Orthogonal polynomials with respect to generalized Jacobi measures

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"All asymptotic formulas have refinements."

Harold Widom

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1 Introduction

One of the first sentences of the seminal paper *Extremal polynomials associated with a system of curves in the complex plane* by Harold Widom is *"All asymptotic formulas have refinements"*. This thesis has been written with a similar mindset. Our aim is to refine, extend and establish asymptotic formulas for orthogonal polynomials with respect to generalized Jacobi measures, i. e. for measures having an algebraic singularity $|x-x_0|^\alpha dx$ around some x_0 in their support. They are the generalizations of the classical Jacobi measure

$$d\mu(x) = (1-x)^\alpha (1+x)^\beta dx, \quad x \in [-1,1]$$

with no restriction made about the support, absolute continuity and the location of the algebraic singularities. In this section first we shall precisely define the mathematical objects of our study, collect the classical results and state our new results.

Let μ be a finite Borel measure supported on the complex plane with infinitely many points in its support and suppose that for all k, we have

$$\int |z|^k d\mu(z) < \infty,$$

that is, all of its moments are finite. Then the polynomials are in $L^2(\mu)$ and using the Gram-Schmidt orthogonalization process, it is easy to see that there is a unique sequence of orthonormal polynomials $\{p_n\}_{n=0}^\infty$ such that

$$p_n(\mu, z) = p_n(z) = \gamma_n z^n + \ldots, \quad \gamma_n > 0.$$

p_n is called the n-th orthonormal polynomial with respect to μ. If we define the so-called Christoffel-Darboux kernel as

$$K_n(z,w) = \sum_{k=0}^{n-1} p_k(z) p_k(w), \tag{1.1}$$

the identity

$$\Pi_{n-1}(x) = \int \Pi_{n-1}(y) K_n(x,y) d\mu(x)$$

holds for every polynomial Π_{n-1} of degree at most $n-1$ for measuers supported on the real line. In this case, $K_n(x,y)$ can be expressed in terms of p_n and p_{n-1} as

$$K_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}, \tag{1.2}$$

where $\gamma_n(\mu) = \gamma_n$ denotes the leading coefficient of p_n. This is called the Christoffel-Darboux formula. Along the real diagonal $x = y$, the Christoffel-Darboux kernel can be written in terms of the so-called Christoffel functions. The n-th Christoffel function with respect to μ is defined as

$$\lambda_n(\mu, z_0) = \inf_{\deg(P_n)<n} \int \frac{|P_n(z)|^2}{|P_n(z_0)|^2} d\mu(z), \tag{1.3}$$

where the infimum is taken for polynomials P_n of degree at most $n-1$ with $|P_n(z_0)| \neq 0$. In other words, $\lambda_n(\mu, z_0)$ is the $(-1/2)$-th power of the norm of the evaluation functional at z_0 defined in $\mathcal{P}_{n-1} \cap L^2(\mu)$, where $\mathcal{P}_n$ denotes the linear subspace of polynomials of degree at most n. It is known that

$$\lambda_n(\mu, z_0) = \frac{1}{\sum_{k=0}^{n-1} |p_k(z_0)|^2}, \tag{1.4}$$

which is very useful, since the Christoffel functions admit to a strong localization principle.

The study of Christoffel functions has started in the beginning of the XXth century, one important early result is due to Gábor Szegő. The theorem of Szegő says that if μ is a measure supported on the unit circle $\mathbb{T}$ which is absolutely continuous with $d\mu(e^{it}) = w(e^{it})dt$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(e^{it}) dt > -\infty$$

holds, which is called Szegő condition, then we have

$$\lim_{n\to\infty} \lambda_n(\mu, z) = (1 - |z|^2) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right] \log w(e^{it}) dt \right), \quad |z| < 1.$$

The influence of this result can be seen in the asymptotic theory of orthogonal polynomials, but it has also served as a motivation to study Hardy spaces. The details can be found in [14] and [15], and for a detailed historical account see [29].

If we study the asymptotics in the points of the unit circle, we have

$$\lim_{n\to\infty} \lambda_n(\mu, z) = \mu(\{z\}), \quad |z| = 1,$$

which is zero, if the measure is absolutely continuous, therefore this does not provide much useful information. The proof of this fact can be found for example at [36, Theorem 2.2.1.] or in [25]. In this case, the main question is to determine the exact order of asymptotics.

A. Máté, P. Nevai and V. Totik proved in the seminal paper [25] that if μ is supported on the unit circle with $d\mu(e^{it}) = w(e^{it})dt + d\mu_s(e^{it})$ there, then if the Szegő condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w(e^{it})dt > -\infty$$

holds, we have

$$\lim_{n\to\infty} n\lambda_n(\mu, e^{it}) = 2\pi w(e^{it})$$

for $t \in [-\pi, \pi)$ almost everywhere. A similar result is known for measures supported on the real line. In the same paper, Máté, Nevai and Totik also proved that it μ is supported on the interval $[-1, 1]$ with $d\mu(x) = w(x)dx + d\mu_s(x)$ there, then if

$$\int_{-1}^{1} \frac{\log w(x)}{\pi\sqrt{1-x^2}}dx > -\infty$$

holds, which is also called Szegő condition, we have

$$\lim_{n\to\infty} n\lambda_n(\mu, x) = \pi\sqrt{1-x^2}w(x)$$

for $x \in [-1, 1]$ almost everywhere.

In this same paper, the authors also studied how the Szegő condition can be weakened. If the Szegő condition is only required for a subinterval $I \subset [-1, 1]$, then global conditions are needed on the measure in order to have similar results. Such a condition is the so-called Stahl-Totik regularity, which plays an important role. A measure μ is said to be *regular in the sense of Stahl and Totik* (or $\mu \in \mathbf{Reg}$ in short), if for every sequence of nonzero polynomials $\{P_n\}_{n=1}^{\infty}$, the estimate

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}}\right)^{1/\deg(P_n)} \le 1 \tag{1.5}$$

holds for all $z \in \mathrm{supp}(\mu) \setminus H$, where H is a set of zero logarithmic capacity. (For the definition of logarithmic capacity, see Section 2.1 below.) Thus Máté, Nevai and Totik proved that if μ is supported on the interval $[-1, 1]$ and regular in the sense of Stahl-Totik, then if the local Szegő condition

$$\int_I \log w(x)dx > -\infty \tag{1.6}$$

holds for some interval $I \subset [-1, 1]$, we have

$$\lim_{n\to\infty} n\lambda_n(\mu, x) = \pi\sqrt{1-x^2}w(x)$$

for $x \in I$ almost everywhere.

For measures supported on a general compact subset of the real line, the above results were extended by Totik in [39] using the polynomial inverse image method developed by him in [42]. He showed that if μ is regular in the sense of Stahl-Totik and it is supported on a compact set $\mathrm{supp}(\mu) = K$ with $d\mu(x) = w(x)dx + d\mu_s(x)$ there, then if the local Szegő condition

$$\int_I \log(w(x))\omega_K(x)dx$$

holds, where ω_K is the density function of the equilibrium measure (see Section 2.1 about the most important potential theoretic concepts), we have

$$\lim_{n\to\infty} n\lambda_n(\mu, x) = \frac{w(w)}{\omega_K(x)}$$

for $x \in I$ almost everywhere. Asymptotics are also established for measures supported on a set of disjoint Jordan arcs and curves. Totik proved in [41] that if μ is supported on a disjoint union of Jordan curves γ lying exterior to each other, then if $z_0 \in \gamma$ and μ is absolutely continuous with respect to the arc length measure s_γ in a small subarc containing z_0 and $d\mu(z) = w(z)ds_\gamma(z)$ there for some continuous and strictly positive weight w, we have

$$\lim_{n\to\infty} n\lambda_n(\mu, z_0) = \frac{w(z_0)}{\omega_\gamma(z_0)}, \tag{1.7}$$

where ω_γ again denotes the Radon-Nikodym derivative of the equilibrium measure with respect to the arc length measure s_γ. Note that although this theorem requires continuity of the weight around z_0, which is stronger than a local Szegő condition, it not only gives an almost everywhere result, it gives the asymptotics at the prescribed point z_0. Totik also managed to prove similar results in [40] when γ contains Jordan arcs and z_0 can be an endpoint.

If, however, $w(z)$ is not continuous or positive at the prescribed point z_0, the asymptotics in (1.7) does not hold anymore. The goal of Section 4 is to generalize this result for measures exhibiting a power-type singularity $d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z)$ around z_0 for some $\alpha > -1$. Our main result in this setting is the following.

Theorem 1.1. *Let γ be a disjoint union of rectifiable Jordan curves lying exterior to each other and let μ be a finite Borel measure regular in the sense of Stahl-Totik with support* $\mathrm{supp}(\mu) = \gamma$. *Suppose that for a $z_0 \in \gamma$, there is an open set U such that $J = U \cap \gamma$*

is a C^2 smooth Jordan arc and μ is absolutely continuous with respect to the arc length measure and

$$d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z), \quad z \in J$$

there for some $\alpha > -1$ and some weight function w which is strictly positive and continuous at z_0. Then

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) \tag{1.8}$$

holds, where $\Gamma(z)$ denotes the Gamma function and ω_γ again denotes the Radon-Nikodym derivative of the equilibrium measure with respect to the arc length measure s_γ.

Note that since $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$, in the special case $\alpha = 0$ the formula (1.8) yields (1.7). Theorem 1.1 is one of the main results of [4]. In there, asymptotics were also established when there can be Jordan arcs present and z_0 can be an endpoint of one of them.

Asymptotics for the Christoffel-Darboux kernel can also be studied off the diagonal, where we no longer have (1.4), therefore Christoffel functions cannot be used directly. One area of interest is the so-called universality limits for random matrices, which is an intensively studied topic of mathematical physics, having several applications even outside mathematics. A detailed account on the various and diverse applications can be found in [11]. For ensembles of $n \times n$ Hermitian random matrices invariant under unitary conjugation, a connection with orthogonal polynomials can be established. If the eigenvalue distribution is given by

$$p(x_1, \dots, x_n) = \frac{1}{Z_n} \prod_{1\le i<j\le n} |x_i - x_j|^2 \prod_{k=1}^{N} w(x_k)dx_k,$$

then the k-point correlation functions defined by

$$R_{k,n}(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int \dots \int p(x_1, \dots, x_n)dx_{k+1}\dots x_n$$

can be expressed as

$$R_{k,n}(x_1, \dots, x_k) = \det\big(\widetilde{K}_n(x_i, x_j)\big)_{i,j=1}^{n}, \tag{1.9}$$

where $\widetilde{K}_n(x,y) = \sqrt{w(x)w(y)}K_n(x,y)$ denotes the normalized Christoffel-Darboux kernel. This was originally shown for Gaussian ensembles by Mehta and Gaudin in [26], but later this technique was developed for more general ensembles, see in particular [7, (4.89)] or for example [6] [8] [9] [31].

Because of (1.9), scaling limits of the type

$$\lim_{n\to\infty} \frac{K_n\big(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\big)}{K_n(x_0, x_0)}, \quad a, b \in \mathbb{R}, \tag{1.10}$$

which are called universality limits, are playing an especially important role in the study of eigenvalue distributions for random matrices. For measures supported on $[-1, 1]$, a new approach for universality limits was developed by D. S. Lubinsky in the seminal papers [19] [20] [21]. In [19] it was shown that if μ is a finite Borel measure supported on $[-1, 1]$ which is regular in the sense of Stahl-Totik (see (1.5)) and absolutely continuous with $d\mu(x) = w(x)dx$ in a neighbourhood of $x_0 \in (-1, 1)$, where $w(x)$ is also continuous and strictly positive, then

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\Big(x_0 + \frac{a}{\widetilde{K}_n(x_0,x_0)}, x_0 + \frac{b}{\widetilde{K}_n(x_0,x_0)}\Big)}{\widetilde{K}_n(x_0, x_0)} = \frac{\sin \pi(b-a)}{\pi(b-a)} \tag{1.11}$$

holds, where $\widetilde{K}_n(x, y) = \sqrt{w(x)w(y)}K_n(x, y)$ denotes the normalized Christoffel-Darboux kernel. Before this result of Lubinsky, analiticity of the weight function was required on the whole support $[-1, 1]$, therefore this was a large step ahead.

An important part of Lubinsky's method is that if one is able to deduce limits of the type (1.10) with $a = b$, then this can be used to obtain (1.10) in general. The analysis was largely based upon Christoffel functions. (1.4) implies that

$$\frac{K_n\big(x_0 + \frac{a}{n}, x_0 + \frac{a}{n}\big)}{K_n(x_0, x_0)} = \frac{\lambda_n(\mu, x_0)}{\lambda_n\big(\mu, x_0 + \frac{a}{n}\big)},$$

holds, therefore this way universality limits can be translated to Christoffel functions. This has proven to be very useful, because Christoffel functions exhibit strong localization properties. Lubinsky's result was simultaneously extended for measures supported on more general subsets of the real line by B. Simon in [35] and by V. Totik in [43], although they used very different methods.

When the measure exhibits singular behavior at the prescribed point x_0, for example it behaves like $|x - x_0|^\alpha dx$ for some $\alpha > -1$, it no longer shows the same behavior and instead of the sinc kernel, something else appears. Generalized Jacobi measures of the form

$$d\mu(x) = (1-x)^\alpha(1+x)^\beta h(x)dx, \quad x \in [-1, 1],$$

where $h(x)$ is positive and analytic, were studied by A. B. J. Kuijlaars and M. Vanlessen in [17]. Using Riemann-Hilbert methods, they showed that

$$\lim_{n\to\infty} \frac{1}{2n^2}\widetilde{K}_n\Big(1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\Big) = \mathbb{J}_\alpha(a,b) \tag{1.12}$$

uniformly for a, b in compact subsets of $(0, \infty)$, where $\mathbb{J}_\alpha(a,b)$ is the so-called Bessel kernel defined as

$$\mathbb{J}_\alpha(a,b) = \frac{J_\alpha(\sqrt{a})\sqrt{b}J'_\alpha(\sqrt{b}) - J_\alpha(\sqrt{b})\sqrt{a}J'_\alpha(\sqrt{a})}{2(a-b)} \tag{1.13}$$

and $J_\alpha(x)$ denotes the Bessel function of the first kind and order α. (Actually, they showed a much stronger result, from which (1.12) follows.) This was extended by Lubinsky in [20]. He proved that if μ is a finite Borel measure supported on $[-1,1]$ which is absolutely continuous on $[1-\varepsilon, 1]$ for some $\varepsilon > 0$ with

$$d\mu(x) = w(x)|x-1|^\alpha, \quad x \in [1-\varepsilon, 1]$$

there, where $w(x)$ is strictly positive and continuous at 1, then

$$\lim_{n\to\infty} \frac{1}{2n^{2\alpha+2}} K_n\Big(1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\Big) = \mathbb{J}^*_\alpha(a,b)$$

holds, where $\mathbb{J}^*_\alpha(a,b) = \frac{\mathbb{J}_\alpha(a,b)}{a^{\alpha/2}b^{\alpha/2}}$ is the entire version of the Bessel kernel. It was also shown by Lubinsky in [21] that if K is a compact subset of the real line and $x_0 \in K$ is a right endpoint of K (i.e. there exists an $\varepsilon > 0$ such that $K \cap (x_0, x_0+\varepsilon) = \varnothing$), then if μ is a finite Borel measure with $\mathrm{supp}(\mu) = K$ which is absolutely continuous in a small left neighbourhood of x_0 with

$$d\mu(x) = |x-x_0|^\alpha dx$$

there for some $\alpha > -1$, then

$$\lim_{n\to\infty} \frac{K_n(x_0 - a\eta_n, x_0 - a\eta_n)}{K_n(x_0,x_0)} = \frac{\mathbb{J}^*_\alpha(a,a)}{\mathbb{J}^*_\alpha(0,0)} \tag{1.14}$$

for all $a \in [0, \infty)$ implies

$$\lim_{n\to\infty} \frac{K_n(x_0 - a\eta_n, x_0 - b\eta_n)}{K_n(x_0,x_0)} = \frac{\mathbb{J}^*_\alpha(a,b)}{\mathbb{J}^*_\alpha(0,0)} \tag{1.15}$$

uniformly for a, b on compact subsets of the complex plane, where the sequence η_n is $\eta_n = (\mathbb{J}^*_\alpha(0,0)/K_n(x_0,x_0))^{1/(\alpha+1)}$. In such a general setting, it was not known if (1.14) holds. Our aim in Sections 5 and 6 is twofold. On the one hand, we will show that (1.14) does indeed hold, hence (1.15) also holds as well. On the other hand, we also aim to establish universality limits in the case when the singularity is located in the interior of the support rather than at the hard edge.

In order to express universality limits for measures exhibiting power-type singularity in the interior of its support, we define the kernel function for $a, b \in \mathbb{R}$ by

$$\mathbb{L}_\alpha(a,b) = \begin{cases} \frac{\sqrt{ab}}{2(a-b)}\Big(J_{\frac{\alpha+1}{2}}(a)J_{\frac{\alpha-1}{2}}(b) - J_{\frac{\alpha+1}{2}}(b)J_{\frac{\alpha-1}{2}}(a)\Big) & \text{if } a,b \geq 0, \\ \frac{\sqrt{a(-b)}}{2(a-b)}\Big(J_{\frac{\alpha+1}{2}}(a)J_{\frac{\alpha-1}{2}}(-b) + J_{\frac{\alpha+1}{2}}(-b)J_{\frac{\alpha-1}{2}}(a)\Big) & \text{if } a \geq 0, b < 0, \\ \mathbb{L}_\alpha(-a,-b) & \text{else,} \end{cases} \tag{1.16}$$

where $J_\nu(x)$ denotes the Bessel functions of the first kind and order ν. Note that

$$\mathbb{L}_\alpha(a,a) = \frac{|a|}{2}\Big(J'_{\frac{\alpha+1}{2}}(|a|)J_{\frac{\alpha-1}{2}}(|a|) - J_{\frac{\alpha+1}{2}}(|a|)J'_{\frac{\alpha-1}{2}}(|a|)\Big).$$

Since $J_\nu(z) = z^\nu G(z)$ where $G(z)$ is an entire function, we can define the entire version of the kernel function for arbitrary complex arguments as

$$\mathbb{L}^*_\alpha(a,b) = \frac{\mathbb{L}_\alpha(a,b)}{a^{\alpha/2}b^{\alpha/2}}, \quad \mathbb{L}^*_\alpha(a) = \frac{\mathbb{L}_\alpha(a,a)}{a^\alpha}, \quad a,b \in \mathbb{C}. \tag{1.17}$$

We emphasize that $\mathbb{L}_\alpha(a,b)$ is defined with different formulas for the cases $ab \geq 0$ and $ab < 0$, and without the normalization in the definition (1.17), this would cause problems. This way however, using that $J_\nu(-z) = (-1)^\nu J_\nu(z)$, we see that in fact the two formulas in (1.16) coincide after normalization.

Our main results in Sections 5 and 6 are the following four theorems. The first two deals with the asymptotics of Christoffel functions when the power type singularity is in the interior (in other words, in the bulk) or at an endpoint (in other words, at the hard edge). The last two theorems are concerned with universality limits in the same cases.

Theorem 1.2. *Let μ be a finite Borel measure which is regular in the sense of Stahl-Totik and suppose that μ is supported on a compact set $K = \operatorname{supp}(\mu)$ on the real line. Let $x_0 \in \operatorname{int}(K)$ be a point from the interior of its support and suppose that on some interval $(x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ containing x_0, the measure μ is absolutely continuous with*

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$$

there for some $\alpha > -1$ and $\alpha \neq 0$, where w is strictly positive and continuous at x_0. Then

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n\Big(\mu, x_0 + \frac{a}{n}\Big) = \frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}}\Big(\mathbb{L}^*_\alpha\big(\pi\omega_K(x_0)a\big)\Big)^{-1} \tag{1.18}$$

holds uniformly for a in compact subsets of the real line, where $\mathbb{L}^*_\alpha(\cdot)$ is defined by (1.17).

The analogue for the edge is the following.

Theorem 1.3. *Let μ be a finite Borel measure which is regular in the sense of Stahl-Totik and suppose that μ is supported on a compact set $K = \operatorname{supp}(\mu)$ on the real line. Let $x_0 \in K$ be a right endpoint of K (i.e. $K \cap (x_0, x_0 + \varepsilon_1) = \varnothing$ for some $\varepsilon_1 > 0$) and assume that on some interval $(x_0 - \varepsilon_0, x_0]$ the measure μ is absolutely continuous with*

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in (x_0 - \varepsilon_0, x_0]$$

there for some $\alpha > -1$, where w is strictly positive and left continuous at x_0. Then

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\Big(\mu, x_0 - \frac{a}{2n^2}\Big) = \frac{w(x_0)}{M(K, x_0)^{2\alpha+2}}\Big(2^{\alpha+1}\mathbb{J}^*_\alpha\big(M(K, x_0)^2 a\big)\Big)^{-1} \tag{1.19}$$

holds uniformly for a in compact subsets of $[0, \infty)$, where $\mathbb{J}^*_\alpha(\cdot)$ is the Bessel kernel defined by (1.13) and $M(K, x_0)$ is defined by

$$M(K, x_0) = \lim_{x\to x_0-} \sqrt{2}\pi|x - x_0|^{1/2}\omega_K(x). \tag{1.20}$$

By symmetry, there is a similar result for left endpoints. Both of these theorems are in agreement with Theorem 1.1 in the case when K is a finite union of intervals and $a = 0$. From the asymptotics for Christoffel functions we obtain universality limits.

Theorem 1.4. *With the assumptions of Theorem 1.2, we have*

$$\lim_{n\to\infty} \frac{K_n\big(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\big)}{K_n(x_0, x_0)} = \frac{\mathbb{L}^*_\alpha(\pi\omega_K(x_0)a, \pi\omega_K(x_0)b)}{\mathbb{L}^*_\alpha(0, 0)} \tag{1.21}$$

uniformly for a, b in compact subsets of the complex plane.

Theorem 1.5. *With the assumptions of Theorem 1.3, we have*

$$\lim_{n\to\infty} \frac{K_n\big(x_0 - \frac{a}{2n^2}, x_0 - \frac{b}{2n^2}\big)}{K_n(x_0, x_0)} = \frac{\mathbb{J}^*_\alpha\big(M(K, x_0)^2 a, M(K, x_0)^2 b\big)}{\mathbb{J}^*_\alpha(0, 0)}. \tag{1.22}$$

uniformly for a, b in compact subsets of the complex plane.

Again by symmetry, there is a similar result for left endpoints.

The proofs of the main results are done in several steps. First, we study the measures μ^b_α and μ^e_α supported on $[-1, 1]$ and defined by

$$d\mu^b_\alpha(x) = |x|^\alpha, \quad x \in [-1, 1]$$

and

$$d\mu_\alpha^e(x) = |x-1|^\alpha, \quad x \in [-1,1].$$

Using the Riemann-Hilbert method, we shall prove (1.18) for μ_α^b and (1.19) for μ_α^e, which shall serve as a model case for our investigations about measures supported on the real line. Then we transform some of the results to obtain (1.8) for the measure $\mu_\alpha^{\mathbb{T}}$ defined by

$$d\mu_\alpha^{\mathbb{T}}(e^{it}) = |e^{it} - i|^\alpha dt, \quad t \in [-\pi, \pi).$$

After this, we shall transform these model cases using the polynomial inverse image method of Totik, to obtain Theorems 1.1, 1.2 and 1.3. Then we set out to prove Theorems 1.4 and 1.5. The latter one is an immediate consequence of Theorem 1.3 using Lubinsky's result [21, Theorem 1.2] which was mentioned earlier, see (1.14) and (1.15). However, this cannot be used to handle the situation where the singularity is located in the interior of the support, therefore we have to build the same machinery. This will be done in Section 6.

2 Mathematical tools

The purpose of this section is to collect the tools used to prove our main results and to provide an overview of the concepts which are important for us.

2.1 Potential theory

To study universality limits and Christoffel functions for measures supported on general compact sets, we shall need a few concepts from logarithmic potential theory, most importantly the concept of equilibrium measures. For a detailed account on logarithmic potential theory, see the books [32] and [34]. If μ is a finite Borel measure supported on the complex plane, its *energy* is defined as

$$I(\mu) = \int\int \log \frac{1}{|z-w|} d\mu(z) d\mu(w).$$

We can define the energy of a *set* $K \subseteq \mathbb{C}$ as the infimum of energies for probability measures supported inside K, i.e.

$$I(K) = \inf_{\mu \in \mathcal{M}_1(K)} I(\mu),$$

where $\mathcal{M}_1(K)$ denotes the set of Borel probability measures with support lying in K. (The quantity $I(K)$ is also known as *Robin's constant.*) The logarithmic capacity of K is defined as

$$\mathrm{cap}(K) = e^{-I(K)}.$$

Sets of zero logarithmic capacity are called *polar sets*. They are playing the role of negligible sets in logarithmic potential theory. If K is a compact subset of the complex plane with nonzero logarithmic capacity, then there is a unique measure denoted by ν_K such that $I(\nu_K) = I(K)$. The measure ν_K is called the *equilibrium measure* for K, and its Radon-Nikodym derivative, if it exists, is denoted by $\omega_K(x)$. For example, we have

$$\omega_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1-x^2}}, \tag{2.1}$$

which is the arcsine distribution.

For a domain $D \subseteq \mathbb{C}_\infty$ which contains a neighbourhood of ∞, the *Green's function* with pole at infinity is the unique function $g_D(\cdot,\infty) : D \to [-\infty,\infty)$ for which
(a) $g_D(z,\infty)$ is harmonic on $D \subseteq \mathbb{C}_\infty$ and bounded outside the neighbourhoods of ∞,
(b) $g_D(z,\infty) = \log|z| + O(1)$ as $z \to \infty$,
(c) $g_D(z,\infty) \to 0$ as $z \to \xi \in \partial D \setminus H$, where H is a set of zero logarithmic capacity.

A compact set K, if Ω denotes the unbounded component of its complement, is said to be *regular with respect to the Dirichlet problem*, if $g_\Omega(z,\infty) \to 0$ as $z \to \xi$ for all $\xi \in \partial\Omega$, i.e. the exceptional set H in property (c) is empty.

Along with local conditions imposed on the measure, (for example the Szegő condition like in [25], continuity of weight function like in [19], or singular behavior of type $|x-x_0|^\alpha dx$ as in our case) some kind of global condition is needed. The so-called Stahl-Totik regularity is such a property. A measure μ is said to be regular in the sense of Stahl-Totik (or $\mu \in \mathbf{Reg}$ in short), if for every sequence of nonzero polynomials $\{P_n\}_{n=0}^\infty$,

$$\limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}} \right)^{1/\deg(P_n)} \leq 1 \tag{2.2}$$

holds for all $z \in \mathrm{supp}(\mu) \setminus H$, where H is a set of zero logarithmic capacity. If $\mathbb{C} \setminus \mathrm{supp}(\mu)$ is regular with respect to the Dirichlet problem, Stahl-Totik regularity is equivalent with the uniform estimate

$$\limsup_{n\to\infty} \left(\frac{\|P_n\|_{\mathrm{supp}(\mu)}}{\|P_n\|_{L^2(\mu)}} \right)^{1/\deg(P_n)} \leq 1. \tag{2.3}$$

There are several criteria for regularity, most notably the Erdős-Turán criterion. In a special case, it says that if μ is a measure supported on the interval $[-1, 1]$ and it is absolutely continuous with $d\mu(x) = w(x)dx$, then "$w(x) > 0$ *almost everywhere on* $[-1, 1]$" implies that μ is regular in the sense of Stahl-Totik. For a detailed account on the **Reg** class for measures, see [37].

