

INTEGRABLE MANY-BODY SYSTEMS OF CALOGERO-RUIJSENAARS TYPE

Summary of the Ph.D. thesis

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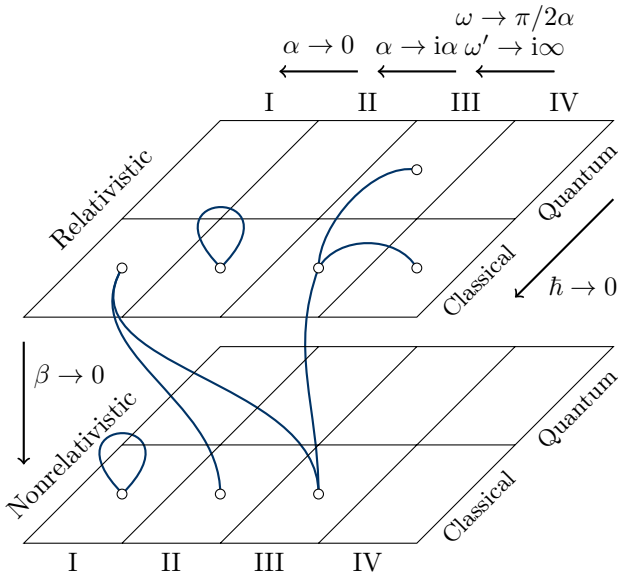


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Introduction

Integrable many-body systems in one spatial dimension form an important class of exactly solvable Hamiltonian systems with their diverse mathematical structure and widespread applicability in physics. Among these models, the systems of Calogero-Ruijsenaars type occupy a central position, due to their intimate relation with soliton theory and since many other interesting models (e.g. Toda lattice) can be obtained from them by taking various limits and analytic continuations. Calogero-Ruijsenaars systems model interacting particles moving on a line or circle. They come in different types called rational (I), hyperbolic (II), trigonometric (III), and elliptic (IV) depending on the functional form of their Hamiltonian. They exist in nonrelativistic and relativistic form, and both at the classical and quantum level. There are other extensions maintaining integrability, such as versions attached to root systems or allowing internal degrees of freedom (spin).



The schematics of Calogero-Ruijsenaars type systems and our work.

Scientific background

Action-angle duality is a relation between two Liouville integrable systems, say (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$, requiring the existence of canonical coordinates (q, p) on M and (\tilde{q}, \tilde{p}) on \tilde{M} (or on dense open submanifolds of M and \tilde{M}) and a global symplectomorphism $\mathcal{R}: M \rightarrow \tilde{M}$, the *action-angle map*, such that $(\tilde{q}, \tilde{p}) \circ \mathcal{R}$ are action-angle variables for the Hamiltonian H and $(q, p) \circ \mathcal{R}^{-1}$ are action-angle variables for the Hamiltonian \tilde{H} . This means that $H \circ \mathcal{R}^{-1}$ depends only on \tilde{q} and $\tilde{H} \circ \mathcal{R}$ only on q . Then one says that the systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ are in *action-angle duality*. In addition, for the systems of our interest it also happens that the Hamiltonian H , when expressed in the coordinates (q, p) , admits interpretation in terms of interacting ‘particles’ with position variables q , and similarly, \tilde{H} expressed in (\tilde{q}, \tilde{p}) describes the interacting points with positions \tilde{q} . Thus q are particle positions for H and action variables for \tilde{H} , and the \tilde{q} are positions for \tilde{H} and actions for H . The significance of this curious property is clear for instance from the fact that it persists at the quantum mechanical level as the bispectral character of the wave functions [2, 20], which are important special functions.

Dual pairs of many-body systems were exhibited by Ruijsenaars in the course of his direct construction [19, 21, 22, 23] of action-angle variables for the many-body systems (of non-elliptic Calogero-Ruijsenaars type and non-periodic Toda type) associated with the root system A_{n-1} . The idea to interpret these dualities in terms of Hamiltonian reduction was put forward in several papers in the 1990s, e.g. [10, 11]. In the last decade or so, Fehér and collaborators undertook the systematic study of dualities within the framework of reduction [6, 5, 1, 7, 8, 4, 9]. It seems natural to expect that action-angle dualities exist for many-body systems associated with other root systems. Substantial evidence in favour of this expectation was given by Pusztai [14, 15, 16, 17, 18].

This thesis presents results (see *Publications*) that were obtained in connection to these earlier developments.

