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# Partial iteration theories

– doctoral dissertation –

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# Introduction

Fixed point operations occur in just about all areas of theoretical computer science including automata and languages, the semantics of programming languages, process algebra, logical theories of computational systems, programming logics, recursive types and proof theory, computational complexity, etc. The equational properties of the fixed point, or dagger operation can be best described in the context of Lawvere theories of functions over a set equipped with structure, or more generally, in the context of abstract Lawvere theories (or just theories), or cartesian or co-cartesian categories, cf. [Law63, Elg75, BE93, SP00].

Iteration theories were introduced in [BEW80a], and independently in [É80] in order to describe the equational properties of the dagger operation in iterative and rational algebraic theories, cf. [WTWG76, Elg75]. In an iterative theory, dagger is defined by unique fixed points, and in rational theories, by least fixed points. In both types of theories, the dagger operation satisfies the same set of identities. These identities define iteration theories. In [SP00], it is argued that any nontrivial fixed point model satisfies the iteration theory identities.

In an iteration theory the fixed point operation takes a morphism $f : n \to n + p$ to a morphism $f^\dagger : n \to p$ which provides a solution to the fixed point equation

$$\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$$

in the morphism variable $\xi : n \to p$. If a theory is equipped with an additional structure, such as an additive structure, then the dagger operation is usually related to some "Kleenean operations".

For example, the theory of matrices over a semiring $S$ has an additive structure. Under a natural condition, cf. [BE93], any dagger operation over a matrix theory determines and is determined by a star operation mapping an $n \times n$ square matrix $A$ (i.e., a morphism $A : n \to n$) to an $n \times n$ square matrix $A^*$. Properties of the dagger operation are then reflected by corresponding properties of the star operation. In **Chapter 2**, which is based on [EH09], we show that this correspondence between the dagger and star operations can naturally be generalized to arbitrary grove theories.

When $S$ is a semiring of formal power series then the usual partially defined star operation determines and is determined by a partially defined dagger operation. But this is not the only example where it is natural to work with a partial dagger operation, since the dagger operation is necessarily a partial operation in (nontrivial) iterative theories.

In [BEW80b, É82] (see also [BE93], Theorem 6.4.5) it was shown that any iterative theory with at least one "constant" (i.e., morphism $1 \to 0$) can be turned into an iteration theory that has a total dagger operation. Moreover, the extension of the dagger operation to a total operation only depends on the choice of the constant that serves as the canonical solution of the fixed point equation associated with the identity morphism $1 \to 1$.

**Chapter 3** is based on [EH11a]. Here we provide a generalization of this construction that is applicable to partial iterative theories. We give a sufficient condition ensuring that a partially defined dagger operation of a partial iterative theory can be extended to a total operation so that the resulting theory becomes an iteration theory. We show that this general result can be instantiated to prove that every iterative theory with at least one constant can be extended to an iteration theory. We also apply our main result to theories equipped with an additive structure. We show that our result implies the Matrix Extension Theorem of [BE93] and the Grove Extension Theorem of [BE03]. In the context of these theories, the extension theorem asserts that if we have unique solutions of certain "guarded" fixed point equations, then under certain conditions, the fixed point operation can be extended in a unique way to provide solutions to all fixed point equations such that the resulting theory becomes an iteration theory. Possible applications of these results include Process Algebra, where one usually deals with unique fixed points of guarded fixed point equations (cf. [Fok07]).

Iteration theories can be axiomatized by the Conway theory identities and a group identity associated with each finite (simple) group, cf. [É99]. Whereas the group identities are needed for completeness, several constructions in automata and language theory and other areas of computer science only require the Conway identities.

In [BE93], a general Kleene type theorem was proved for all Conway theories. However, in many models of interest, the dagger operation is only partially defined. **Chapter 4** is based on [EH11b]. Here we provide a Kleene theorem for partial Conway theories. We also discuss several application of this generic result.

**Chapter 5** of this thesis is based on [EH14]. Here we give a description of the free iteration semirings using a simple congruence. However, at the time of the writing of this thesis we do not yet have a decidability result for the equational theory of iteration semirings. Moreover, the contents of Chapter 5 are unpublished at this time.

The publications that were used in the writing of this thesis are [EH11b], [EH11a], [EH09] and the forthcoming [EH14]. I have contributed to one more publication. This is [HH13].

