Bijective enumeration of lattice walks

Outline of Ph.D. thesis

Gábor Nagy

Supervisor:
Péter Hajnal
associate professor

Doctoral School of Mathematics and Computer Science
University of Szeged
Faculty of Science and Informatics
Bolyai Institute

Szeged
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1 Introduction

Using the methods of bijective combinatorics, this thesis investigates two topics which are related to enumeration problems on lattice walks.

The main result of Chapter 2 is a bijective proof of Shapiro's convolution formula involving Catalan numbers of even index, as asked by Stanley. As one of the consequences of our argument, we obtain an elementary proof of the alternating convolution formula of the central binomial coefficients, too.

In Chapter 3, we prove a convexity property of the hitting distribution of the x-axis for symmetric planar random walks, then we consider the higher dimensional analogue of the problem. This work has been motivated by the fact that the main theorem implies a new proof of a recent result on the density function of certain harmonic measures.

The dissertation is based on the author's papers [1]–[4]. We use the same numberings and notations in the outline as in the thesis (except for the references).

2 Convolution of Catalan numbers with even index

We say that $\frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, and it is denoted by C_n. Stanley has collected more than 200 equivalent combinatorial definitions of them [16, 15]. This number also indicates that the Catalan numbers are fundamental combinatorial quantities [10], for which a reason is that they satisfy the recurrence relation

$$C_0 = 1; \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k},$$

which often arises in combinatorial problems.

In 2002, Shapiro observed [10; p. 123] that the following elegant identity can be easily deduced from the closed form of the generating function of the Catalan numbers:

Theorem 2.1 (Shapiro [15; 6.C18], Nagy [2], Hajnal–Nagy [1])

$$\sum_{k=0}^{n} C_{2k} C_{2n-2k} = 4^n C_n. \tag{2.2}$$

However, in spite of its simple form and its short non-combinatorial proofs, this formula is difficult to prove combinatorially, it is listed in Stanley's Bijective Proof Problems as unsolved [14]. (We note that Andrews formulated a q-analog of the identity and, together with other proofs, he gave a combinatorial proof of it [5].) The main result of Chapter 2 is that we verify (2.2) first by an elementary combinatorial reasoning in Section 2.2, following the argument of paper [2]; then we present a completely bijective proof of it in Section 2.3, based on our joint paper with Péter Hajnal [1].

We will double count certain paths, where a *path* is a sequence of up-steps and down-steps. Although formally we defined one-dimensional walks here, we will visualize paths in the plane in the usual way: Unless otherwise stated, they start from the origin, an up-step is interpreted as a step $(1,1)$, and a down-step is interpreted as a step $(1,-1)$. We also need the following notion: A path is *even-zeroed* if it never hits the x-axis at a point of the form $(4k+2,0)$, for $k \in \mathbb{Z}$. Our counting problem associated to Shapiro's identity is based on the observation that the number of even-zeroed paths from the origin to the point $(4n,0)$ is given by the Catalan number C_{2n} (see Lemma 2.4). This has been posed as an American Mathematical Monthly problem [12] in 1981, and a bijective proof has been published [11] in 1987, in which an explicit bijection is given between the set of paths in question and the set of Dyck paths of length $4n$. It is well known that the latter set has C_{2n} elements. (A *Dyck path* is a path that ends on the x-axis but never goes below it, provided that it is started from the origin. Of course, the length of a path is the number of its steps.) This yields a combinatorial interpretation of the left-hand side of (2.2), together with an analogous proposition (throughout the thesis, B_n denotes the central binomial coefficient $\binom{2n}{n}$):

Corollary 2.5 (Nagy [2])

a) $\sum_{k=0}^{n} C_{2k}C_{2n-2k}$ *counts the number of even-zeroed paths from the origin to* $(4n+1,1)$.

b) $\sum_{k=0}^{n} C_{2k}B_{2n-2k}$ *counts the number of even-zeroed paths of length* $4n$.

