left(\frac{1}{(1+f'(t)^2)^{1/2}}, \frac{f'(t)}{(1+f'(t)^2)^{1/2}}\right)$$

and the arc length element is $ds = (1 + f'(t)^2)^{1/2} dt$. Now a similar computation that was done in Lemma 11 gives that the curvature is

$$\frac{f''(t)}{(1+f'(t)^2)^{3/2}} = \left(\frac{f'(t)}{(1+f'(t)^2)^{1/2}}\right)'.$$

Therefore, the assumption is that

$$\left(\frac{F'(t)}{(1+F'(t)^2)^{1/2}}\right)' \geq \left(\frac{f'(t)}{(1+f'(t)^2)^{1/2}}\right)',$$

and upon using $f'(0) = F'(0) = 0$, integration from 0 to t gives

$$\frac{F'(t)}{(1+F'(t)^2)^{1/2}} \geq \frac{f'(t)}{(1+f'(t)^2)^{1/2}}.$$

Since the function $x/(1 + x^2)^{1/2}$ is increasing, this implies $F'(t) \geq f'(t)$, and another integration yields $F(t) \geq f(t)$ on $[0, a]$. $\square$

3.7 Proof of Theorems 9 and 10

First we verify a lemma.

Lemma 14. *Let D be a simply connected region with $C^{1+\alpha}$ boundary for some $\alpha > 0$, let J be a closed arc on the boundary of D and let $z_0 \in J$ be an inner point of this arc. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of continuous functions on $\overline{D}$ such that they are harmonic in D, vanish on J and uniformly tend to 0 on every compact subset of the complementary open arc $\partial D \setminus J$. Then*

$$\frac{\partial u_n(z_0)}{\partial \mathbf{n}} \to 0, \qquad \text{as } n \to \infty,$$

where $\partial/\partial\mathbf{n}$ denotes normal derivative in the direction of the inner normal to D.

Proof. . Let $|u_n| \le M$, and fix a conformal map of D onto the upper half plane $\mathbf{C}_+$ that maps z_0 into the origin. This map can be extended to a C^1 function to the boundary ([50, Theorems 3.5–3.6]), and the normal direction is preserved under this map. Hence, it is sufficient to show the result in the case $D = \mathbf{C}_+$ and $z_0 = 0$ (see also the estimate (3.40) below).

Let J be the interval $[-x_1, x_2]$ with $x_1, x_2 > 0$. The assumption is that for every $\varepsilon > 0$ the sequence $\{u_n\}$ tends uniformly to 0 on $\mathbf{R} \setminus (-x_1 - \varepsilon, x_2 + \varepsilon)$, thus there is an N_ε such that $|u_n| \le \varepsilon$ there for $n \ge N_\varepsilon$. If $\omega(z, E, \mathbf{C}_+)$ denotes the harmonic measure of the set $E \subset \mathbf{R}$ at z relative to $\mathbf{C}_+$, then for $n \ge N_\varepsilon$

$$\begin{aligned}|u_n(z)| \le M\omega(z, [-x_1 - \varepsilon, -x_1], \mathbf{C}_+) \;&+\; M\omega(z, [x_2, x_2 + \varepsilon], \mathbf{C}_+)\\ &+\; \varepsilon\omega(z, \mathbf{R} \setminus (-x_1 - \varepsilon, x_2 + \varepsilon), \mathbf{C}_+).\end{aligned}$$

Seeing that the harmonic measure $\omega(z, E, \mathbf{C}_+)$ is $1/\pi$-times the angle that the set E is seen from z (see [51, Theorem 4.3.13] or [1, Example 3-1, p. 38]), for $|z| < x_1/2$ the first term is easily seen to be at most

$$\arctan \frac{|z|}{x_1/2} - \arctan \frac{|z|}{x_1/2 + \varepsilon} \le C_1 \varepsilon |z|$$

with some constant C_1 depending only on x_1, and a similar estimate is true for the second term with the same constant C_1. Finally, the third term is at most

$$\varepsilon \left(\arctan \frac{|z|}{x_1/2} + \arctan \frac{|z|}{x_2/2} \right) \le C_2 \varepsilon |z|.$$

