Introduction

We study the local dynamics of the time-periodic scalar delay differential equation

$$\dot{x}(t) = \gamma\big(a(t)x(t) + f(t, x(t-1))\big),$$

in a neighborhood of the critical values of the parameter. Such equations arise very naturally in several applications (neural networks, population dynamics, mechanical engineering). At certain critical values of the parameter the dynamics dramatically change, for instance the equilibrium loses its stability, appearance of periodic orbits can be observed, etc. The classical theory of bifurcations was established for one- and two-dimensional dynamical systems. Using center manifold reduction and projection methods, one can generalize bifurcation theory to higher dimensional systems. For delay differential equations the usual phase space is the infinite dimensional Banach space of continuous functions on the initial interval. In the critical case, all the essential qualitative features of our dynamical system are captured by the center manifold. Unfortunately, the classical process of computing the dynamical system restricted to the center manifold using Hale bilinear forms can not be applied directly to periodic equations. Recently a normal form theory was presented for general periodic functional differential equations, but that works only for equations with autonomous linear part.

The main achievement of this dissertation is that we give a complete bifurcation analysis for a wide class of periodic delay differential equations. When the delay is the same as the period, then we are able to build up the entire theory of Neimark-Sacker bifurcations for the infinite dimensional case without any additional restrictions on the nonlinearity. The main technical difficulty is that we need explicit computation of normal forms and finite dimensional manifolds in our Banach space. To perform this, we use a functional analytic approach, the spectral projection method. All the results are explicit, we can determine the bifurcation points and the direction of bifurcations by the right hand side of our equation, this is important for specific applications. We observe the appearance of invariant tori in an extended phase space.

The equation

$$\dot{x}(t) = \gamma f(t, x(t-1))$$

is not only a special case of the previous one, taking $a(t) \equiv 0$, but qualitatively new phenomena may appear. In this situation we have bifurcations with strong 1:4 resonance and the invariant torus does not necessarily exist. Strong resonances have an extended theory, in this dissertation we also generalize the case of 1:4 strong resonance to periodic delay equations. All our results are explicit again.

We can apply our theorems to many well-known equations. Below we discuss the main results of the dissertation and the applied mathematical tools and techniques. All the statements are new results.

Neimark-Sacker Bifurcation

Consider the equation

$$\dot{x}(t) = \gamma\big(a(t)x(t) + f(t, x(t-1))\big), \tag{1}$$

where γ is a real parameter, $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are C^4-smooth functions satisfying $a(t+1) = a(t)$, $f(t+1, \xi) = f(t, \xi)$ and $f(t, 0) = 0$ for all $t, \xi \in \mathbb{R}$. As usual, the Banach-space $C := C([-1, 0], \mathbb{R})$ of continuous functions on the initial interval $[-1, 0]$ serves as state space, equipped with the supremum norm.

For every initial function $\phi \in C$ there is a unique continuous function $x^\phi : [-1, \infty) \to \mathbb{R}$, which is differentiable on $(0, \infty)$, satisfies the equation for all $t > 0$ and $x^\phi(t) = \phi(t)$ for all $t \in [-1, 0]$. We call such a function x^ϕ the solution of the delay differential equation. The time-one map $F : C \to C$ is defined by the relations

$$F(\phi) = x_1^\phi, x_t(s) = x(t+s), s \in [-1, 0].$$

Using Floquet theory ([3]) we deduce the characteristic function $h(\lambda) = \gamma\alpha + \gamma\beta e^{-\lambda} - \lambda$, where $\alpha = \int_{-1}^0 a(t)dt$, $\beta = \int_{-1}^0 f_\xi(t, 0)dt$. The zeros of $h(\lambda)$ are the Floquet exponents. The Floquet multipliers are the eigenvalues of the monodromy operator U, which is the derivative of the time-one map at the equilibrium 0. Studying the characteristic equation we find the number of Floquet multipliers inside the complex unit circle. We can detect the critical values of the parameter when this number changes and we have Floquet multipliers on the unit circle. We show that the conditions of the bifurcation theorem are fulfilled. Varying the parameter, at the critical values a pair of conjugate Floquet multipliers crosses the unit circle and an invariant curve occurs on the center manifold. This is a Neimark-Sacker bifurcation. In an extended phase space we can consider this as the appearance of an invariant torus. To compute the restricted map on the center manifold and the direction of the bifurcation, we generalize the projection method. The theorem of Riesz and Schauder on spectral decompositions is an important tool to do this. The spectral projection operator can be expressed by a Riesz-Dunford integral and can be computed by solving a boundary value problem. Then we are able to perform the complete bifurcation analysis. We give an explicit formula for the coefficient that determines the direction of the bifurcation and the stability properties of the bifurcated invariant curve. Combining all the information we can get by the characteristic equation, the Neimark-Sacker bifurcation theorem ([6]) and the smooth center manifold theorem for maps in Banach spaces ([2], [5]), we obtain the following bifurcation theorem.