2.2 The polynomial inverse image method

The main idea of the polynomial inverse image method, developed by Geronimo and Van Assche in [23] and Totik in [42], is that for some special sets, most frequently the unit interval $[-1, 1]$ or the unit circle $\mathbb{T}$, we have strong tools and special results, however similar tools are unavailable for general sets. For example, Bernstein's inequality for algebraic polynomials says that if P_n is a polynomial of degree n, then

$$|P_n'(x)| \leq n \frac{1}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in (-1, 1).$$

What happens if we replace $[-1, 1]$ with a more general set? It can be difficult to find a proof which works on general sets, but using polynomial inverse images, sometimes the *result* can be transformed. Using this method, Totik proved in [42] that if K is an arbitrary compact subset of the real line, then

$$|P_n'(x)| \leq n\pi\omega_K(x)\|P_n\|_K, \quad x \in \mathrm{int}(K),$$

where ω_K is the equilibrium density of K, which was defined in Section 2.1.

A crucial step in the method is approximating general sets with polynomial inverse images, i.e. sets of the form $T_N^{-1}([-1, 1])$ or $T_N^{-1}(\mathbb{T})$. First we talk about the approximability of unions of Jordan curves. Let $T_N(z)$ be a polynomial of degree N. The set

$$\sigma = \{z \in \mathbb{C} : |T_N(z)| = 1\},$$

which is the level line of T_N, is called a lemniscate. The domain $L = \{z \in \mathbb{C} : |T_N(z)| \leq 1\}$ is called the enclosed lemniscate domain. Assume that σ has no self-intersections. Since the normal derivative of the Green's function with pole at infinity of the outer domain to σ is $|T_N'(z)|/N$ (see [41, (2.2)]) and this normal derivative is 2π times the equilibrium density (see [41, Theorem 3.2]), it follows that the equilibrium density of σ can be written as

$$\omega_\sigma(z) = \frac{|T_N'(z)|}{2\pi N}. \tag{2.4}$$

For every $z \in \sigma$, we introduce the notation $T_N^{-1}(z) = \{z_1, \dots, z_N\}$. Then for every integrable f, we have (see [41, (2.12)])

$$\int_\sigma \left(\sum_{i=1}^{N} f(z_i) \right) |T_N'(z)| ds_\sigma(z) = N \int_\sigma f(z) |T_N'(z)| ds_\sigma(z). \tag{2.5}$$

Furthermore, if $g : \mathbb{T} \to \mathbb{C}$ is arbitrary, we have (see [41, (2.14)])

$$\int_\sigma g(T_N(z)) |T_N'(z)| ds_\sigma(z) = N \int_0^{2\pi} g(e^{it}) dt. \tag{2.6}$$

Lemniscates can be used to approximate unions of Jordan curves in a precise way. The following theorem was proved in [28].

Theorem 2.1. *Let γ consist of finitely many Jordan curves lying exterior to each other, let $z_0 \in \gamma$, and assume that in a neighborhood of z_0 the curve γ is C^2 smooth. Then, for every $\varepsilon > 0$, there is a lemniscate σ_{z_0} consisting of Jordan curves such that σ_{z_0} touches γ at z_0, the lemniscate σ_{z_0} contains γ in its interior except for the point z_0, every component of σ_{z_0} contains in its interior precisely one component of γ, and*

$$\omega_\gamma(z_0) \le \omega_{\sigma_{z_0}}(z_0) + \varepsilon. \tag{2.7}$$

Also, for every $\varepsilon > 0$, there exists another lemniscate σ_{z_0} consisting of Jordan curves such that σ touches γ at z_0, the lemniscate σ_{z_0} lies strictly inside γ except for the point z_0, σ_{z_0} has exactly one component lying inside every component of γ, and

$$\omega_{\sigma_{z_0}}(z_0) \le \omega_\gamma(z_0) + \varepsilon. \tag{2.8}$$

This method also has an analogue for sets on the real line. Let $T_N(x)$ be a polynomial of degree N. T_N is called *admissible* if all of its zeros are real and simple, moreover if $T_N'(x_0) = 0$ for some x_0 (i.e. x_0 is a local extrema for T_N), then $|T_N(x_0)| \ge 1$. The inverse images of the interval $[-1, 1]$ taken with respect to admissible polynomials (sometimes called real lemniscates) enjoy many pleasant approximating properties. If we define $E_N = T_N^{-1}([-1, 1])$, then it is easy to see that $E_N = \cup_{k=0}^{N-1} [a_k, b_k]$, where T_N restricted to $[a_k, b_k]$

is a bijection between it and $[-1,1]$. Moreover, for every integrable function f and for each $k \in \{0,1,\dots,N-1\}$ the formula

$$\begin{aligned}\int_{-1}^{1} f(x)dx &= \int_{a_k}^{b_k} f(T_N(x))|T_N'(x)|dx \\ &= \frac{1}{N}\int_{E_N} f(T_N(x))|T_N'(x)|dx\end{aligned} \tag{2.9}$$

also holds. For example, if P_n is a polynomial of degree N, we have

$$\begin{aligned}\int_{-1}^{1} |P_n(x)|^2|x|^\alpha dx &= \int_{a_k}^{b_k} |P_n(T_N(x))|^2|T_N(x)|^\alpha|T_N'(x)|dx \\ &= \frac{1}{N}\int_{E_N} |P_n(T_N(x))|^2|T_N(x)|^\alpha|T_N'(x)|dx,\end{aligned} \tag{2.10}$$

which will be especially useful to us. In addition, the equilibrium density for E_N is given by the formula

$$\omega_{E_N}(x) = \frac{|T_N'(x)|}{N\pi\sqrt{1-T_N(x)^2}}, \tag{2.11}$$

see [42] for details. (Or use (2.9) and the fact that the equilibrium measure on E_N is the pullback of the equilibrium measure on $[-1,1]$ with respect to the mapping T_N.)

Regarding the approximation properties of polynomial inverse images on the real line, we have the following.

Lemma 2.2. *Let K be a compact set. Suppose that $x_0 \in \mathrm{int}(K)$ is a point in its interior and let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. There exists a set $E_N = \cup_{k=0}^{N-1}[a_k,b_k]$, $b_k \le a_{k+1}$ such that*

(a) $E_N = T_N^{-1}([-1,1])$, where T_N is an admissible polynomial of degree N with $T_N(x_0) = 0$ and $T_N'(x_0) > 0$,

(b) $K \subseteq E_N$,

(c) $\mathrm{dist}(K,E_N) < \varepsilon$, where $\mathrm{dist}(K,E_N)$ denotes the Hausdorff distance of K and E_N,

(d) $\frac{1}{1+\eta}\omega_K(x_0) \le \omega_{E_N}(x_0) \le \omega_K(x_0)$, where $\omega_S(x)$ denotes the equilibrium density of a set S.

Moreover, we have

$$\omega_{E_N}(x_0) = \frac{|T_N'(x_0)|}{N\pi}. \tag{2.12}$$

Proof. Since K is a compact subset, its complement can be obtained as

$$\mathbb{R}\setminus K = (-\infty,a^*) \cup (b^*,\infty) \cup \Big(\bigcup_{k=0}^{\infty}(a_k^*,b_k^*)\Big),$$

where $a^* \leq a_k^*$ and $b_k^* \leq b^*$ for all $k \in \{1, 2, \dots\}$. Hence the set

$$F_m = \mathbb{R} \setminus \Big((-\infty, a^*) \cup (b^*, \infty) \cup \Big(\bigcup_{k=0}^{m-1} (a_k^*, b_k^*) \Big) \Big)$$

is a finite union of intervals. If m is large enough, $\operatorname{dist}(K, F_m) \leq \varepsilon/2$ and, as [43, Lemma 4.2] implies,

$$\left(\frac{1}{1+\eta} \right)^{1/2} \omega_K(x_0) \leq \omega_{F_m}(x_0) \leq \omega_K(x_0)$$

holds. Now [42, Theorem 2.1] gives an admissible polynomial T_N and an inverse image set $E_N = T_N^{-1}([-1,1]) = \cup_{k=0}^{N-1} [a_k, b_k]$ such that the endpoints of E_N are arbitrarily close to the endpoints of F_m. Using Chebyshev polynomials $\mathcal{T}_n(x)$ and replacing $T_N(x)$ with $T_N(\mathcal{T}_n(x))$ and introducing a very small shift if necessary it can be achieved that $T_N(x_0) = 0$. (For details on this idea, see [44].) By multiplying with (-1) if necessary, it can also be achieved that $T_N'(x_0) > 0$, therefore the conditions (a)-(c) are satisfied. If the endpoints of E_N are close enough to the endpoints of F_m, then [43, Lemma 4.2] again implies that E_N satisfies the condition (d). The identity (2.12) is a direct consequence of (2.11). $\square$

If x_0 is an endpoint of our compact set K (i.e., for example, there is a $\varepsilon_0 > 0$ such that $K \cap (x_0, x_0 + \varepsilon_0) = \varnothing$), the previous lemma has an analogue. In this case, the equilibrium density of K is not defined at x_0, but a related quantity takes its place. The behavior of the equilibrium density $\omega_K(x)$ at an endpoint can be quantified as

$$M(K, x_0) = \lim_{x \to x_0 -} \sqrt{2}\pi |x - x_0|^{1/2} \omega_K(x).$$

This quantity is finite and well defined in our case. (The constant $\sqrt{2}\pi$ is usually not incorporated in the definition of $M(K, x_0)$, but we have found it more convenient to do so.) It has appeared several times in the literature, for example it was shown by Totik that this is the asymptotically best possible constant in Markov inequalities for polynomials in several intervals, see [42, Theorem 4.1]. The analogue of Lemma 2.2 is the following.

Lemma 2.3. *Let K be a compact subset of the real line and let $x_0 \in K$ be a point such that $K \cap (x_0, x_0 + \varepsilon_0) = \varnothing$ and $[x_0 - \varepsilon_0, x_0] \subseteq K$ for some $\varepsilon_0 > 0$. Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. There exists a set $E_N = \cup_{k=0}^{N-1} [a_k, b_k]$, $b_k \leq a_{k+1}$ such that*
(a) $E_N = T_N^{-1}([-1,1])$, where T_N is an admissible polynomial of degree N, x_0 is a right endpoint of E_N with $T_N(x_0) = 1$ and $T_N'(x_0) > 0$,

(b) $K \subseteq E_N$,

(c) $\mathrm{dist}(K, E_N) < \varepsilon$, *where* $\mathrm{dist}(K, E_N)$ *denotes the Hausdorff distance of* K *and* E_N,

(d) $\frac{1}{1+\eta} M(K, x_0) \leq M(E_N, x_0) \leq M(K, x_0)$.

Moreover, we have

$$|T'_N(x_0)| = N^2 M(E_N, x_0)^2. \tag{2.13}$$

Proof. The proof of (a)-(d) is almost identical to the proof of Lemma 2.2, except where we select the set $F_m = \cup_{k=0}^{m-1}[a_k^*, b_k^*]$ which is a finite union of intervals containing K, we make sure that x_0 is a right endpoint of F_m. Then we select E_N using [42, Theorem 2.1], again in such a way that x_0 remains a right endpoint of E_N.

To prove (d), first note that the convergence of $\omega_{E_N}(x)$ is locally uniform in $(x_0 - \varepsilon_0, x_0)$ as granted by [43, Lemma 4.2]. Now [27, Lemma 2.1] says that if S is any compact subset of the real line for which x_0 is a right endpoint, then for any $\varepsilon^* > 0$ there is a δ_0 (independent of S) such that

$$|\sqrt{2}\pi\omega_S(x)|x - x_0|^{1/2} - M(S, x_0)| \leq \varepsilon^*, \quad x \in (x_0 - \delta_0, x_0)$$

holds. If we select E_N such that x_0 is still an endpoint and $\mathrm{cap}(E_N)$ is sufficiently close to $\mathrm{cap}(K)$, we have

$$\begin{aligned} |M(K, x_0) - M(E_N, x_0)| &\leq \left| M(K, x_0) - \sqrt{2}\pi\omega_K(x)|x - x_0|^{1/2} \right| \\ &\quad + |\sqrt{2}\pi\omega_K(x) - \sqrt{2}\pi\omega_{E_N}(x)||x - x_0|^{1/2} \\ &\quad + \left| M(E_N, x_0) - \sqrt{2}\pi\omega_{E_N}(x)|x - x_0|^{1/2} \right| \\ &\leq 3\varepsilon^*. \end{aligned}$$

If we fix an $x \in (x_0 - \delta_0, x_0)$ and select a sufficiently small ε^*, (d) follows. Finally, the formula (2.13) is just [42, (4.10)]. $\square$

2.3 Polynomial inequalities

To prove our main results, we often need to compare different norms of the same polynomial or same norm of different polynomials, for which some basic polynomial inequalities will be our aid. We shall collect them in this section without proofs. First we start with a weighted Nikolskii-type inequality.

Lemma 2.4. *Let P_n be a polynomial of degree n and let $\alpha > -1$. Then there exists a constant C_α such that the inequality*

$$\|P_n\|_{[-1,1]} \leq C_\alpha n^{(1+\alpha^*)/2} \left(\int_{-1}^{1} |P_n(x)|^2 |x|^\alpha dx \right)^{1/2}$$

holds with $\alpha^* = \max\{1, \alpha\}$, where $\|P_n\|_{[-1,1]}$ denotes the supremum norm.

The proof of this lemma can be found at [4, Lemma 2.8]. Note that if $\alpha = 0$, this is a special case of the classical Nikolskii-inequality. Nikolskii-type inequalities are also needed on different sets than intervals, for example Jordan arcs and curves.

Lemma 2.5. *Let J be a C^{1+} smooth Jordan arc (i.e. it is $C^{1+\theta}$ smooth for some $\theta > 0$) and let $J^* \subset J$ be a subarc of J which has no common endpoint with J. Let $z_0 \in J$ be a fixed point and define the measure*

$$d\nu_\alpha(z) = w(z)|z - z_0|^\alpha ds_J(z), \quad z \in J,$$

where $\alpha > -1$, the function $w(z)$ is strictly positive and continuous, and $s_J(z)$ denotes the arc length measure with respect to J. Then there is a constant C depending on α, J and J^* such that for any polynomials P_n of degree at most n, we have

$$\|P_n\|_{J^*} \leq C n^{(1+\alpha^*)/2} \|P_n\|_{L^2(\nu_\alpha)},$$

where $\alpha^* = \max\{0, \alpha\}$.

The proof can be found in [4, Lemma 2.7]. Next, a Bernstein type inequality.

Lemma 2.6. *Let J be a C^2 smooth closed Jordan arc and let J_1 be a closed subarc of J not having common endpoint with J. Then for every $D > 0$ there is a constant C_D such that*

$$|P_n'(z)| \leq C_D n \|P_n\|_J, \quad \operatorname{dist}(z, J_1) \leq D/n$$

holds for every polynomial P_n of degree n.

For proof, see [41, Corollary 7.4].

2.4 Fast decreasing polynomials

Recall that the n-th Christoffel function with respect to a measure μ was defined by

$$\lambda_n(\mu, z_0) = \inf_{\deg(P_n)<n} \int \frac{|P_n(z)|^2}{|P_n(z_0)|^2} d\mu(z).$$

By using test polynomials P_n such that P_n is small outside some small neighborhood of z_0, this definition suggests that the asymptotic behavior of the Christoffel functions might be localized. Indeed, as we shall see later this is the case. To use this "localization argument" precisely, we need such fast decreasing polynomials on both Jordan curves and on the real line. In this section we collect the most important constructions used by us.

For compact subsets of the complex plane, we have the following lemmas.

Lemma 2.7. *Let K be a compact set on $\mathbb{C}$, let Ω denote the unbounded component of $\mathbb{C} \setminus K$ and let $z_0 \in \partial\Omega$. Suppose that there is a disk in Ω that contains z_0 on its boundary. Then, for every $\gamma > 1$ there are constants c_γ, C_γ and for every n polynomials $S_{n,z_0,K}$ of degree at most n such that $S_{n,z_0,K}(z_0) = 1$, $|S_{n,z_0,K}(z)| \leq 1$ for all $z \in K$ and*

$$|S_{n,z_0,K}(z)| \leq C_\gamma e^{-nc_\gamma|z-z_0|^\gamma}, \quad z \in K.$$

For proof, see [41, Theorem 4.1]. As an immediate corollary, we have the following, which will be also useful to us.

Lemma 2.8. *With the assumptions of Lemma 2.7, for every $0 < \tau < 1$ there exists positive constants c_τ, C_τ, τ_0 and for every n a polynomial $S_{n,z_0,K}$ of degree $o(n)$ such that $S_{n,z_0,K}(z_0) = 1$, $|S_{n,z_0,K}(z)| \leq 1$ holds for all $z \in K$, and*

$$|S_{n,z_0,K}(z)| \leq C_\tau e^{-c_\tau n^{\tau_0}}, \quad |z - z_0| \geq n^{-\tau}.$$

Proof. Let $\varepsilon > 0$ be sufficiently small and select $\gamma > 1$ such that $1 - \varepsilon - \tau\gamma > 0$. Lemma 2.7 tells us that there is a polynomial $S_{n,z_0,K}$ with $\deg(S_{n,z_0,K}) \leq n^{1-\varepsilon}$ such that

$$|S_{n,z_0,K}(z)| \leq C_\gamma e^{-c_\gamma n^{1-(\varepsilon+\tau\gamma)}}, \quad |z - z_0| \geq n^{-\tau}.$$

This $S_{n,z_0,K}$ satisfies our requirements. $\square$

There is a version of Lemma 2.7 where the decrease is not exponentially small, but starts much earlier than in Lemma 2.7.

Lemma 2.9. *Let K be as in Lemma 2.7. Then for every $\beta < 1$ there are constants $c_\beta, C_\beta > 0$ and for every $n = 1, 2, \ldots$ polynomials $S_{n,z_0,K}$ of degree at most n such that $S_{n,z_0,K}(z_0) = 1$, $|S_{n,z_0,K}(z)| \leq 1$ for all $z \in K$ and*

$$|S_{n,z_0,K}(z)| \leq C_\beta e^{-c_\beta(n|z-z_0|)^\beta}, \quad z \in K.$$

For proof, see [45, Lemma 4]. The situation is somewhat simpler on the real line, because there is no maximum modulus principle there. For example, if we consider the annulus $\{z \in \mathbb{C} : 1 \leq z \leq 2\}$, there are no fast decreasing polynomials for this set in $z_0 = 1$. If K is a compact subset of the real line and $x_0 \in K$, polynomials of the type

$$S_{n,x_0,K}(x) = \left(1 - \left(\frac{x - x_0}{2\operatorname{diam}(K)}\right)^2\right)^{\lfloor \eta n \rfloor},$$

where $\eta > 0$ is an arbitrary number and $\operatorname{diam}(K)$ denotes the diameter of K, will be sufficient for us. This $S_{n,x_0,K}$ is nonnegative, fast decreasing on K, moreover $\deg(S_{n,x_0,K}) = 2\lfloor \eta n \rfloor$, which can be made arbitrarily small (though always $O(n)$) by choosing η accordingly.

2.5 Apriori estimates for Christoffel functions

When establishing asymptotics for $\lambda_n(\mu, x)$, we often do it only for some subsequence n_k. It is enough to study special subsequences from which the asymptotic behavior of the complete sequence is implied. This is summarized in the following lemma, which will be used frequently.

Lemma 2.10. *Let $\{n_k\}_{k=1}^\infty$ be a subsequence of $\mathbb{N}$ such that $n_{k+1}/n_k \to 1$ as $k \to \infty$. Then for every $\kappa > 0$,*

$$\liminf_{k\to\infty} n_k^\kappa \lambda_{n_k}(\mu, x) = \liminf_{n\to\infty} n^\kappa \lambda_n(\mu, x)$$

and

$$\limsup_{k\to\infty} n_k^\kappa \lambda_{n_k}(\mu, x) = \limsup_{n\to\infty} n^\kappa \lambda_n(\mu, x)$$

holds.

Proof. If k is selected such that $n_k \leq n \leq n_{k+1}$, the monotonicity of $\lambda_n(\mu, x)$ in n implies

$$\left(\frac{n}{n_k}\right)^\kappa n_k^\kappa \lambda_{n_k}(\mu, x) \leq n^\kappa \lambda_n(\mu, x) \leq \left(\frac{n}{n_{k+1}}\right)^\kappa n_{k+1}^\kappa \lambda_{n_{k+1}}(\mu, x).$$

Since $n/n_k \to 1$ as $k \to \infty$, this implies what we need to prove. □

Next we prove a simple bound for Christoffel functions on Jordan curves, which will be useful later.

Lemma 2.11. *Let μ be a measure as in Theorem 1.1. Then*

$$\lambda_n(\mu, z_0) \leq C n^{-\alpha-1}.$$

Proof. Let $S_{n,z_0,\gamma}$ be the fast decreasing polynomial given by Lemma 2.9 with $\beta = 1/2$ and $K = \gamma$. Let $\delta > 0$ be so small that in the δ-neighbourhood of z_0 we have $d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z)$. Outside this neighbourhood

$$|S_{n,z_0,\gamma}(z)| \leq C_\beta e^{-c_\beta (n\delta)^{1/2}},$$

therefore

$$\int |S_{n,z_0,\gamma}(z)|^2 d\mu(z) \leq C \int e^{-2c_\beta (n|t|)^{1/2}} |t|^\alpha dt + Ce^{-2c_\beta (n\delta)^{1/2}} \leq Cn^{-\alpha-1},$$

which is what we needed to prove. □

2.6 The Riemann-Hilbert method

The Riemann-Hilbert problem is a boundary value problem for functions analytic on the complex plane except at points of a contour, where the function has a prescribed jump. It is connected with Hilbert's 21th problem regarding proof of the existence of linear differential equations having a prescribed monodromic group. Recently it has been discovered that certain orthogonal polynomials can be characterized as a solution of a 2×2 matrix valued Riemann-Hilbert problem, and since then, it was used to solve several previously untouchable problems. The method was pioneered by P. Deift and X. Zhou in [10]. The purpose of this section is to provide a very brief overview about the use of Riemann-Hilbert method for orthogonal polynomials on $[-1, 1]$. For more details, see [6] and [16].

Let $\gamma \subset \mathbb{C}$ be a disjoint union of oriented arcs and curves on the complex plane and denote γ^o its points except points of self-intersection and endpoints. The orientation defines a positive and negative side of the curves. (For example, if our curve is the unit circle oriented clockwise, the positive side is the outside of the circle, the negative side is the inside of the circle.) Denote the positive and negative side with γ^+ and γ^- respectively. Suppose that $V(z) : \gamma^o \to \mathbb{C}^{2\times 2}$ is a given matrix valued function. We say that $Y(z)$ is a solution of the Riemann-Hilbert problem with respect to γ and $V(z)$ if the following holds.

(MRH1) $Y(z) : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2\times 2}$ is analytic. (That is, its elements are analytic.)

(MRH2) The limits

$$Y_+(z_0) = \lim_{\substack{z\to z_0 \\ z\in\gamma^+}} Y(z), \quad Y_-(z_0) = \lim_{\substack{z\to z_0 \\ z\in\gamma^-}} Y(z)$$

exist and the jump condition $Y_+(z_0) = Y_-(z_0)V(z_0)$ holds for all $z_0 \in \gamma^o$.
(MRH3) $Y(z) \to I$ as $z \to \infty$, where I denotes the 2×2 identity matrix.

If the jump matrix $V(z)$ satisfies some general conditions, the solution exists, however it may not be unique. If the behavior of $Y(z)$ is prescribed around infinity and around the points of $\gamma \setminus \gamma^o$, we may have unicity.

If the contour γ is the interval $[-1, 1]$ with a special jump matrix and the solution is normalized around infinity, $Y(z)$ can be written in terms of orthogonal polynomials. Let $w(x)$ be a function analytic and stictly positive on the interval $[-1, 1]$. We seek a 2×2 matrix valued function $Y(z) : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{2\times 2}$ which satisfies the following.
(OPRH1) $Y(z)$ is analytic on $\mathbb{C} \setminus [-1, 1]$.
(OPRH2) For all $x \in (-1, 1)$ the limits

$$Y_+(x) = \lim_{\substack{z\to x \\ \mathrm{Im}(z)>0}} Y(z), \quad Y_-(x) = \lim_{\substack{z\to x \\ \mathrm{Im}(z)<0}} Y(z)$$

exist and the jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$$

holds.
(OPRH3) Near infinity, the behavior of $Y(z)$ is

$$Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},$$

as $z \to \infty$.
(OPRH4) Near the endpoints -1 and 1, the behavior of $Y(z)$ is

$$Y(z) = \begin{pmatrix} 1 & \log|z - (-1)^k| \\ 1 & \log|z - (-1)^k| \end{pmatrix}$$

as $z \to (-1)^k$ in $\mathbb{C} \setminus [-1, 1]$.

It can be shown that $Y(z)$ can be written in terms of orthogonal polynomials, i. e. we have

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i}\int_{-1}^{1} \frac{\pi_n(x)}{x-z} d\mu_\alpha^b(x) \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_{-1}^{1} \frac{\pi_{n-1}(x)}{x-z} d\mu_\alpha^b(x) \end{pmatrix},$$

where $\pi_n(z)$ denotes the monic orthogonal polynomials with respect to the measure $w(x)dx$ and γ_n denotes the leading coefficient of the n-th orthonormal polynomial $p_n(z)$. This is useful, because by transforming the Riemann-Hilbert problem adequately, we can obtain an asymptotic formula for π_n, which is due to the following theorem, see [16, Theorem 3.1].

Theorem 2.12. *Let γ be a positively oriented simple closed contour and let Ω be an open neighbourhood of γ. Then there exists constants C and δ such that if $Y(z)$ is a solution of the Riemann-Hilbert problem (MRH1) - (MRH3) with a jump matrix V that is analytic on Ω, then*

$$\|Y(z) - I\| \le C\|V - I\|_\Omega$$

holds for every $z \in \mathbb{C} \setminus \gamma$, where the norm is defined by the maximum row sum norm $\|R(z)\| = \max\{|R_{11}(z)| + |R_{12}(z)|, |R_{21}(z)| + |R_{22}(z)|\}$ and $\|R\|_\Omega = \sup_{z\in\Omega} \|R(z)\|$.

In other words, if the jump matrix is small, the solution is also small. Although the jump matrix in (OPRH2) is not small, we can transform the problem into an other one with small jump matrix, therefore obtaining an asymptotic formula like

$$M(z)Y(z)T(z) = I + O(1/n),$$

where $M(z)$ and $T(z)$ are some 2×2 transform matrices.

3 Model cases

Our goal in this section is to prove Theorems 1.1, 1.2 and 1.3 for very special cases. These results then will be used to prove the mentioned theorems in their full generality.

3.1 Measures on [-1,1]

First we study measures supported on the interval $[-1, 1]$. Define the measures μ_α^b and μ_α^e by

$$d\mu_\alpha^b(x) = |x|^\alpha, \quad x \in [-1, 1] \tag{3.1}$$

and

$$d\mu_\alpha^e(x) = |x-1|^\alpha, \quad x \in [-1,1]. \tag{3.2}$$

First we prove (1.21) for μ_α^b. Although $\lambda_n(\mu_\alpha^b, 0)$ was studied in [30] (along with $\lambda_n(\mu_\alpha^e, 1)$), we need to study $\lambda_n(\mu_\alpha^b, a/n)$ for an arbitrary a. To do this, we use the Riemann-Hilbert method. For a brief introduction, see Section 2.6. During this part we follow closely the lines of [17] and [46]. Although the Riemann-Hilbert analysis for generalized Jacobi measures was carried out by M. Vanlessen in [46], it does not cover the asymptotics for the Christoffel-Darboux kernels. First we define a Riemann-Hilbert problem for the 2×2 matrix-valued function $Y(z) : \mathbb{C} \to \mathbb{C}^{2\times 2}$, which can be expressed in terms of the orthogonal polynomials. Then via a series of transformations $Y \mapsto T \mapsto S \mapsto R$ a 2×2 matrix-valued function $R(z)$ can be obtained for which asymptotic behavior is known. These transformations can be unraveled to obtain strong asymptotics for the orthogonal polynomials for μ_α^b which will yield (1.21) in this special case.