These together give for $n \ge N_\varepsilon$ and $|z| < \min(x_1/2, x_2/2)$

$$|u_n(z)| \le (C_1 M + C_1 M + C_2)\varepsilon |z|, \tag{3.40}$$

from which the claim immediately follows, as $\varepsilon > 0$ is arbitrary here. $\square$

Proof. Theorem 9. Let $U_1, \ldots, U_j, \ldots$ be the connected components of the interior $\mathrm{Int}(K)$ of K (their number may be finite). Every U_j is a simply connected Jordan domain (i.e. its boundary is a Jordan curve – a homeomorphic image of a circle), hence $\overline{U_j}$ is homeomorphic to the closed unit disk (see e.g. [50, Theorem 2.6]). It easily follows from the Jordan curve theorem that for $k \neq j$ the boundaries ∂U_j and ∂U_k may have at most one common point. Let M_0 be so large that z_0 belongs to $K_M = \overline{U_1 \cup \ldots \cup U_M}$ for $M \geq M_0$. We claim that for any $\varepsilon > 0$

$$\frac{\partial g(K_M, z_0)}{\partial \mathbf{n}} < \frac{\partial g(K, z_0)}{\partial \mathbf{n}} + \varepsilon \tag{3.41}$$

for all sufficiently large M. Indeed, select a simply connected domain D with C^2 boundary in the complement of K in such a way that z_0 is on its boundary, and for some small disk Δ with center at z_0 the set $K \cap \partial\Delta$ coincides with $\Delta \cap \partial K_{M_0}$ (recall that this is a C^2 Jordan arc for small Δ). Now $\mathrm{cap}(K_M) \to \mathrm{cap}(K)$ as $M \to \infty$ (in case there are infinitely many U_M's), and hence $g(K_M, z) \to g(K, z)$ locally uniformly in the complement of K (just apply Harnack's inequality in $\mathbf{C}_\infty \setminus K$ to the nonnegative harmonic functions $g(K_M, z) - g(K, z)$ that take the value $\log(\mathrm{cap}(K)/\mathrm{cap}(K_M))$ at infinity, cf. the proof of Theorem 8)). Hence, if D is a domain in $\mathbf{C} \setminus K$ with C^2 boundary so that $\overline{D} \cap K_M = \overline{\Delta} \cap \partial K$, then $g(K_M, z) - g(K, z) \to 0$ as $M \to \infty$ locally uniformly inside the complementary arc $\partial D \setminus (\overline{\Delta} \cap \partial K)$ to $\Delta \cap \partial K$ on the boundary of D. Now (3.41) follows for large M from Lemma 14 applied to the functions $g(K_M, z) - g(K, z)$.

Any two of the $U_1, \ldots, U_M$ can touch each other only in a single point (necessarily different from z_0), so we have altogether only finitely many touching points for these domains. If we remove from $U_1, \ldots, U_M$ some tiny parts around these touching points then we get domains $U_1^*, \ldots, U_M^*$ bounded by disjoint Jordan curves. The parts removed can be so small that for $K_M^* = \overline{U_1^* \cup \ldots \cup U_M^*}$ we have

$$\frac{\partial g(K_M^*, z_0)}{\partial \mathbf{n}} < \frac{\partial g(K_M, z_0)}{\partial \mathbf{n}} + \varepsilon, \tag{3.42}$$

– just repeat the proof of (3.41).

Now approximate K_M^* from the inside as in Theorem 8 by a lemniscate σ touching ∂K_M^* from the inside at z_0 in such a way that for the region L enclosed by σ we have the analogue of (3.1) (recall that L is the closed region enclosed by σ):

$$\frac{\partial g(L, z_0)}{\partial \mathbf{n}} \leq \frac{\partial g(K_M^*, z_0)}{\partial \mathbf{n}} + \varepsilon \leq \frac{\partial g(K, z_0)}{\partial \mathbf{n}} + 3\varepsilon. \tag{3.43}$$

Finally, apply Theorem 2 to conclude

$$|P_n'(z_0)| \le n(1+o(1))\frac{\partial g(L,z_0)}{\partial \mathbf{n}}\|P_n\|_L \le n(1+o(1))\Big(\frac{\partial g(K,z_0)}{\partial \mathbf{n}}+3\varepsilon\Big)\|P_n\|_K,$$

where we used that, because of $L\subset K$, we have $\|P_n\|_L \le \|P_n\|_K$. Since here $\varepsilon>0$ is arbitrary small, this is the same as (3.3). □