So, in order to prove (2.2), we have to show that the right-hand side, $4^n C_n$, also counts the paths of part a) of the corollary. First, we establish a connection between the two parts of the corollary. A straightforward calculation shows that the convolution in part b) is $(n+1)$ times the convolution in part a):

Lemma 2.6 (Nagy [2])

$$\sum_{k=0}^{n} C_{2k}B_{2n-2k} = (n+1)\sum_{k=0}^{n} C_{2k}C_{2n-2k}.$$

In other words, Corollary 2.5 interprets combinatorially both the left-hand side of Shapiro's identity and the $(n+1)$st multiple of it. Using this, one can enumerate the paths in the corollary in a recursive way:

Lemma 2.7 (Nagy [2])

a) *The number of even-zeroed paths from the origin to* $(4n+1,1)$ *is* $4^n C_n$.

b) *The number of even-zeroed paths of length* $4n$ *is* $4^n B_n$.

By Corollary 2.5, Lemma 2.6, and the fact $B_n = (n+1)C_n$, the two statements are equivalent. It is not hard to prove part b) by induction, using also the equivalence of the two parts. In sum, we proved Shapiro's formula by double counting, and also the following equivalent form, which is just (2.2), multiplied by $(n+1)$:

$$\sum_{k=0}^{n} C_{2k} B_{2n-2k} = 4^n B_n. \tag{2.4}$$

In Section 2.3, we present the bijective proof of Shapiro's identity. (In this proof, we view C_n as the number of Dyck paths of length $2n$.) Taking into account that Corollary 2.5 has been proved bijectively, to achieve our goal it suffices to find a bijective proof of Lemma 2.7.a, i.e. to show bijectively that the number of $(0,0) \rightsquigarrow (4n+1,1)$ even-zeroed paths in part a) of Corollary 2.5 is $4^n C_n$. Denoting the set of these paths by $\mathcal{H}$, we construct a bijection $\psi\colon \mathcal{H} \to \mathcal{D}_4$, where $\mathcal{D}_4$ is an appropriate set with $4^n C_n$ elements: $\mathcal{D}_4$ is the set of 4-labeled Dyck paths of length $2n$, where a *4-labeled Dyck path* is a Dyck path in which each step in even position is labeled with a number from $\{0,1,2,3\}$ and the steps in odd positions are unlabeled. The definition of ψ is a long process, the milestones are indicated by lemmas (Lemmas 2.8–2.10). In preparation of the major step, we apply a simple technical conversion on the paths first, then we "compress" them, which yields a bijection $\mathcal{H} \to \mathcal{E}_3^*$, where $\mathcal{E}_3^*$ denotes the set of (unlabeled, generalized) paths that have the following three properties:

(i) They start from the point $(0,1)$ and end at $(2n,1)$ or $(2n,-1)$;

(ii) they contain either an unlabeled "long" step $(1,\pm 3)$ or a "short" step $(1,\pm 1)$ labeled with '1', '2', or '3' in each even position (and each step is an unlabeled short step in odd position);

(iii) they never hit the x-axis (but they are allowed to "jump over" it).

We will see that in some sense the labels 1,2,3 of the paths of $\mathcal{E}_3^*$ and $\mathcal{D}_4$ can be neglected. So, let $\mathcal{E}^*$ and $\mathcal{D}_2$ be the sets that contain the same paths as $\mathcal{E}_3^*$ and $\mathcal{D}_4$, respectively, but with the labels omitted. Accordingly, $\mathcal{E}^*$ consists of those $(0,1) \rightsquigarrow (2n,\pm 1)$ (unlabeled) paths that can have long steps $(1,\pm 3)$, but only in even positions, while all their other steps are conventional steps $(1,\pm 1)$, and that never hit the x-axis; and $\mathcal{D}_2$ consists of such Dyck paths of length $2n$ in which each step in even position is either marked (labeled with '0') or unmarked (unlabeled). The paths of $\mathcal{D}_2$ are called *marked Dyck paths* (of length $2n$). It is not hard to see that a desired bijection $\mathcal{E}_3^* \to \mathcal{D}_4$ can be constructed from a bijection $\mathcal{E}^* \to \mathcal{D}_2$ satisfying the following requirement:

Lemma 2.10 (Hajnal–Nagy [1]) *There exists a bijection $\phi\colon \mathcal{E}^* \to \mathcal{D}_2$ such that, for all $E \in \mathcal{E}^*$, the number of marked steps in $\phi(E)$ is equal to the number of long steps in E.*