Theorem 1 *The one-parameter family of time-one maps $F_\gamma : C \to C$ corresponding to equation (1) has at the critical values $\gamma = \gamma_j$ the fixed point $\phi = 0$ with exactly two simple Floquet-multipliers ($e^{i\theta}$ and $e^{-i\theta}$) on the unit circle. There is a neighborhood of 0 in which a unique closed invariant curve bifurcates from 0 as γ passes through γ_j, providing that the non-resonance conditions $\mu_j^4 \neq 1, \mu_j^3 \neq 1$ hold. The transversality (Hopf) condition $\frac{\partial|\mu(\gamma)|}{\partial\gamma}\ |_{\gamma_j}\neq 0$ is always fulfilled for (1), furthermore $\mu_j^4 = 1$ if and only if $\alpha = 0$, $\mu_j^3 = 1$ if and only if $\beta = 2\alpha$. The two critical Floquet multipliers $\mu_j = e^{\lambda_j} = e^{i\gamma_j\sqrt{\beta^2-\alpha^2}} = -\frac{\alpha}{\beta} - i\sqrt{1-\frac{\alpha}{\beta}^2}$ and $\bar{\mu}_j = e^{\bar{\lambda}_j} = e^{-i\gamma_j\sqrt{\beta^2-\alpha^2}} = -\frac{\alpha}{\beta} + i\sqrt{1-\frac{\alpha^2}{\beta^2}}$ are simple. The critical values of the parameter are $\gamma_{\pm n} = -\frac{\pm\arccos(-\frac{\alpha}{\beta})+2n\pi}{\pm\beta\sin\left(\arccos(-\frac{\alpha}{\beta})\right)}, n \in \mathbb{N}$.*

For simplicity, let $b(t) = \gamma f_\xi(t, 0)$ and $c(t) = \gamma a(t)$. With this notation the linear variational equation takes the form

$$\dot{y}(t) = c(t)y(t) + b(t)y(t-1),$$

the eigenfunction corresponding to a simple eigenvalue is

$$\chi_\mu(t) : [-1, 0] \ni t \mapsto e^{\int_{-1}^{t}[c(s)+\frac{b(s)}{\mu}]ds} \in \mathbb{C}.$$

Solving a boundary value problem we can compute the resolvent of the monodromy operator. The spectral projection operator is the residuum of the resolvent.

Theorem 2 *The resolvent of the monodromy operator can be expressed as*

$$
\begin{aligned}
(zI-U)^{-1}(\psi)(t) &= e^{\int_{-1}^{t}[c(u)+\frac{b(u)}{z}]du} \\
&\times \left(\left(\frac{1}{z}\psi(0) + e^{\int_{-1}^{0}[c(u)+\frac{b(u)}{z}]du} \int_{-1}^{0} \frac{1}{z^2} e^{-\int_{-1}^{s}[c(u)+\frac{b(u)}{z}]du} b(s)\psi(s)ds \right) \right. \\
&\times \left(z - e^{\int_{-1}^{0}[c(u)+\frac{b(u)}{z}]du}\right)^{-1} + \frac{1}{z} e^{-\int_{-1}^{t}[c(u)+\frac{b(u)}{z}]du}\psi(t) \\
&+ \left. \int_{-1}^{t} \frac{1}{z^2} e^{-\int_{-1}^{s}[c(u)+\frac{b(u)}{z}]du} b(s)\psi(s)ds \right), \quad t\in[-1,0], z\in\mathbb{C}, \psi\in C.
\end{aligned} \tag{2}
$$

Theorem 3 *For a simple eigenvalue μ, the spectral projection operator has the representation*

$$P_\mu(\psi) = \chi_\mu R_\mu(\psi),$$

where

$$R_\mu(\psi) = \left(\frac{1}{\mu+\gamma\beta}\right)\left(\psi(0) + \int_{-1}^{0} \frac{b(s)\psi(s)}{\chi_\mu(s)}ds\right).$$

By the spectral projection we can generalize the projection method. An elaborative calculation leads to the restricted map on the center manifold and a coefficient that determines the nature of the bifurcation. Define the multilinear operators $V := D^2F(0)$ and $W := D^3F(0)$.