Proof. Theorem 10. Let Δ be a small neighborhood of z_0, and approximate K from the outside by compact sets K_l, $l=1,2,\ldots$ such that each K_l is bounded by finitely many Jordan curves, $K_l\cap\overline{\Delta}=K\cap\overline{\Delta}$, the Hausdorff distance between K and K_l tends to 0, and so $\mathrm{cap}(K_l)\to\mathrm{cap}(K)$ as $l\to\infty$. These imply (see the preceding proof) $g(K_l,z)\to g(K,z)$ as $l\to\infty$ locally uniformly in the complement of K, and then Lemma 14 gives by an argument similar to what we did in the preceding proof that for large l

$$\frac{\partial g(K,z_0)}{\partial \mathbf{n}} \le \frac{\partial g(K_l,z_0)}{\partial \mathbf{n}}+\varepsilon.$$

Select such a large l.

K_l is such that Theorem 8 can be applied to it, so let us approximate K_l from the outside by a lemniscate touching ∂K_l at z_0 as in Theorem 8 so that (3.2) holds in the form

$$\frac{\partial g(K_l,z_0)}{\partial \mathbf{n}} \le \frac{\partial g(L,z_0)}{\partial \mathbf{n}}+\varepsilon.$$

(i.e. now Γ^* plays the role of γ^* in Theorem 8, and the outer curve is at our disposal). Let $\sigma=\{z \mid |T_N(z)|=1\}$. Green's function for $\mathbf{C}_\infty\setminus L$ with pole at infinity is $g(L,z)=\frac{1}{N}\log|T_N(z)|$, and its normal derivative on the level curve σ is the gradient of $z\mapsto \frac{1}{N}\log|T_N(z)|$, i.e. at $z_0\in\sigma$ it is $|T_N'(z_0)|/N$. Now let n be large and $k=[n/N]$ the integral part of n/N. For $P_n(z)=T_N^k(z)$, which is a polynomial of degree at most $Nk\le n$, we have

$$\begin{aligned}|P_n'(z_0)| &= k|T_N'(z_0)| = kN\frac{\partial g(L,z_0)}{\partial \mathbf{n}} \ge kN\Big(\frac{\partial g(K,z_0)}{\partial \mathbf{n}}-2\varepsilon\Big)\\ &\ge n(1-o(1))\Big(\frac{\partial g(K,z_0)}{\partial \mathbf{n}}-2\varepsilon\Big)\|P_n\|_K\end{aligned}$$

because $\|P_n\|_K\le\|P_n\|_L=1$, and this is (3.4). □

Chapter 4

Higher order sharpness of the generalized Hilbert's lemniscate theorem

4.1 Curves touching each other

As above, if γ is a system of closed curves, we denote the complement of the unbounded component of $\mathbf{C} \setminus \gamma$ by $\operatorname{Int}\gamma$. That is, $\operatorname{Int}\gamma$ is the set what γ encloses. E.g. if γ is the unit circle, then $\operatorname{Int}\gamma$ is the closed unit disk.

Suppose that we have two C^1 smooth curves, γ_1, γ_2 which pass through the same point, $z_0 = \gamma_1(0) = \gamma_2(0)$ and have the same tangent line at z_0. We can assume that $z_0 = 0$ and their common tangent line is the real axis. So we can parametrize them near the origin as follows: $\gamma_j(t) = t + ig_j(t)$, $j = 1, 2$ where the smoothness of γ_j's imply that g_j's are C^1 smooth.

So we have defined some functions using the original curves. Using these functions:

Definition 15. *Suppose we have two C^1 smooth curves, γ_1, γ_2 and the functions g_1, g_2 as above. We say that γ_1 and γ_2 touch each other at $\gamma_1(0) = \gamma_2(0)$ in order s ($s \geq 1$), if $\big|g_1(t) - g_2(t)\big| \sim |t|^s$. That is, for some constants $C_1 > c_1 > 0$, we have*

$$c_1|t|^s \leq \big|g_1(t) - g_2(t)\big| \leq C_1|t|^s. \tag{4.1}$$

Remarks.
The order s can be a real number. This translation ($z_0 = 0$) and rotation (their tangent line is the real axis) is needed so that we could easily compare the two curves pointwise.