# Chapter 1

# Basic Definitions

In this section, we review the basic concepts used in the thesis. For more details, the reader is referred to [BE93]. In any category whose objects are the nonnegative integers we will denote the composite of the morphisms $f : n \to p$ and $g : p \to q$ in diagrammatic order as $f \cdot g$. The identity morphism corresponding to object $p$ will be denoted $\mathbf{1}_p$. When $n$ is a nonnegative integer, we will denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Thus, $[0]$ is the empty set. Throughout this thesis we will assume that the reader has some familiarity with the concept of formal power series and rational power series. See [BR10] and [BR82] for an introduction to this subject.

## 1.1 Theories

Let us recall from [BE93] that a *(Lawvere) theory* $T$ is a small category with objects the nonnegative integers such that each nonnegative integer $n$ is the $n$-fold coproduct of the object 1 with itself. We assume that each theory $T$ comes with distinguished coproduct injections $i_n : 1 \to n$, $i \in [n]$, called *distinguished morphisms*, turning $n$ to an $n$-fold coproduct of object 1 with itself. By the coproduct property, for each finite sequence of *scalar morphisms* $f_1, \ldots, f_n : 1 \to p$ there is a unique morphism $f : n \to p$ such that $i_n \cdot f = f_i$, for each $i \in [n]$. This unique morphism is denoted $\langle f_1, \ldots, f_n \rangle$. The operation implicitly defined by the coproduct property is called *tupling*. In particular, when $n = 0$, tupling defines a unique morphism $0_p : 0 \to p$, for each $p \geq 0$. Note that $\mathbf{1}_n = \langle 1_n, \ldots, n_n \rangle$ for all nonnegative integers $n$. In addition, we will always assume that $\mathbf{1}_1 = 1_1$, so that $\langle f \rangle = f$ for each $f : 1 \to p$. A theory $T$ is termed *trivial* if $1_2 = 2_2$. In a trivial theory, there is at most one morphism $n \to p$, for each $n, p \geq 0$.

Tuplings of distinguished morphisms are called *base morphisms*. For example, $0_n$ and $\mathbf{1}_n$ are base morphisms. When $\rho$ is a mapping $[n] \to [p]$, there is an associated base morphism $n \to p$, the tupling $\langle (1\rho)_p, \ldots, (n\rho)_p \rangle$ of the distinguished morphisms $(1\rho)_p, \ldots, (n\rho)_p$. A *base permutation* is a base morphism associated with a bijective mapping. Note that in any theory, a base permutation corresponding to a bijection $\pi : [n] \to [n]$ is an isomorphism with inverse the base permutation corresponding to the inverse function of $\pi$.

When $f : n \to p$ and $g : m \to p$ in a theory $T$, we define $\langle f, g \rangle$ to be the morphism $h : n + m \to p$ with $i_{n+m} \cdot h = i_n \cdot f$ and $(n + j)_{n+m} \cdot h = j_m \cdot g$ for all $i \in [n]$ and $j \in [m]$. Moreover, for each $f : n \to p$ and $g : m \to q$, we define $f \oplus g = \langle f \cdot \kappa, g \cdot \lambda \rangle : n + m \to p + q$, where $\kappa$ is the base morphism corresponding to the inclusion $[p] \hookrightarrow [p + q]$ and $\lambda$ is the base morphism corresponding to the translated inclusion $[q] \hookrightarrow [p + q]$ mapping $j$ in $[q]$ to $p + j$ in $[p + q]$, for all $j \in [q]$. Note that the *pairing* operation $\langle f, g \rangle$ and the *separated sum* operation $f \oplus g$ are associative. Moreover, $\langle f, 0_p \rangle = f = \langle 0_p, f \rangle$ and $f \oplus 0_0 = f = 0_0 \oplus f$ for all $f : n \to p$. Also, $\langle f, g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle$ for all $f : n \to p$, $g : m \to p$ and $h : p \to q$, and $(f \oplus g) \cdot \langle h, k \rangle = \langle f \cdot h, g \cdot k \rangle$ for all $f : n \to p$, $g : m \to q$, $h : p \to r$ and $k : q \to r$. Finally, $(f \oplus g) \cdot (h \oplus k) = (f \cdot h) \oplus (g \cdot k)$ for all appropriate morphisms $f, g, h, k$.

A *morphism* $T \to T'$ between theories $T$ and $T'$ is a functor which preserves the objects and the distinguished morphisms. It follows that any theory morphism preserves the pairing, tupling and separated sum operations. A theory $T$ is a *subtheory* of a theory $T'$ if $T$ is a subcategory of $T'$ and has the same distinguished morphisms as $T$, so that the inclusion $T \hookrightarrow T'$ is a theory morphism.