Theorem 4 *The direction of the appearance of the invariant curve is determined by the sign of the coefficient*

$$
\begin{aligned}
\delta(\gamma_j) = \frac{1}{2}\mathrm{Re}\, &\left(\frac{1}{\mu} R_\mu \Big(W(\chi_\mu,\chi_\mu,\bar{\chi}_\mu) + 2V\big(\chi_\mu,(1-U)^{-1}V(\chi_\mu,\bar{\chi}_\mu)\big) \right. \\
&\left. + V\big(\bar{\chi}_\mu,(\mu^2-U)^{-1}V(\chi_\mu,\chi_\mu)\big)\Big)\right),
\end{aligned}
$$

where all the terms can be computed explicitly from (1).

Let us mention that the cases $\delta(\gamma_j) < 0$ and $\delta(\gamma_j) > 0$ are called supercritical and subcritical Neimark-Sacker bifurcations. In the supercritical case a stable (only in a restricted sense, inside the center manifold) invariant curve appears for $\gamma > \gamma_j$, while in the subcritical case an unstable invariant curve disappears when γ increasingly crosses γ_j. When $\delta(\gamma_j) = 0$, we need further investigations. Here we suppose that the nondegeneracy condition $\delta(\gamma_j) \neq 0$ is fulfilled. The smoothness of F_γ is guaranteed by the smoothness of $a(t)$ and $f(t, \xi)$.

In the extended phase space $C \times S^1$ the semi-dynamical system generated by the periodic delay differential equation can be considered as an autonomous system. Denote the corresponding solution operators by $G(t)$.

Theorem 5 *If the conditions of the Neimark-Sacker bifurcation theorem hold, then for the one-parameter family of dynamical systems generated by the solution maps* $G_\gamma(t) : C \times S^1 \to C \times S^1$ *corresponding to equation (1), a unique invariant torus bifurcates from the periodic solution* $(0, t)$ *as the parameter* γ *passes through the critical value* γ_j*. The direction of the appearance of the invariant torus is determined by the sign of the coefficient* $\delta(\gamma_j)$*, which can be computed explicitly.*

Applications

Our results can be applied to the periodic versions of many notable equations in the theory of delay differential equations, such as the Mackey-Glass, Nicholson and Krisztin-Walther equations. In this way we prove several new bifurcation theorems. In the sequel $r(t)$ denotes a real function, which satisfies $r(t) = r(t+1) > 0$ for all $t \in \mathbb{R}$. Similarly, $m(t), q(t)$ are also non-negative 1-periodic functions. We use the notations R, M and Q, respectively, for the integral of $r(r), m(t)$ and $q(t)$ on intervals of length 1. Let us start with a rather general periodic form of the Krisztin-Walther equation.

Theorem 6 *Suppose that* $0 < R$, $g'(0) < 0$, $g''(0) = 0$ *and* $g'''(0) \neq 0$*. Then the one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to the equation*

$$\dot{z}(t) = \gamma r(t)\big(- mz(t) + g(z(t-1)\big) \tag{3}$$

undergoes a Neimark-Sacker bifurcation, as the parameter γ *increasingly passes through* γ_0*. If* $g'''(0) < 0$ *then the bifurcation is subcritical, if* $g'''(0) > 0$ *then the bifurcation is supercritical.*

The next two theorems concern with the periodic variants of the Mackey-Glass and the Nicholson equations.

Theorem 7 *Suppose* $0,9 < \frac{M}{Q} < 1$. *Then the one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to equation*

$$\dot{x}(t) = \gamma\Big(- m(t)x(t) + \frac{q(t)x(t-1)}{1 + x(t-1)^2}\Big), \tag{4}$$

undergoes a supercritical Neimark-Sacker bifurcation as the parameter γ *increasingly passes through* γ_n.

Assume that the constants p, d, c, β satisfy $p > d > 0$, $c > 0$, $\beta > 0$.

Theorem 8 *The one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to equation*

$$\dot{N}(t) = \gamma r(t)\big(- dN(t) + pN(t-1)e^{-cN(t-1)}\big) \tag{5}$$

undergoes a supercritical Neimark-Sacker bifurcation in the neighborhood of the positive equilibrium as the parameter γ *increasingly crosses the critical value* γ_n*, where* $n \geq 1$.