Aims

The aims of the research presented in the thesis can be summarized as follows:

- I. Prove Sklyanin's formula using reduction methods to obtain explicit expressions for the action-angle variables of the rational Calogero-Moser.
- II. Study in detail the action-angle duality for the trigonometric BC_n Sutherland system within the framework of Hamiltonian reduction.
- III. Generalize the results of the previous point and Marshall's earlier work to derive an integrable deformation of the trigonometric BC_n Sutherland model.
- IV. Extend the Lax formalism to hyperbolic Ruijsenaars-Schneider systems with more than one coupling parameters.
- V. Construct new compactified elliptic Ruijsenaars-Schneider models.

The above-mentioned research proposal has been successfully realized, moreover further, originally not anticipated developments have been made.

Applied methods

We applied *Hamiltonian reduction* as well as direct methods to achieve the above-mentioned goals.

In a nutshell, the complicated motion of these many-body systems is derived from a carefully chosen projection of a free particle moving in some higher dimensional space.

The reduction procedure starts with choosing a ‘big phase space’ of group-theoretic origin. This might be, say, the cotangent bundle $P = T^*X$ of a matrix Lie group or algebra X . The natural symplectic structure Ω of the cotangent bundle P permits one to define a Hamiltonian system (P, Ω, \mathcal{H}) by specifying a Hamiltonian $\mathcal{H}: P \rightarrow \mathbb{R}$. If \mathcal{H} is simple enough, then the equations of motion can be solved, or even better, a family of Poisson commuting functions $\{\mathcal{H}_j\}$ be found, which \mathcal{H} is a member/function of. Then by choosing an appropriate group action (of some group G) on X (hence P), under which \mathcal{H}_j are invariant, one can construct the momentum map $\Phi: P \rightarrow \mathfrak{g}^*$ corresponding to this action. Fixing the value μ of the momentum map Φ produces a level surface $\Phi^{-1}(\mu)$ in the ‘big phase space’. This constraint surface is foliated by the orbits of the isotropy/gauge group $G_\mu \subset G$ of the momentum value. The reduced phase space $(P_{\text{red}}, \omega_{\text{red}})$ consists of these orbits. The point is that the flows of the commuting ‘free’ Hamiltonians $\{\mathcal{H}_j\}$ preserve the momentum surface and are constant along orbits. Therefore they admit reduced versions $H_j: P_{\text{red}} \rightarrow \mathbb{R}$, which still Poisson commute and the resulting Hamiltonian system $(P_{\text{red}}, \omega_{\text{red}}, H)$ is Liouville integrable. In practice, we model the reduced phase space by a smooth slice S of the gauge orbits. This slice S is obtained by solving the momentum equation $\Phi = \mu$. Systems in action-angle duality can emerge in this picture if one has two sets of invariant functions and two models S, \tilde{S} of the reduced phase space.

New scientific results

Here we collect the main results of the thesis, going chapter by chapter. In each title, we cite our related contribution.

I. Spectral coordinates of the rational Calogero-Moser system [P6]

- + The canonical variables given by Falqui and Mencattini [3] were identified in terms of the reduction picture.
- + A relation conjectured by Falqui and Mencattini [3] was proved.
- + Sklyanin's formula [24] providing spectral Darboux coordinates for the rational Calogero-Moser system was attained as corollary.

II. Action-angle duality for the trigonometric BC_n Calogero-Moser-Sutherland system [P1, P8, P5]

- + Using Hamiltonian we derived the action-angle dual of the trigonometric BC_n Sutherland system, in which we recognized a real form of the rational BC_n Ruijsenaars-Schneider system.
- + We proved that the coordinates used in the local description of the dual system form a canonical coordinate system [P8].
- + We gave an explicit expression for the Lax matrix of the dual system.
- + A global characterization of the phase space and Lax matrix of the dual model was presented [P1].
- + In terms of action-angle duality, we found the equilibrium configuration of the trigonometric BC_n Sutherland model.

- + As an additional application of action-angle duality, we showed the existence of $(n - 1)$ extra constants of motion in the dual system, hence proving its maximal superintegrability.
- + Finally, we proved that the first integrals of the hyperbolic BC_n Sutherland model in involution that were constructed by Pusztai [16] and the Poisson commuting set of functions found by van Diejen [25] generate the same Abelian algebra [P5]. We established a linear relation between these aforementioned family of functions.

III. A Poisson-Lie deformation of the trigonometric BC_n Calogero-Moser-Sutherland system [P2, P3]

- + Generalising earlier work of Marshall on the hyperbolic case [12], we derived a 1-parameter deformation of the trigonometric BC_n Sutherland system by applying generalised Marsden-Weinstein reduction to the Heisenberg double of the Poisson-Lie group of $2n \times 2n$ unitary matrices with determinant one
- + We solved the momentum equations by observing that previous work of Fehér and Klimčík [7] can be applied to our situation.
- + The main result of this point is the global description of the reduced system [P2]. Consequently, we proved the reduced system to be Liouville integrable.
- + In addition, we showed that our model can be obtained as a certain limit of van Diejen's [25] system containing five couplings. Hence we placed the reduced system into the scheme of Calogero-Ruijsenaars type integrable systems.
- + Finally, we completed Marshall's [12] recent work on the hyperbolic analogue of our system [P3].