Proposition 3.1. *Let μ_α^b be the measure supported on $[-1,1]$ defined as*

$$d\mu_\alpha^b(x) = |x|^\alpha dx, \quad x \in [-1,1],$$

where $\alpha > -1$ and $\alpha \neq 0$. Then for the normalized Christoffel-Darboux kernel,

$$\frac{1}{n}\widetilde{K}_n\left(\frac{a}{n}, \frac{b}{n}\right) = \mathbb{L}_\alpha(a,b) + O\left(\frac{|a|^{\alpha/2}|b|^{\alpha/2}}{n}\right) \tag{3.3}$$

holds uniformly for a, b in bounded subsets of $(-\infty, 0) \cup (0, \infty)$, where $\mathbb{L}_\alpha(a,b)$ is defined by (1.16). Moreover, for the non-normalized Christoffel-Darboux kernel, we have

$$\frac{1}{n^{\alpha+1}} K_n\left(\frac{a}{n}, \frac{b}{n}\right) = \mathbb{L}_\alpha^*(a,b) + O(1/n) \tag{3.4}$$

uniformly for a, b in compact subsets of the real line, where $\mathbb{L}_\alpha^*(a,b)$ is defined by (1.17).

Proof. First we show (3.3) using the Riemann-Hilbert method following the steps of Kuijlaars and Vanlessen [17] and Vanlessen [46], then we show (3.4) by normalizing and appealing to known results for $a = b = 0$. We define the Riemann-Hilbert problem for the 2×2 matrix valued function $Y(z) = (Y_{ij}(z))_{i,j=1}^2$ as in [46]. Suppose that

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus [-1,1]$.

(b) For all $x \in (-1,0) \cup (0,1)$ the limits

$$Y_+(x) = \lim_{\substack{z \to x \\ \mathrm{Im}(z) > 0}} Y(z), \quad Y_-(x) = \lim_{\substack{z \to x \\ \mathrm{Im}(z) < 0}} Y(z)$$

exist and the jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & |x|^\alpha \\ 0 & 1 \end{pmatrix}, \quad x \in (-1,0) \cup (0,1)$$

holds.

(c) For the behavior of $Y(z)$ near infinity we have

$$Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},$$

as $z \to \infty$.

(d) For the behavior of $Y(z)$ near 0 we have

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z|^{\alpha/2} \\ 1 & |z|^{\alpha/2} \end{pmatrix} & \text{if } \alpha < 0 \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha > 0 \end{cases}$$

as $z \to 0$ in $\mathbb{C} \setminus [-1,1]$.

(e) For the behavior of $Y(z)$ near the endpoints $(-1)^k$ for $k = 1, 2$ we have

$$Y(z) = \begin{pmatrix} 1 & \log|z-(-1)^k| \\ 1 & \log|z-(-1)^k| \end{pmatrix}$$

as $z \to (-1)^k$ in $\mathbb{C} \setminus [-1,1]$.

The unique solution for this Riemann-Hilbert problem can be expressed in terms of orthogonal polynomials. If $\pi_n(\mu_\alpha^b, z) = \pi_n(z)$ denotes the monic orthogonal polynomial of degree n with respect to the measure μ_α^b and $\gamma_n(\mu_\alpha^b) = \gamma_n$ denotes the leading coefficient of the orthonormal polynomial $p_n(\mu_\alpha^b, z) = p_n(z)$, then, see [46, Theorem 2.2], $Y(z)$ takes the form

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i}\int_{-1}^{1} \frac{\pi_n(x)}{x-z} d\mu_\alpha^b(x) \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_{-1}^{1} \frac{\pi_{n-1}(x)}{x-z} d\mu_\alpha^b(x) \end{pmatrix}. \tag{3.5}$$

To give an asymptotic formula for $Y(z)$, we need to introduce some special functions. Let $w(z)$ be an analytic continuation of the function $|x|^\alpha$ to the two half-planes defined by

$$w(z) = \begin{cases} (-z)^\alpha & \text{if } \operatorname{Re}(z) < 0, \\ z^\alpha & \text{if } \operatorname{Re}(z) > 0. \end{cases} \tag{3.6}$$

as in [46, (3.4)], and define the function $W(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ similarly by

$$W(z) = \begin{cases} z^{\alpha/2} & \text{if } \operatorname{Re}(z) < 0, \\ (-z)^{\alpha/2} & \text{if } \operatorname{Re}(z) > 0 \end{cases} \tag{3.7}$$

so that overall, we have

$$W^2(z) = \begin{cases} w(z)e^{-i\pi\alpha} & \text{if } \operatorname{Re}(z)\operatorname{Im}(z) \geq 0, \\ w(z)e^{i\pi\alpha} & \text{if } \operatorname{Re}(z)\operatorname{Im}(z) < 0. \end{cases} \tag{3.8}$$

The function $\varphi(z) = z + \sqrt{z^2 - 1}$ denotes the conformal mapping of $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit circle and the auxiliary function $f(z)$ is defined by

$$f(z) = \begin{cases} i \log \varphi(z) - i \log \varphi_+(0) & \text{if } \operatorname{Im}(z) > 0, \\ -i \log \varphi(z) - i \log \varphi_+(0) & \text{if } \operatorname{Im}(z) < 0, \end{cases} \tag{3.9}$$

where $\varphi_+(0) = \lim_{z\to 0, \operatorname{Im}(z)>0} \varphi(z) = i$. Now we divide the complex plane into eight congruent octants defined by

$$O_i = \left\{ z : \frac{(k-1)\pi}{4} \leq \arg(z) \leq \frac{k\pi}{4} \right\}, \quad i = 1, 2, \ldots, 8.$$

Define a 2×2 matrix valued function $\psi(z)$ in the first and fourth octants O_1 and O_4 by

$$\psi(z) = \begin{cases} \frac{1}{2}\sqrt{\pi} z^{1/2} \begin{pmatrix} e^{-i\frac{2\alpha+1}{4}\pi} H^{(2)}_{\frac{\alpha+1}{2}}(z) & -ie^{i\frac{2\alpha+1}{4}\pi} H^{(1)}_{\frac{\alpha+1}{2}}(z) \\ e^{-i\frac{2\alpha+1}{4}\pi} H^{(2)}_{\frac{\alpha-1}{2}}(z) & -ie^{i\frac{2\alpha+1}{4}\pi} H^{(1)}_{\frac{\alpha-1}{2}}(z) \end{pmatrix} & z \in O_1, \\ \frac{1}{2}\sqrt{\pi}(-z)^{1/2} \begin{pmatrix} ie^{i\frac{2\alpha+1}{4}\pi} H^{(1)}_{\frac{\alpha+1}{2}}(-z) & -e^{-i\frac{2\alpha+1}{4}\pi} H^{(2)}_{\frac{\alpha+1}{2}}(-z) \\ -ie^{i\frac{2\alpha+1}{4}\pi} H^{(1)}_{\frac{\alpha-1}{2}}(-z) & e^{-i\frac{2\alpha+1}{4}\pi} H^{(2)}_{\frac{\alpha-1}{2}}(-z) \end{pmatrix} & z \in O_4, \end{cases} \tag{3.10}$$

where $H^{(1)}_\gamma$ and $H^{(2)}_\gamma$ denotes the Hankel functions of the first and second kind of order γ. For more on the Hankel functions, see [1, 9.1.3, 9.1.4]. The 2×2 matrix σ_3 denotes the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and for a function $h(z)$ the symbol $h(z)^{\sigma_3}$ denotes

$$h(z)^{\sigma_3} = \begin{pmatrix} h(z) & 0 \\ 0 & h(z)^{-1} \end{pmatrix}.$$

The definition of $\psi(z)$ can be extended to the whole complex plane, but to avoid complications, we shall define it only on O_1 and O_4, because this will be sufficient for our purposes. For the complete definition, see [46, Section 4.2]. The function $N(z)$, which is a solution of a so-called model Riemann-Hilbert problem, is defined as

$$N(z) = D_\infty^{\sigma_3} \begin{pmatrix} \frac{a(z)+a(z)^{-1}}{2} & \frac{a(z)-a(z)^{-1}}{2i} \\ \frac{a(z)-a(z)^{-1}}{-2i} & \frac{a(z)+a(z)^{-1}}{2} \end{pmatrix} D(z)^{\sigma_3}, \tag{3.11}$$

where $a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}$ and $D(z)$ is the Szegő function (associated to the measure μ_α^b), which is given by

$$D(z) = \frac{z^{\alpha/2}}{\varphi(z)^{\alpha/2}}$$

and $D_\infty = \lim_{z\to\infty} D(z)$. Finally we define the auxiliary function $E_n(z)$ on the first and fourth octant O_1 and O_4 as

$$E_n(z) = \begin{cases} N(z)W(z)^{\sigma_3} e^{\frac{\alpha}{4}\pi i\sigma_3} i^{n\sigma_3} e^{-\frac{\pi}{4}i\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} & z \in O_1, \\ N(z)W(z)^{\sigma_3} e^{-\frac{\alpha}{4}\pi i\sigma_3} i^{n\sigma_3} e^{-\frac{\pi}{4}i\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} & z \in O_4. \end{cases} \tag{3.12}$$

$E_n(z)$ can also be defined on the whole complex plane, but we shall only work on the first and fourth octant. For the complete definition see [46, Section 4.3].

In order to prove (3.3), we shall need two lemmas which are the analogues of [17, Lemmas 3.3 and 3.5].

Lemma 3.2. *For every $x \in (0, \delta)$, where δ is small enough, the first column of $Y(z)$ given by (3.5) takes the form*

$$\begin{aligned} \begin{pmatrix} Y_{11}(x) \\ Y_{21}(x) \end{pmatrix} &= \sqrt{\frac{\pi n}{w(x)}} 2^{-n\sigma_3} M_+(x)(\pi/2 - \arccos x)^{1/2} \\ &\qquad \times \begin{pmatrix} e^{-i\frac{\pi}{4}} J_{\frac{\alpha+1}{2}}\big(n(\pi/2 - \arccos x)\big) \\ e^{-i\frac{\pi}{4}} J_{\frac{\alpha-1}{2}}\big(n(\pi/2 - \arccos x)\big) \end{pmatrix}, \end{aligned} \tag{3.13}$$

and for every $x \in (-\delta, 0)$, it takes the form

$$\begin{pmatrix} Y_{11}(x) \\ Y_{21}(x) \end{pmatrix} = \sqrt{\frac{\pi n}{w(x)}} 2^{-n\sigma_3} M_+(x)(\arccos x - \pi/2)^{1/2} \\ \times \begin{pmatrix} -e^{-i\frac{\pi}{4}} J_{\frac{\alpha+1}{2}}\left(n(\arccos x - \pi/2)\right) \\ e^{-i\frac{\pi}{4}} J_{\frac{\alpha-1}{2}}\left(n(\arccos x - \pi/2)\right) \end{pmatrix}, \tag{3.14}$$

where $M(z)$ denotes

$$M(z) = R(z)E_n(z) \tag{3.15}$$

and $M_+(x) = \lim_{z \to x, \mathrm{Im}(z)>0} M(z)$. Moreover, $\det M(z) \equiv 1$, $M(z)$ is analytic in $O_1 \cup O_4$, and also $M(z)$ and $\frac{d}{dz}M(z)$ are uniformly bounded in $(O_1 \cup O_4) \cap \{z : |z| \leq \delta\}$.

Proof. Unraveling the series of transformations $Y \mapsto T \mapsto S \mapsto R$ described in [46] it can be obtained that in the first and fourth octant O_1, O_4 and near the origin, $Y(z)$ takes the form

$$Y(z) = 2^{-n\sigma_3} R(z)E_n(z)\psi(nf(z))W(z)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{1}{w(z)} & 1 \end{pmatrix}, \tag{3.16}$$

where $R(z)$ is analytic and $R(z) = I + O(1/n)$ uniformly in a small neighbourhood of the origin. From now on, we have to distinguish between the cases $x \in (0, \delta)$ and $x \in (-\delta, 0)$.

First case: $x \in (0, \delta)$. To obtain (3.13), we work in O_1 and let $z \to x$ in there. Since (3.8) gives that $W(z) = w^{1/2}(z)e^{-i\pi\alpha/2}$ for $z \in O_1$,

$$W(z)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{1}{w(z)} & 1 \end{pmatrix} = \begin{pmatrix} w(z)^{-1/2}e^{i\pi\alpha/2} & 0 \\ w(z)^{-1/2}e^{-i\pi\alpha/2} & w(z)^{1/2}e^{-i\pi\alpha/2} \end{pmatrix}.$$

Combining this with (3.16) we obtain that for $z \in O_1$,

$$\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = w(z)^{-1/2} 2^{-n\sigma_3} R(z)E_n(z)\psi(nf(z)) \begin{pmatrix} e^{i\pi\alpha/2} \\ e^{-i\pi\alpha/2} \end{pmatrix}$$

holds. Now we aim to express $\psi(nf(z)) \begin{pmatrix} e^{i\pi\alpha/2} \\ e^{-i\pi\alpha/2} \end{pmatrix}$ in terms of Bessel functions. Since $H_\nu^{(1)}(z) + H_\nu^{(2)}(z) = 2J_\nu(z)$, see [1, 9.1.2 and 9.1.3], we have

$$\psi(z) \begin{pmatrix} e^{i\pi\alpha/2} \\ e^{-i\pi\alpha/2} \end{pmatrix} = \sqrt{\pi} z^{1/2} \begin{pmatrix} e^{-i\pi/4} J_{\frac{\alpha+1}{2}}(z) \\ e^{-i\pi/4} J_{\frac{\alpha-1}{2}}(z) \end{pmatrix},$$

which gives

$$\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = w(z)^{-1/2}\sqrt{\pi}2^{-n\sigma_3}R(z)E_n(z)(nf(z))^{1/2}\begin{pmatrix} e^{-i\pi/4}J_{\frac{\alpha+1}{2}}(nf(z)) \\ e^{-i\pi/4}J_{\frac{\alpha-1}{2}}(nf(z)) \end{pmatrix}.$$

For the $f(z)$ defined by (3.9) we have $f_+(x) = \pi/2 - \arccos x$, therefore the above identity gives (3.13). Since $\det R(z) \equiv 1$ (use that $\det R(z)$ is analytic and $R(z) = I + O(1/z)$ around infinity, see [46, Section 3.3]), it follows easily from (3.11) that $\det M(z) \equiv 1$. Moreover, $M(z)$ is analytic in the octant O_1 near the origin. The boundedness of $R(z)$ is implied by [46, (3.30)], and since it is actually analytic in some small neighbourhood of 0, the Cauchy integral formula gives that $\frac{d}{dz}R(z)$ is also bounded in some small disk with center at the origin. The same can be said about $E_n(z)$, see [46, Proposition 4.5], which implies that $M(z) = R(z)E_n(z)$ and $\frac{d}{dz}M(z)$ is uniformly bounded in a small disk around 0 as $n \to \infty$.

Second case: $x \in (-\delta, 0)$. Calculating similarly as in the first case, we obtain that for $z \in O_4$, we have

$$\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = w(z)^{-1/2}2^{-n\sigma_3}R(z)E_n(z)\psi(nf(z))\begin{pmatrix} e^{-i\pi\alpha/2} \\ e^{i\pi\alpha/2} \end{pmatrix}.$$

We also get

$$\psi(z)\begin{pmatrix} e^{-i\pi\alpha/2} \\ e^{i\pi\alpha/2} \end{pmatrix} = \sqrt{\pi}(-z)^{1/2}\begin{pmatrix} -e^{-i\frac{\pi}{4}}J_{\frac{\alpha+1}{2}}(-z) \\ e^{-i\frac{\pi}{4}}J_{\frac{\alpha-1}{2}}(-z) \end{pmatrix},$$

which gives

$$\begin{pmatrix} Y_{11}(z) \\ Y_{12}(z) \end{pmatrix} = w(z)^{-1/2}\sqrt{\pi}2^{-n\sigma_3}R(z)E_n(z)(-nf(z))^{1/2}\begin{pmatrix} -e^{-i\frac{\pi}{4}}J_{\frac{\alpha+1}{2}}(-nf(z)) \\ e^{-i\frac{\pi}{4}}J_{\frac{\alpha-1}{2}}(-nf(z)). \end{pmatrix}$$

Similarly as in the first case, letting $z \to x$ through O_4 yields (3.14). □

The next lemma, which is the analogue of [17, Lemma 3.5], studies the asymptotic behavior of $J_\nu\big(n\big(\frac{\pi}{2} - \arccos\frac{a}{n}\big)\big)$.

Lemma 3.3. *Let $a \in \mathbb{R} \setminus \{0\}$. Define $a_n = a/n$ and $\tilde{a}_n = n\big(\frac{\pi}{2} - \arccos|a_n|\big)$. Then*

$$\tilde{a}_n = |a| + O\bigg(\frac{|a|^3}{n^2}\bigg) \tag{3.17}$$

and

$$J_\alpha(\tilde{a}_n) = J_\alpha(|a|) + O\bigg(\frac{|a|^{2+\alpha}}{n^2}\bigg) \tag{3.18}$$

holds.

Proof. Without the loss of generality we can assume that $a > 0$. Since $\pi/2 - \arccos x = x + O(x^3)$ (just use the Taylor expansion of $\pi/2 - \arccos x = \arcsin x$), the asymptotic formula (3.17) easily follows. As for the behavior of the Bessel functions, [1, 9.1.10] says that $J_\alpha(z) = z^\alpha G(z)$, where $G(z)$ is an entire function. It follows that

$$\begin{aligned} J_\alpha(\tilde{a}_n) &= \left(a + O\left(\frac{a^3}{n^2}\right)\right)^\alpha \left(G(a) + O\left(\frac{a^3}{n^2}\right)\right) \\ &= a^\alpha \left(1 + O\left(\frac{a^2}{n^2}\right)\right)\left(G(a) + O\left(\frac{a^3}{n^2}\right)\right) \\ &= J_\alpha(a) + O\left(\frac{a^{2+\alpha}}{n^2}\right), \end{aligned}$$

which is what we needed to show. $\square$

To show (3.3), we shall distinguish between the four cases (i) $a, b \geq 0$, (ii) $a \geq 0, b < 0$, (iii) $a < 0, b \geq 0$, (iv) $a, b < 0$. Because of the symmetry of μ_α^b, we have $K_n(x, y) = K_n(-x, -y)$, therefore it is enough to deal with the first two cases.

First case: $a, b \geq 0$. Let $a, b \in (0, \infty)$ and define $a_n = a/n$, $b_n = b/n$, moreover let $\tilde{a}_n = n\left(\frac{\pi}{2} - \arccos a_n\right)$, $\tilde{b}_n = n\left(\frac{\pi}{2} - \arccos b_n\right)$. First we shall express $\widetilde{K}_n(x, y)$ in terms of $Y_{11}(x)$ and $Y_{21}(x)$. (3.5) gives that $Y_{11}(x) = \frac{1}{\gamma_n} p_n(x)$ and $Y_{21}(x) = -2\pi i \gamma_{n-1} p_{n-1}(x)$. Using the Christoffel-Darboux formula (1.2) with Lemma 3.2 yields

$$\begin{aligned} \frac{1}{n}\widetilde{K}_n(a_n, b_n) &= \frac{1}{2\pi i(b-a)}\sqrt{w(a_n)w(b_n)}\Big(Y_{11}(a_n)Y_{21}(b_n) - Y_{11}(b_n)Y_{21}(a_n)\Big) \\ &= \frac{1}{2\pi i(b-a)}\sqrt{w(a_n)w(b_n)}\det\begin{pmatrix} Y_{11}(a_n) & Y_{11}(b_n) \\ Y_{21}(a_n) & Y_{21}(b_n)\end{pmatrix} \\ &= \frac{1}{2\pi i(b-a)}\det\begin{pmatrix} \sqrt{w(a_n)}Y_{11}(a_n) & \sqrt{w(b_n)}Y_{11}(b_n) \\ \sqrt{w(a_n)}Y_{21}(a_n) & \sqrt{w(b_n)}Y_{21}(b_n)\end{pmatrix} \\ &= \frac{n}{2(a-b)}\det\Bigg[M_+(a_n)(\tilde{a}_n/n)^{1/2}\begin{pmatrix} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) & 0 \\ J_{\frac{\alpha-1}{2}}(\tilde{a}_n) & 0\end{pmatrix} \\ &\qquad + M_+(b_n)(\tilde{b}_n/n)^{1/2}\begin{pmatrix} 0 & J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ 0 & J_{\frac{\alpha-1}{2}}(\tilde{b}_n)\end{pmatrix}\Bigg] \end{aligned}$$

$$= \frac{n}{2(a-b)} \det\Bigg[M_+(b_n) \Bigg\{ \begin{pmatrix} (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) & (\tilde{b}_n/n)^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{a}_n) & (\tilde{b}_n/n)^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{b}_n) \end{pmatrix}$$

$$+ M_+(b_n)^{-1}(M_+(a_n) - M_+(b_n))$$

$$\times \begin{pmatrix} (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) & 0 \\ (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{a}_n) & 0 \end{pmatrix} \Bigg\} \Bigg].$$

Since $M(z)$ is uniformly bounded and its determinant is 1 (see Lemma 3.2), $M(z)^{-1}$ is also uniformly bounded, moreover the uniform boundedness of $\frac{d}{dz}M(z)$ imply that $M_+(a_n) - M_+(b_n) = O(\frac{a-b}{n})$. We also have $J_\alpha(\tilde{a}_n) = O(a^\alpha)$ (use (3.18) and the fact that $J_\alpha(z) = z^\alpha G(z)$, where $G(z)$ is an entire function), which gives

$$M_+(b_n)^{-1}(M_+(a_n) - M_+(b_n)) \begin{pmatrix} (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) & 0 \\ (\tilde{a}_n/n)^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{a}_n) & 0 \end{pmatrix} = \begin{pmatrix} O\big(\frac{a-b}{n^{3/2}} a^{\frac{\alpha+2}{2}}\big) & 0 \\ O\big(\frac{a-b}{n^{3/2}} a^{\frac{\alpha}{2}}\big) & 0 \end{pmatrix}.$$

From these and $\det M(z) \equiv 1$ it follows that

$$\begin{aligned} \frac{1}{n}\widetilde{K}_n(a_n, b_n) &= \frac{1}{2(a-b)} \det \begin{pmatrix} \tilde{a}_n^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) + O\big(\frac{a-b}{n} a^{\frac{\alpha+2}{2}}\big) & \tilde{b}_n^{1/2} J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ \tilde{a}_n^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{a}_n) + O\big(\frac{a-b}{n} a^{\frac{\alpha}{2}}\big) & \tilde{b}_n^{1/2} J_{\frac{\alpha-1}{2}}(\tilde{b}_n) \end{pmatrix} \\ &= \frac{\tilde{a}_n^{1/2} \tilde{b}_n^{1/2}}{2(a-b)} \det \begin{pmatrix} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) & J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ J_{\frac{\alpha-1}{2}}(\tilde{a}_n) & J_{\frac{\alpha-1}{2}}(\tilde{b}_n) \end{pmatrix} + O\bigg(\frac{a^{\alpha/2} b^{\alpha/2}}{n}\bigg) \\ &= \frac{\tilde{a}_n^{1/2} \tilde{b}_n^{1/2} a^{\frac{\alpha-1}{2}} b^{\frac{\alpha-1}{2}}}{2(a-b)} \det \begin{pmatrix} a^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) - b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(\tilde{b}_n) & b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ a^{-\frac{\alpha-1}{2}} J_{\frac{\alpha-1}{2}}(\tilde{a}_n) - b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha-1}{2}}(\tilde{b}_n) & b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha-1}{2}}(\tilde{b}_n) \end{pmatrix} \\ &\qquad + O\bigg(\frac{a^{\alpha/2} b^{\alpha/2}}{n}\bigg), \end{aligned} \tag{3.19}$$

where the error term is uniform for a, b on bounded subsets of $(0, \infty)$, even intervals of the form $(0, c]$. Lemma 3.3 gives

$$a^{-\frac{\alpha-1}{2}} \big(J_{\frac{\alpha+1}{2}}(\tilde{a}_n) - J_{\frac{\alpha+1}{2}}(a)\big) = O\bigg(\frac{a^3}{n^2}\bigg)$$

and

$$a^{-\frac{\alpha-1}{2}} \big(J_{\frac{\alpha-1}{2}}(\tilde{a}_n) - J_{\frac{\alpha-1}{2}}(a)\big) = O\bigg(\frac{a^2}{n^2}\bigg),$$

which imply

$$\begin{aligned} &a^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(\tilde{a}_n) - b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(\tilde{b}_n) \\ &\qquad = a^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(a) - b^{-\frac{\alpha-1}{2}} J_{\frac{\alpha+1}{2}}(b) + O\bigg(\frac{a^3 - b^3}{n^2}\bigg) \end{aligned}$$

and

$$a^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(\tilde{a}_n)-b^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(\tilde{b}_n)$$
$$=a^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(a)-b^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(b)+O\Big(\frac{a^2-b^2}{n^2}\Big).$$

Continuing (3.19) with these, we have

$$\begin{aligned}\frac{1}{n}\widetilde{K}_n(a_n,b_n)=&\frac{\tilde{a}_n^{1/2}\tilde{b}_n^{1/2}}{2(a-b)}\det\begin{pmatrix}J_{\frac{\alpha+1}{2}}(a) & J_{\frac{\alpha+1}{2}}(b)\\ J_{\frac{\alpha-1}{2}}(a) & J_{\frac{\alpha-1}{2}}(b)\end{pmatrix}\\ &+\frac{\tilde{a}_n^{1/2}\tilde{b}_n^{1/2}a^{\frac{\alpha-1}{2}}b^{\frac{\alpha-1}{2}}}{2(a-b)}\det\begin{pmatrix}a^{-\frac{\alpha-1}{2}}J_{\frac{\alpha+1}{2}}(a)-b^{-\frac{\alpha-1}{2}}J_{\frac{\alpha+1}{2}}(b) & O\big(\frac{b^2}{n^2}\big)\\ a^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(a)-b^{-\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(b) & O\big(\frac{b^2}{n^2}\big)\end{pmatrix}\\ &+O\Big(\frac{a^{\alpha/2}b^{\alpha/2}}{n}\Big)\end{aligned}$$

In the second term, since $J_\nu(z)=z^\nu G(z)$ where $G(z)$ is an entire function, by the mean value theorem we have

$$\frac{a^{-\frac{\alpha-1}{2}}J_{\frac{\alpha\pm1}{2}}(a)-b^{-\frac{\alpha-1}{2}}J_{\frac{\alpha\pm1}{2}}(b)}{a-b}=O(1), \tag{3.20}$$

hence the second term is $O\big(\frac{a^{\alpha/2}b^{\alpha/2}}{n}\big)$. In the first term $\tilde{a}_n=a+O\big(\frac{a^3}{n^2}\big)$ and $\tilde{b}_n=b+O\big(\frac{b^3}{n^2}\big)$ can be replaced with a and b respectively, because the resulting error terms can be absorbed into the previous error term. Overall, we have

$$\frac{1}{n}\widetilde{K}_n\Big(\frac{a}{n},\frac{b}{n}\Big)=\mathbb{L}_\alpha(a,b)+O\Big(\frac{a^{\alpha/2}b^{\alpha/2}}{n}\Big),$$

which is uniform for $a,b\in(0,\infty)$ in bounded sets, even when $a-b$ is small.

Second case: $a\geq 0, b<0$. Let $a\geq 0, b<0$ and define $a_n=a/n$, $b_n=b/n$, moreover let $\tilde{a}_n=n\big(\frac{\pi}{2}-\arccos a_n\big)$, $\tilde{b}_n=n\big(\arccos b_n-\frac{\pi}{2}\big)=n\big(\frac{\pi}{n}-\arccos|b_n|\big)$. With similar calculations as before but with (3.14) instead of (3.13) we obtain (3.20). (Note that the definition of $\mathbb{L}_\alpha(a,b)$ differs when $a\geq 0$ and $b<0$, see (1.16).)