Investigate this definition in the following geometric case.

Theorem 16. *Suppose that we have two C^2 curves γ_1, γ_2 and the corresponding functions g_1, g_2 as above.*

Then γ_1 and γ_2 touch each other in order 2 if and only if the their tangent line at $t = 0$ coincide but their curvature at $t = 0$ are different.

Proof. We use the following well-known relation. The curvature of the curve $t \mapsto t + if(t)$ at t_0 is

$$\frac{f''(t_0)}{\left(1 + \left(f'(t_0)\right)^2\right)^{3/2}}. \tag{4.2}$$

The γ_j $j = 1, 2$ are C^2 smooth curves so if we reparametrize them with the functions g_j $j = 1, 2$ such that they describe the same curve, that is, $\{\gamma_j(s) : |s| < s_0 \text{ for some } s_0 > 0\} = \{t + ig_j(t) : |t| < t_0 \text{ for some } t_0 > 0\}$ $j = 1, 2$, then the functions g_1, g_2 will be C^2 smooth too.

If they touch each other in order 2, then $\left|g_1(t) - g_2(t)\right| \sim |t|^2$. This, with their C^2 smoothness, give that $g_1'(0) = g_2'(0)$ and $g_1''(0) \neq g_2''(0)$. Using the curvature formula (4.2), we immediately obtain that their curvature is different and we also know that their tangent line coincide (it will be the real axis).

On the other hand, suppose that their tangent line coincide and their curvature are different. We can assume that their common tangent line is the real axis and they pass through the origin. As above, we have that g_1, g_2 are C^2 smooth. Since their tangent line coincide at the origin, this implies that $g_1'(0) = g_2'(0)$. Using that their curvature are different at the origin, we obtain with the curvature formula (4.2) that $g_1''(0) \neq g_2''(0)$. These two facts immediately imply that $\left|g_1(t) - g_2(t)\right| \sim |t|^2$, that is, they touch each other in order 2.

$\square$

This criteria describes the touching of order 2 with a geometric property.

4.2 Examples

Now we show two examples where (a badly required) higher order of touching excludes the existence of in-between lemniscate (see Theorem 7).

Example 1.

Let s be a noninteger, real number ($s \in \mathbf{R} \setminus \mathbf{Z}$), $s > 2$. We use the upper integral part of a real number, $\lceil x \rceil := \min\{k \in \mathbf{Z} : x \leq k\}$. Consider the functions $f_1(t) := |t|^s, f_2(t) := 1/2 f_1(t)$. These functions can be differentiated $\lceil s-1 \rceil$ times and actually $f_1^{(\lceil s-1 \rceil)}(t) = s(s-1)\dots(s - \lceil s-1 \rceil) \cdot |t|^{s - \lceil s-1 \rceil}$

where $s - \lceil s-1 \rceil \le 1$. So $f_1, f_2 \in C^{\lceil s-1 \rceil}$, but $f_j^{(\lceil s-1 \rceil)}(t)$, $j = 1, 2$ can not be differentiated at $t = 0$.

Consider their graphs. Let $\gamma^* : t \mapsto t + if_1(t)$ and $\Gamma^* : t \mapsto t + if_2(t)$. Let γ be the following closed Jordan curve $\{\gamma^*(t) : -1/2 \le t \le 1/2\} \cup \{t + i/2^s : -1/2 \le t \le 1/2\}$ such that $\gamma(0) = 0$. Define Γ in a similar way: $\Gamma := \{\Gamma^*(t) : -1 \le t \le 1\} \cup \{t + i/2 : -1 \le t \le 1\}$ such that $\Gamma(0) = 0$.

By definition, $\big|f_1(t) - f_2(t)\big| \sim |t|^s$, so γ and Γ touch each other at 0 in order s.

On the other hand, any lemniscate $L = r^{-1}[C]$ away from its critical points (where $r' = 0$) is locally an analytic curve because of the inverse function theorem (for holomorphic functions). At the critical points, the lemniscates branch off, so r can have no critical points at $z = 0$ because of higher order of touching (of γ and Γ).