**Example 1.1.1** A basic example of a theory is $\mathbf{Fun}_A$ the *theory of functions over a set* $A$. In this theory, a morphism $n \to p$ is a function $f : A^p \to A^n$. Note the reversal of the arrow. The composite of morphisms $f : n \to p$ and $g : p \to q$ is their function composition written from right to left, which is a function $A^q \to A^n$. The distinguished morphisms are the projection functions.

**Example 1.1.2** Let $S = (S, +, \cdot, 0, 1)$ be a semiring [Gol99]. The *theory of matrices* $\mathbf{Mat}_S$ *over* $S$ has as morphisms $n \to p$ all $n \times p$ matrices in $S^{n \times p}$. Composition is matrix multiplication defined in the usual way. For each $i \in [p]$, $p \geq 0$, the distinguished morphism $i_p : 1 \to p$ is the $1 \times p$ row matrix with a 1 on the $i$th position and 0's elsewhere. It is known that in each matrix theory, each object $n$ is also the $n$-fold product of the object 1 with itself. The transposes $i_n^T$ of the distinguished morphisms serve as the projection morphisms $n \to 1$, see [Elg76, BE93]. The theory $\mathbf{Mat}_S$ comes with a sum operation $+$ defined on each hom-set $\mathbf{Mat}_S(n,p) = S^{n\times p}$. For each $n,p \geq 0$, $(\mathbf{Mat}_S(n,p), +, 0_{n,p})$ is a commutative monoid, where $0_{n,p}$ is the $n \times p$ matrix whose entries are all 0. Moreover, composition distributes over finite sums both on the left and on the right. Thus, for each $n$, $(\mathbf{Mat}_S(n,n), +, \cdot, \mathbf{1}_n, 0_{n,n})$ is itself a semiring, since the product of two $n \times n$ matrices is an $n \times n$ matrix. In particular, $\mathbf{Mat}_S(1,1)$ is isomorphic to $S$. We will usually identify a morphism $1 \to 1$ with the corresponding element of $S$.

**Example 1.1.3** Suppose that $S$ is a semiring and $V = (V, +, 0)$ is a (left) $S$-semimodule, cf. [Gol99]. Then the *matricial theory* $\mathbf{Matr}_{S,V}$ [Elg76, BE93] over $(S,V)$ has as morphisms $n \to p$ all ordered pairs $(A; v)$ consisting of a matrix $A : n \to p$ in $\mathbf{Mat}_S$ and an $n$-dimensional column vector $v \in V^n$. When $p = 0$, we usually write $(;v)$ or just $v$. Composition is defined by the rule

$$(A;v) \cdot (B;w) = (AB; v + Aw)$$

for all $(A;v) : n \to p$ and $(B;w) : p \to q$. For each $i \in [p]$, $p \geq 0$, the distinguished morphism $i_p$ is the ordered pair $(i_p; 0)$, where somewhat ambiguously, $i_p$ also denotes the corresponding distinguished morphism in $\mathbf{Mat}_S$. The theory $\mathbf{Matr}_{S,V}$ comes with the pointwise sum operation and the zero morphisms $0_{n,p} = (0_{n,p}; 0^n)$, where $0^n$ denotes the $n$-dimensional column vector of 0's in $V^n$. Each hom-set $\mathbf{Matr}_{S,V}(n,p) = (\mathbf{Matr}_{S,V}(n,p), +, 0_{n,p})$ is a commutative monoid and composition distributes over finite sums on the right, but usually not on the left. Note that $\mathbf{Mat}_S$ may be identified with the subtheory of $\mathbf{Matr}_{S,V}$ determined by the morphisms of the sort $(A; 0^n) : n \to p$, $n,p \geq 0$. We call $\mathbf{Mat}_S$ the *underlying matrix theory* of $\mathbf{Matr}_{S,V}$.