Finally we consider a Krisztin-Walther equation that describes the dynamics of a periodically excited neuron, and a more general class of equations as well.

Theorem 9 *Assume that* $0,9 < \frac{M}{\beta Q} < 1$. *Then the one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to equation*

$$\dot{x}(t) = \gamma\big(- m(t)x(t) + q(t)\tanh(\beta x(t-1))\big), \tag{6}$$

undergoes a supercritical Neimark-Sacker bifurcation as the parameter γ *increasingly crosses the critical value* γ_n.

Theorem 10 *Consider the equation*

$$\dot{x}(t) = \gamma\big(- m(t)x(t) + f(t, x(t-1))\big), \tag{7}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a* C^4*-smooth function satistying* $f(t+1, \xi) = f(t, \xi)$ *and* $f(t, 0) = 0$ *for all* $t, \xi \in \mathbb{R}$. *Suppose that* $f_\xi(t, 0) > 0$ *for all* $t \in \mathbb{R}$ *and* $0,9 < \frac{M}{\int_{-1}^{0} f_\xi(s,0)ds} < 1$. *Then if* $f_{\xi\xi\xi}(t, 0) < 0$ *for all* $t \in \mathbb{R}$, *the one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to equation (7) undergoes a supercritical (if* $f_{\xi\xi\xi}(t, 0) > 0$ *for all* $t \in \mathbb{R}$, *a subcritical) Neimark-Sacker bifurcation, as the parameter* γ *increasingly crosses the critical value* γ_n.

Resonant Bifurcations

In this section we consider the equation

$$\dot{x}(t) = \gamma f(t, x(t-1)), \tag{8}$$

which is the same as equation (1) taking $a(t) \equiv 0$. However, in this case the non-resonance condition is not fulfilled and the bifurcation theorem is not valid anymore. In the case of $1:4$ strong resonance, it is possible, that the invariant curve does not exist at all, but 4-periodic points bifurcate ([4]). We give explicit conditions to detect all the possibilities. Stable and unstable invariant curves, stable and unstable families of 4-periodic points may bifurcate. We find the resonant Poincaré normal form and we conclude that in the case of periodic coefficient, the strong resonance has no effect on the bifurcations. We illustrate this result on the specific example of the Wright equation with periodic coefficient.

Theorem 11 *Let $g = g_\gamma : \mathbb{C} \to \mathbb{C}$ be a map depending on a parameter, of the form*

$$g(z) = \mu z + \frac{\rho_{20}}{2} z^2 + \rho_{11} z\bar{z} + \frac{\rho_{02}}{2}\bar{z}^2 + \frac{\rho_{30}}{6} z^3 + \frac{\rho_{21}}{2} z^2\bar{z} + \frac{\rho_{12}}{2} z\bar{z}^2 + \frac{\rho_{03}}{6}\bar{z}^3 + \mathcal{O}(|z|^4) \tag{9}$$

where $\mu = \mu(\gamma)$ and the coefficients $\rho_{kl} = \rho_{kl}(\gamma)$ are smooth functions of the parameter, moreover $\mu(\gamma_j) = i$ at the critical values $\gamma = \gamma_j$. Then by a coordinate transformation, depending smoothly on the parameter γ, the map g can be transformed into

$$\tilde{g}(w) = iw + c_1 w^2\bar{w} + c_2\bar{w}^3 + \mathcal{O}(|w|^4),$$

where

$$c_1 = \frac{1+3i}{4}\rho_{20}\rho_{11} + \frac{1-i}{2}\rho_{11}\bar{\rho}_{11} + \frac{-1-i}{4}\rho_{02}\bar{\rho}_{02} + \frac{\rho_{21}}{2}$$

and

$$c_2 = \frac{i-1}{4}\rho_{11}\rho_{02} + \frac{-i-1}{4}\rho_{02}\bar{\rho}_{20} + \frac{\rho_{03}}{6}.$$

The coefficient c_1 is the same as in the nonresonant case, but the coefficient of the resonant term $\bar{w}^3$ can not be removed in case of strong resonance.

The formula for the coefficient c_1 is widely known, one can find c_2 in [6] also, but without the detailed computations. Nevertheless, in the literature one can find miscalculated formulae, and some uses the false ones in applications. For that reason we present in the dissertation the computations for the correct formula with all the details. In specific applications it is important to deal with the exact value of c_2.