Indirectly, assume that there is a lemniscate $L = r^{-1}[C]$ in between γ and Γ. We can assume that $r(0) = 1$. Parametrize its subarc near $z = 0$ by λ that is, $\big|r\big(\lambda(t)\big)\big| \equiv 1$ for small values of t and $\lambda(0) = 0$. Since the lemniscate L is analytic near $z = 0$, we can assume that $\lambda' \ne 0$ and $\mathrm{Re}\ \lambda(t) = t$. The tangent lines of γ and Γ at $z = 0$ coincide with the real axis, so the same holds for λ (i.e. $\lambda'(0) \in \mathbf{R} \setminus \{0\}$). Now introduce the real function g as follows

$$\lambda(t) = t + ig(t) \qquad \text{(for small } t) \ ,$$

where actually $g(t) = \mathrm{Im}\ \lambda(t)$, so $g(0) = 0$, g is a real valued function and g is C^∞ smooth.

So we have 3 functions

$$f_2(t) \le g(t) \le f_1(t) \qquad \text{(for small } t). \tag{4.3}$$

$g^{(\lceil s-1 \rceil)}(0)$ is necessarily 0. So $g(t) = O\big(|t|^{\lceil s-1 \rceil + 1}\big)$, but $f_1(t), f_2(t) \sim |t|^s$. Since $\lceil s-1 \rceil + 1 > s$ ($s \notin \mathbf{Z}$), for some small $t > 0$, we have $g(t) < f_2(t)$ which contradicts (4.3).

Example 2.
This is very similar to the previous example, but a small change is needed because e.g. $t \mapsto |t|^k$ is analytic if k is an even integer.

Let k be an integer, $k \ge 3$. Let $f_3(t) := \mathrm{sign}\, t \cdot t^k$,

$$f_4(t) := \begin{cases} \frac{|t|^k}{2} & \text{if } t \ge 0, \\ \frac{|t|^k}{2} & \text{if } t < 0 \text{ and } k \text{ is odd}, \\ -2|t|^k & \text{if } t < 0 \text{ and } k \text{ is even}. \end{cases}$$

This definition immediately implies $f_4 \le f_3$ and $\big|f_3(t) - f_4(t)\big| \sim |t|^k$. It is easy to see that f_3, f_4 are C^{k-1} smooth functions but neither $f_3^{(k-1)}(t)$, nor $f_4^{(k-1)}(t)$ is differentiable at $t = 0$.

Now consider their graphs. Let $\gamma^* : t \mapsto t + if_3(t)$ and $\Gamma^* : t \mapsto t + if_4(t)$. Let γ be the following closed Jordan curve $\{\gamma^*(t) : -1/2 \le t \le 1/2\} \cup \{t + i/2^k : -1/2 \le t \le 1/2\}$ such that $\gamma(0) = 0$. Let Γ be the union of the following curves: $\{\Gamma^*(t) : -1 \le t \le 1\}$, $\{t + if_4(1) : -1 \le t \le 1\}$ and $\{-1 + it : f_4(-1) \le t \le f_4(1)\}$. We can also assume that $\Gamma(0) = 0$.

By definition, γ and Γ touch each other at $t = 0$ in order k.

Indirectly, assume that there is a lemniscate in between γ and Γ. Exactly as above, we introduce the notations λ and g. Again, we have 3 functions

$$f_4(t) \le g(t) \le f_3(t) \qquad \text{(for small } t). \tag{4.4}$$

We argue as follows. It is easy to verify that $f_j'(0) = \ldots = f_j^{(k-1)} = 0$, $j = 1, 2$, so $g'(0) = \ldots = g^{(k-1)}(0) = 0$.

If k is even, we have 3 subcases depending on the sign of $g^{(k)}(0)$. If $g^{(k)}(0) > 0$, then for small $t < 0$, we have $g(t) > 0$, which is a contradiction with (4.3) because for small $t < 0$ we already have $f_3(t), f_4(t) < 0$.

If $g^{(k)}(0) < 0$, then we have contradiction the same way for small $t > 0$.