By normalizing (3.3) with $a^{\alpha/2}b^{\alpha/2}$, we obtain (3.4) uniformly for a,b in bounded subsets of $\mathbb{R}\setminus\{0\}$. Using [4, Theorem 1.1] we see that this actually holds uniformly for a,b in compact subsets of $\mathbb{R}$, and this is what we had to show. □

By letting $b\to a$ in (3.4) we obtain the formula

$$\frac{1}{n^{\alpha+1}}K_n\Big(\frac{a}{n},\frac{a}{n}\Big)=\mathbb{L}_\alpha^*(a)+O(1/n),$$

which can be written in terms of Christoffel functions as

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\alpha^b, a/n) = \Big(\mathbb{L}_\alpha^*(a)\Big)^{-1}. \tag{3.21}$$

By using that $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}(\frac{x}{2})^{2n+\nu}$, we obtain by letting a to 0 in (3.21) that

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\alpha^b, 0) = 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big). \tag{3.22}$$

This last identity can also be established by considering (3.23) for $a = 0$ and using the transformation $x \mapsto 2x^2 - 1$.

To establish the model case when the singularity is at an endpoint of the support, we shall study the measure μ_α^e defined by (3.2). For this, we have the following.

Proposition 3.4. *Let μ_α^b be the measure supported on $[-1, 1]$ defined as*

$$d\mu_\alpha^e(x) = |x-1|^\alpha dx, \quad x \in [-1, 1],$$

where $\alpha > -1$. Then

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\left(\mu_\alpha^e, 1 - \frac{a}{2n^2}\right) = \Big(2^{\alpha+1}\mathbb{J}_\alpha^*(a)\Big)^{-1} \tag{3.23}$$

holds.

Since classical Jacobi measures are very well studied, this result along with much stronger results were already known, see [17]. (3.23) is a special case of [17, Theorem 1.3 (c)].

3.2 Measures on the unit circle

To prove Theorem 1.1, first we shall study a special measure on the unit circle. We will use the transformation $e^{it} \mapsto \cos t$ to establish asymptotics of the Christoffel function for the pullback measure of μ_α^b which was defined by (3.1), then we use this to study the measure

$$d\mu_\alpha^{\mathbb{T}}(e^{it}) = |e^{it} - i|^\alpha dt,$$

which will be our model case.

Proposition 3.5. *Let $\mu_\alpha^{\mathbb{T}}$ be the measure supported on the unit circle $\mathbb{T}$ defined as*

$$d\mu_\alpha^{\mathbb{T}}(e^{it}) = |e^{it} - i|^\alpha dt, \quad t \in [-\pi, \pi), \tag{3.24}$$

where $\alpha > -1$. Then we have

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2}) = 2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big). \tag{3.25}$$

Proof. First we prove a result analogous to (3.25) for the pullback measure of μ_α^b defined by (3.1). Let $\eta_\alpha^{\mathbb{T}}$ be the measure on the unit circle $\mathbb{T}$ defined by

$$d\eta_\alpha^{\mathbb{T}}(e^{it}) = \frac{|e^{2it}+1|^\alpha}{2^\alpha}\frac{|e^{2it}-1|}{2}dt.$$

For arbitrary integrable function f, we have

$$\int_{-\pi}^{\pi} f(\cos t)\frac{|e^{2it}+1|^\alpha}{2^\alpha}\frac{|e^{2it}-1|}{2}dt = 2\int_{-1}^{1} f(x)|x|^\alpha dx,$$

which means that $\eta_\alpha^{\mathbb{T}}$ is indeed the pullback measure of μ_α^b with respect to the mapping $e^{it} \mapsto \cos t$. First our aim is to prove that

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) = 2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).$$

To do this, it is enough to show the upper and lower estimates

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) \le 2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big)$$

and

$$\liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) \ge 2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).$$

Upper estimate. Let P_n be the extremal polynomial for $\lambda_n(\mu_\alpha^b, 0)$ and define

$$S_n(e^{it}) = P_n(\cos t)\left(\frac{1+e^{i(t-\pi/2)}}{2}\right)^{\lfloor \eta n \rfloor} e^{in(t-\pi/2)},$$

where $\eta > 0$ is arbitrary. S_n is a polynomial of degree $2n + \lfloor \eta n \rfloor$ (which can be seen by using that $\cos t = (e^{it} + e^{-it})/2$), moreover we have $S_n(e^{i\pi/2}) = 1$.

On the one hand, for any fixed $0 < \delta < 1$, we have

$$\begin{aligned}\int_{\pi/2-\delta}^{\pi/2+\delta} |S_n(e^{it})|^2 d\nu_\alpha^{\mathbb{T}}(e^{it}) &\le \int_{\pi/2-\delta}^{\pi/2+\delta} |P_n(\cos t)|^2 d\nu_\alpha^{\mathbb{T}}(e^{it}) \\ &\le \int_{-1}^{1} |P_n(x)|^2 |x|^\alpha dx \\ &= \lambda_n(\mu_\alpha^b, 0).\end{aligned}$$

On the other hand, to estimate the corresponding integral over $[-\pi,\pi/2-\delta]$ and over $[\pi/2+\delta,\pi]$, we notice that

$$\sup_{t\in[-\pi,\pi]\setminus[\pi/2-\delta,\pi/2+\delta]}\left|\frac{1+e^{i(t-\pi/2)}}{2}\right|^{\lfloor\eta n\rfloor}=O(q^n) \tag{3.26}$$

for some $q<1$. It follows from Lemma 2.4 that

$$\|P_n\|_{[-1,1]}\le Cn^{1+|\alpha|/2}\|P_n\|_{L^2(\mu_\alpha^b)}\le Cn^{1+|\alpha|/2},$$

where in the last step we used that $\|P_n\|_{L^2(\mu_\alpha^b)}\le 1$, which follows by considering that P_n is extremal and using 1 in the definition (1.3). Thus, it follows that

$$\left(\int_{-\pi}^{\pi/2-\delta}+\int_{\pi/2+\delta}^{\pi}\right)|S_n(e^{it})|^2d\nu_\alpha^{\mathbb{T}}(e^{it})\le Cq^nn^{1+|\alpha|/2}=o(n^{-(\alpha+1)}).$$

It follows from these preceding estimates that

$$\lambda_{\deg(S_n)}(\nu_\alpha^{\mathbb{T}},e^{i\pi/2})\le\lambda_n(\mu_\alpha^b,0)+o(n^{-(\alpha+1)}),$$

and by this and (3.22), we have

$$\begin{aligned}\limsup_{n\to\infty}&(2n+\lfloor\eta n\rfloor)^{\alpha+1}\lambda_{2n+\lfloor\eta n\rfloor}(\eta_\alpha^{\mathbb{T}},e^{i\pi/2})\\&\le\limsup_{n\to\infty}(2+\lfloor\eta n\rfloor/n)^{\alpha+1}n^{\alpha+1}\lambda_n(\mu_\alpha^b,0)\\&=(1+\eta/2)^{\alpha+1}2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).\end{aligned}$$

Since $\eta>0$ was arbitrary, Lemma 2.10 gives that

$$\limsup_{n\to\infty}n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}},e^{i\pi/2})\le2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big), \tag{3.27}$$

which is the desired upper estimate.

Lower estimate. To prove the matching lower estimate, let $S_{2n}(e^{it})$ be the extremal polynomial for $\lambda_{2n}(\eta_\alpha^{\mathbb{T}},e^{i\pi/2})$. Define

$$P_n^*(e^{it})S_{2n}(e^{it})\left(\frac{1+e^{i(t-\pi/2)}}{2}\right)^{2\lfloor\eta n\rfloor}e^{-(n+\lfloor\eta n\rfloor)i(t-\pi/2)}$$

and

$$P_n(\cos t)=P_n^*(e^{it})+P_n^*(e^{-it}).$$

$P_n(\cos t)$ is a polynomial in $\cos t$, moreover $\deg(P_n) \le n + \lfloor \eta n \rfloor$ and $P_n(0) = 1$. With this, we have

$$\lambda_{\deg(P_n)}(\mu_\alpha^b, 0) \le \int_{-1}^{1} |P_n(x)|^2 |x|^\alpha dx = \frac{1}{2} \int_{-\pi}^{\pi} |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}). \tag{3.28}$$

This P_n has some useful localization properties. Let $0 < \delta < 1$ be fixed. Then we claim that

$$\begin{aligned} |P_n(\cos t)|^2 &= |P_n^*(e^{it})|^2 + O(q^n), \quad t \in [\pi/2 - \delta, \pi/2 + \delta], \\ |P_n(\cos t)|^2 &= |P_n^*(e^{-it})|^2 + O(q^n), \quad t \in [-\pi/2 - \delta, -\pi/2 + \delta], \\ |P_n(\cos t)|^2 &= O(q^n) \quad \text{otherwise} \end{aligned} \tag{3.29}$$

holds for some $q < 1$. To show this, consider that

$$\begin{aligned} |P_n(\cos t)|^2 &= |P_n^*(e^{it}) + P_n^*(e^{it})|^2 \\ &\le |P_n^*(e^{it})|^2 + 2|P_n^*(e^{it})||P_n^*(e^{-it})| + |P_n^*(e^{-it})|^2. \end{aligned} \tag{3.30}$$

Now if we apply Lemma 2.5 to two subarcs of $\mathbb{T}$ that contains the upper and respectively the lower half of the unit circle and in addition they cover the whole circle, we obtain that

$$\|P_n^*\|_{\mathbb{T}} \le \|S_{2n}\|_{\mathbb{T}} \le C n^{(1+|\alpha|)/2} \|S_{2n}\|_{L^2(\eta_\alpha^{\mathbb{T}})} \le C n^{(1+|\alpha|)/2},$$

which, along with (3.26), implies

$$|P_n^*(e^{it})| \le C q^n n^{(1+|\alpha|)/2} \quad t \in [-\pi, \pi) \setminus [\pi/2 - \delta, \pi/2 + \delta].$$

This gives that the terms in (3.30) are exponentially small with possibly only one exception, which gives (3.29). Now we have

$$\begin{aligned} \int_{-\pi}^{\pi} |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) &= \left(\int_{\pi/2-\delta}^{\pi/2+\delta} + \int_{-\pi/2-\delta}^{-\pi/2+\delta} \right) |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) \\ &+ \left(\int_{-\pi}^{-\pi/2-\delta} + \int_{-\pi/2+\delta}^{\pi/2-\delta} + \int_{\pi/2+\delta}^{\pi} \right) |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}). \end{aligned}$$

(3.29) implies that the last three terms are of magnitude $O(q^n)$, while for the other two terms we have

$$\begin{aligned} \int_{\pi/2-\delta}^{\pi/2+\delta} |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) &= \int_{\pi/2-\delta}^{\pi/2+\delta} |P_n^*(e^{it})|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) + O(q^n) \\ &\le \int_{\pi/2-\delta}^{\pi/2+\delta} |S_{2n}(e^{it})|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) + O(q^n) \\ &\le \lambda_{2n}(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) + O(q^n), \end{aligned}$$

and similarly

$$\int_{-\pi/2-\delta}^{-\pi/2+\delta} |P_n(\cos t)|^2 d\eta_\alpha^{\mathbb{T}}(e^{it}) \le \lambda_{2n}(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) + O(q^n).$$

Combining these with (3.28), we have

$$\lambda_{\deg(P_n)}(\mu_\alpha^b, 0) \le \lambda_{2n}(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) + O(q^n),$$

thus

$$\begin{aligned}\liminf_{n\to\infty} \deg(P_n)^{\alpha+1}\lambda_{\deg(P_n)}(\mu_\alpha^b, 0) &\le \liminf_{n\to\infty}(n+\lfloor \eta n\rfloor)^{\alpha+1}\Big(\lambda_{2n}(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) + O(q^n)\Big)\\ &\le \liminf_{n\to\infty}(1+\lfloor \eta n\rfloor/n)^{\alpha+1}\frac{1}{2^{\alpha+1}}(2n)^{\alpha+1}\lambda_{2n}(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}).\end{aligned}$$

Now by letting η to 0, Lemma 2.10 and (3.22) implies that

$$2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) \le \liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}). \tag{3.31}$$

(3.27) and (3.31) implies that

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2}) = 2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big),$$

which is almost what we wanted. We need the same for $\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2})$ instead of $\lambda_n(\eta_\alpha^{\mathbb{T}}, e^{i\pi/2})$.

Proof of (3.25). First notice that

$$d\mu_\alpha^{\mathbb{T}}(e^{i\pi/2}) = w(e^{it})d\eta_\alpha^{\mathbb{T}}(e^{it}),$$

where w is continuous in a neighbourhood of $e^{i\pi/2}$ and $w(e^{i\pi/2}) = 1$. For an arbitrary $\tau > 0$, select $\delta > 0$ in a way that

$$\frac{1}{1+\tau} \le w(e^{it}) \le 1+\tau, \quad t \in [\pi/2-\delta, \pi/2+\delta]$$

holds. If we carry out the preceding arguments with this δ and $\eta_\alpha^{\mathbb{T}}$ replaced with $\mu_\alpha^{\mathbb{T}}$, we obtain that

$$\begin{aligned}\frac{1}{1+\tau}2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) &\le \liminf_{n\to\infty} \lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2})\\ &\le \liminf_{n\to\infty} \lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2})\\ &\le (1+\tau)2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).\end{aligned}$$

Since $\tau > 0$ was arbitrary, (3.25) follows.

□

4 Asymptotics for the Christoffel functions with respect to generalized Jacobi measures supported on a system of Jordan curves

4.1 Lemniscates

Now we prove Theorem 1.1 on lemniscates. For the definition and basic properties of lemniscates, see Section 2.2.

Let $T_N(z)$ be a polynomial of degree N and consider the lemniscate defined by the set $\sigma = \{z \in \mathbb{C} : |T_N(z)| = 1\}$. Let $z_0 \in \sigma$ be arbitrary and define the measure

$$d\mu_\sigma(z) = |z - z_0|^\alpha ds_\sigma(z), \quad z \in \sigma \tag{4.1}$$

for some $\alpha > -1$, where s_σ denotes the arc length measure with respect to σ. Without loss of generality we can assume that $T_N(z_0) = e^{i\pi/2}$. Our plan is to compare the Christoffel functions for the measure μ_σ with that for the measure $\mu_\alpha^{\mathbb{T}}$, which is supported on the unit circle and was defined by (3.24). Our aim is to prove

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\sigma, z_0) = \frac{1}{(\pi\omega_\sigma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big). \tag{4.2}$$

To do this, it is enough to prove the upper and lower estimates like in the proof of Proposition 3.5.

Upper estimate. Let $\eta > 0$ be an arbitrary small number and select $\delta > 0$ such that for every z with $|z - z_0| < \delta$, we have

$$\begin{aligned} &(1) \quad \frac{1}{1+\eta}|T_N'(z_0)| \le |T_N'(z)| \le (1+\eta)|T_N'(z_0)| \\ &(2) \quad \frac{1}{1+\eta}|T_N'(z_0)||z - z_0| \le |T_N'(z) - T_N(z_0)| \le (1+\eta)|T_N'(z_0)||z - z_0|. \end{aligned} \tag{4.3}$$

Note that $T_N'(z_0) \neq 0$, because σ has no self-intersections. Let Q_n be the extremal polynomial for $\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2})$, where $\mu_\alpha^{\mathbb{T}}$ is defined by (3.24). Define R_n as

$$R_n(z) = Q_n(T_N(z))S_{n,z_0,L}(z),$$

where $S_{n,z_0,L}$ is the fast decreasing polynomial given by Lemma 2.8 for the enclosed lemniscate domain $L = \{z \in \mathbb{C} : |T_N(z)| \le 1\}$. (The $0 < \tau < 1$ in Lemma 2.8 can be anything.) R_n is a polynomial of degree $nN + o(n)$ with $R_n(z_0) = 1$. Since $S_{n,z_0,L}$ is fast decreasing, we have

$$\sup_{z\in L\setminus z:|z-z_0|<\delta} |S_{n,z_0,L}(z)| = O(q^{n^{\tau_0}})$$

for some $q < 1$ and $\tau_0 > 0$. The Nikolskii-type inequality in Lemma 2.5 applied to two subarcs of $\mathbb{T}$ containing the upper and respectively the lower part of the unit circle yields

$$\|Q_n\|_{\mathbb{T}} \leq Cn^{(1+|\alpha|)/2}\|Q_n\|_{L^2(\mu_\alpha^{\mathbb{T}})} \leq Cn^{(1+|\alpha|)/2},$$

therefore

$$\sup_{z\in L\setminus z:|z-z_0|<\delta} |R_n(z)| = O(q^{n^{\tau_0/2}}),$$

from which it follows that

$$\int_{|z-z_0|>\delta} |R_n(z)|^2|z-z_0|^\alpha ds_\sigma(z) = O(q^{n^{\tau_0/2}}). \tag{4.4}$$

On the other hand, with (4.3), we have

$$\begin{aligned}
&\int_{|z-z_0|<\delta} |R_n(z)|^2|z-z_0|^\alpha ds_\sigma(z)\\
&\quad\leq \int_{|z-z_0|<\delta} |Q_n(T_N(z))|^2|z-z_0|^\alpha ds_\sigma(z)\\
&\quad\leq \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}} \int_{|z-z_0|<\delta} |Q_n(T_N(z))|^2|T_N(z)-T_N(z_0)|^\alpha|T_N'(z)|ds_\sigma(z)\\
&\quad\leq \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}} \int_0^{2\pi} |Q_n(e^{it})|^2|e^{it}-e^{i\pi/2}|^\alpha dt\\
&\quad= \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}}\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2}).
\end{aligned}$$

This and (4.4) imply that

$$\lambda_{\deg(R_n)}(\mu_\sigma, z_0) \leq \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}}\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2}) + O(q^{n^{\tau_0}}),$$

from which the model case (3.25) gives

$$\begin{aligned}
&\limsup_{n\to\infty} \deg(R_n)^{\alpha+1}\lambda_{\deg(R_n)}(\mu_\sigma, z_0)\\
&\qquad\leq \limsup_{n\to\infty}(nN+o(n))^{\alpha+1}\frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}}\lambda_n(\mu_\alpha^{\mathbb{T}}, e^{i\pi/2})\\
&\qquad= (1+\eta)^{|\alpha|+1}\frac{N^{\alpha+1}}{|T_N'(z_0)|^{\alpha+1}}2^{2\alpha+2}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).
\end{aligned}$$

Since $\eta > 0$ is arbitrary, Lemma 2.10 and (2.4) yields the desired upper estimate

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\sigma, z_0) \leq \frac{1}{(\pi\omega_\sigma(z_0))^{\alpha+1}}2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big). \tag{4.5}$$

Lower estimate. Let P_n be the extremal polynomial for $\lambda_n(\mu_\sigma, z_0)$ and let $S_{n,z_0,L}$ be the fast decreasing polynomial given by Lemma 2.8 for the enclosed lemniscate domain σ, where $0 < \tau < 1$ is fixed. Lemma 2.5 again implies that

$$\|P_n\|_\sigma = O\big(n^{(1+|\alpha|)/2}\big). \tag{4.6}$$

Define

$$R_n(z) = P_n(z) S_{n,z_0,L}(z).$$

R_n is a polynomial of degree at most $n + o(n)$ with $R_n(z_0) = 1$. Similarly as in the upper estimate, we have

$$\sup_{z \in L \setminus z : |z - z_0| < \delta} |R_n(z)| = O(q^{n^{\tau_0/2}}), \tag{4.7}$$

for some $|q| < 1$ and $\tau_0 > 0$. Since the expression $\sum_{k=1}^N R_n(z_k)$, where $\{z_1, \ldots, z_N\} = T_N^{-1}(T_N(z))$, is symmetric in the variables z_k, it is a sum of elementary symmetric polynomials. For more details on this idea, see [44]. Therefore there is a polynomial Q_n of degree at most $\deg(R_n)/N = (n + o(n))/N$ such that

$$Q_n(T_N(z)) = \sum_{k=1}^N R_n(z_k), \quad z \in \sigma.$$

We claim that for every $z \in \sigma$, we have

$$|Q_n(T_N(z))|^2 \le \sum_{k=1}^N |R_n(z_k)|^2 + O(q^{n^{\tau_0/2}}). \tag{4.8}$$

Indeed, since σ has no self intersection, $|z_k - z_l|$ cannot be arbitrarily small for distinct k and l. As a consequence, for every z at most one z_j belongs to the set $\{z : |z - z_0| < \delta\}$ if δ is sufficiently small, and hence (4.6) and (4.7) implies that in the sum

$$|Q_n(T_N(z))|^2 \le \sum_{k=1}^N \sum_{l=1}^N |R_n(z_k)||R_n(z_l)|$$

every term with $k \ne l$ is $O(q^{n^{\tau_0/2}})$.

Now let $\delta > 0$ be so small such that (4.3) holds for every z with $|z - z_0| < \delta$. Then

(2.5) and (4.8) yield

$$\begin{aligned}
&\int_\sigma |Q_n(T_N(z))|^2 |T_N(z)|' |T_N(z) - T_N(z_0)|^\alpha ds_\sigma(z) \\
&\quad \le O(q^{n^{\tau_0/2}}) + \int_\sigma \left(\sum_{k=1}^{N} |R_n(z_k)|^2 \right) |T_N(z) - T_N(z_0)|^\alpha |T_N'(z)| ds_\sigma(z) \\
&\quad = O(q^{n^{\tau_0/2}}) + \int_\sigma \left(\sum_{k=1}^{N} |R_n(z_k)|^2 |T_N(z) - T_N(z_0)|^\alpha \right) |T_N'(z)| ds_\sigma(z) \\
&\quad = O(q^{n^{\tau_0/2}}) + N \int_\sigma |R_n(z)|^2 |T_N(z) - T_N(z_0)|^\alpha |T_N'(z)| ds_\sigma(z) \\
&\quad \le O(q^{n^{\tau_0/2}}) + (1+\eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} N \int_{|z-z_0|<\delta} |P_n(z)|^2 |z - z_0|^\alpha ds_\sigma(z) \\
&\quad \le O(q^{n^{\tau_0/2}}) + (1+\eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} N \lambda_n(\mu_\sigma, z_0).
\end{aligned}$$

Since σ has no self-intersections, $Q_n(T_N(z_0)) = 1 + o(1)$, this along with (2.6) implies that

$$\begin{aligned}
\int_\sigma |Q_n(T_N(z))|^2 |T_N(z) - T_N(z_0)|^\alpha |T_N'(z)| ds_\sigma(z) &= N \int_0^{2\pi} |Q_n(e^{it})|^2 |e^{it} - e^{i\pi/2}|^\alpha dt \\
&\le (1 + o(1)) N \lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2}),
\end{aligned}$$

where we recall that $T_N(z_0) = e^{i\pi/2}$. These estimates give the inequality

$$(1 + o(1)) \lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2}) \le O(q^{n^{\tau_0/2}}) + (1+\eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} \lambda_n(\mu_\sigma, z_0).$$

Using that $\deg(Q_n) \le (n + o(n))/N$, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \deg(Q_n)^{\alpha+1} \lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2}) \\
&\qquad \le (1+\eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} \liminf_{n\to\infty} \left(\frac{n + o(n)}{N} \right)^{\alpha+1} \lambda_n(\mu_\sigma, z_0) \\
&\qquad \le (1+\eta)^{|\alpha|+1} \frac{|T_N'(z_0)|^{\alpha+1}}{N^{\alpha+1}} \liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu_\sigma, z_0).
\end{aligned}$$

Since $\eta > 0$ was arbitrary, (2.4) and (3.25) implies the desired lower estimate

$$\frac{1}{(\pi\omega_\sigma(z_0))^{\alpha+1}} 2^{\alpha+1} \Gamma\Big(\frac{\alpha+1}{2}\Big) \Gamma\Big(\frac{\alpha+3}{2}\Big) \le \liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu_\sigma, z_0). \tag{4.9}$$

(4.5) and (4.9) give (4.2), and this is what we wanted to show in this section.

4.2 Union of Jordan curves

Now we set out to prove Theorem 1.1, the first main result of the thesis. Let therefore γ be a disjoint union of rectifiable Jordan curves lying exterior to each other and let μ be a finite Borel measure with support γ regular in the sense of Stahl-Totik. Suppose that for a $z_0 \in \gamma$ there is an open set U such that $J = U \cap \gamma$ is a C^2 smooth Jordan arc and μ is absolutely continuous there with

$$d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z), \quad z \in J,$$

where $\alpha > -1$ and w is strictly positive and continuous at z_0. Our aim is to show

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big),$$

for which, similarly to the previous sections, enough to show the matching upper and lower estimates.

Lower estimate. Let P_n be the extremal polynomial for $\lambda_n(\mu, z_0)$ and for some $\tau > 0$ let $S_{\tau n,z_0,K}$ be the fast decreasing polynomial given by Lemma 2.7 with some $\gamma > 1$ to be determined below, where K denotes the set enclosed by γ. Let $\sigma = \sigma_{z_0}$ be the lemniscate given by the second part of Theorem 2.1, i.e. σ lies inside γ and $\omega_{\sigma_{z_0}}(z_0) \leq \omega_\gamma(z_0)+\varepsilon$, where $\varepsilon > 0$ is arbitrary. Without the loss of generality, we can assume that $\sigma = \{z : |T_N(z)| = 1\}$ and $T_N(z_0) = e^{i\pi/2}$. Similarly as before, define

$$R_n(z) = P_n(z)S_{\tau n,z_0,K}(z).$$

Again, R_n is a polynomial of degree at most $(1 + \tau)n$ with $R_n(z_0) = 1$. These will be our test polynomials in estimating the Christoffel functions. Before we carry out our estimations, we shall need two lemmas.

Lemma 4.1. *Let $\frac{1}{2} < \beta < 1$ be fixed. For each $z \in \gamma$ with $|z-z_0| \leq 2n^{-\beta}$, let $z^* \in \sigma$ be the point such that $s_\sigma([z_0, z^*]) = s_\gamma([z_0, z])$ (there are in fact two such points, we choose the one which lies closer to z). Then the mapping $q(z) = z^*$ is one to one, $|q(z) - z| \leq C|z - z_0|^2$, $ds_\gamma(z) = d_\sigma(z^*)$, $|q'(z_0)| = 1$, and with the notation $I_n = \{z^* \in \sigma : |z^* - z_0| \leq n^{-\beta}\}$ we have*

$$\left| \int_{z^*\in I_n} |R_n(z^*)|^2|z - z_0|^\alpha ds_\sigma(z^*) - \int_{z^*\in I_n} |R_n(z)|^2|z - z_0|^\alpha ds_\gamma(z) \right| = o(n^{-\alpha-1}). \quad (4.10)$$

Proof. By the selection of σ and the C^2 smoothness of the curves is clear that $q(z) = z + O(|z - z_0|^2)$. It follows that $|q'(z_0)| = 1$, which implies that if $|z - z_0|$ is small enough, we have

$$1 - \varepsilon \leq \frac{|q(z) - z_0|}{|z - z_0|} \leq 1 + \varepsilon.$$

We proceed to prove (4.10).

$$\begin{aligned}\Big|\int_{z^*\in I_n}|R_n(z^*)|^2|z-z_0|^\alpha ds_\sigma(z^*)-\int_{z^*\in I_n}|R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\Big|\\ \le\Big|\int_{z^*\in I_n}\Big(|R_n(z^*)|^2-|R_n(z)|^2\Big)|z-z_0|^\alpha ds_\gamma(z)\Big|\\ \le\int_{z^*\in I_n}\Big||R_n(z^*)|^2-|R_n(z)|^2\Big||z-z_0|^\alpha ds_\gamma(z)=:A.\end{aligned}$$

Using the Hölder and Minkowski inequalities we can continue as

$$\begin{aligned}A\le&\Big(\int_{z^*\in I_n}|R_n(z^*)-R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\Big)\\ &\times\Big\{\Big(\int_{z^*\in I_n}|R_n(z^*)|^2|z-z_0|^\alpha ds_\gamma(z)\Big)+\Big(\int_{z^*\in I_n}|R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\Big)\Big\}.\end{aligned}\tag{4.11}$$

We estimate these integrals term by term. Since P_n is extremal for $\lambda_n(\mu,z_0)$ and $\lambda_n(\mu,z_0)=O(n^{-\alpha-1})$ (see Lemma 2.11), we have

$$\Big(\int_{z^*\in I_n}|R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\Big)^{1/2}\le Cn^{-\frac{\alpha+1}{2}},\tag{4.12}$$

which estimates the third term. To estimate the other terms, we shall differentiate between the cases $\alpha\ge 0$ and $\alpha<0$.