**Example 1.1.4** A *ranked alphabet* $\Sigma$ is a family of pairwise disjoint sets $(\Sigma_n)_n$, where $n$ ranges over the nonnegative integers. We assume that the reader is familiar with the notion of (total) $\Sigma$-*trees* over a set $X_p = \{x_1, \ldots, x_p\}$ of variables, defined as usual, see e.g. [BE93]. Below we will denote the collection of finite and infinite $\Sigma$-trees over $X_p$ by $T^\omega_\Sigma(X_p)$ and the collection of just the finite trees by $T_\Sigma(X_p)$. We call a tree *proper* if it is not one of the trees $x_i$. $\Sigma$-trees form a theory $\Sigma\mathrm{TR}$ whose morphisms $n \to p$ are all $n$-tuples of trees in $T^\omega_\Sigma(X_p)$. Composition is defined by substitution for the variables $x_i$, and for $i \in [p]$, the tree with a single vertex labeled $x_i$ serves as the $i$th distinguished morphism $1 \to p$. Thus, if $t : 1 \to n$ and $t'_1, \ldots, t'_n : 1 \to p$ in $\Sigma\mathrm{TR}$, then $t \cdot \langle t'_1, \ldots, t'_n \rangle : 1 \to p$ is the tree obtained by substituting a copy of $t'_i$ for each leaf of $t$ labeled $x_i$, for $i \in [n]$. See [BE93] for details. A tree is called *regular* if up to isomorphism it has a finite number of subtrees. The subtheory of $\Sigma$TR containing only the regular $\Sigma$-trees is denoted $\Sigma$tr, and the subtheory containing only the finite $\Sigma$-trees is denoted $\Sigma$Term. As usual, we identify each letter $\sigma$ in $\Sigma_n$ with the corresponding *atomic tree* $\sigma(x_1, \ldots, x_n)$ in $T_\Sigma(X_n)$ whose root is labeled $\sigma$ and has $n$ immediate successors labeled $x_1, \ldots, x_n$, respectively.

It is known that $\Sigma$Term is freely generated by $\Sigma$ in the category of theories. In particular, when $\Sigma$ is is empty, $\Sigma$Term is an initial theory.

**Example 1.1.5** The theory $\Theta$ has as morphisms $n \to p$ all functions $[n] \to [p]$. Composition is defined by function composition written from left to right. For each $n$ and $i \in [n]$, the distinguished morphism $1 \to n$ is the function $[1] \to [n]$ selecting the integer $i$. When $T$ is a nontrivial theory, the base morphisms form a subtheory of $T$ isomorphic to $\Theta$. Moreover, when $\Sigma$ is the empty ranked alphabet, the theory $\Sigma$TR (or $\Sigma$Term) is isomorphic to $\Theta$.

## 1.2 Partial Conway and iteration theories

Let $T$ be a theory. A nonempty collection of morphisms $I$ is an *ideal* [BE93] of $T$ if it is closed under tupling, composition with base morphisms on the left, and composition with arbitrary morphisms on the right. $I$ is *proper* iff $\mathbf{1}_1 \notin I$. In other words, $I$ is proper iff there is no base morphism in $I$, or equivalently, $I \neq T$. Note that every ideal contains the morphisms $0_p$, $p \geq 0$.

**Definition 1.2.1** *A* partial dagger theory *is a theory $T$ equipped with a distinguished ideal $D(T)$ and a partially defined dagger operation*

$$^\dagger : T(n, n+p) \to T(n,p),\ n,p \geqslant 0$$

*defined on morphisms $n \to n+p$ in $D(T)$.*

Let $T, T'$ be partial dagger theories. A *partial dagger theory morphism* $\varphi : T \to T'$ is a theory morphism $T \to T'$ which preserves the distinguished ideal and the dagger operation, i.e. $D(T)\varphi \subseteq D(T')$ and $(f^\dagger)\varphi = (f\varphi)^\dagger$ for all $f : n \to n+p$ in $D(T)$ and $n, p \geqslant 0$. We say that $T$ is a *partial sub-dagger theory* of $T'$ if $T$ is a subtheory of $T'$, $D(T) = D(T') \cap T$, and the dagger operation of $T$ is the restriction of the dagger operation of $T'$.

In the sequel, we will consider partial dagger theories satisfying certain identities that we will define now. For the origins of these identities, the reader is referred to [dBS69, WTWG76, É80, Bek84, Plo85, Niw85, Niw86] and [É99].