If $g^{(k)}(0) = 0$, then for small t, we know that $f_3(t), f_4(t) \sim |t|^k$, but $g(t) = o(|t|^k)$, so for some small t, $g(t) < f_4(t)$ which contradicts (4.4).

If k is odd, then the same idea can be applied except that the first two subcases change place.

Again, these contradictions show that there is no such in-between lemniscate.

Remarks.

The second example can be extended to $k = 2$, but in that case, γ and Γ are not C^2 smooth, so they do not give counterexamples to Theorem 7.

In the proof of Theorem 7 we actually used the fact that there is an order s (actually $s = 2$) such that the derivates of the curves up to order $s - 1$ coincide and they have derivatives of order s which are different. This is the same as that we can insert two different analytic curves ϕ and Φ in between them such a way that $\gamma \subset \operatorname{Int} \phi$, $\phi \subset \operatorname{Int} \Phi$, $\Phi \subset \operatorname{Int} \Gamma$. These ϕ and Φ in [44] were (subarcs of) properly chosen circles.

On the other hand, one of the common features of these two examples is the fact that all the derivatives (at $t = 0$) they have coincide.

We conjecture the following.

Suppose we have two curves Γ, γ as above: γ, Γ closed Jordan curves, their interiors are fat, $\gamma \subset \operatorname{Int} \Gamma$, they have finite number of common points $z_1, \ldots, z_N$ and they are C^{n_j} smooth near z_j ($j = 1, \ldots, N$). Furthermore assume that their derivatives coincide at z_j up to order $n_j - 1$ but their derivatives at z_j of order n_j differ. Then we conjecture that we can put a lemniscate in between γ and Γ.

Note that the assumption on their derivatives implies that locally one can construct a lemniscate.

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Summary

The well-known Bernstein's inequality states that

$$\left|P_n'(z_0)\right| \leq n||P_n||_D \ , \tag{1.1}$$

where P_n is an arbitrary complex polynomial with degree n, $||P_n||_D$ denotes its supremum norm over the unit disk $D = \{z \in \mathbf{C} : |z| \leq 1\}$ and $|z_0| = 1$. The subject of this dissertation is to extend this inequality.

If $K \subset \mathbf{C}$ is compact, then Green's function of the complement of K with pole at infinity is denoted by $g_K(z) = g(K, z)$. We say that the compact set K is *Jordan fat*, if the boundary of every connected component of its interior $\mathrm{Int}(K)$ is a Jordan curve and K is the closure of its interior: $K = \overline{\mathrm{Int}(K)}$.

One of the main results is

Theorem (9). *Let K be a Jordan fat compact set on the plane with connected complement. Let z_0 be a point on the boundary of K and let us suppose that this boundary is a twice continuously differentiable Jordan arc in a neighborhood of z_0. Then*

$$|P_n'(z_0)| \leq n(1 + o(1))\frac{\partial g(K, z_0)}{\partial \mathbf{n}}\|P_n\|_K, \tag{3.3}$$

where the $o(1)$ tends to 0 uniformly in the polynomials P_n of degree at most n as $n \to \infty$ and $\frac{\partial g(K,z_0)}{\partial \mathbf{n}}$ denotes the normal derivative of the Green's function of $\mathbf{C}_\infty \setminus K$ in the (outward) normal direction $\mathbf{n}$ (at z).

The proof of Theorem 9 is based on the the following two notions and two theorems:

Definition (1). *The set $L \subset \mathbf{C}$ is a* lemniscate *if for some complex polynomial r, $L = r^{-1}[\partial D]$, that is, $z \in L \Leftrightarrow |r(z)| = 1$. The set $r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \leq 1\}$ is called the* interior of the lemniscate *L.*

Let γ^* and Γ^* be some finite systems of Jordan curves γ^* lying inside Γ^*. We assume that γ^* and Γ^* are twice continuously differentiable in a neighborhood of P and touching each other at P. We say that they $\mathcal{K}$-*touch*

each other if their (signed) curvature at P is different (signed curvature is seen from the outside of Γ^*). Equivalently we can say that in a neighborhood of P the two curves are separated by two circles one of them lying in the interior of the other one.