First case: $\alpha\ge 0$. From Lemma 2.5 we get that for any closed subarc $J_1\subset J$

$$\|R_n\|_{J_1}\le Cn^{(\alpha+1)/2}\|R_n\|_{L^2(\mu)}\le C,$$

where we used Lemma 2.11 and $|R_n(z)|\le|P_n(z)|$. Choose this J_1 so that it contains z_0 in its interior. Next we note that if $z^*\in I_n$, then $|z^*-z|\le Cn^{-2\beta}$, therefore $\operatorname{dist}(z^*,z)\le C/n$. Therefore an application of Lemma 2.6 yields for such z that

$$\frac{|R_n(q(z))-R_n(z)|}{|q(z)-z|}\le Cn\|R_n\|_{J_1}$$

holds and so we have

$$|R_n(q(z))-R_n(z)|\le Cn|q(z)-z|\le Cn^{1-2\beta}.\tag{4.13}$$

Since $s_\sigma(I_n)\le Cn^{-\beta}$ is also true, we have (recall that $z^*=q(z)$)

$$\begin{aligned}\Big(\int_{z^*\in I_n}|R_n(z^*)-R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\Big)^{1/2}&\le C\Big(n^{-\beta}n^{2-4\beta}n^{-\alpha\beta}\Big)^{1/2}\\ &=Cn^{1-\frac{5+\alpha}{2}\beta}.\end{aligned}$$

This is the required estimate for the first term in (4.11). For the middle term, we have

$$
\begin{aligned}
&\left(\int_{z^*\in I_n} |R_n(z^*)|^2|z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \\
&\qquad\qquad = \left(\int_{z^*\in I_n} \Big||R_n(z^*)|^2 - |R_n(z)|^2 + |R_n(z)|^2\Big||z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \\
&\qquad\qquad \le \left(\int_{z^*\in I_n} \Big||R_n(z^*)|^2 - |R_n(z)|^2\Big||z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \\
&\qquad\qquad\qquad + \left(\int_{z^*\in I_n} |R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \\
&\qquad\qquad \le A^{1/2} + Cn^{-\frac{\alpha+1}{2}},
\end{aligned}
$$

where A is the left-hand side in (4.11) and we also used (4.12). Combining these estimates we get

$$
\begin{aligned}
A &\le Cn^{1-\frac{\alpha+5}{2}\beta}(A^{1/2} + Cn^{-\frac{\alpha+1}{2}}) \le CA^{1/2}n^{1-\frac{\alpha+5}{2}\beta} + Cn^{\frac{1}{2}-\frac{\alpha}{2}-\frac{\alpha+5}{2}\beta} \\
&\le C\max\{A^{1/2}n^{1-\frac{\alpha+5}{2}\beta}, n^{\frac{1}{2}-\frac{\alpha}{2}-\frac{\alpha+5}{2}\beta}\}.
\end{aligned}
$$

Therefore $A \le Cn^{2-(\alpha+5)\beta}$ or $A \le Cn^{\frac{1}{2}-\frac{\alpha}{2}-\frac{\alpha+5}{2}\beta}$. If $\beta < 1$ is sufficiently close to 1, both imply $A = o(n^{-\alpha-1})$.

Second case: $\alpha < 0$. From Lemma 2.5 we get that for any closed subarc $J_1 \subset J$

$$
\|R_n\|_{J_1} \le \|P_n\|_{J_1} \le Cn^{1/2}\|P_n\|_{L^2(\mu)} \le Cn^{-\alpha/2}
$$

holds, and we may assume that here J_1 is such that it contains the neighbourhood of z_0. Therefore in this case (4.13) takes the form

$$
|R_n(z^*) - R_n(z)| \le Cn^{1-\alpha/2-2\beta}.
$$

Since

$$
\int_{z^*\in I_n} |z-z_0|^\alpha ds_\gamma(z) \le Cn^{-\alpha\beta-\beta},
$$

we obtain

$$
\left(\int_{z^*\in I_n} |R_n(z^*) - R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \le Cn^{1-\frac{\alpha}{2}-2\beta-\frac{\alpha+1}{2}\beta},
$$

which is the required estimate for the first term in (4.11). Finally, for the middle term in (4.11), we get

$$
\left(\int_{z^*\in I_n} |R_n(z^*)|^2|z-z_0|^\alpha ds_\gamma(z)\right)^{1/2} \le A^{1/2} + Cn^{-\frac{\alpha+1}{2}}.
$$

As previously, we can conclude from these that

$$A \le C n^{1-\frac{\alpha}{2}-2\beta-\frac{\alpha+1}{2}\beta}(A^{1/2}+n^{-\frac{\alpha+1}{2}}),$$

which implies

$$A \le C \max\{n^{2-\alpha-4\beta-(\alpha+1)\beta}, n^{\frac{1}{2}-\alpha-2\beta-\frac{\alpha+1}{2}\beta}\}.$$

If $\beta < 1$ is sufficiently close to 1, then this yields again $A = o(n^{-\alpha-1})$, and this is what we needed to prove. $\square$

In the following lemma we keep the notations from the preceeding proof. Denote the disk of radius δ about z_0 with $\Delta_\delta(z_0)$. Note that up to this point the $\gamma > 1$ in Lemma 2.7 was arbitrary. Now we specify how close it should be to 1.

Lemma 4.2. *If $0 < \beta < 1$ is fixed and $\gamma > 1$ is chosen so that $\beta\gamma < 1$ holds, then*

$$\|R_n\|_{K\setminus\Delta_{n^{-\beta}/2}}(z_0) = O(n^{-\alpha-1}), \tag{4.14}$$

where K is the set enclosed by γ.

Proof. Let us fix a $\delta > 0$ such that the intersection $\gamma \cap \Delta_\delta(z_0)$ lies in the interior of the arc J from Theorem 1.1. Since μ is regular in the sense of Stahl-Totik and the trivial estimate $\|P_n\|_{L^2(\mu)} = O(1)$ holds, we get that no matter how small $\varepsilon > 0$ is given, for sufficiently large n we have $\|P_n\|_\gamma \le (1+\varepsilon)^n$. On the other hand, in view of Lemma 2.7, for $z \notin \Delta_\delta(z_0)$ and $z \in K$ we have

$$|S_{\tau n, z_0, K}(z)| \le C_\gamma e^{-c_\gamma \tau n \delta^2},$$

so

$$\|R_n\|_{K\setminus\Delta_\delta(z_0)} = o(n^{-\alpha-1}) \tag{4.15}$$

holds.

Consider now $K \cap \Delta_\delta(z_0)$. Its boundary consists of the arc $\gamma \cap \Delta_\delta(z_0)$, which is a part of J, and of an arc on the boundary of $\Delta_\delta(z_0)$, where we already know the bound (4.15). On the other hand, on $\gamma \cap \Delta_\delta(z_0)$, by Lemma 2.5 we have

$$|P_n(z)| \le C n^{(1+|\alpha|)/2} \|P_n\|_{L^2(\mu)} \le C n^{(1+|\alpha|)/2}.$$

Therefore, by the maximum principle, we obtain the same bound (for large n) on the whole set $K \cap \Delta_\delta(z_0)$. As a consequence, for $z \in K \setminus \Delta_{n^{-\beta}/2}$

$$|R_n(z)| \le C n^{(1+|\alpha|)/2} e^{-c_\gamma \tau n (n^{-\beta}/2)^\gamma} = o(n^{-\alpha-1})$$

holds, if $\gamma > 1$ is choosen in Lemma 2.7 such that $\beta\gamma < 1$, and this is what we had to show. $\square$

After these preliminary lemmas we are ready to prove the lower estimate for Theorem 1.1.

Let $\eta > 0$ be arbitrary and let n be so large that (recall that $z^* = q(z)$)

$$\begin{aligned}
&(1) \quad \frac{1}{1+\eta} w(z_0) \le w(z) \le (1+\eta) w(z_0),\\
&(2) \quad \frac{1}{1+\eta}|z - z_0| \le |q(z) - z_0| \le (1+\eta)|z - z_0|
\end{aligned}$$

hold for all $z^* \in I_n$, where I_n is defined in Lemma 4.1. Then using this Lemma, we obtain

$$\begin{aligned}
&\int_{z^* \in I_n} |R_n(z^*)|^2 |z^* - z_0|^\alpha ds_\sigma(z^*)\\
&\qquad \le (1+\eta)^{|\alpha|} \int_{z^*} |R_n(z^*)|^2 |z - z_0|^\alpha ds_\gamma(z)\\
&\qquad \le (1+\eta)^{|\alpha|} \int_{z^*} |R_n(z)|^2 |z - z_0|^\alpha ds_\gamma(z) + o(n^{-\alpha-1})\\
&\qquad \le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \int_{z^*} |R_n(z)|^2 w(z) |z - z_0|^\alpha ds_\gamma(z) + o(n^{-\alpha-1})\\
&\qquad \le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \lambda_n(\mu, z_0) + o(n^{-\alpha-1}).
\end{aligned}$$

On the other hand, if we note that if for some $z \in \sigma$ we have $z^* \notin I_n$ then necessarily $|z - z_0| \ge n^{-\beta}/2$, we obtain from Lemma 4.2 that

$$\int_{z^* \in \sigma \setminus I_n} |R_n(z^*)|^2 |z^* - z_0|^\alpha ds_\sigma(z^*) = o(n^{-\alpha-1}).$$

Combining these estimates it follows that

$$\begin{aligned}
\lambda_{\deg(R_n)}(\mu_\sigma, z_0) &\le \int_{z \in \sigma} |R_n(z^*)|^2 |z^* - z_0|^\alpha ds_\sigma(z^*)\\
&\le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \lambda_n(\mu, z_0) + o(n^{-\alpha-1}).
\end{aligned}$$

Since $\deg(R_n) \le (1+\tau)n$, we can conclude from (4.2) that

$$\begin{aligned}
\frac{1}{(\pi\omega_\sigma(z_0))^{\alpha+1}} 2^{\alpha+1} \Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) &= \liminf_{n\to\infty} \deg(R_n)^{\alpha+1} \lambda_{\deg(R_n)}(\mu_\sigma, z_0)\\
&\le \liminf_{n\to\infty} (1+\tau)^{\alpha+1} \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} n^{\alpha+1} \lambda_n(\mu, z_0).
\end{aligned}$$

But since $\tau, \eta > 0$ are arbitrary, we have

$$\frac{w(z_0)}{(\pi\omega_\sigma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) \le \liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0).$$

As $\omega_\sigma(z_0) \le \omega_\gamma(z_0) + \varepsilon$ (see (2.8)), by letting $\varepsilon \to 0$ we finally obtan the desired lower estimate

$$\frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big) \le \liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0). \tag{4.16}$$

Upper estimate. Now let σ be the lemniscate given by the first part of Theorem 2.1 and let P_n be the extremal polynomial for $\lambda_n(\mu_\sigma, z_0)$. For some $\tau > 0$ define

$$R_n(z) = P_n(z)S_{\tau n, z_0, L}(z),$$

where $S_{\tau n, z_0, L}$ is the fast decreasing polynomial given by Lemma 2.7 for the lemniscate domain L enclosed by γ. (The γ in Lemma 2.7 can be arbitrary.) Let $\eta > 0$ be arbitrary, let $1/2 < \beta < 1$ as before, and suppose that n is so large such that

$$\begin{aligned}
&(1) \quad \frac{1}{1+\eta} w(z_0) \le w(z) \le (1+\eta)w(z_0),\\
&(2) \quad \frac{1}{1+\eta} \le |q'(z)| \le 1+\eta,\\
&(3) \quad \frac{1}{1+\eta}|z-z_0| \le |q(z)-z_0| \le (1+\eta)|z-z_0|
\end{aligned}$$

hold for all $|z - z_0| \le n^{-\beta}$. Using Lemma 4.1 (more precisely its version when σ encloses γ) we have

$$\begin{aligned}
&\int_{z^*\in I_n} |R_n(z)|^2 w(z)|z-z_0|^\alpha ds_\gamma(z)\\
&\qquad \le (1+\eta)w(z_0)\int_{z^*\in I_n} |R_n(z)|^2|z-z_0|^\alpha ds_\gamma(z)\\
&\qquad \le (1+\eta)w(z_0)\int_{z^*\in I_n} |R_n(z^*)|^2|z-z_0|^\alpha ds_\sigma(z^*) + o(n^{-\alpha-1})\\
&\qquad \le (1+\eta)^{|\alpha|+1}w(z_0)\int_{z^*\in I_n} |R_n(z^*)|^2|z^*-z_0|^\alpha ds_\sigma(z^*) + o(n^{-\alpha-1})\\
&\qquad \le (1+\eta)^{|\alpha|+1}w(z_0)\lambda_n(\mu_\sigma, z_0) + o(n^{-\alpha-1}).
\end{aligned}$$

On the other hand, Lemma 4.2 (but now applied to σ rather than γ) implies, as before, that

$$\int_{\gamma\setminus\Delta_{n^{-\beta}/2}(z_0)} |R_n(z)|^2|z-z_0|^\alpha d\mu(z) = o(n^{-\alpha-1})$$

holds, therefore

$$\lambda_{\deg(R_n)}(\mu, z_0) \le (1+\eta)^{|\alpha|+1}w(z_0)\lambda_n(\mu_\sigma, z_0) + o(n^{-\alpha-1}),$$

which, similarly to the lower estimate, by using (4.9) and letting $\tau, \eta \to 0$ implies

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) \le \frac{w(z_0)}{(\pi\omega_\sigma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big).$$

But, as $\omega_\gamma(z_0) \le \omega_\sigma(z_0) + \varepsilon$ (see (2.7)), by letting $\varepsilon \to 0$ we obtain the desired upper estimate

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) \le \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big). \tag{4.17}$$

(4.16) and (4.17) give the proof of Theorem 1.1.

□

5 Christoffel functions on the real line for generalized Jacobi measures

5.1 Small perturbations in Christoffel functions

In order to study the asymptotic behavior of $\lambda_n(\mu, x_0 + a/n)$ for measures supported on general compact sets, we use polynomial inverse images to transform the results (3.21) and (3.21) obtained as model cases. Because of this, using the polynomial $T_N(x)$, the point a/n will be transformed to something like $x_0 + \frac{a}{|T_N'(x_0)|n} + o(n^{-1})$. Because we want to study the asymptotics of the Christoffel functions only at $x_0 + \frac{a}{|T_N'(x_0)|n}$, we shall need a tool to control small perturbations. The next lemma takes care of this when the power-type singularity is in the bulk of the support.

Lemma 5.1. *Let μ be a finite Borel measure and suppose that μ is supported on a compact set $K = \operatorname{supp}(\mu)$ on the real line. Let $x_0 \in \operatorname{int}(K)$ be a point from the interior of its support and suppose that for some $\varepsilon > 0$ the measure μ is absolutely continuous on $(x_0 - \varepsilon, x_0 + \varepsilon)$ with*

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$

there for some $\alpha > -1$, where w is strictly positive and continuous at x_0. Then for a given sequence $\varepsilon_n = o(n^{-1})$,

$$\lim_{n\to\infty} \frac{\lambda_n(\mu, x_0 + a/n)}{\lambda_n(\mu, x_0 + a/n + \varepsilon_n)} = 1$$

holds uniformly for $a \in \mathbb{R}$ in compact subsets of the real line.

Proof. During the proof, constants are denoted with C and their value often varies from line to line. We can assume without the loss of generality that $x_0 = 0$. The classical bound of Nevai [30, p. 120 Theorem 28] says that there is a constant C independent of x such that

$$\frac{1}{Cn}\left(|x| + \frac{1}{n}\right)^{\alpha} \leq \lambda_n(\mu_\alpha^b, x) \leq \frac{C}{n}\left(|x| + \frac{1}{n}\right)^{\alpha}, \quad x \in (-1/2, 1/2) \tag{5.1}$$

holds, where μ_α^b is defined by (3.1). We wish to establish the same bounds for $\lambda_n(\mu, x)$. Let $\delta > 0$ be so small such that $\delta < \varepsilon$ and

$$\frac{w(0)}{2} \leq w(x) \leq 2w(0), \quad x \in (-\delta, \delta)$$

holds. Suppose that $P_n(x)$ is extremal for $\lambda_n(\mu, x_1)$ for some $x_1 \in (-\delta, \delta)$, i.e. $P_n(x)$ is a polynomial of degree less than n with $P_n(x_1) = 1$ and $\int |P_n|^2 d\mu = \lambda_n(\mu, x_1)$. Then

$$\lambda_n(\mu, x_1) \geq \frac{w(0)}{2}\int_{-\delta}^{\delta} |P_n(x)||x|^\alpha dx \geq \frac{w(0)}{2}\lambda_n\big(|x|^\alpha \chi_{[-\delta,\delta]}(x)dx, x_1\big),$$

where $\chi_H(x)$ denotes the characteristic function of the set H. After scaling the measures appropriately, the bound (5.1) can also be applied for the Christoffel function $\lambda_n\big(|x|^\alpha \chi_{[-\delta,\delta]}(x)dx, x_1\big)$, thus there is a constant C such that

$$\lambda_n(\mu, x_1) \geq \frac{1}{Cn}\left(|x_1| + \frac{1}{n}\right)^{\alpha}, \quad x_1 \in (-\delta/2, \delta/2). \tag{5.2}$$

On the other hand, let $b > 0$ be so large such that $K \subseteq [-b, b]$, let $P_n(x)$ be extremal for $\lambda_n\big(|x|^\alpha \chi_{[-b,b]}(x)dx, x_1\big)$ and define the polynomial

$$R_n(x) = P_n(x)\left(1 - \frac{(x - x_1)^2}{2b}\right)^n.$$

$R_n(x)$ is a polynomial of degree at most $3n$, moreover $R_n(x_1) = 1$ and $|R_n(x)| \leq |P_n(x)|$ for all $x \in [-b, b]$. Then

$$\begin{aligned}
\lambda_{3n}(\mu, x_1) &\leq \int |R_n(x)|^2 d\mu(x) \\
&\leq 2w(0)\int_{-\delta}^{\delta} |P_n(x)|^2 |x|^\alpha dx + \gamma^n \int_{K\setminus[-\delta,\delta]} |P_n(x)|^2 d\mu(x) \qquad (5.3) \\
&\leq 2w(0)\lambda_n(|x|^\alpha \chi_{[-b,b]}(x)dx, x_1) + \mu(K)\gamma^n \|P_n\|_K,
\end{aligned}$$

where

$$\gamma = \sup_{x \in K\setminus[-\delta,\delta]} \left|1 - \frac{(x - x_1)^2}{2\,\mathrm{diam}(K)}\right|^n < 1.$$

As the Erdős-Turán criterion implies, the measure $|x|^\alpha \chi_{[-b,b]} dx$ is regular in the sense of Stahl and Totik moreover its support $[-b,b]$ is regular with respect to the Dirichlet problem. Then (2.3) gives that for every $\tau > 0$,

$$\|P_n\|_{[-b,b]} \leq (1+\tau)^n \lambda_n\big(|x|^\alpha \chi_{[-b,b]}(x)dx, x_1\big)^{1/2}$$

holds if n is large enough. Since $\lambda_n\big(|x|^\alpha \chi_{[-b,b]}(x)dx, x_1\big) = O(1)$ (use the constant polynomial 1 in the definition (1.3)), then it follows that if τ is chosen such that $q = \gamma(1+\tau) < 1$, we have

$$\mu(K)\gamma^n \|P_n\|_K = O(q^n).$$

This together with Lemma 2.10, (5.1) and (5.3) imply that there is a constant C such that

$$\lambda_n(\mu, x_1) \leq \frac{C}{n}\left(|x_1| + \frac{1}{n}\right)^\alpha, x_1 \in (-\delta/2, \delta/2). \tag{5.4}$$

Now define the polynomial $Q_n(x)$ as

$$Q_n(x) = \frac{\lambda_n(\mu, a/n)}{\lambda_n(\mu, a/n + x)}.$$

$Q_n(x)$ is indeed a polynomial of degree $2n-2$ as implied by (1.4), moreover $Q_n(0) = 1$. (5.2) and (5.4) gives that

$$|Q_n(x)| \leq C\left(\frac{|a/n| + 1/n}{|a/n + x| + 1/n}\right)^\alpha \leq C, \quad x \in [-\delta/4, \delta/4] \tag{5.5}$$

that is $Q_n(x)$ is bounded on the small but fixed interval $[-\delta/4, \delta/4]$, moreover the bound holds uniformly for a in compact subsets of the real line. The iterated Bernstein inequality for $[-\delta/4, \delta/4]$, see [3, p. 260 Exercise 5e], gives that

$$|Q_n^{(k)}(0)| \leq CM^k n^k \tag{5.6}$$

holds for some constants C and M. Overall, since $\varepsilon_n = o(n^{-1})$, we have

$$|Q_n(\varepsilon_n)| \leq \sum_{k=0}^{2n-2} \frac{|Q_n^{(k)}(0)|}{k!}\varepsilon_n^k \leq 1 + C\sum_{k=1}^{2n-2} \frac{M^k n^k \varepsilon_n^k}{k!} \leq 1 + o(1), \tag{5.7}$$

and since (5.5) holds uniformly for a in compact subsets of the real line, the above bound is also uniform. This implies

$$\limsup_{n\to\infty} \frac{\lambda_n(\mu, a/n)}{\lambda_n(a/n + \varepsilon_n)} \leq 1,$$

which is half of what we need. To obtain the matching estimate

$$\limsup_{n\to\infty} \frac{\lambda_n(a/n+\varepsilon_n)}{\lambda_n(\mu, a/n)} \le 1,$$

define the polynomial

$$Q_n(x) = \frac{\lambda_n(a/n+\varepsilon_n)}{\lambda_n(\mu, a/n+\varepsilon_n+x)}$$

and repeat the argument given in (5.5) - (5.7) to see that we have $|Q_n(-\varepsilon_n)| \le 1+o(1)$. $\square$

The analogue of the previous lemma for the edge of the support is the following.

Lemma 5.2. *Let μ be a finite Borel measure and suppose that μ is supported on a compact set $K = \operatorname{supp}(\mu)$ on the real line. Let $x_0 \in K$ be a right endpoint of K (i.e. $K \cap (x_0, x_0 + \varepsilon_0) = \varnothing$ for some $\varepsilon_0 > 0$) and assume that for some $\varepsilon > 0$ the measure μ is absolutely continuous on $(x_0 - \varepsilon, x_0]$ with*

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in (x_0 - \varepsilon, x_0]$$

there for some $\alpha > -1$, where w is strictly positive and left continuous at x_0. Then for a given sequence $\varepsilon_n = o(n^{-2})$ for which $x_0 - a/n^2 + \varepsilon_n \in K$,

$$\lim_{n\to\infty} \frac{\lambda_n(\mu, x_0 - a/n^2)}{\lambda_n(\mu, x_0 - a/n^2 + \varepsilon_n)} = 1$$

holds for all $a \in [0, \infty)$.

Proof. The proof follows in a similar tune to Lemma 5.1, with a few differences. Without the loss of generality we can assume that $x_0 = 1$. Let $\delta > 0$ be so small such that

$$\frac{w(1)}{2} \le w(x) \le 2w(1), \quad x \in (1-\delta, 1]$$

holds and $K \cap (1, 1+\delta) = \varnothing$. The classical bound of Nevai [30, p. 120 Theorem 28] once more says that there is a constant C independent of x such that

$$\frac{1}{Cn}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1} \le \lambda_n(\mu_\alpha^e, x) \le \frac{C}{n}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1}, \quad x \in (1/2, 1]$$

holds. Similarly like in the proof of Lemma 5.1, we shall show that this holds if we replace μ_α^e with μ. The proof of the lower estimate

$$\frac{1}{Cn}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1} \le \lambda_n(\mu, x), \quad x \in (1-\delta, 1]$$

goes through verbatim as in the proof of Lemma 5.1, though the upper estimate is slightly different. Let $b > 0$ be so large such that $K \subseteq [-b, b]$ and let $P_n(x)$ be extremal for $\lambda_n(|x-1|^\alpha \chi_{[-b,1]}(x)dx, x_1)$. Define the polynomial $R_n(x)$ as

$$R_n(x) = P_n(x)\left(1 - \frac{(x-x_1)^2}{4b^2}\right)^{kn},$$

where k is an integer yet to be determined. The degree of R_n is at most $(2k+1)n$ and $R_n(x_1) = 1$. Now we have

$$\begin{aligned}\lambda_{(2k+1)n}(\mu, x_1) &\leq \int |R_n(x)|^2 d\mu(x) \\ &\leq 2w(1)\int_{1-\delta}^{1} |P_n(x)|^2|x-1|^\alpha dx + \int_{K\setminus[1-\delta,1]} |R_n(x)|^2 d\mu(x).\end{aligned}$$

On the one hand, the extremality of P_n implies that

$$\int_{1-\delta}^{1} |P_n(x)|^2|x-1|^\alpha dx \leq \lambda_n(|x-1|^\alpha \chi_{[-b,1]}(x)dx, x_1).$$

On the other hand, since

$$\sup_{x\in K\setminus[1-\delta,1]} \left|1 - \frac{(x-x_1)^2}{2b}\right|^{kn} \leq \gamma^n$$

for some $|\gamma| < 1$ depending on k, we have

$$\int_{K\setminus[1-\delta,1]} |R_n(x)|^2 d\mu(x) \leq \mu(K)\gamma^n \|P_n\|_{[-b,b]}.$$

Since the measure $|x-1|^\alpha \chi_{[-b,1]}(x)dx$ is regular in the sense of Stahl and Totik, (2.3) implies that for all $\tau > 0$

$$\|P_n\|_{[-b,1]} \leq (1+\tau)^n \lambda_n(|x-1|^\alpha \chi_{[-b,1]}(x)dx, x_1)^{1/2}$$

holds if n is large enough. In addition, the Bernstein-Walsh lemma, see [32, Theorem 5.5.7a], gives that

$$\|P_n\|_{[-b,b]} \leq c^n \|P_n\|_{[-b,1]}$$

holds for some, possibly very large constant c. Overall, we have

$$\int_{K\setminus[1-\delta,1]} |R_n(x)|^2 d\mu(x) \leq \mu(K)\gamma^n c^n (1+\tau)^n \lambda_n(|x-1|^\alpha \chi_{[-b,1]}(x)dx, x_1)^{1/2}.$$

If the integer k in the definition of $R_n(x)$ is selected such that $\gamma c(1+\tau) < 1$ holds (recall that γ depends on k), the above integral is small, that is,

$$\int_{K\setminus[1-\delta,1]} |R_n(x)|^2 d\mu(x) = O(q^n)$$

for some $|q| < 1$. These estimates give that

$$\lambda_{(2k+1)n}(\mu, x_1) \leq C\lambda_n(|x-1|^{\alpha}\chi_{[-b,1]}(x)dx, x_1),$$

where is C is a fixed constant. Now Lemma 2.10 yields

$$\lambda_n(\mu, x) \leq \frac{C}{n}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1}, \quad x \in (1-\delta, 1]$$

for some possibly different constant C. Overall, we have

$$\frac{1}{Cn}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1} \leq \lambda_n(\mu, x) \leq \frac{C}{n}\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha+1}, \quad x \in (1-\delta, 1]. \tag{5.8}$$

Now define the polynomial $Q_n(x)$ as

$$Q_n(x) = \frac{\lambda_n(\mu, 1-a/n^2)}{\lambda_n(\mu, 1-a/n^2-x)}.$$

The bound (5.8) gives that

$$|Q_n(x)| \leq C, \quad x \in [-a/n^2, \delta/2],$$

holds uniformly for a in compact subsets of $[0, \infty)$. The classical Markov inequality for $[-a/n^2, \delta/2]$, see [12, Chapter 4, Theorem 1.4], implies that

$$|Q_n^{(k)}(x)| \leq CM^k n^{2k}, \quad x \in [-a/n^2, \delta/2],$$

where M is some fixed constant. Now we have

$$|Q_n(-\varepsilon_n)| \leq \sum_{k=0}^{2n-2} \frac{|Q_n^{(k)}(0)|}{k!}\varepsilon_n^k \leq 1 + C\sum_{k=1}^{2n-2} \frac{M^k n^{2k}\varepsilon_n^k}{k!} = 1 + o(1),$$

which yields

$$\limsup_{n\to\infty} \frac{\lambda_n(\mu, 1-a/n^2)}{\lambda_n(\mu, 1-a/n^2+\varepsilon_n)} \leq 1.$$

To obtain the matching bound

$$\limsup_{n\to\infty} \frac{\lambda_n(\mu, 1-a/n^2+\varepsilon_n)}{\lambda_n(\mu, 1-a/n^2)} \leq 1,$$

define the polynomial

$$Q_n(x) = \frac{\lambda_n(\mu, 1-a/n^2+\varepsilon_n)}{\lambda_n(\mu, 1-a/n^2+\varepsilon_n-x)}$$

and repeat the same argument as above to see that $|Q_n(-\varepsilon_n)| \leq 1 + o(1)$. □

5.2 Christoffel functions in the bulk

Throughout this section, let K be a compact set and let $x_0 \in \mathrm{K}$ be an element in its interior. Let μ be a measure with $\operatorname{supp}(\mu) = K$ and suppose that μ is absolutely continuous in a small neighbourhood of x_0 with

$$d\mu(x) = w(x)|x - x_0|^\alpha dx$$

there, where $\alpha > -1$ and $w(x)$ is strictly positive and continuous in x_0. Our aim in this section is to prove Theorem 1.2, for which it is enough to show that the upper and lower estimates

$$\limsup_{n\to\infty} \lambda_n\left(\mu, x_0 + \frac{a}{n}\right) \leq \frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}} \Big(\mathbb{L}_\alpha^*\big(\pi\omega_K(x_0)a\big)\Big)^{-1}$$

and

$$\frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}} \Big(\mathbb{L}_\alpha^*\big(\pi\omega_K(x_0)a\big)\Big)^{-1} \leq \liminf_{n\to\infty} \lambda_n\left(\mu, x_0 + \frac{a}{n}\right)$$

hold.