IV. Lax representation of the hyperbolic BC_n Ruijsenaars-Schneider-van Diejen system [P7]

- + We proved that our Lax matrix is an element in the Lie group of pseudo unitary matrices with signature (n, n) .
- + Using an earlier result of Puztai [15], we showed that our Lax matrix is positive definite.
- + By relying on Puztai's results on the scattering properties of the system, we verified the equivalence between van Diejen's [25] commuting family of Hamiltonians and the coefficients of the characteristic polynomial of our Lax matrix.
- + Based on this technical result, we inferred that the eigenvalues of the proposed Lax matrix provide a commuting family of first integrals for the Hamiltonian system

V. Trigonometric and elliptic Ruijsenaars-Schneider models on the complex projective space [P4]

- + We examined the the compactified Ruijsenaars-Schneider systems with so-called *type (i)* couplings discovered by Fehér and Kluck [9], and using only direct, elementary methods, we reconstructed the corresponding compactification of the trigonometric systems on the complex projective space.
- + Based on the trigonometric case, we explained that the direct method is applicable to obtain type (i) compactifications of the elliptic Ruijsenaars-Schneider system as well. This new result extends the previous results of Ruijsenaars [20, 23].

Publications

Refereed research papers:

- [P1] L. Fehér and **T.F. Görbe**, *Duality between the trigonometric BC_n Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system*, J. Math. Phys. **55** (2014) 102704; doi:10.1063/1.4898077; arXiv:1407.2057 [math-ph]
- [P2] L. Fehér and **T.F. Görbe**, *On a Poisson-Lie deformation of the BC_n Sutherland system*, Nucl. Phys. B **901** (2015) 85-114; doi:10.1016/j.nuclphysb.2015.10.008; arXiv:1508.04991 [math-ph]
- [P3] L. Fehér and **T.F. Görbe**, *The full phase space of a model in the Calogero-Ruijsenaars family*, J. Geom. Phys. (2016); doi:10.1016/j.geomphys.2016.04.018; arXiv:1603.02877 [math-ph]
- [P4] L. Fehér and **T.F. Görbe**, *Trigonometric and elliptic Ruijsenaars-Schneider systems on the complex projective space*, Lett. Math. Phys. **106** (2016) 1429-1449; doi:10.1007/s11005-016-0877-z; arXiv:1605.09736 [math-ph]
- [P5] **T.F. Görbe** and L. Fehér, *Equivalence of two sets of Hamiltonians associated with the rational BC_n Ruijsenaars-Schneider-van Diejen system*, Phys. Lett. A **379** (2015) 2685-2689; doi:10.1016/j.physleta.2015.08.014; arXiv:1503.01303 [math-ph]
- [P6] **T.F. Görbe**, *A simple proof of Sklyanin's formula for canonical spectral coordinates of the rational Calogero-Moser system*, SIGMA **12** (2016) 027; doi:10.3842/SIGMA.2016.027; arXiv:1601.01181 [math-ph]
- [P7] B.G. Puztai and **T.F. Görbe**, *Lax representation of the hyperbolic van Diejen dynamics with two coupling parameters*, to appear in Commun. Math. Phys. (2017); arXiv:1603.06710 [math-ph]

Research paper in conference proceedings:

- [P8] **T.F. Görbe**, *On the derivation of Darboux form for the action-angle dual of trigonometric BC_n Sutherland system*, J. Phys.: Conf. Ser. **563** (2014) 012012; doi:10.1088/1742-6596/563/1/012012; arXiv:1410.0301 [math-ph]

Additional refereed research paper:

- [P9] V. Ayadi, L. Fehér, and **T.F. Görbe**, *Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals*, J. Geom. Symmetry Phys. **27** (2012) 27-44; doi:10.7546/jgsp-27-2012-27-44; arXiv:1209.1314 [nlin.SI]

Conference posters:

- [P10] **T.F. Görbe**, *The trigonometric BC_n Sutherland system: action-angle duality and applications*, 3-rd Conference on Finite Dimensional Integrable Systems in Geometry and Mathematical Physics 2015 (FDIS2015), 12-17 July 2015, Będlewo (Poland)
- [P11] **T.F. Görbe**, *Lax representation of the hyperbolic van Diejen system with two coupling parameters*, Integrable Systems Conference 2016, 19-24 June 2016, CSF Ascona (Switzerland)

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