This is the key lemma of our proof. There are two main phases of the conversion $\mathcal{E}^* \ni E \mapsto \phi(E)$. First, those long steps in E that jump over the x-axis are replaced with marked (short) steps, and the "negative" parts of E are reflected across the x-axis to obtain a path E^+ that stays above the x-axis. (This is the easy part.) Next, we show that all possible paths E^+ can be built up from fairly simple "building blocks"; the building process is an extension of the decomposition of the sequences of parentheses, that can be identified with Dyck paths, into matching pairs of parentheses. In the second phase, the building blocks of the path E^+ are transformed by applying a predefined conversion method, and the image of E is defined to be the obtained signed Dyck path.

The next two corollaries can be read off from the the details of the proof:

Corollary 2.11 (Hajnal–Nagy [1]) *Let $\mathcal{E}^*(n)$ denote the set of those $(0,1) \rightsquigarrow (2n, \pm 1)$ (unlabeled) paths that never hit the x-axis and that can contain long steps in even positions. (This set is just $\mathcal{E}^*$, with a notation indicating n.)*

a) *The number of all paths in $\mathcal{E}^*(n)$ is $2^n C_n$.*
b) *The number of paths in $\mathcal{E}^*(n)$ with k long steps is $\binom{n}{k} C_n$.*
c) *Consequently, C_n counts the number of paths in $\mathcal{E}^*(n)$ with n long steps (i.e. with alternating short and long steps).*
d) *If $n \geq 1$, C_n counts the number of such $(0,0) \rightsquigarrow (n,1)$ paths of length n in which every step is either $(1, \pm 1)$ or $(1, \pm 2)$ and that never hit the x-axis after the starting point.*

Corollary 2.12 *Let $\mathcal{E}'(n)$ denote the set of those $2n$-length (unlabeled) paths starting from $(0,1)$ that never hit the x-axis and that can contain long steps in even positions.*

a) *The number of all paths in $\mathcal{E}'(n)$ is $2^n B_n$.*
b) *The number of paths in $\mathcal{E}'(n)$ with k long steps is $\binom{n}{k} B_n$.*

In Section 2.4, we present some further consequences of the foregoing. (Our goal is always to find a purely combinatorial proof, it is not hard to prove these results using generating functions.) We establish a technical convolution formula first, from which (2.4), an equivalent form of Shapiro's identity, can be deduced.

Lemma 2.14 (Nagy [2]) *For arbitrary fixed n,*

$$2 \cdot \sum_{i+j+k=n} C_{2i} C_{2j} B_{2k} = B_{2n+1},$$

where the indices i, j, k are nonnegative integers.

In part c) of Theorem 2.15, we determine the closed form of the convolution of the numbers B_n with even index:

$$\sum_{k=0}^{n} B_{2k} B_{2n-2k} = \frac{16^n + 4^n B_n}{2}.$$

We give a combinatorial proof of an equivalent form, which is called the alternating convolution formula of central binomial coefficients:

Theorem 2.16 (Spivey [13], Nagy [2])

$$\sum_{k=0}^{n} B_{2k}B_{2n-2k} - \sum_{k=0}^{n-1} B_{2k+1}B_{2n-2k-1} = 4^n B_n.$$

By double counting, we obtain the form

$$\sum_{k=0}^{n} B_{2k}B_{2n-2k} - \sum_{k=0}^{n-1} B_{2k+1}B_{2n-2k-1} = \sum_{k=0}^{n} C_{2k}B_{2n-2k},$$

thus, the problem is reduced to (2.4). Our proof differs from Spivey's elegant combinatorial proof [13] that interprets the identity with the help of random colored permutations.