Let us define $a_1 = \frac{c_1}{i}$, $a_2 = \frac{c_2}{i}$, $d = \frac{\partial |\mu(\gamma)|}{\partial \gamma} |_{\gamma=\gamma_j}$ and

$$\delta = |\text{Im }(a_1) - B\text{Re }(a_1)| - |a_2|\sqrt{1+B^2}, \tag{10}$$

where B is the integral of $b(s)$ on an interval of length 1.

Theorem 12 *For the time-one map corresponding to equation (8),*

$$\begin{aligned} a_1 = -\frac{i}{2}\Big[& R_i(W(\chi_i, \chi_i, \bar{\chi}_i)) + 2R_i(V(\chi_i, (1-U)^{-1}V(\chi_i, \bar{\chi}_i))) \\ & + R_i(V(\bar{\chi}_i, (i^2 - U)^{-1}V(\chi_i, \chi_i)))\Big], \end{aligned} \tag{11}$$

$$a_2 - \frac{i}{6}\Big[R_i(W(\bar{\chi}_i, \bar{\chi}_i, \bar{\chi}_i)) + 3R_i\Big(V(\bar{\chi}_i, (i^{-2}I - U)^{-1}\big(V(\bar{\chi}_i, \bar{\chi}_i)\big)\Big)\Big]. \tag{12}$$

We extend the 1 : 4 resonant bifurcation theorem ([4]) to periodic delay differential equations.

Theorem 13 *The one-parameter family of time-one maps $F_\gamma : C \to C$ corresponding to equation (8) has the fixed point $\phi = 0$ with exactly two critical, simple Floquet multipliers $\mu_j = i$ and $\bar{\mu}_j = -i$ on the unit circle. This is a 1:4 strong resonance. The transversality condition holds. There is a neighborhood of the equilibrium* 0*, in which a unique invariant curve bifurcates (and no* 4*-periodic points), providing that $\delta > 0$. The direction of the bifurcation is determined by the sign of* Re (a_1)*. If $\delta < 0$, then two families of 4-periodic points (and no invariant curve) bifurcate from the equilibrium* 0*. Furthermore, if $|a_1| > |a_2|$, then the two families appearing on the same side and at least one of them is unstable. If $|a_1| < |a_2|$, then the two families appear on opposite sides and both of them are unstable.*

Apparently, there are several different possible outcomes of the bifurcation. Since our conditions are explicit, we are able to determine the type of the bifurcation for any given equation. We show that in the case of periodic coefficients the resonance does not affect the bifurcation.

Theorem 14 *Consider the equation*

$$\dot{x}(t) = \gamma r(t) f(x(t-1)), \tag{13}$$

where $f(\xi) = \xi + \frac{S}{2}\xi^2 + \frac{T}{6}\xi^3 + \mathcal{O}(\xi^4)$ *is a* C^4*-smooth function,* $r(t)$ *is* 1*-periodic. The one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to equation (13) undergoes a bifurcation and a unique invariant curve bifurcates from the equilibrium* 0*, as the parameter* γ *passes through the critical value* γ_j*. The bifurcation is supercritical if* $T < S^2\left(\frac{11B+2}{5B}\right)$ *and subcritical if* $T > S^2\left(\frac{11B+2}{5B}\right)$.

The previous theorem can be applied to the celebrated Wright equation with periodic coefficient. As an immediate consequence we find our bifurcation theorem, being consistent with the well known bifurcation results for the classical Wright equation.

Theorem 15 *The one-parameter family of time-one maps* $F_\gamma : C \to C$ *corresponding to the equation*

$$\dot{z}(t) = -\alpha r(t)(e^{z(t-1)} - 1)$$

undergoes a supercritical bifurcation and a unique invariant curve bifurcates from the equilibrium 0 *as the parameter* α *crosses* $\frac{\pi}{2}$.

Some further questions are discussed in the final part of the dissertation, such as the possible dynamics on the invariant tori, the problem of global existence of invariant tori, the case of higher order equations and various delays and periods.

The dissertation is based on the following publications of the author:

- Röst, G. , *Neimark-Sacker Bifurcation for Periodic Delay Differential Equations*, Nonlinear Analysis Theor., 2005, vol. 60, issue 6, pp. 1025-1044

- Röst, G. , *Some Applications of Bifurcation Formulae to the Period Map of Delay Differential Equations*, in: Dynamical Systems and Applications (eds.: Akca H., Boucherif A. and Covachev V.), GBS Publishers, Delhi, 2005, pp. 624-641

- Röst, G. , *Bifurcation for Periodic Delay Differential Equations at Points of 1:4 Resonance*, to appear in Functional Differential Equations, pp. 1-17

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