Upper estimate. Let $\eta > 0$ be arbitrary and let $E_N = \cup_{k=0}^{N-1}[a_k, b_k] = T_N^{-1}([-1,1])$ and T_N be the approximating set and the matching admissible polynomial granted by Lemma 2.2. (For the purpose of the upper estimate, $\varepsilon > 0$ in Lemma 2.2 can be chosen arbitrarily. However, this will not be the case for the lower estimate.) It can be assumed without the loss of generality that $x_0 \in (a_0, b_0)$ and $T_N'(x_0) > 0$. Select a $\delta > 0$ so small such that

$$\begin{aligned}
&(1) \quad \frac{1}{1+\eta} w(x) \leq w(x_0) \leq (1+\eta) w(x),\\
&(2) \quad \frac{1}{1+\eta}|T_N(x)| \leq |T_N'(x_0)||x - x_0| \leq (1+\eta)|T_N(x)|,\\
&(3) \quad \frac{1}{1+\eta}|T_N'(x)| \leq |T_N'(x_0)| \leq (1+\eta)|T_N'(x)|.
\end{aligned} \tag{5.9}$$

holds for all $[x_0 - \delta, x_0 + \delta]$. (This can be achieved since w is continuous and T_N is continuously differentiable at x_0.) Let $\xi \in \mathbb{R}$ be arbitrary and let $x_0 + \xi_n$ be the unique element of $[a_0, b_0]$ such that $T_N(x_0 + \xi_n) = \xi/n$. Since T_N is a polynomial, $\xi_n = O(n^{-1})$ and

$$\frac{\xi}{n} = T_N(x_0 + \xi_n) = T_N(x_0) + T_N'(x_0)\xi_n + O(n^{-2}) = T_N'(x_0)\xi_n + O(n^{-2}),$$

which implies

$$\xi_n = \frac{\xi}{|T_N'(x_0)|n} + o(n^{-1}). \tag{5.10}$$

Assume that P_n is extremal for $\lambda_n(\mu_\alpha^b, \xi/n)$ and define

$$R_n(x) = P_n(T_N(x))S_{n,x_0+\xi_n,K}(x),$$

where $S_{n,x_0+\xi_n,K}(x)$ is defined by

$$S_{n,x_0+\xi_n,K}(x) = \left(1 - \left(\frac{x_0+\xi_n-x}{\operatorname{diam}(K)}\right)^2\right)^{\lfloor \eta n \rfloor} \tag{5.11}$$

(The reason why we choose an arbitrary $\xi \in \mathbb{R}$ instead of the a appearing in (1.18) will become apparent at the end of our calculations, where it will be clear that some scaling is necessary.) This way R_n is a polynomial of degree less than $nN + 2\lfloor \eta n \rfloor$ with $R_n(x_0 + \xi_n) = 1$. Now we have

$$\begin{aligned}
\lambda_{nN+2\lfloor \eta n \rfloor}(\mu, x_0+\xi_n) &\leq \int |R_n(x)|^2 d\mu(x) \\
&= \int_{x_0-\delta}^{x_0+\delta} |R_n(x)|^2 w(x)|x-x_0|^\alpha dx \\
&\qquad + \int_{K\setminus[x_0-\delta, x_0+\delta]} |R_n(x)|^2 d\mu(x).
\end{aligned}$$

On the one hand, (2.10) and (5.9) gives

$$\begin{aligned}
&\int_{x_0-\delta}^{x_0+\delta} |R_n(x)|^2 w(x)|x-x_0|^\alpha dx \\
&\quad\leq \int_{x_0-\delta}^{x_0+\delta} |P_n(T_N(x))|^2 w(x)|x-x_0|^\alpha dx \\
&\quad= \int_{x_0-\delta}^{x_0+\delta} |P_n(T_N(x))|^2 \frac{|T_N'(x_0)|^{\alpha+1}}{|T_N'(x_0)|^{\alpha+1}} w(x)|x-x_0|^\alpha dx \\
&\quad\leq (1+\eta)^{\alpha+2} \frac{w(x_0)}{|T_N'(x_0)|^{\alpha+1}} \int_{x_0-\delta}^{x_0+\delta} |P_N(T_N(x))|^2 |T_N'(x)||T_N(x)|^\alpha dx \\
&\quad\leq (1+\eta)^{\alpha+2} \frac{w(x_0)}{|T_N'(x_0)|^{\alpha+1}} \int_{a_0}^{b_0} |P_N(T_N(x))|^2 |T_N'(x)||T_N(x)|^\alpha dx \\
&\quad= (1+\eta)^{\alpha+2} \frac{w(x_0)}{|T_N'(x_0)|^{\alpha+1}} \int_{-1}^{1} |P_n(x)|^2 |x|^\alpha dx \\
&\quad= (1+\eta)^{\alpha+2} \frac{w(x_0)}{|T_N'(x_0)|^{\alpha+1}} \lambda_n(\mu_\alpha^b, \xi/n).
\end{aligned}$$

On the other hand, as implied by the Erdős-Turán criterion, μ_α^b is regular in the sense of Stahl-Totik, hence for every $\tau > 0$

$$\|P_n\|_{[-1,1]} \leq (1+\tau)^n \|P_n\|_{L^2(\mu_\alpha^b)} \leq C(1+\tau)^n$$

holds for all large n, where we used the extremality of P_n with respect to $\lambda_n(\mu_\alpha^b, \xi/n)$ and (3.21). The polynomial $S_{n,x_0+\xi_n,K}(x)$ defined by (5.11) is decreasing exponentially fast, that is

$$\|S_{n,x_0+\xi_n,K}\|_{K\setminus[x_0-\delta,x_0+\delta]} \le \gamma^n$$

holds for some $|\gamma| < 1$ if n is large enough. If τ is selected so small that $q = (1+\tau)\gamma < 1$, then

$$\int_{K\setminus[x_0-\delta,x_0+\delta]} |R_n(x)|^2 d\mu(x) = O(q^n),$$

that is, this residual integral is also decreasing exponentially fast. Combining these estimates, it follows that

$$\lambda_{nN+2\lfloor\eta n\rfloor}(\mu, x_0+\xi_n) \le O(q^n) + (1+\eta)^{\alpha+2}\frac{w(x_0)}{|T_N'(x_0)|^{\alpha+1}}\lambda_n(\mu_\alpha^b, \xi/n).$$

This is almost what we need. Since

$$\begin{aligned}\frac{\xi}{|T_N'(x_0)|n} &= \frac{\xi N}{|T_N'(x_0)|(nN+2\lfloor\eta n\rfloor)}(1+2\eta/N) + \frac{2\xi}{|T_N'(x_0)|}\frac{\frac{\lfloor\eta n\rfloor}{n}-\eta}{nN+2\lfloor\eta n\rfloor} \\ &= \frac{\xi N}{|T_N'(x_0)|(nN+2\lfloor\eta n\rfloor)}(1+2\eta/N) + o(n^{-1}),\end{aligned} \tag{5.12}$$

it follows from Lemma 5.1, (5.10) and (5.12) that

$$\begin{aligned}&\limsup_{n\to\infty}(nN+2\lfloor\eta n\rfloor)^{\alpha+1}\lambda_{nN+2\lfloor\eta n\rfloor}(\mu, x_0+\xi_n) \\ &\qquad = \limsup_{n\to\infty}(nN+2\lfloor\eta n\rfloor)^{\alpha+1} \\ &\qquad\qquad \times \lambda_{nN+2\lfloor\eta n\rfloor}\left(\mu, x_0 + \frac{\xi N(1+2\eta/N)}{|T_N'(x_0)|(nN+2\lfloor\eta n\rfloor)}\right).\end{aligned} \tag{5.13}$$

If k is selected such that $nN+2\lfloor\eta n\rfloor \le k \le (n+1)N+2\lfloor\eta(n+1)\rfloor$, we have

$$\begin{aligned}&\left(\frac{nN+2\lfloor\eta n\rfloor}{k}\right)^{\alpha+1}k^{\alpha+1}\lambda_k\left(\mu, x_0+\frac{\xi N(1+2\eta/N)}{|T_N'(x_0)|(nN+2\lfloor\eta n\rfloor)}\right) \\ &\quad \le (nN+2\lfloor\eta n\rfloor)^{\alpha+1}\lambda_{nN+2\lfloor\eta n\rfloor}\left(\mu, x_0+\frac{\xi N(1+2\eta/N)}{|T_N'(x_0)|(nN+2\lfloor\eta n\rfloor)}\right).\end{aligned} \tag{5.14}$$

Since $(nN+2\lfloor\eta n\rfloor)/k = 1+o(1)$, these estimates, along with Lemma 5.1, (2.12) and (3.21) imply

$$\begin{aligned}&\limsup_{k\to\infty} k^{\alpha+1}\lambda_k\left(\mu, x_0+\frac{\xi N(1+2\eta/N)}{|T_N'(x_0)|k}\right) \\ &\quad \le \limsup_{k\to\infty}(nN+2\lfloor\eta n\rfloor)^{\alpha+1}\lambda_{nN+2\lfloor\eta n\rfloor}(\mu, x_0+\xi_n) \\ &\quad \le \limsup_{k\to\infty}\left(1+\frac{2\lfloor\eta n\rfloor}{nN}\right)^{\alpha+1}(1+\eta)^{\alpha+2}\frac{w(x_0)N^{\alpha+1}}{|T_N'(x_0)|^{\alpha+1}}n^{\alpha+1}\lambda_n(\mu_\alpha^b, \xi/n) \\ &\quad = (1+2\eta/N)^{\alpha+1}(1+\eta)^{\alpha+2}\frac{w(x_0)}{(\pi\omega_{E_N}(x_0))^{\alpha+1}}\left(\mathbb{L}_\alpha^*(\xi)\right)^{-1},\end{aligned} \tag{5.15}$$

which, by selecting $a = \frac{\xi(1+2\eta/N)}{\pi\omega_{E_n}(x_0)}$, gives

$$\limsup_{k\to\infty} \; k^{\alpha+1}\lambda_k\left(\mu, x_0+\frac{a}{k}\right) \\ \leq (1+2\eta/N)^{\alpha+1}(1+\eta)^{\alpha+2}\frac{w(x_0)}{(\pi\omega_{E_N}(x_0))^{\alpha+1}}\left(\mathbb{L}_\alpha^*\left(\frac{\pi\omega_{E_N}(x_0)a}{1+2\eta/N}\right)\right)^{-1}. \tag{5.16}$$

Since η was arbitrary and $\mathbb{L}_\alpha^*(\cdot)$ is continuous, we have

$$\limsup_{k\to\infty} k^{\alpha+1}\lambda_k\left(\mu, x_0+\frac{a}{k}\right) \leq \frac{w(x_0)}{(\pi\omega_{E_N}(x_0))^{\alpha+1}}\Big(\mathbb{L}_\alpha^*\big(\pi\omega_{E_N}(x_0)a\big)\Big)^{-1}. \tag{5.17}$$

The approximating set E_N was selected such that $\omega_{E_N}(x_0)$ is arbitrarily close to $\omega_K(x_0)$, therefore this gives us the desired upper estimate

$$\limsup_{k\to\infty} k^{\alpha+1}\lambda_k\left(\mu, x_0+\frac{a}{k}\right) \leq \frac{w(x_0)}{(\pi\omega_{K}(x_0))^{\alpha+1}}\Big(\mathbb{L}_\alpha^*\big(\pi\omega_{K}(x_0)a\big)\Big)^{-1}. \tag{5.18}$$

Note that since (3.21) is uniform for a in compact subsets of the real line, this upper estimate is also uniform.

Lower estimate for sets regular with respect to the Dirichlet problem. For the upper estimate the Stahl-Totik regularity of μ was not used. However, it will be needed for the lower estimate, therefore we prove it first for sets regular with respect to the Dirichlet problem to reduce technical difficulties. If a set is such, the Stahl-Totik regularity for a measure supported there gives us the uniform estimate (2.3). Therefore assume that K is regular with respect to the Dirichlet problem. Let $\eta > 0$ be arbitrary but fixed, moreover let $\delta_1 > 0$ so small such that

$$(1) \quad \frac{1}{1+\eta}w(x_0) \leq w(x) \leq (1+\eta)w(x_0) \tag{5.19}$$

holds for all $x \in [x_0-\delta_1, x_0+\delta_1]$. Now let $E_N = \cup_{k=0}^{N-1}[a_k, b_k]$ be the approximating set for K and T_N be the matching admissible polynomial given by Lemma 2.2. We can assume without loss of generality that $x_0 \in (a_0, b_0)$. At the moment, the ε which controls the distance of E_N and K is arbitrary, but soon we'll select this parameter according to our purpose. Assume that P_n is extremal for $\lambda_n(\mu, x_0+a/n)$. Let

$$R_n(x) = P_n(x)S_{n,x_0+a/n,E}(x),$$

where $S_{n,x_0+a/n,E}(x)$ is defined similarly as in (5.11), i.e. let $E = [-m, m]$ be an interval so large such that $K \subseteq [-m/2, m/2]$ and for an arbitrary $\eta > 0$ define

$$S_{n,x_0+a/n,E}(x) = \left(1-\left(\frac{x_0+a/n-x}{2m}\right)^2\right)^{\lfloor \eta n\rfloor}.$$

The large interval $E = [-m, m]$ is needed to avoid dependence of $S_{n,x_0+a/n,E}$ on the approximating set E_N. We only need $S_{n,x_0+a/n,E}$ to be fast decreasing on E_N, but we also want to make sure that the rate of decrease does not depend on E_N, because actually the set E_N will be choosen to fit the rate of decrease of $S_{n,x_0+a/n,E}(x)$.

Because μ is regular in the sense of Stahl-Totik, (2.3) gives that for arbitrary $\tau > 0$,

$$\|P_n\|_K \leq (1+\tau)^n \|P_n\|_{L^2(\mu)}$$

holds if n is large enough. Now the Bernstein-Walsh lemma, see [32, Theorem 5.5.7a], says that if E_N is selected accordingly (that is, the Hausdorff distance $\operatorname{dist}(E_N, K)$ is small enough), we have

$$\|P_n\|_{E_N} \leq (1+\tau)^n \|P_n\|_K$$

Overall, since $\sup_{x \in E_N \setminus [x_0-\delta_1, x_0+\delta_1]} |S_{n,x_0+a/n,E}(x)| \leq \gamma^n$ for some $\gamma < 1$,

$$\|R_n\|_{E_N \setminus [x_0-\delta_1, x_0+\delta_1]} \leq (1+\tau)^{2n} \gamma^n \|P_n\|_{L^2(\mu)} \leq (1+\tau)^{2n} \gamma^n \tag{5.20}$$

holds, where in the final step we used the extremality of P_n. Now select τ such that $q = (1+\tau)^2 \gamma < 1$. Note that this means fixing E_N, because small τ can be achieved if $\operatorname{dist}(E_N, K)$ is small enough in Lemma 2.2.

Let $\delta_2 > 0$ be so small such that $\delta_2 < \delta_1$, moreover $[x_0 - \delta_2, x_0 + \delta_2] \subseteq [a_0, b_0]$ and

$$\begin{aligned} &(2) \quad \frac{1}{1+\eta} |T_N'(x_0)| \leq |T_N'(x)| \leq (1+\eta) |T_N'(x_0)|, \\ &(3) \quad \frac{1}{1+\eta} |T_N'(x_0)||x - x_0| \leq |T_N(x)| \leq (1+\eta) |T_N'(x_0)||x - x_0| \end{aligned} \tag{5.21}$$

holds for all $x \in [x_0 - \delta_2, x_0 + \delta_2]$. Since $w(x)|x - x_0|^\alpha$ is bounded from above and below on the intervals $[x_0 - \delta_1, x_0 - \delta_2]$ and $[x_0 + \delta_2, x_0 + \delta_1]$, Nikolskii's inequality can be used, see [12, Chapter 4, Theorem 2.6], which gives

$$\|P_n\|_{[x_0-\delta_1, x_0+\delta_1] \setminus [x_0-\delta_2, x_0+\delta_2]} \leq Cn \|P_n\|_{L^2(\mu)} \leq Cn,$$

for some constant C, where again the extremality of P_n was used. It follows that we have

$$\|R_n\|_{[x_0-\delta_1, x_0+\delta_1] \setminus [x_0-\delta_2, x_0+\delta_2]} \leq Cr^n n \tag{5.22}$$

for some $|r| < 1$, which is dependent on E_N through δ_2. Inside the interval $[x_0 - \delta_2, x_0 + \delta_2]$, the Nikolskii-type inequality [4, Lemma 2.7] for generalized Jacobi weights can be used to obtain

$$\|R_n\|_{[x_0-\delta_2, x_0+\delta_2]} \leq Cn^{\max\{1/2, (1+\alpha)/2\}}. \tag{5.23}$$

For arbitrary $y \in E_N$ we introduce the notation

$$\{y_0, y_1, \dots, y_{N-1}\} = T_N^{-1}(T_N(y)).$$

It can be assumed without loss of generality that $y_k \in [a_k, b_k]$. Define

$$R_n^*(y) = \sum_{k=0}^{N-1} R_n(y_k).$$

R_n^* is a polynomial of degree less than $n + 2\lfloor \eta n \rfloor$ and there exists a polynomial V_n of degree at most $(n + 2\lfloor \eta n \rfloor)/N$ such that

$$R_n^*(y) = V_n(T_N(y)). \tag{5.24}$$

The proof of this fact can be found, for example in [44, Section 5]. For R_n^* it is also true that

$$\begin{aligned} &|R_n^*(y)|^2 = |R_n(y)|^2 + O(q^n), \quad y \in [x_0 - \delta_2, x_0 + \delta_2] \\ &|R_n^*(y)|^2 = O(q^n), \quad y \in [a_0, b_0] \setminus [x_0 - \delta_2, x_0 + \delta_2] \end{aligned} \tag{5.25}$$

holds for some $|q| < 1$. Indeed, in general, we have

$$|R_n^*(y)|^2 \le \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} |R_n(y_l)||R_n(y_k)|. \tag{5.26}$$

Because among the values $\{y_0, y_1, \dots, y_{N-1}\} = T_N^{-1}(T_N(y))$ only $y = y_0$ is contained in $[a_0, b_0]$, the estimates (5.20) and (5.22) gives that all terms $|R_n(y_l)|$ with the possible exception of $|R_n(y_0)|$ are exponentially small, which gives (5.25).

Now we can proceed to estimate the Christoffel functions. Let α_n be the unique element in $[-1, 1]$ such that $T_N(x_0 + a/n) = \alpha_n$. Using the Taylor expansion of $T_N(x)$, it is clear that

$$\alpha_n = T_N'(x_0)\frac{a}{n} + o(n^{-1}).$$

For the polynomial $V_n(x)$ defined in (5.24), according to (5.25) we have $V_n(\alpha_n) = 1 + o(1)$, which implies

$$\begin{aligned} (1 + o(1))\lambda_{\deg(V_n)}(\mu_\alpha^b, \alpha_n) &\le \int_{-1}^{1} |V_n(x)|^2 |x|^\alpha dx \\ &= \int_{a_0}^{b_0} |V_n(T_N(x))|^2 |T_N(x)|^\alpha |T_N'(x)| dx \\ &= \int_{x_0-\delta_2}^{x_0+\delta_2} |R_n^*(x)|^2 |T_N(x)|^\alpha |T_N'(x)| dx \\ &\quad + \left(\int_{a_0}^{x_0-\delta_2} + \int_{x_0+\delta_2}^{b_0} \right) |R_n^*(x)|^2 |T_N(x)|^\alpha |T_N'(x)| dx. \end{aligned}$$

On the one hand, using (5.19), (5.21) and (5.25) we have

$$\begin{aligned}
&\int_{x_0-\delta_2}^{x_0+\delta_2} |R_n^*(x)|^2 |T_N(x)|^\alpha |T_N'(x)| dx \\
&\qquad \le (1+\eta)^{\alpha+2} \frac{|T_N'(x_0)|^{\alpha+1}}{w(x_0)} \int_{x_0-\delta_2}^{x_0+\delta_2} \big(O(q^n) + |R_n(x)|^2\big) w(x) |x-x_0|^\alpha dx \\
&\qquad \le O(q^n) + (1+\eta)^{\alpha+2} \frac{|T_N'(x_0)|^{\alpha+1}}{w(x_0)} \int_{x_0-\delta_2}^{x_0+\delta_2} |P_n(x)|^2 w(x) |x-x_0|^\alpha dx \\
&\qquad \le O(q^n) + (1+\eta)^{\alpha+2} \frac{|T_N'(x_0)|^{\alpha+1}}{w(x_0)} \lambda_n(\mu, x_0 + a/n).
\end{aligned}$$

On the other hand, (5.25) also implies that

$$\left(\int_{a_0}^{x_0-\delta_2} + \int_{x_0+\delta_2}^{b_0} \right) |R_n^*(x)|^2 |T_N(x)|^\alpha |T_N'(x)| dx = O(q^n),$$

therefore the combination of these two estimates gives

$$(1+o(1)) \lambda_{\deg(V_n)}(\mu_\alpha^b, \alpha_n) \le O(q^n) + (1+\eta)^{\alpha+2} \frac{|T_N'(x_0)|^{\alpha+1}}{w(x_0)} \lambda_n(\mu, x_0 + a/n).$$

Similarly as in (5.12) - (5.18), this implies the lower estimate

$$\frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}} \Big(\mathbb{L}_\alpha^*\big(\pi\omega_K(x_0)a\big) \Big)^{-1} \le \liminf_{n\to\infty} \lambda_n\left(\mu, x_0 + \frac{a}{n}\right),$$

which holds if K is regular with respect to the Dirichlet problem. Note again that since (3.21) is uniform for a in compact subsets of the real line, this upper estimate is also uniform.

Lower estimate for general sets. Now we omit the assumption that $\mathbb{C}\setminus K$ is regular with respect to the Dirichlet problem. To overcome the problem caused by this, we apply an idea from [43]. For every $\tau > 0$ and $m \in \mathbb{N}$ define the the set

$$K_{\tau,m} = \left\{ x \in K : \sup_{\deg(Q_n)=n} \frac{|Q_n(x)|}{\|Q_n\|_{L^2(\mu)}} \le (1+\tau)^n, n \ge m \right\}.$$

$F_{m,\tau}$ is compact, $F_{m,\tau} \subset F_{m+1,\tau}$, moreover, since μ is regular in the sense of Stahl-Totik, we have $\cup_{m=1}^\infty F_{m,\tau} = K \setminus H$, where H is a set of zero logarithmic capacity. Let $\theta > 0$ be arbitrary and choose m so large such that $\mathrm{cap}(F_{m,\tau}) > \mathrm{cap}(K) - \theta/2$. Ancona's theorem says, see [2], that there is a set $K_\theta^* \subseteq F_{m,\tau}$ such that K_θ^* is regular with respect to the Dirichlet problem and

$$\mathrm{cap}(K_m^*) > \mathrm{cap}(F_{m,\tau}) - \theta/2 > \mathrm{cap}(K) - \theta$$

holds. Define $K_\theta = K_\theta^* \cup [x_0 - \varepsilon, x_0 + \varepsilon]$, where $\varepsilon > 0$ is so small such that μ is absolutely continuous there and $K_\theta \subseteq K$. Now K_θ is regular with respect to the Dirichlet problem, and due to the construction of K_θ,

$$\frac{\|Q_n\|_{K_\theta}}{\|Q_n\|_{L^2(\mu)}} \leq (1+\tau)^{\deg(Q_n)}$$

holds for an arbitrary sequence of nonzero polynomials $\{Q_n\}_{n=1}^\infty$ if n is large enough. From this point, proceeding similarly as in the case of sets regular with respect to the Dirichlet problem, we obtain

$$\frac{w(x_0)}{(\pi\omega_{K_\theta}(x_0))^{\alpha+1}}\Big(\mathbb{L}_\alpha^*\big(\pi\omega_{K_\theta}(x_0)a\big)\Big)^{-1} \leq \liminf_{n\to\infty} \lambda_n\Big(\mu, x_0 + \frac{a}{n}\Big).$$

Since [43, Lemma 4.2] implies that $\omega_{K_\theta}(x_0) \to \omega_K(x_0)$ as $\theta \to 0$, and since $\theta > 0$ was arbitrary, the desired lower estimate

$$\frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}}\Big(\mathbb{L}_\alpha^*\big(\pi\omega_K(x_0)a\big)\Big)^{-1} \leq \liminf_{n\to\infty} \lambda_n\Big(\mu, x_0 + \frac{a}{n}\Big) \tag{5.27}$$

follows. (5.18) and (5.27) gives (1.18), which completes the proof of Theorem 1.2.

□

5.3 Christoffel functions at the edge

Our aim now is to prove Theorem 1.3. Let K be a compact subset of the real line and suppose that $x_0 \in K$ is a right endpoint, i.e. there is an $\varepsilon_1 > 0$ such that $K \cap (x_0, x_0+\varepsilon_1) = \varnothing$ and $K \cap [x_0 - \varepsilon_1, x_0] = [x_0 - \varepsilon_1, x_0]$. Let μ be a measure with $\operatorname{supp}(\mu) = K$ and suppose that μ is absolutely continuous in $(x_0 - \varepsilon_0, x_0]$ for some $\varepsilon_0 > 0$ and

$$d\mu(x) = w(x)|x - x_0|^\alpha dx, \quad x \in (x_0 - \varepsilon_0, x_0]$$

there, where $\alpha > -1$ and $w(x)$ is strictly positive and left-continuous in x_0. When x_0 is a right endpoint, the density of the equilibrium measure is undefined there, but a related quantity takes its place instead. The behavior of the equilibrium density $\omega_K(x)$ at an endpoint of K can be quantified as

$$M(K, x_0) = \lim_{x \to x_0-} \sqrt{2}\pi|x - x_0|^{1/2}\omega_K(x).$$

This quantity is finite and well defined in our case. (The constant $\sqrt{2}\pi$ is usually not incorporated in the definition of $M(K, x_0)$, but we have found it more convenient to do so.) It has appeared several times in the literature, for example it was shown by Totik that this is the asymptotically best possible constant in Markov inequalities for polynomials in several intervals, see [42, Theorem 4.1]. To show (1.19), we shall again prove matching upper and lower estimates. In order to avoid excessive repetition, we only discuss the upper estimate, with an emphasis on the differences. The lower estimate works similarly, aside from the same differences.