**Definition 1.2.2** *We say that the partial dagger theory $T$ satisfies:*

1. *the* fixed point identity*, if*

$$f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$$

*for each $f : n \to n + p$ in $D(T)$,*

2. *the* left zero identity*, if*

$$(0_n \oplus f)^\dagger = f$$

*for each $f : n \to p$ in $D(T)$,*

3. *the* right zero identity*, if*

$$(f \oplus 0_q)^\dagger = f^\dagger \oplus 0_q$$

*for each $f : n \to n + p$ in $D(T)$,*

4. *the (*base*)* parameter identity*, if*

$$f^\dagger \cdot g = (f \cdot (\mathbf{1}_n \oplus g))^\dagger$$

*for each $f : n \to n + p$ in $D(T)$ and $g : p \to q$ in $T$ (such that $g$ is a base morphism),*

5. *the* permutation identity*, if*

$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger = \pi \cdot f^\dagger$$

*for each $f : n \to n + p$ in $D(T)$ and base permutation $\pi : n \to n$ with inverse $\pi^{-1}$,*

6. *the* pairing identity *(or* Bekić identity*), if for all $f : n \to n + m + p$ and $g : m \to n + m + p$ in $D(T)$,*

$$\langle f, g \rangle^\dagger = \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle$$

*where $h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \to m + p$,*

7. *the* double dagger identity*, if*

$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger = (f \cdot \langle f^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger$$

*for each* $f : n \to n + n + p$ *in* $D(T)$,

8. *the* composition identity, *if*

$$f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle = (f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger$$

*for each* $f : n \to m + p$ *and* $g : m \to n + p$ *in* $D(T)$,

9. *the* simplified composition identity, *if*

$$(f \cdot g)^\dagger = f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger$$

*for each* $f : n \to m$ *and* $g : m \to n + p$ *in* $D(T)$,

10. *the* group identity $C(S)$ *associated with the finite group* $S = ([n], \circ)$, *if*

$$g_S^\dagger = \tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger$$

*for each* $g : 1 \to n + p$ *in* $D(T)$, *where*

$$g_S = \langle g \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, g \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle,$$

$\tau_n$ *is the unique base morphism* $n \to 1$ *and* $\rho_i^S = \langle (i \circ 1)_n, \ldots, (i \circ n)_n \rangle$ *for each* $i \in [n]$, *where* $\circ$ *is the group operation of* $S$,

11. *the* commutative identity[1], *if*

$$((\rho \cdot f) \| (\rho_1, \ldots, \rho_m))^\dagger = \rho \cdot (f \| (\tau_1, \ldots, \tau_n))^\dagger$$

*for each* $f : n \to k + p$ *in* $D(T)$, *surjective base morphism* $\rho : m \to n$ *and base morphisms* $\rho_i : k \to m$, $\tau_j : k \to n$, $i \in [m]$, $j \in [n]$, *such that* $\rho_i \cdot \rho = \tau_{i\rho}$ *for each* $i \in [m]$.

**Remark 1.2.3** Note that the composition identity implies the simplified composition identity and the fixed point identity implies the left zero identity. If the fixed point identity holds, then $f^\dagger$ is in $D(T)$ for each $f : n \to n + p$ in $D(T)$, since $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$. Thus, when the fixed point identity holds, $^\dagger$ is a function $D(T) \to D(T)$. Then we say that $T$ satisfies

12. *the* simplified form of the double dagger identity, *if*

$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger = f^{\dagger\dagger}$$

*for each* $f : n \to n + n + p$ *in* $D(T)$.

[1] In [BE93] this identity is called the *generalized commutative identity*.

If the fixed point identity holds, then the simplified form of the double dagger identity is equivalent to the double dagger identity.

The *scalar versions* of the fixed point, left zero, right zero, (base) parameter and double dagger identities are obtained by taking $n = 1$ in the corresponding identity. The scalar versions of the composition and simplified composition identities are obtained by taking $n = m = 1$ in those identities.

**Definition 1.2.4** *A* partial Conway theory *is a partial dagger theory satisfying the fixed point, right zero, pairing, double dagger and permutation identities. A* partial iteration theory *is a partial Conway theory satisfying the group identity $C(S)$ for each finite group $S$. A morphism of partial Conway or iteration theories is a partial dagger theory morphism.*

A partial Conway theory $T$ is a *partial sub-Conway theory* of the partial Conway theory $T'$ if $T$ is a partial sub-dagger theory of $T'$. Similarly, a partial iteration theory $T$ is a *partial subiteration theory* of the partial iteration theory $T'$ if $T$ is a partial sub-dagger theory of $T'$.

In earlier work [BE93] partial iteration theories were defined using the commutative identities instead of the group identities, see Definition 5.3.8 and Proposition 5.3.26 from [BE93]. Later, it was proven in [É99] that a Conway theory (see Definition 1.2.12) satisfies the group identities iff it satisfies the commutative identity, but the proof does not carry through to the partial case seamlessly. Indeed, in Lemma 14.1 of [É99] the following corollary of the (simplified) composition identity is applied:

$$(\tau_n \cdot f)^\dagger = \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger \tag{1.1}$$

where $\tau_n$ is the unique base morphism $n \to 1$ and $f : 1 \to n + p$ is an arbitrary morphism in a Conway theory. If $T$ is a partial Conway theory with a proper distinguished ideal then (1.1) is not an instance of the (simplified) composition identity, since $\tau_n$ is not in $D(T)$. To fix this we have to prove that whenever $T$ is a partial Conway theory then (1.1) holds for every $f : 1 \to n + p$ in $D(T)$.