Theorem (7). *Let $\gamma^* = \cup_{j=1}^m \gamma_j$ and $\Gamma^* = \cup_{j=1}^m \Gamma_j$ be as above, and let γ^* $\mathcal{K}$-touch Γ^* in finitely many points $P_1, \ldots, P_k$ in a neighborhood of which both curves are twice continuously differentiable. Then there is a lemniscate σ that separates γ^* and Γ^* and $\mathcal{K}$-touches both γ^* and Γ^* at each P_j.*

Furthermore, σ lies strictly in between γ^* and Γ^* except for the points $P_1, \ldots, P_k$, and has precisely one connected component in between each γ_j and Γ_j, $j = 1, \ldots, m$, and these m components are Jordan curves.

This is a sharpening of a celebrated theorem of David Hilbert claiming the same but for untouching curves.

Theorem (8). *Let Γ^*, γ^* and $P_1, \ldots, P_k \in \Gamma^*$ be as in Theorem 7. Then for every $\varepsilon > 0$ there is a lemniscate σ as in Theorem 7 such that for each P_j we have*

$$\frac{\partial g(L, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(K, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.1}$$

where $\partial(\cdot)/\partial\mathbf{n}$ denotes (outward) normal derivative and K is the compact set enclosed by Γ^*.

In a similar manner, for every $\varepsilon > 0$ there is a lemniscate σ as in Theorem 7 such that for each P_j we have

$$\frac{\partial g(K_0, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(L, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.2}$$

where K_0 is the compact set enclosed by γ^*.

Theorem 9 follows from Theorem 7 and its special case when K is enclosed by a lemniscate (it is formulated as Theorem 2, as a very important special case of Theorem 9). Theorem 9 is sharp regarding the constant $\partial g(K, z_0)/\partial\mathbf{n}$:

Theorem (10). *Let K and z_0 be as in Theorem 9. Then for every n there is a polynomial P_n of degree at most n such that*

$$|P_n'(z_0)| > n(1 - o(1))\frac{\partial g(K, z_0)}{\partial \mathbf{n}}\|P_n\|_K. \tag{3.4}$$

It is sharp also in the sense that in general the inequality

$$|P_n'(z_0)| \leq n\frac{\partial g(K, z_0)}{\partial \mathbf{n}}\|P_n\|_K$$

i.e. (3.3) without the term $1 + o(1)$ is not true.

Összefoglaló

A jól ismert Bernstein egyelőtlenség azt állítja, hogy

$$\left|P_n'(z_0)\right| \leq n||P_n||_D \ , \tag{1.1}$$

ahol P_n egy tetszőleges n-ed fokú komplex polinom, $||P_n||_D$ jelöli a szuprémum normáját a $D = \{z \in \mathbf{C} : |z| \leq 1\}$ egységkörlapon és $|z_0| = 1$. Ennek az egyelőtlenségnek kiterjesztése ezen disszertáció tárgya.

Ha $K \subset \mathbf{C}$ kompakt, akkor $g_K(z) = g(K, z)$-val jelöljük a komplementerének Green függvényét végtelenbeli pólussal. Azt mondjuk, hogy a K kompakt halmaz *Jordan kövér*, ha a határa minden összefüggő komponensének egy Jordan görbe, és K a belsejének a lezártja: $K = \overline{\mathrm{Int}(K)}$.

Az egyik fő eredmény a következő

Tétel (9). *Legyen K egy Jordan kövér kompakt halmaz a síkon összefüggő komplementerrel. Legyen z_0 egy pont K határán és tegyük fel, hogy K határa kétszer folytonosan differenciálható Jordan ív a z_0 egy környezetében. Ekkor*

$$|P_n'(z_0)| \leq n(1 + o(1))\frac{\partial g(K, z_0)}{\partial \mathbf{n}}\|P_n\|_K, \tag{3.3}$$

ahol $o(1)$ tart 0-hoz egyenletesen a legfeljebb n-ed fokú polinomokon amint $n \to \infty$ és $\frac{\partial g(K,z_0)}{\partial \mathbf{n}}$ jelöli az $\mathbf{n}$ külső normális menti deriváltját a $\mathbf{C}_\infty \setminus K$ Green függvényének a z pontban.