As in the bulk, let $\eta > 0$ be arbitrary and let $E_N = \cup_{k=0}^{N-1}[a_k, b_k] = T_N^{-1}([-1,1])$ and T_N be the approximating set and the matching admissible polynomial granted by Lemma 2.3. We can assume without the loss of generality that $x_0 = b_0$, as it is implied by [42, Theorem 2.1] and the remark after it. Select a $\delta > 0$ so small such that

$$\begin{aligned} &(1) \quad \frac{1}{1+\eta} w(x_0) \leq w(x) \leq (1+\eta) w(x_0), \\ &(2) \quad \frac{1}{1+\eta} |T_N(x_0) - 1| \leq |T_N'(x)||x - x_0| \leq (1+\eta)|T_N(x_0)|, \\ &(3) \quad \frac{1}{1+\eta} |T_N'(x_0)| \leq |T_N'(x)| \leq (1+\eta)|T_N'(x_0)| \end{aligned} \tag{5.28}$$

holds for all $x \in [x_0 - \delta, x_0]$. Let $\xi \in [0, \infty)$ be arbitrary and let $x_0 - \xi_n$ be the unique element of $[a_0, b_0]$ such that $T_N(x_0 - \xi_n) = 1 - \xi/(2n^2)$. Since T_N is a polynomial, we have

$$1 - \frac{\xi}{2n^2} = T_N(x_0 - \xi_n) = 1 - T_N'(x_0)\xi_n + o(n^{-2}),$$

which implies

$$\xi_n = \frac{\xi}{|T_N'(x_0)| 2n^2} + o(n^{-2}).$$

Assume that P_n is extremal for $\lambda_n\big(\mu_\alpha^e, 1 - \frac{\xi}{2n^2}\big)$ and define

$$R_n(x) = P_n(T_N(x)) S_{n, x_0 - \xi_n, K}(x),$$

where $S_{n,x_0-\xi_n,K}(x)$ is defined as

$$S_{n,x_0+\xi_n,K}(x) = \left(1 - \left(\frac{x_0 - \xi_n - x}{\operatorname{diam}(K)}\right)^2\right)^{\lfloor \eta n \rfloor},$$

as usual. Then R_n is a polynomial of degree less than $nN + 2\lfloor \eta n \rfloor$ with $R_n(x_0 - \xi_n) = 1$. Then, similarly as before, (5.28) gives

$$\begin{aligned} \lambda_{nN+2\lfloor \eta n \rfloor}(\mu, x_0 - \xi_n) \leq & \int_{x_0-\delta}^{x_0} |R_n(x)|^2 w(x) |x - x_0|^\alpha dx \\ & + \int_{K \setminus [x_0 - \delta, x_0]} |R_n(x)|^2 w(x) |x - x_0|^\alpha dx \\ \leq & \, O(q^n) + \frac{(1+\eta)^{\alpha+1} w(x_0)}{|T_N'(x_0)|^{\alpha+1}} \lambda_n\left(\mu_\alpha^e, 1 - \frac{\xi}{2n^2}\right). \end{aligned}$$

Now the application of Lemma 2.10 and Lemma 5.2 yields that

$$\limsup_{n\to\infty}(nN+2\lfloor\eta n\rfloor)^{2\alpha+2}\lambda_{nN+2\lfloor\eta n\rfloor}(\mu,x_0-\xi_n)$$
$$=\limsup_{k\to\infty}k^{2\alpha+2}\lambda_k\left(\mu,x_0-\frac{(1+\eta/N)^2\xi}{2M(E_N,x_0)^2k^2}\right).$$

which, along with the previous estimate, by selecting $a=\frac{(1+2\eta/N)^2\xi}{M(E_N,x_0)^2}$ implies

$$\limsup_{k\to\infty}k^{2\alpha+2}\lambda_k\left(\mu,x_0-\frac{a}{2k^2}\right)$$
$$\leq\frac{(1+\eta)^{\alpha+1}(1+2\eta/N)^{2\alpha+2}w(x_0)}{M(E_N,x_0)^{2\alpha+2}}\left(2^{\alpha+1}\mathbb{J}^*_\alpha\left(\frac{M(E_N,x_0)^2}{(1+2\eta/N)^2}a\right)\right)^{-1}.$$

Since η was arbitrary and E_N was choosen such that Lemma 2.2 (d) holds, this implies the desired upper estimate

$$\limsup_{k\to\infty}k^{2\alpha+2}\lambda_k\left(\mu,x_0-\frac{a}{2k^2}\right)\leq\frac{w(x_0)}{M(K,x_0)^{2\alpha+2}}\left(2^{\alpha+1}\mathbb{J}^*_\alpha\big(M(K,x_0)^2a\big)\right)^{-1}. \tag{5.29}$$

The lower estimate

$$\frac{w(x_0)}{M(K,x_0)^{2\alpha+2}}\left(2^{\alpha+1}\mathbb{J}^*_\alpha\big(M(K,x_0)^2a\big)\right)^{-1}\leq\liminf_{k\to\infty}k^{2\alpha+2}\lambda_k\left(\mu,x_0-\frac{a}{2k^2}\right) \tag{5.30}$$

can be obtained as we did in Theorem 1.2, except of course with the same differences which also appeared at the upper estimate as well. Finally, (5.29) and (5.30) gives (1.19), and this is what we had to prove.

□

6 Universality limits

Our aim in this section is to prove Theorems 1.4 and 1.5. Theorem 1.5 is a direct corollary of Theorem 1.3 using the result [21, Theorem 1.2]. To prove Theorem 1.4, we employ the second method of Lubinsky which is based upon the theory of entire functions of exponential type. We say that an entire function $g(z)$ is of *order* ρ if

$$\rho=\limsup_{r\to\infty}\frac{\log\big(\log\big(\sup_{|z|=r}|g(z)|\big)\big)}{\log r}.$$

An entire function of order 1 is said to be of the *exponential type* σ if

$$\sigma=\limsup_{r\to\infty}\frac{\sup_{|z|=r}\log|g(z)|}{r}.$$

If $g(z)$ is of the exponential type, it belongs to the *Cartwright class* if

$$\int_{-\infty}^{\infty} \frac{\log^+ |g(x)|}{1+x^2} dx < \infty.$$

A sequence of entire functions $\{g_n(z)\}_{n=1}^{\infty}$ is said to be *normal*, if every subsequence contains a subsequence which converges uniformly on compact subsets of the complex plane. It is known, see [33, Theorem 14.6], that if $\{g_n(z)\}_{n=1}^{\infty}$ is uniformly bounded on each compact subset of the complex plane, then it is normal.

In this section we follow the lines of [21]. First we develop reproducing identities for the kernel function $\mathbb{L}_\alpha^*$, then we use the theory of entire functions of exponential type to deduce universality limits from Theorem 1.2.

6.1 Reproducing kernel identities for $\mathbb{L}_\alpha^*$

Theorem 6.1. *Let g be an entire function of exponential type 1 and suppose that we have $|x|^{\alpha/2} g(x) \in L^2(\mathbb{R})$ for some $\alpha > -1$. Then*

$$g(z) = \int_{-\infty}^{\infty} g(s) \mathbb{L}_\alpha^*(z,s) |s|^\alpha ds \tag{6.1}$$

holds for all $z \in \mathbb{C}$.

The proof of Theorem 6.1, given in the next lemma, is almost verbatim to the proof of [21, Theorem 6.1], therefore we shall not carry it out in detail. It depends on Lemma 6.2, which is an analogue of [21, Lemma 6.2].

Lemma 6.2. *Let $\alpha > -1$.*

(a) For all $a, b \in \mathbb{R}$ we have

$$\mathbb{L}_\alpha^*(a,b) = \int_{-\infty}^{\infty} \mathbb{L}_\alpha^*(a,s) \mathbb{L}_\alpha^*(s,b) |s|^\alpha ds. \tag{6.2}$$

(b) If $\{j_{\alpha,k}\}_{k=-\infty}^{\infty}$ denotes the zeros of $x^{-\alpha} J_\alpha(x)$, then

$$\int_{-\infty}^{\infty} \mathbb{L}_\alpha^*(j_{\frac{\alpha-1}{2},k}, x) \mathbb{L}_\alpha^*(x, j_{\frac{\alpha-1}{2},l}) |x|^\alpha dx = \delta_{k,l} \mathbb{L}_\alpha^*(j_{\frac{\alpha-1}{2},k}, j_{\frac{\alpha-1}{2},l}).$$

(c) Let $\{c_k\}_{k=-\infty}^{\infty} \in l^2(\mathbb{Z})$. Then

$$\int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_k \frac{\mathbb{L}_\alpha^*(j_{\frac{\alpha-1}{2},k}, x)}{\sqrt{\mathbb{L}_\alpha^*(j_{\frac{\alpha-1}{2},k}, j_{\frac{\alpha-1}{2},k})}} \right)^2 |x|^\alpha dx = \sum_{k=-\infty}^{\infty} c_k^2.$$

(d) Let g be an entire function of exponential type 1. If $|x|^{\alpha/2}g(x) \in L^2(\mathbb{R})$, then

$$g(z) = \sum_{k=-\infty}^{\infty} g(j_{\frac{\alpha-1}{2},k}) \frac{\mathbb{L}^*_\alpha(j_{\frac{\alpha-1}{2},k}, z)}{\mathbb{L}^*_\alpha(j_{\frac{\alpha-1}{2},k}, j_{\frac{\alpha-1}{2},k})}$$

holds for all $z \in \mathbb{C}$, and the series converge uniformly on compact sets.

Proof. (a) This proof was kindly provided to us by D. S. Lubinsky [22]. Using the reproducing kernel relations for $K_n(\mu^b_\alpha, x, y)$, where μ^b_α is defined by (3.1), we have

$$K_n(\mu^b_\alpha, a/n, b/n) = \int_{-1}^{1} K_n(\mu^b_\alpha, a/n, x) K_n(\mu^b_\alpha, x, b/n)|x|^\alpha dx.$$

Substituting $x = s/n$, the asymptotic formula (3.4) implies

$$\begin{aligned}
\mathbb{L}^*_\alpha(a, b) &= \frac{1}{n^{\alpha+1}} \int_{-1}^{1} K_n(\mu^b_\alpha, a/n, x) K_n(\mu^b_\alpha, x, b/n)|x|^\alpha dx + o(1) \\
&= \frac{1}{n^{\alpha+1}} \Big(\int_{-1}^{-r/n} + \int_{-r/n}^{r/n} + \int_{r/n}^{1} \Big) K_n(\mu^b_\alpha, a/n, x) K_n(\mu^b_\alpha, x, b/n)|x|^\alpha dx + o(1) \\
&= \int_{-r}^{r} \mathbb{L}^*_\alpha(a, s)\mathbb{L}^*_\alpha(s, b) ds \\
&\quad + \frac{1}{n^{\alpha+1}} \Big(\int_{-1}^{-r/n} + \int_{r/n}^{1} \Big) K_n(\mu^b_\alpha, a/n, x) K_n(\mu^b_\alpha, x, b/n)|x|^\alpha dx + o(1).
\end{aligned}$$

We will show that the last integrals are small in terms of n and r. To do this, we shall use Pollard's decomposition of the Christoffel-Darboux kernel. According to [48, (4.6)-(4.8)] and the formula after, we have

$$K_n(\mu^b_\alpha, x, y) = K_{n,1}(\mu^b_\alpha, x, y) + K_{n,2}(\mu^b_\alpha, x, y) + K_{n,3}(\mu^b_\alpha, x, y),$$

where

$$\begin{aligned}
K_{n,1}(\mu^b_\alpha, x, y) &= a_n p_n(x) p_n(y), \\
K_{n,2}(\mu^b_\alpha, x, y) &= b_n \frac{(1-y^2)p_n(x)q_{n-1}(y)}{x-y}, \\
K_{n,3}(\mu^b_\alpha, x, y) &= b_n \frac{(1-x^2)p_n(y)q_{n-1}(x)}{y-x},
\end{aligned}$$

where a_n, b_n are bounded constants, $p_n(x)$ is the n-th orthonormal polynomial with respect to the measure $|x|^\alpha dx$ and $q_n(x)$ is the n-th orthonormal polynomial with respect to the measure $(1-x^2)|x|^\alpha dx$. Using [30, Lemma 29, p. 170], we obtain the estimates

$$\begin{aligned}
p_n(x)^2 &\le (\sqrt{1-x} + 1/n)^{-1} (\sqrt{1+x} + 1/n)^{-1} (|x| + 1/n)^{-\alpha}, \\
q_n(x)^2 &\le (\sqrt{1-x} + 1/n)^{-3} (\sqrt{1+x} + 1/n)^{-3} (|x| + 1/n)^{-\alpha}.
\end{aligned}$$

Now the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_n(\mu_\alpha^b, a/n, x)K_n(\mu_\alpha^b, x, b/n)|x|^\alpha dx \\
&\quad\le \left(\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_n(\mu_\alpha^b, a/n, x)^2|x|^\alpha dx\right)^{1/2} \\
&\qquad\times \left(\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_n(\mu_\alpha^b, x, b/n)^2|x|^\alpha dx\right)^{1/2}.
\end{aligned}$$

Suppose that $r > \max\{a, b\}$. We have the following estimates. (In the following calculations the constant c often varies from line to line.)

$$\begin{aligned}
&\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_{n,1}(\mu_\alpha^b, a/n, x)^2|x|^\alpha dx \\
&\quad\le \frac{c}{n^{\alpha+1}}\int_{r/n}^{1} |p_n(a/n)|^2|p_n(x)|^2|x|^\alpha dx \\
&\quad\le \frac{c}{n}\int_{r/n}^{1} |p_n(x)|^2|x|^\alpha dx \\
&\quad\le \frac{c}{n}
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_{n,2}(\mu_\alpha^b, a/n, x)^2|x|^\alpha dx \\
&\quad\le \frac{c}{n^{\alpha+1}}\int_{r/n}^{1} |p_n(a/n)|^2\left|\frac{(1-x^2)q_{n-1}(x)}{x-a/n}\right|^2|x|^\alpha dx \\
&\quad\le \frac{c}{n}\int_{r/n}^{1/2} |x|^{-2}dx + \frac{c}{n}\int_{1/2}^{1} |1-x|^{1/2}dx \\
&\quad\le \frac{c}{r}+\frac{c}{n}
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
&\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_{n,3}(\mu_\alpha^b, a/n, x)dx \\
&\quad\le \frac{c}{n^{\alpha+1}}\int_{r/n}^{1} |1-a^2/n^2|^2|q_{n-1}(a/n)|^2\left|\frac{p_n(x)}{x-a/n}\right|^2|x|^\alpha dx \\
&\quad\le \frac{c}{n}\int_{r/n}^{1/2} |x|^{-2}dx + \frac{c}{n}\int_{1/2}^{1} |1-x|^{-1/2}dx \\
&\quad\le \frac{c}{r}+\frac{c}{n}
\end{aligned} \tag{6.5}$$

These altogether give that

$$\frac{1}{n^{\alpha+1}}\int_{r/n}^{1} K_n(\mu_\alpha^b, a/n, x)K_n(\mu_\alpha^b, x, b/n)|x|^\alpha dx \le c\Big(\frac{1}{n}+\frac{1}{r}\Big).$$

Overall, we have

$$\mathbb{L}_\alpha^*(a,b) = \int_{-r}^{r} \mathbb{L}_\alpha^*(a,s)\mathbb{L}_\alpha^*(s,b)|s|^\alpha ds + O(1)\Big(\frac{1}{n}+\frac{1}{r}\Big),$$

from which (6.2) follows by letting first n then r to infinity.

(b) is a simple consequence of (a) and the proofs of (c)-(d) go through verbatim as in [21, Lemma 6.2]. $\square$

6.2 Limits of K_n

From now on, $K_n(z,w)$ will always denote the n-th Christoffel-Darboux kernel with respect to the measure μ in Theorem 1.2. Define

$$f_n(a,b) = \frac{K_n\big(x_0 + \frac{a}{\pi\omega_K(x_0)n}, x_0 + \frac{b}{\pi\omega_K(x_0)n}\big)}{K_n(x_0,x_0)}, \quad a,b \in \mathbb{C}. \tag{6.6}$$

For convenience we shall use the notation

$$z^* = \frac{z}{\pi\omega_K(x_0)} \tag{6.7}$$

for all z in the complex plane, so this way $f_n(a,b)$ takes the form

$$f_n(a,b) = \frac{K_n(x_0 + a^*/n, x_0 + b^*/n)}{K_n(x_0,x_0)}.$$

First shall prove that $\{f_n(a,b)\}_{n=1}^\infty$ is a normal family of entire functions in both variable and then we will study its possible limits.

Lemma 6.3. *For all $a,b \in \mathbb{C}$, we have*

$$|f_n(a,b)| \le c_1 e^{c_2(|\operatorname{Im}(a)|+|\operatorname{Im}(b)|)}(1+|\operatorname{Re}(a)|)^{-\alpha/2}(1+|\operatorname{Re}(b)|)^{-\alpha/2} \tag{6.8}$$

for some positive constants c_1, c_2. In particular, $\{f_n(a,b)\}_{n=1}^\infty$ is a normal family of functions for a,b in compact subsets of the complex plane.

Proof. Since by definition (note that the complex conjugate has been left off for purpose) we have

$$K_n(z,w) = \sum_{k=0}^{n-1} p_k(z)p_k(w),$$

the Cauchy-Schwarz inequality in $\mathbb{R}^n$ implies

$$\begin{aligned}
\left|K_n\Big(x_0 + \frac{a^*}{n}, x_0 + \frac{b^*}{n}\Big)\right|^2 &\le \left(\sum_{k=0}^{n-1}\left|p_k\Big(x_0+\frac{a^*}{n}\Big)\right|\left|p_k\Big(x_0+\frac{b^*}{n}\Big)\right|\right)^2 \\
&\le \left(\sum_{k=0}^{n-1}\left|p_k\Big(x_0+\frac{a^*}{n}\Big)\right|^2\right)\left(\sum_{k=0}^{n-1}\left|p_k\Big(x_0+\frac{b^*}{n}\Big)\right|^2\right) \qquad (6.9)\\
&= \left(\lambda_n\Big(\mu, x_0+\frac{a^*}{n}\Big)\right)^{-1}\left(\lambda_n\Big(\mu, x_0+\frac{b^*}{n}\Big)\right)^{-1}.
\end{aligned}$$

(Recall that $\lambda_n(\mu, z) = (\sum_{k=0}^{n-1} |p_k(z)|^2)$.) Similarly as in the proof of Lemma 5.1, define the polynomial

$$P_n(z) = \frac{\lambda_n(\mu, x_0 + \xi/n)}{\lambda_n(\mu, x_0 + \xi/n + z)}$$

for an arbitrary $\xi \in \mathbb{R}$. As (1.4) implies, P_n is indeed a polynomial with $P_n(0) = 1$. Therefore for an arbitrary $\eta \in \mathbb{R}$ we have

$$P_n(i\eta/n) = 1 + \sum_{k=1}^{2n-2} \frac{P_n^{(k)}(0)}{k!} \Big(i\frac{\eta}{n}\Big)^k.$$

According to (5.6), $|P_n^{(k)}(0)| \leq CM^k n^k$ for some constants C and M, therefore

$$|P_n(i\eta/n)| \leq 1 + \sum_{k=1}^{2n-2} \frac{|P_n^{(k)}(0)|}{k!} \Big|\frac{\eta}{n}\Big|^k \leq 1 + \sum_{k=0}^{2n-2} C\frac{M^k|\eta|^k}{k!} \leq Ce^{M|\eta|}.$$

Together with this and (6.9), we have

$$\begin{aligned}\Big|K_n\Big(x_0 + \frac{a^*}{n}, x_0 + \frac{b^*}{n}\Big)\Big|^2 &\leq \Big(\lambda_n\Big(\mu, x_0 + \frac{a^*}{n}\Big)\Big)^{-1}\Big(\lambda_n\Big(\mu, x_0 + \frac{b^*}{n}\Big)\Big)^{-1} \\ &\leq Ce^{M|\operatorname{Im}(a^*)|}K_n\Big(x_0 + \frac{\operatorname{Re}(a^*)}{n}, x_0 + \frac{\operatorname{Re}(a^*)}{n}\Big) \\ &\quad \times e^{M|\operatorname{Im}(b^*)|}K_n\Big(x_0 + \frac{\operatorname{Re}(b^*)}{n}, x_0 + \frac{\operatorname{Re}(b^*)}{n}\Big).\end{aligned}$$

Now, Theorem 1.2 says that

$$\frac{K_n\big(x_0 + \frac{\xi}{n}, x_0 + \frac{\xi}{n}\big)}{K_n(x_0, x_0)} = (1 + o(1))\mathbb{L}_\alpha^*(\pi\omega_K(x_0)\xi)$$

uniformly for ξ in compact subsets of the real line. Using the formulas [1, 9.1.27], we have

$$\begin{aligned}\mathbb{L}_\alpha^*(a) &= \frac{1}{2a^{\alpha-1}}\Big(J'_{\frac{\alpha+1}{2}}(a)J_{\frac{\alpha-1}{2}}(a) - J'_{\frac{\alpha-1}{2}}(a)J_{\frac{\alpha+1}{2}}(a)\Big) \\ &= \frac{1}{2a^{\alpha-1}}\Big(\Big(J_{\frac{\alpha-1}{2}}(a) - \frac{\alpha+1}{2a}J_{\frac{\alpha+1}{2}}(a)\Big)J_{\frac{\alpha-1}{2}}(a) \\ &\qquad - \Big(-J_{\frac{\alpha+1}{2}}(a) + \frac{\alpha-1}{2a}J_{\frac{\alpha-1}{2}}(a)\Big)J_{\frac{\alpha+1}{2}}(a)\Big).\end{aligned}$$

With this and some elementary trigonometric identities, [1, 9.2.1] gives that for large ξ we have

$$J_{\frac{\alpha+1}{2}}(a) = \Big(\frac{2}{\pi a}\Big)^{1/2}\Big(\cos\Big(a - \frac{(\alpha+1)\pi}{4} - \frac{\pi}{4}\Big) + O(1/a)\Big)$$

and

$$J_{\frac{\alpha-1}{2}}(a) = \Big(\frac{2}{\pi a}\Big)^{1/2}\Big(\sin\Big(a - \frac{(\alpha+1)\pi}{4} - \frac{\pi}{4}\Big) + O(1/a)\Big),$$

which yields that

$$\mathbb{L}_\alpha^*(a) = \frac{1}{\pi|a|^\alpha}(1 + o(1)), \tag{6.10}$$

therefore $\mathbb{L}_\alpha^*(\pi\omega_K(x_0)\xi) \le C|\xi|^{-\alpha}$ holds for large ξ. Since $\mathbb{L}_\alpha^*(\pi\omega_K(x_0)\xi) \le C(1+|\xi|)^{-\alpha}$ obviously holds in any bounded set containing 0 for some constant, we have

$$\left|\frac{K_n\left(x_0+\frac{a^*}{n}, x_0+\frac{b^*}{n}\right)}{K_n(x_0,x_0)}\right|^2 \le Ce^{c(|\operatorname{Im}(a)|+|\operatorname{Im}(b)|)}(1+|\operatorname{Re}(a)|)^{-\alpha}(1+|\operatorname{Re}(b)|)^{-\alpha}$$

for some constants c, C and this implies (6.8). $\square$

Now we study the possible limits of $\{f_n(a,b)\}_{n=1}^\infty$. In the next lemmas we prove that a limit of its subsequence is an entire function of exponential type belonging to the Cartwright class and we take a look at its zeros. The exponential type and the behavior of the zeros are connected, because if $g(z)$ is an entire function of exponential type σ belonging to the Cartwright class, then

$$\lim_{r\to\infty} \frac{n(g,r)}{2r} = \frac{\sigma}{\pi}$$

holds, where $n(g,r)$ is the number of zeros of g in a disk of radius r centered at zero. (See [18, Theorem 17.2.1] for details.) Before we state our next lemma, we fix some notations about the zeros of some frequently used functions.

First define the function

$$\psi_n(z,w) = p_n(z)p_{n-1}(w) - p_n(w)p_{n-1}(z). \tag{6.11}$$

For real ξ, the zeros of $\psi_n(\xi,\cdot)$ will be denoted as

$$\cdots < t_{-1,n}(\xi) < t_{0n}(\xi) = \xi < t_{1n}(\xi) \dots . \tag{6.12}$$

Note that these zeros are indeed real, see [14, Theorem 3.1], and they are centered around ξ, moreover $t_{0n}(\xi) = \xi$ is indeed a zero of $\psi_n(\xi,\cdot)$. The zeros of $K_n(x_0+a/n,\cdot)$ are denoted as

$$\cdots < x_{-1,n}(a) < x_0 + \frac{a}{n} < x_{1n}(a) < \dots . \tag{6.13}$$

For convenience, we write $x_{0n}(a) = x_0 + a/n$. Note again that since $K_n(\xi,\xi)$ is strictly positive, $x_{0n}(a)$ cannot be a zero of $K_n(x_0+a/n,\cdot)$. The Christoffel-Darboux formula (1.2) says that

$$K_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n}\frac{\psi_n(x,y)}{x-y},$$

therefore

$$x_{kn}(a) = t_{kn}(x_0 + a/n) \tag{6.14}$$

holds for all integer k for which the above expression makes sense. In our case, the zeros of $f_n(a,\cdot)$ are also important, thus they will be denoted as

$$\cdots < \rho_{-1,n}(a) < a < \rho_{1n}(a) < \ldots,$$

and again we write $\rho_{0n}(a) = a$ for convenience. Since $f_n(a,a)$ is strictly positive, $\rho_{0n}(a)$ cannot be a zero of $f_n(a,\cdot)$. The definition of $f_n(a,b)$ implies that

$$\rho_{kn}(a) = n\pi\omega_K(x_0)(x_{kn}(a^*) - x_0) \tag{6.15}$$

holds for all integers k for which the above expression makes sense, where $a^* = a/(\pi\omega_K(x_0))$.

Lemma 6.4. *Let $f(a,b) = \lim_{k\to\infty} f_{n_k}(a,b)$ for some subsequence n_k.*
(a) If $a \in \mathbb{R}$, then all the zeros of $f(a,\cdot)$ are real. Moreover, if $n(f(a,\cdot),r)$ denotes the number of zeros of $f(a,\cdot)$ in the disk of center 0 with radius r, then

$$n(f(a,\cdot),r) - n(f(0,\cdot),r) = O(1). \tag{6.16}$$

(b) Let

$$\cdots \le \rho_{-2} \le \rho_{-1} < 0 < \rho_1 \le \rho_2 \le \ldots$$

denote the zeros of $f(0,\cdot)$ ordered around zero and write $\rho_0 = 0$ for convenience. Then

$$\rho_{kn}(0) \to \rho_k, \quad n \to \infty \tag{6.17}$$

holds for all $k \ge 0$ and there are positive constants c_1, c_2 such that

$$\begin{aligned} \rho_k - \rho_{k-1} &\le c_1, \\ \rho_k - \rho_{k-2} &\ge c_2. \end{aligned} \tag{6.18}$$

In particular, the zeros of $f(0,\cdot)$ are at most double.