In Section 2.5, we pose two conjectures, based on computational experience. We need the following notation for the formulation of them, which can be used to define classes of paths in the way the set of even-zeroed paths are defined: For a 0-1-2-sequence $b_0, b_1, \dots, b_n$, let $\mathcal{P}[b_0b_1\dots b_n]$ denote the set of such (conventional) paths of length $2n$ that start from the origin and avoid the points $\{(2i,0) : b_i = 0\}$ on the x-axis but visit at least one point of $\{(2i,0) : b_i = 2\}$. Our first conjecture is a generalization of Lemma 2.7 (the second equations are obvious in both parts, the real question is the validity of the first equations):

Conjecture 2.19 (Hajnal–Nagy [1])

a) $\left|\mathcal{P}[(1^k0^k)^{n-1}1^k2^k]\right| = \left|\mathcal{P}[10^{n-1}21^{2kn-n-1}]\right| = 4^{2kn-n-1}2C_{n-1}$,

b) $\left|\mathcal{P}[(1^k0^k)^n]\right| = \left|\mathcal{P}[10^n1^{2kn-n-1}]\right| = 4^{2kn-n-1}B_n$.

We showed the equivalence of the two parts, so it is enough to prove one of them. In the second conjecture both the sequences defining the set $\mathcal{P}$ and the conjectured cardinalities are similar to previous ones:

Conjecture 2.20

a) $\left|\mathcal{P}[1(1^k0^{k+1})^{n-1}1^k2^{k+1}]\right| = \left|\mathcal{P}[10^{n-1}21^{2kn}]\right| = 4^{2kn}2C_{n-1}$,

b) $\left|\mathcal{P}[1(1^k0^{k+1})^n]\right| = \left|\mathcal{P}[10^n1^{2kn}]\right| = 4^{2kn}B_n$.

3 A convexity property of discrete random walks

We deal with symmetric random walks in Chapter 3. The research was motivated by a 2012 result on certain planar harmonic measures [6], of which we gave a new proof in our joint paper with Vilmos Totik [4], using a discrete approach. (Harmonic measure [8], a fundamental tool in harmonic analysis, has an interpretation via Brownian motion, and the Brownian motion can be approximated by discrete random walks.) The chapter presents the discrete, combinatorial results of the submitted paper [4], together with further, joint results with Attila Szalai [3] which generalize the base theorems.

Let us introduce the required concepts. A *walk* on $\mathbb{Z}^2$ is finite or infinite sequence $Q_0, Q_1, Q_2 \ldots$ $(Q_i \in \mathbb{Z}^2)$ such that the vector $Q_{i+1} - Q_i$ is either $(0,1)$, $(0,-1)$, $(1,0)$, or $(-1,0)$ for all i. (These vectors are called the steps of the walk.) The walks on $\mathbb{Z}^d$ are defined analogously: in d dimensions, there are $2d$ permitted steps, the standard base vectors, and their opposites. For a given point Q_0, the (symmetric) *random walk* with starting point Q_0 is an infinite walk starting from Q_0 whose steps are chosen uniformly and independently at random. The basic result of the chapter is the main lemma of [4]:

Theorem 3.1 (Nagy–Totik [4]) *For $k \in \mathbb{Z}$, let p_k be the probability that a symmetric random walk on $\mathbb{Z}^2$, started from the point $(0,1)$, first hits the x-axis at the point $(k,0)$. Then the sequence $(p_k)_{k=0}^{\infty}$ is convex, that is, $p_k \le \frac{1}{2}(p_{k-1} + p_{k+1})$ holds for all $k \ge 1$.*

In Section 3.2, we present an elementary proof of the theorem that does not involve any calculations. Clearly, when determining the probability p_k, we only have to deal with the initial parts of the walks, the parts preceding the first intersection with the x-axis; this is the reason for the following definition: We say that a $(k_1, h) \rightsquigarrow (k_2, 0)$ walk is *positive*, if it stays strictly above the x-axis before its last step. $\mathcal{W}_{k_2}^{(k_1,h)}$ denotes the set of positive $(k_1, h) \rightsquigarrow (k_2, 0)$ walks, and the set $\mathcal{W}_{k_2}^{(k_1,h)}[l]$ consists of the l-length walks of $\mathcal{W}_{k_2}^{(k_1,h)}$. By conditioning on the first step and the initial part of the walks that contribute to the probability p_k, it can be easily seen that the following lemma implies Theorem 3.1:

Lemma 3.2 (Nagy–Totik [4]) *For all integers k, there exists an injective length-preserving map $\mathcal{W}_k^{(0,2)} \to \mathcal{W}_k^{(1,1)} \cup \mathcal{W}_k^{(-1,1)}$. This means that, for all $k \in \mathbb{Z}$ and $l \in \mathbb{N}$,*

$$\left|\mathcal{W}_k^{(0,2)}[l]\right| \le \left|\mathcal{W}_k^{(1,1)}[l]\right| + \left|\mathcal{W}_k^{(-1,1)}[l]\right|.$$

In the proof, we give a simple injection such that the image of a path of $\mathcal{W}_k^{(0,2)}$ is obtained by interchanging some right-steps and up-steps, or some right-steps and down-steps. (It is somewhat surprising that we have not found any injection that could be easier to visualize geometrically.) In addition, we sketch some other proofs of Theorem 3.1: After presenting the first steps of the original argument by Vilmos Totik that is based on expressing p_k as the kth Fourier-coefficient of an elementary function, we also outline two other combinatorial proof of Lemma 3.2 which involve more or less algebraic manipulations.

In Section 3.3, we investigate the convexity of the hitting distribution of the x-axis for random walks with arbitrary starting point, and we establish an analogous result:

Theorem 3.3 (Nagy–Szalai [3]) *Let p_k^h be the probability that a symmetric random walk on $\mathbb{Z}^2$, started from the point $(0,h)$, first hits the x-axis at the point $(k,0)$. Then, for arbitrary fixed $h \geq 2$, the sequence $\left(p_k^h\right)_{k=h-2}^{\infty}$ is convex, that is, $p_k^h \leq \frac{1}{2}\left(p_{k-1}^h + p_{k+1}^h\right)$ holds for all $k \geq h-1$.*

Similarly to the case $h = 1$, now it is enough to prove the following analogue of Lemma 3.2:

Lemma 3.4 (Nagy–Szalai [3]) *Let h, k be integers such that $k \geq h-1$ and $h \geq 2$. Then there exists a length-preserving injection $\mathcal{W}_k^{(0,h-1)} \cup \mathcal{W}_k^{(0,h+1)} \to \mathcal{W}_k^{(1,h)} \cup \mathcal{W}_k^{(-1,h)}$. This means that, for all $l \in \mathbb{N}$,*

$$\left|\mathcal{W}_k^{(0,h-1)}[l]\right| + \left|\mathcal{W}_k^{(0,h+1)}[l]\right| \leq \left|\mathcal{W}_k^{(1,h)}[l]\right| + \left|\mathcal{W}_k^{(-1,h)}[l]\right|.$$

When defining the image of $W \in \mathcal{W}_k^{(0,h-1)} \cup \mathcal{W}_k^{(0,h+1)}$ in the proof, we distinguish two cases, depending on whether W hits a diagonal or a side of the square with vertices $(h,0)$, $(h,2h)$, $(-h,2h)$ and $(-h,0)$ first. The simpler case is when it hits a diagonal first; in this case we perform a reflection. In the case when W hits a side of the square first (necessarily the top side), we utilize the following sublemma when defining the image of W to obtain a suitable injection. The sublemma is a generalization of Lemma 3.2, it can be also proved in a similar way:

Lemma 3.5 (Nagy–Szalai [3]) *For integers h, k, m such that $h \geq 1$ and $-h < m < h$, there exists an injective length-preserving map from $\mathcal{W}_k^{(m,2h)}$ into $\mathcal{W}_k^{(h,h+m)} \cup \mathcal{W}_k^{(-h,h-m)}$.*

The continuous version of the problem is easy to solve, and its solution suggests that Theorem 3.3 is not sharp. That is why we investigate the question whether we can improve it with our method (by strengthening Lemma 3.4), i.e. whether we can replace $h-2$ with a better convexity threshold for some h, or prove concavity on an interval. The answer is no, so one needs more sophisticated methods. However, the attempt resulted in two interesting observations, for which we do not know any combinatorial proof. (We can prove them by elementary but tedious calculation based on the fact that the number of positive $(a,b) \rightsquigarrow (c,d)$ walks of length n has a simple closed form by [7] and [9], see Lemma 3.7.) The obtained results are the following:

Theorem 3.6 (Nagy–Szalai [3]) *Let $h \geq 2$ and k be fixed, and let V_l [and F_l] denote the number of l-length walks in $\mathcal{W}_k^{(0,h)}$ that start with a horizontal (left or right) step [or vertical (up or down) step].*

- *If $l = h^2 - k^2$, then $V_l = F_l$.*
- *If $l \geq h^2 - k^2$, then $V_l \geq F_l$.*
- *If $l \leq h^2 - k^2$, then $V_l \leq F_l$.*

Moreover, if $l \neq h^2 - k^2$, then $V_l = F_l$ can occur only if $V_l = F_l = 0$, i.e. if l is such that $\mathcal{W}_k^{(0,h)}[l] = \emptyset$.

Lemma 3.9 (Nagy–Szalai [3]) *Let $h \geq 2$ and k be fixed, and let J_l [and L_l] denote the number of l-length walks in $\mathcal{W}_k^{(0,h)}$ that start with a right step [or down step].*

- *If $l = (h-k)(2h-1)$, then $J_l = L_l$;*
- *If $l \geq (h-k)(2h-1)$ then $J_l \geq L_l$;*
- *If $l \leq (h-k)(2h-1)$ then $J_l \leq L_l$.*

Moreover, if $l \neq (h-k)(2h-1)$, then $J_l = L_l$ can occur only if $J_l = L_l = 0$.

Finally, we consider the higher dimensional analogue of the problem in Section 3.4. We will need the following generalization of the convexity of sequences: We say that the discrete function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is (locally) *subharmonic* at the point $\boldsymbol{k} \in \mathbb{Z}^n$, if

$$f(\boldsymbol{k}) \leq \frac{1}{2n} \sum_{\boldsymbol{j} \in N(\boldsymbol{k})} f(\boldsymbol{j}),$$

where $N(\boldsymbol{k})$ denotes set of $2n$ neighbors of the point $\boldsymbol{k}$ in $\mathbb{Z}^n$, i.e. $N(\boldsymbol{k}) := \{\boldsymbol{k} \pm \boldsymbol{e_i} : i = 1, \ldots, n\}$, using the notation $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}$ for the standard basis vectors.

We investigate the following probabilities in an arbitrary fixed dimension $d \geq 2$: For given $h \in \mathbb{N}$ and $\boldsymbol{k} = (k_1, \ldots, k_{d-1}) \in \mathbb{Z}^{d-1}$, let $p_{\boldsymbol{k}}^h$ be the probability that a symmetric random walk on $\mathbb{Z}^d$, started from the point $(0, \ldots, 0, h)$, first hits the hyperplane $x_d = 0$ at the point $(k_1, \ldots, k_{d-1}, 0)$. The following analogue of Theorem 3.1 has been proved by Vilmos Totik:

Theorem 3.10 (Nagy–Totik [4]) *The function $\mathbb{Z}^{d-1} \ni \boldsymbol{k} \mapsto p_{\boldsymbol{k}}^1$ is subharmonic at all $\boldsymbol{k} \neq \boldsymbol{0}$.*

The theorem can be deduced from the already proven planar case. We conclude the chapter with an analogous extension of Theorem 3.3:

Theorem 3.11 (Nagy–Szalai [3]) *For all $h \geq 2$, the function $\mathbb{Z}^{d-1} \ni \boldsymbol{k} \mapsto p_{\boldsymbol{k}}^h$ is subharmonic at all points of $[h-1, \infty)^{d-1} \cap \mathbb{Z}^{d-1}$.*

The dissertation is based on the following papers

[1] P. Hajnal & G. V. Nagy: A bijective proof of Shapiro's Catalan convolution, *Electron. J. Combin.* **21** (2014), Issue 2, Paper #P2.42, 1–10.

[2] G. V. Nagy: A combinatorial proof of Shapiro's Catalan convolution, *Adv. in Appl. Math.* **49** (2012), 391–396.

[3] G. V. Nagy & A. Szalai: On the convexity of a hitting distribution for discrete random walks, *Acta Sci. Math. (Szeged)*, *accepted* (2014).

[4] G. V. Nagy & V. Totik: A convexity property of discrete random walks, *submitted* (2014).

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