A 9. Tétel bizonyítása a következő két fogalmon és két tételen alapul:

Definíció (1). *Az $L \subset \mathbf{C}$ halmaz egy* lemniszkáta, *ha valamely r komplex polinomra $L = r^{-1}[\partial D]$, vagyis $z \in L \Leftrightarrow |r(z)| = 1$. Az $r^{-1}[D] = \{z \in \mathbf{C} : |r(z)| \leq 1\}$ halmazt hívjuk az L* lemniszkáta belsejének.

Legyen γ^* és Γ^* két, véges sok zárt Jordan görbéből álló rendszer. Feltesszük, hogy γ^* és Γ^* kétszer folytonosan differenciálható egy P pont valamely környezetében és érintik egymást P-ben. Azt mondjuk, hogy $\mathcal{K}$-*érintik* egymást, ha az (előjeles) görbületük P-ben különbözik (az előjeles görbületet Γ^* külsejéből tekintve). Ezzel egyenértékű, ha P egy környezetében a két görbét szét lehet választani két körvonallal úgy, hogy az egyik körvonal a másik körvonal belsejében fekszik.

Tétel (7). *Legyen* $\gamma^* = \cup_{j=1}^m \gamma_j$, $\Gamma^* = \cup_{j=1}^m \Gamma_j$ *mint fentebb,* γ^* *és* Γ^* $\mathcal{K}$-*érintse egymást véges sok* $P_1, \ldots, P_k$ *pontokban, amelyek környezeteiben kétszer folytonosan differenciálhatóak. Ekkor létezik egy* σ *lemniszkáta, amely elválasztja* γ^**-t és* Γ^**-t, valamint* $\mathcal{K}$*-érinti* γ^**-t és* Γ^**-t mindegyik* P_j*-nél.*

Továbbá, σ *szigorúan* γ^* *és* Γ^* *közt helyezkedik el, kivéve a* $P_1, \ldots, P_k$ *pontokat, pontosan egy komponense van minden egyes* γ_j *és* Γ_j *közt,* $j = 1, \ldots, m$*, és ez az* m *komponens mindegyike Jordan görbe.*

Ez élesítése David Hilbert egy híres tételének, amely hasonlót állít, de nem-érintő görbékre.

Tétel (8). *Legyen* γ^*, Γ^* *és* $P_1, \ldots, P_k \in \Gamma^*$ *mint a 7. Tételben. Ekkor minden* $\varepsilon > 0$*-ra létezik egy lemniszkáta, olyan mint a 7. Tételben, úgy, hogy mindegyik* P_j*-nél fennáll, hogy*

$$\frac{\partial g(L, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(K, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.1}$$

ahol $\partial(\cdot)/\partial\mathbf{n}$ *jelöli a (külső) normális szerinti deriváltat és* K *az a kompakt halmaz, amit* Γ^* *közrefog.*

Hasonló módon, minden $\varepsilon > 0$*-ra létezik egy lemniszkáta, olyan mint a 7. Tételben, úgy, hogy mindegyik* P_j*-nél fennáll, hogy*

$$\frac{\partial g(K_0, P_j)}{\partial \mathbf{n}} \leq \frac{\partial g(L, P_j)}{\partial \mathbf{n}} + \varepsilon, \tag{3.2}$$

ahol K_0 *az a kompakt halmaz, amit* γ^* *közrefog.*

A 9. Tétel következik a 7. Tételből és annak speciális esetéből, amikor K-t egy lemniszkáta fogja közre (ez a 2. Tétel, mint a 9. Tétel nagyon fontos speciális esete). A 9. Tétel éles a $\partial g(K, z_0)/\partial\mathbf{n}$ konstansot tekintve:

Tétel (10). *Legyen* K *és* z_0 *olyan, mint 9. Tételben. Ekkor minden* n*-re létezik egy legfeljebb* n*-ed fokú* P_n *polinom úgy, hogy*

$$|P_n'(z_0)| > n(1 - o(1)) \frac{\partial g(K, z_0)}{\partial \mathbf{n}} \|P_n\|_K. \tag{3.4}$$

Abban az értelemben is éles, hogy a következő egyenlőtlenség

$$|P_n'(z_0)| \leq n \frac{\partial g(K, z_0)}{\partial \mathbf{n}} \|P_n\|_K,$$

vagyis (3.3) az $1 + o(1)$ szorzó nélkül nem igaz.