Proof. (a) The Christoffel-Darboux formula (1.2) gives that

$$f_n(a,z) = \frac{\pi\omega_K(x_0)n}{K_n(x_0,x_0)} \frac{\gamma_{n-1}}{\gamma_n} \frac{\psi_n(x_0 + \frac{a^*}{n}, x_0 + \frac{z^*}{n})}{a - z},$$

where $\psi_n(z,w)$ is defined by (6.11). As we mentioned earlier, for real ξ, all zeros of $\psi_n(\psi,\cdot)$ are real, see for example [14, Theorem 3.1]. Hence Hurwitz's theorem implies that the zeros of $f(a,\cdot)$ are also real for all $a \in \mathbb{R}$. The proof of (6.16) goes exactly as in [21, Lemma 4.3], which we include for completeness. It is known that if $x_{1n} < x_{2n} < \cdots < x_{nn}$ denotes the zeros of the orthonormal polynomial p_n, then if $p_n(\xi)p_{n-1}(\xi) \neq 0$, the function $\psi_n(\xi, \cdot)$ has a simple zero in each of the intervals

$$(x_{1n}, x_{2n}), \ldots, (x_{n-1,n}, x_{nn})$$

plus one zero outside $[x_{1n}, x_{nn}]$, and if $p_n(\xi)p_{n-1}(\xi) = 0$, then $\psi_n(\xi, \cdot)$ is a multiple of p_{n-1} or p_n, hence the interlacing property of the zeros of orthogonal polynomials imply that in the former case $\psi_n(\xi, \cdot)$ has a zero in each of the intervals

$$(x_{1n}, x_{2n}), \ldots, (x_{n-1,n}, x_{nn}),$$

and in the latter case the zeros of $\psi_n(\xi, \cdot)$ coincide with the zeros of p_n. For these facts, see Theorem 2.3 and the proof of Theorem 3.1 in [14]. Therefore, if $n(\psi_n(a, \cdot), [c, d])$ denotes the zeros of $\psi_n(a, \cdot)$ in the interval $[c, d]$, then

$$|n(\psi_n(a, \cdot), [x_{mn}, x_{kn}]) - (m - k)| \leq 1.$$

Now, if $\{x_{jn}(a)\}$ denotes the zeros of $K_n(x_0 + a/n, \cdot)$ centered around $x_0 + a/n$ as in (6.13), then $\rho_{kn}(a) = n\pi\omega_K(x_0)(x_{kn}(a^*) - x_0)$, where $a^* = a/(\pi\omega_K(x_0))$. (Recall that the definition of $f_n(a, b)$ included the scaling constant $\pi\omega_K(x_0)$.) This, together with the previous observations about the location of the zeros of $\psi_n(a, \cdot)$, means that if r is fixed and n is large,

$$|n(f_n(a, \cdot), r) - n(f_n(0, \cdot), r)| \leq M$$

for some constant M. Hurwitz's theorem implies again that the above holds for f, therefore we have (6.16).

(b) First note that ρ_0 can never be a zero of $f(0, \cdot)$, since $f_n(0, 0) = 1$ for all n. Now (6.17) is immediate from Hurwitz's theorem. Since μ is a doubling measure (i.e. there is a constant L such that $\mu([x - 2\delta, x + 2\delta]) \leq L\mu([x - \delta, x + \delta])$ holds for all x and δ) in a small neighborhood $(x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ of x_0 (note that $d\mu(x) = w(x)|x - x_0|^\alpha$ there for a continuous and positive w), [47, Theorem 1.1] says that if $x_{kn}, x_{k+1,n}, \ldots, x_{l,n} \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$, then

$$\frac{c}{n} \leq x_{m+1,n} - x_{mn} \leq \frac{C}{n}, \quad m = k, k + 1, \ldots, l - 1 \tag{6.19}$$

holds for some constants c and C independent of m and n. Together with (6.19) and the above observation about the location of the zeros of $f_n(a, \cdot)$, we have

$$x_{kn}(0) - x_{k-1,n}(0) \leq \frac{C}{n}$$

and

$$x_{kn}(0) - x_{k-2,n}(0) \geq \frac{c}{n}$$

for some possibly different constants, therefore, since $\rho_{kn}(0) = n\pi\omega_K(x_0)(x_{kn}(0) - x_0)$, using Hurwitz's theorem once more gives (6.18). □

Lemma 6.5. *Let $f(a,b) = \lim_{k\to\infty} f_{n_k}(a,b)$ for some subsequence n_k.*

(a) $f(a,\cdot)$ is entire of exponential type σ_a and

$$\int_{-\infty}^{\infty} |f(a,t)|^2 |t|^\alpha dx \leq \frac{f(a,\overline{a})}{\mathbb{L}_\alpha^*(0)} \tag{6.20}$$

holds.

(b) $f(a,\cdot)$ belongs to the Cartwright class.

(c) The exponential type σ_a of the entire function $f(a,\cdot)$ is independent of a.

Proof. (a) It is clear that $f(a,b)$ is entire in both variables, since it is a locally uniform limit of entire functions. Moreover, the bound (6.8) holds for $f(a,b)$ as well, which implies that $f(a,\cdot)$ is of exponential type. We shall denote its exponential type with σ_a for the time being. (In fact, we shall show later that the type is independent of a and it is 1.) As for the proof of (6.20), we proceed similarly as in [21, Lemma 4.2 (b)]. For all $z \in \mathbb{C}$, we have

$$K_n(z,\overline{z}) = \int |K_n(z,x)|^2 d\mu(x) \geq \int_{x_0-r/n}^{x_0+r/n} |K_n(z,x)|^2 w(x)|x-x_0|^\alpha dx$$

for large n. After the substitution $z = x_0 + a^*/n$, $x = x_0 + t^*/n$, we have

$$\begin{aligned} &K_n\Big(x_0 + \frac{a^*}{n}, x_0 + \frac{\overline{a^*}}{n}\Big) \\ &\qquad \geq \frac{1}{(\pi\omega_K(x_0))^{\alpha+1} n^{\alpha+1}} \int_{-r}^{r} \Big|K_n\Big(x_0 + \frac{a^*}{n}, x_0 + \frac{t^*}{n}\Big)\Big|^2 w(x_0 + t^*/n)|t|^\alpha dt \end{aligned}$$

which gives

$$(\pi\omega_K(x_0))^{\alpha+1} \geq \int_{-r}^{r} \frac{|f_n(a,t)|^2}{f_n(a,\overline{a})} \frac{K_n(x_0,x_0)}{n^{\alpha+1}} w(x_0 + t/n)|t|^\alpha dt.$$

By letting $n \to \infty$ through the subsequence n_k, (1.18) gives

$$(\pi\omega_K(x_0))^{\alpha+1} \geq \int_{-r}^{r} \frac{|f(a,t)|^2}{f(a,\overline{a})} (\pi\omega_K(x_0))^{\alpha+1} \mathbb{L}_\alpha^*(0)|t|^\alpha dt,$$

from which (6.20) follows.

(b) To prove that $f(a,\cdot)$ belongs to the Cartwright class, we have to show that

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(a,t)|}{1+t^2} dt < \infty.$$

Since $f(a,\cdot)$ is entire, it is clear that $\int_{-1}^{1} \frac{\log^+ |f(a,t)|}{1+t^2} dt < \infty$. Next, using the well known facts $\log^+ ab \le \log^+ a + \log^+ b$ and $\log^+ a^b = b \log^+ a$, we have

$$\int_{1}^{\infty} \frac{\log^+ |f(a,t)|}{1+t^2} dt \le C\Big(\int_{1}^{\infty} \frac{\log^+ |f(a,t)|^2 |t|^\alpha}{1+t^2} dt + \int_{1}^{\infty} \frac{\log^+ |t|}{1+t^2} dt \Big).$$

The second integral is finite. For the first one, define

$$A_n = \{t \in \mathbb{R} : e^n \le |f(a,t)|^2 |t|^\alpha < e^{n+1}\}.$$

The bound (6.20) and Markov's inequality about the measure of level sets of L^1 functions gives that

$$\begin{aligned}
\int_{1}^{\infty} \frac{\log^+ |f(a,t)|^2 |t|^\alpha}{1+t^2} dt &= \sum_{n=1}^{\infty} \int_{A_n} \frac{\log^+ |f(a,t)|^2 |t|^\alpha}{1+t^2} dt \\
&\le \sum_{n=1}^{\infty} (n+1)|A_n| \\
&\le C \sum_{n=1}^{\infty} (n+1) e^{-n} < \infty.
\end{aligned}$$

The estimation of the integral $\int_{-\infty}^{-1} \frac{\log^+ |f(a,t)|^2 |t|^\alpha}{1+|t|^2} dt$ can be done in the same way, which shows that $f(a,\cdot)$ belongs to the Cartwright class.

(c) This proof is identical to the one in [21, Lemma 4.3]. Because $f(a,\cdot)$ belongs to the Cartwright class, we have

$$\frac{\sigma_a}{\pi} = \lim_{r\to\infty} \frac{n(f(a,\cdot),r)}{2r},$$

which, combined with (6.16), yields that σ_a is independent of a. □

From now on, since Lemma 6.5 (c) gives that σ_a is independent of a, we shall denote the exponential type of $f(a,\cdot)$ with σ.

Lemma 6.6. *For all $a \in \mathbb{R}$, we have*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Big(\frac{f(a/\sigma, t/\sigma)}{f(a/\sigma, a/\sigma)} - \frac{\mathbb{L}_\alpha^*(a,t)}{\mathbb{L}_\alpha^*(a,a)} \Big)^2 |t|^\alpha dt \\
&\qquad \le \frac{\sigma^{\alpha+1}}{f(a/\sigma, a/\sigma)\mathbb{L}_\alpha^*(0,0)} - \frac{1}{\mathbb{L}_\alpha^*(a,a)}.
\end{aligned} \tag{6.21}$$

Moreover,

$$\sigma \ge 1. \tag{6.22}$$

Proof. (6.20) implies that $|x|^{\alpha/2} f(a/\sigma, t/\sigma) \in \mathbb{L}^2(\mathbb{R})$, therefore after expanding the left hand side of (6.21), (6.1) and (6.2) gives (6.21). Using that the left hand side of (6.21) is nonnegative, substituting $a = 0$ and keeping in mind that $f(0,0) = 1$ gives $\sigma \geq 1$. □

The inequality (6.21) and (1.18) imply that if $\sigma = 1$, then $f(a,t) = \frac{\mathbb{L}_\alpha^*(a,t)}{\mathbb{L}_\alpha^*(0,0)}$ for all $a, t \in \mathbb{R}$, which, since $f(a,b)$ is entire in both variables, would imply Theorem 1.4.

Lemma 6.7. *Let $k > l$ be given integers. Then*

$$\sum_{j=l+1}^{k-1} \frac{1}{f(\rho_j, \rho_j)} \leq \mathbb{L}_\alpha^*(0,0) \frac{\rho_k^{\alpha+1} - \rho_l^{\alpha+1}}{\alpha+1} \leq \sum_{j=l}^{k} \frac{1}{f(\rho_j, \rho_j)}. \tag{6.23}$$

Proof. The Markov-Stieltjes inequalities along with (1.4) imply, as in [14, p. 33 (5.10)], that

$$\sum_{j=l+1}^{k-1} \frac{1}{K_n(t_{jn}(x_0), t_{jn}(x_0))} \leq \int_{t_{ln}(x_0)}^{t_{kn}(x_0)} d\mu(x) \leq \sum_{j=l}^{k} \frac{1}{K_n(t_{jn}(x_0), t_{jn}(x_0))}, \tag{6.24}$$

where $t_{jn}(x_0)$ denotes the zeros of $\psi_n(x_0, \cdot) = p_n(x_0)p_{n-1}(\cdot) - p_n(\cdot)p_{n-1}(x_0)$ centered around x_0 such that $t_{0n}(x_0) = x_0$. Now suppose that $t_{ln}(x_0)$ and $t_{kn}(x_0)$ belongs to $(x_0 - \varepsilon_0, x_0 + \varepsilon_0)$, where ε_0 is so small that μ is absolutely continuous in this interval. Then, substituting $x = x_0 + s^*/n$ (recall that $s^* = s/(\pi\omega_K(x_0))$ by definition) and using (6.14) with (6.15), the integral in the middle takes the form

$$\begin{aligned} \int_{t_{ln}(x_0)}^{t_{kn}(x_0)} d\mu(x) &= \int_{t_{ln}(x_0)}^{t_{kn}(x_0)} w(x)|x - x_0|^\alpha dx \\ &= \frac{1}{n^{\alpha+1}} \int_{\rho_{ln}(0)}^{\rho_{kn}(0)} \frac{w(x_0 + s^*/n)}{(\pi\omega_K(x_0))^{\alpha+1}} |s|^\alpha ds. \end{aligned}$$

On the other hand, by definition $\frac{K_n(t_{jn}(x_0), t_{jn}(x_0))}{K_n(x_0,x_0)} = f_n(\rho_{jn}(0), \rho_{jn}(0))$. Multiplying with $K_n(x_0, x_0)$ in (6.24) we obtain

$$\begin{aligned} \sum_{j=l+1}^{k-1} \frac{1}{f_n(\rho_{jn}(0), \rho_{jn}(0))} &\leq \frac{K_n(x_0, x_0)}{n^{\alpha+1}} \int_{\rho_{ln}(0)}^{\rho_{kn}(0)} \frac{w(x_0 + s/n)}{(\pi\omega_K(x_0))^{\alpha+1}} |s|^\alpha ds \\ &\leq \sum_{j=l}^{k} \frac{1}{f_n(\rho_{jn}(0), \rho_{jn}(0))}, \end{aligned}$$

which, after letting n to infinity and using (1.18) with (6.17), yields (6.23). □

The next lemma is an analogue of [21, Lemma 5.3], for which the proof also goes in an identical way.

Lemma 6.8. *Let $\delta > 0$ be arbitrary.*

(a) There exists a positive integer L_+ such that if $k > l > L_+$ are selected in a way that

$$\rho_k \leq (1+\delta)\rho_l \tag{6.25}$$

holds, we have

$$k - l - 1 \leq (1+\delta)^{|\alpha|+1} \frac{\rho_k - \rho_l}{\pi}. \tag{6.26}$$

Similarly, there exists a negative integer L_- such that if $L_- > l > k$ are selected in a way that

$$|\rho_k| \leq (1+\delta)|\rho_l|$$

holds,

$$l - k - 1 \leq (1+\delta)^{|\alpha|+1} \frac{|\rho_k| - |\rho_l|}{\pi}$$

follows.

(b) For the function $f(a,\cdot)$,

$$\limsup_{r\to\infty} \frac{n(f(a,\cdot), r)}{2r} \leq \frac{1}{\pi}$$

holds. In particular, we have

$$\sigma \leq 1 \tag{6.27}$$

Proof. (a) We only show the existence of L_+, the existence of L_- follows similarly. (Or by reflecting the measure μ around x_0.) Since (1.18) gives that $f(a,a) = \mathbb{L}^*_\alpha(a,a)/\mathbb{L}^*_\alpha(0,0)$, the number $k-l-1$ can be written as

$$k - l - 1 = \sum_{j=l+1}^{k-1} \frac{\mathbb{L}^*_\alpha(\rho_j,\rho_j)}{\mathbb{L}^*_\alpha(0,0) f(\rho_j,\rho_j)}.$$

If L is large enough, then (6.10) implies that for all $j \geq L$,

$$\mathbb{L}^*_\alpha(\rho_j,\rho_j) \leq \frac{1+\delta}{\pi\rho_j^\alpha}$$

holds. Combining these with (6.23), we obtain

$$\begin{aligned}
k - l - 1 &= \sum_{j=l+1}^{k-1} \frac{\mathbb{L}^*_\alpha(\rho_j,\rho_j)}{\mathbb{L}^*_\alpha(0,0) f(\rho_j,\rho_j)} \\
&\leq \frac{1+\delta}{\pi\mathbb{L}^*_\alpha(0,0)} \frac{1}{\min\{\rho_l^\alpha, \rho_k^\alpha\}} \sum_{j=l+1}^{k-1} \frac{1}{f(\rho_j,\rho_j)} \\
&\leq \frac{1+\delta}{\pi(1+\alpha)} \frac{\rho_k^{\alpha+1} - \rho_l^{\alpha+1}}{\min\{\rho_l^\alpha, \rho_k^\alpha\}} \\
&\leq (1+\delta)\frac{\rho_k - \rho_l}{\pi} \frac{\max\{\rho_l^\alpha, \rho_k^\alpha\}}{\min\{\rho_l^\alpha, \rho_k^\alpha\}},
\end{aligned}$$

where the mean value theorem was used in the last step. (6.25) gives that

$$\frac{\max\{\rho_l^\alpha, \rho_k^\alpha\}}{\min\{\rho_l^\alpha, \rho_k^\alpha\}} \leq (1+\delta)^{|\alpha|},$$

therefore overall we have

$$k - l - 1 \leq (1+\delta)^{|\alpha|+1} \frac{\rho_k - \rho_l}{\pi},$$

which gives (6.26). The proof of (b) goes exactly as [21, Lemma 5.3 (b)]. $\square$

Proof of Theorem 1.4. (6.22) and (6.27) gives that the exponential type of $f(a, \cdot)$ is $\sigma = 1$. Substituting this back to the inequality (6.21), we obtain that for all real b, we have

$$f(a,b) = \frac{\mathbb{L}_\alpha^*(a,b)}{\mathbb{L}_\alpha^*(0,0)}, \quad a, b \in \mathbb{R}.$$

Since $f(a,b)$ is entire in both variables, it follows that the above equality holds for complex a, b. Because the family $\{f_n(a,b)\}_{n=1}^\infty$ is normal and the above inequality is independent of the particular subsequence (recall that $f(a,b) = \lim_{k\to\infty} f_{n_k}(a,b)$ for some subsequence n_k), it follows that $\lim_{n\to\infty} f_n(a,b)$ exists and it is $f(a,b)$. Moreover, the convergence is uniform for a, b in compact subsets of the complex plane, as stated.

$\square$

7 Summary

In this thesis we are studying the asymptotic behavior of orthogonal polynomials with respect to generalized Jacobi measures, i.e. measures having an algebraic singularity of the type $d\mu(x) = |x - x_0|^\alpha dx$ for some x_0 in the support. Our study is focused on the asymptotics of the Christoffel-Darboux kernel defined as

$$K_n(z,w) = \sum_{k=0}^{n-1} p_k(z)\overline{p_k(w)},$$

where $p_n(x)$ denotes the n-th orthonormal polynomial with respect to the μ, and the Christoffel functions defined as

$$\lambda_n(\mu, z_0) = \Big(\sum_{k=0}^{n-1} |p_k(z_0)|^2\Big)^{-1}.$$

In Section 1 we briefly review the classical results of the subject and set the stage for our investigations. In Section 2 we collect the mathematical tools which will be used in the proofs. In Section 3 we prove the special cases of our main theorems. These so-called model cases will serve as our starting point, and our strategy is to transfer these special results to more general ones using the polynomial inverse image method. After this we first study generalized Jacobi measures supported on a system of Jordan curves in Section 4. There we will prove that if μ is a measure which is supported on a system of Jordan curves γ, regular in the sense of Stahl and Totik, behaving like

$$d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z)$$

in the neighbourhood of some interior point $z_0 \in \gamma$ for some $\alpha > -1$ and some weight function $w(z)$ which is strictly positive and continuous at z_0, then

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big)$$

holds, where $\omega_\gamma(z)$ denotes the density of the equilibrium measure for γ.

In Section 5 we move over to the real line. First we show an analogous, but slightly more general result than what we have for Jordan curves. Suppose that μ is a measure supported on an arbitrary compact subset K of the real line, μ is regular in the sense of Stahl and Totik, behaving like

$$d\mu(x) = w(x)|x - x_0|^\alpha dx$$

in the neighbourhood of some x_0 in the support, where $\alpha > -1$ and $w(x)$ is continuous and strictly positive at x_0. Then if x_0 is an interior point of the support,

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n\Big(\mu, x_0+\frac{a}{n}\Big) = \frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}}\Big(\mathbb{L}^*_\alpha\big(\pi\omega_K(x_0)a\big)\Big)^{-1}$$

holds uniformly for a in compact subsets of the real line, where $\mathbb{L}^*_\alpha(\cdot)$ is the Bessel kernel for the bulk defined by (1.17). Analogously, if x_0 is a right endpoint of the support,

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\Big(\mu, x_0-\frac{a}{2n^2}\Big) = \frac{w(x_0)}{M(K,x_0)^{2\alpha+2}}\Big(2^{\alpha+1}\mathbb{J}^*_\alpha\big(M(K,x_0)^2a\big)\Big)^{-1}$$

holds uniformly for a in compact subsets of $[0,\infty)$, where $\mathbb{J}^*_\alpha(\cdot)$ is the Bessel kernel for the edge defined by (1.13) and $M(K,x_0)$ is defined by

$$M(K,x_0) = \lim_{x\to x_0-} \sqrt{2}\pi|x-x_0|^{1/2}\omega_K(x).$$

These results are further generalized in Section 6, where we study the so-called universality limits. Under the same conditions for the measure as in the previous section, we prove that if x_0 in the interior of the support,

$$\lim_{n\to\infty} \frac{K_n\big(x_0+\frac{a}{n}, x_0+\frac{b}{n}\big)}{K_n(x_0,x_0)} = \frac{\mathbb{L}^*_\alpha(\pi\omega_K(x_0)a, \pi\omega_K(x_0)b)}{\mathbb{L}^*_\alpha(0,0)}$$

holds uniformly for a, b in compact subsets of the complex plane, and if x_0 is an endpoint of the support, we have

$$\lim_{n\to\infty} \frac{K_n\big(x_0-\frac{a}{2n^2}, x_0-\frac{b}{2n^2}\big)}{K_n(x_0,x_0)} = \frac{\mathbb{J}^*_\alpha\big(M(K,x_0)^2a, M(K,x_0)^2b\big)}{\mathbb{J}^*_\alpha(0,0)}$$

also uniformly for a, b in compact subsets of the complex plane.

8 Összefoglalás

A doktori disszertációmban olyan ortogonális polinomok aszimptotikájával kapcsolatos tételeket bizonyítok, ahol az ortogonalitás mértéke ún. általánosított Jacobi-mérték, azaz olyan, melyre a tartó valamely x_0 belső pontjában $d\mu(x) = |x - x_0|^\alpha dx$ típusú algebrai szingularitás található. A vizsgálatok középpontjában a

$$K_n(z, w) = \sum_{k=0}^{n-1} p_k(z)\overline{p_k(w)},$$

formulával definiált Christoffel-Darboux kernel áll, ahol $p_n(x)$ az n-edik ortonormált polinomot jelöli. Másik vizsgált mennyiség a

$$\lambda_n(\mu, z_0) = \Big(\sum_{k=0}^{n-1} |p_k(z_0)|^2\Big)^{-1}.$$

formulával definiált Christoffel-függvény.

Az 1. Fejezetben röviden áttekintjük a terület klasszikus eredményeit és megalapozzuk a későbbi vizsgálatokat, kimondjuk a tézis főbb eredményeit. A 2. Fejezetben összegyűjtjük a fő tételek bizonyításaihoz szükséges eszközöket. A 3. Fejezetben a fő tételek speciális eseteit bizonyítjuk, amik az általános esetben adott bizonyítások kiindulópontjaként fognak szolgálni. Ezen ún. modell-esetekre meglevő bizonyítások nem alkalmazhatóak minden esetben, de a polinom inverkép módszerrel az *eredmények* átvihetők az általános esetre. Mi ezt az utat fogjuk követni. A modell-esetek elkészítése után a 4. Fejezetben olyan általánosított Jacobi-mértékeket vizsgálunk, amely tartója Jordan-görbék uniója. Ebben a fejezetben azt bizonyítjuk, hogy ha μ egy Stahl-Totik értelemben vett reguláris mérték, amelynek γ tartója Jordan-görbék véges uniója és valamely $z_0 \in \gamma$ esetén

$$d\mu(z) = w(z)|z - z_0|^\alpha ds_\gamma(z),$$

ahol $\alpha > -1$, s_γ az ívhossz-mértéket jelöli valamint $w(z)$ szigorúan pozitív és folytonos z_0-ban, akkor

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\Big(\frac{\alpha+1}{2}\Big)\Gamma\Big(\frac{\alpha+3}{2}\Big),$$

ahol $\omega_\gamma(z)$ a γ egyensúlyi mértékének súlyfüggvényét jelöli.

Az 5. Fejezetben áttérünk a valós egyenesen értelmezett mértékekre. Először bizonyítunk egy, a Jordan-görbéken látottakhoz analóg, de kissé általánosabb tételt. Legyen tehát μ egy véges Borel-mérték, mely tartója egy tetszőleges $K \subseteq \mathbb{R}$ kompakt halmaz, valamint μ Stahl-Totik értelemben reguláris és

$$d\mu(x) = w(x)|x - x_0|^\alpha dx$$

az $x_0 \in K$ valamely környezetében, ahol $\alpha > -1$, valamint $w(x)$ szigorúan pozitív és folytonos x_0-ban. Azt állítjuk, hogy ha x_0 a tartó egy belső pontja, akkor

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n\Big(\mu, x_0 + \frac{a}{n}\Big) = \frac{w(x_0)}{(\pi\omega_K(x_0))^{\alpha+1}}\Big(\mathbb{L}_\alpha^*\big(\pi\omega_K(x_0)a\big)\Big)^{-1}$$

a-ban kompakt halmazokon egyenletesen, ahol $\mathbb{L}_\alpha^*(\cdot)$ a belső ponthoz tartozó Bessel magfüggvény (lásd (1.17)); illetve ha x_0 a K halmaz egy végpontja, akkor

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\Big(\mu, x_0 - \frac{a}{2n^2}\Big) = \frac{w(x_0)}{M(K,x_0)^{2\alpha+2}}\Big(2^{\alpha+1}\mathbb{J}_\alpha^*\big(M(K,x_0)^2a\big)\Big)^{-1}$$

egyenletesen a-ban $[0,\infty)$-beli kompakt halmazokon, ahol $\mathbb{J}_\alpha^*(\cdot)$ a végponthoz tartozó Bessel magfüggvény (lásd (1.13)), illetve $M(K,x_0)$ pedig a

$$M(K,x_0) = \lim_{x\to x_0-} \sqrt{2}\pi|x - x_0|^{1/2}\omega_K(x)$$

formula által definiált mennyiség.

A fenti tételeket tovább általánosítjuk a 6. Fejezetben, ahol ún. univerzalitást igazolunk a Christoffel-Darboux kernelre x_0 körül. Ebben az esetben azt állítjuk, hogy ha x_0 a K tartó egy belső pontja, akkor

$$\lim_{n\to\infty} \frac{K_n\big(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\big)}{K_n(x_0,x_0)} = \frac{\mathbb{L}_\alpha^*(\pi\omega_K(x_0)a, \pi\omega_K(x_0)b)}{\mathbb{L}_\alpha^*(0,0)}$$

egyenletesen a, b-ben $\mathbb{C}$-beli kompakt halmazokon; illetve ha x_0 a K tartó valamely végpontja, akkor

$$\lim_{n\to\infty} \frac{K_n\big(x_0 - \frac{a}{2n^2}, x_0 - \frac{b}{2n^2}\big)}{K_n(x_0,x_0)} = \frac{\mathbb{J}_\alpha^*\big(M(K,x_0)^2a, M(K,x_0)^2b\big)}{\mathbb{J}_\alpha^*(0,0)},$$

szintén egyenletesen a, b-ben $\mathbb{C}$-beli kompakt halmazokon.

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List of publications

Publications used in the thesis:

[1] T. Danka, Universality limits for generalized Jacobi measures, *submitted for consideration for publication*, available at arXiv with identifier arXiv:1605.04275

[2] T. Danka and V. Totik, Christoffel functions with power type weights, *to appear in J. Eur. Math. Soc.*, available at arXiv with identifier arXiv:1504.03968

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[3] T. Danka, Christoffel functions on Jordan curves with respect to measures with jump singularity, *Jaen J. Approx.*, Vol 7. No 2. (2015), available at arXiv with identifier arXiv:1504.06245

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