**Notation** From this point we will indicate the identities used in a computation as subscripts of the corresponding equality sign, i.e. given expressions $t, t'$ denoting morphisms in partial dagger theories, by writing $t =_i t'$ we will mean that $t = t'$ follows by application of the $i$th identity in Definition 1.2.2 or the identity in Remark 1.2.3 when $i = 12$.

**Lemma 1.2.5** *Let $T$ be a partial dagger theory satisfying the fixed point, (simplified) double dagger and pairing identities. Then (1.1) holds for every $f : 1 \to n + p$ in $D(T)$.*

**Proof** We proceed by induction on $n$. When $n = 0$,

$$(0_1 \cdot f)^\dagger = 0_p = 0_1 \cdot (f \cdot (0_1 \oplus \mathbf{1}_p))^\dagger.$$

Suppose that (1.1) holds for $n - 1$, for some $n > 0$. We prove that it holds for $n$. Indeed,

$$\begin{array}{rcl} \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger & = & \tau_n \cdot (f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}) \cdot (\tau_2 \oplus \mathbf{1}_p))^\dagger \\ & =_{12} & \tau_n \cdot (f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}))^{\dagger\dagger} \\ & = & \langle \tau_{n-1} \cdot k^{\dagger\dagger}, k^{\dagger\dagger} \rangle \end{array}$$

where $k = f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}) : 1 \to 2 + p$. We continue as follows.

$$\begin{array}{rcl} \langle \tau_{n-1} \cdot k^{\dagger\dagger}, k^{\dagger\dagger} \rangle & =_1 & \langle \tau_{n-1} \cdot k^\dagger \cdot \langle k^{\dagger\dagger}, \mathbf{1}_p \rangle, k^{\dagger\dagger} \rangle \\ & = & \langle \tau_{n-1} \cdot k^\dagger, \mathbf{1}_1 \oplus 0_p \rangle \cdot \langle k^{\dagger\dagger}, \mathbf{1}_p \rangle \\ & = & \langle (\tau_{n-1} \cdot f)^\dagger, \mathbf{1}_1 \oplus 0_p \rangle \cdot \langle k^{\dagger\dagger}, \mathbf{1}_p \rangle \\ & = & \langle f'^\dagger \cdot \langle h'^\dagger, \mathbf{1}_p \rangle, h'^\dagger \rangle \end{array}$$

where $h' = k^\dagger$ and $f' = \tau_{n-1} \cdot f : (n-1) \to n + p$. We used the induction hypothesis in the third equation.

Now we calculate as follows.

$$\begin{array}{rcl} h' & = & k^\dagger \\ & = & (f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}))^\dagger \\ & =_1 & f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}) \cdot \langle (f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}))^\dagger, \mathbf{1}_{1+p} \rangle \\ & = & f \cdot \langle \tau_{n-1} \cdot (f \cdot (\tau_{n-1} \oplus \mathbf{1}_{1+p}))^\dagger, \mathbf{1}_{1+p} \rangle \\ & = & f \cdot \langle (\tau_{n-1} \cdot f)^\dagger, \mathbf{1}_{1+p} \rangle \\ & = & g' \cdot \langle f'^\dagger, \mathbf{1}_{1+p} \rangle \end{array}$$

where $g' = f$. Here we used the induction hypothesis in the fifth equation.

Using the pairing identity we get

$$\begin{array}{rcl} \langle f'^\dagger \cdot \langle h'^\dagger, \mathbf{1}_p \rangle, h'^\dagger \rangle & =_6 & \langle f', g' \rangle^\dagger \\ & = & (\tau_n \cdot f)^\dagger \end{array}$$

and with this the claim is proven. □

**Corollary 1.2.6** *Let $T$ be a partial Conway theory. Then (1.1) holds for every $f : 1 \to n + p$ in $D(T)$.*

The following corollary justifies our definition of partial iteration theory.

**Corollary 1.2.7** *Let $T$ be a partial Conway theory. $T$ satisfies the group identities iff $T$ satisfies the commutative identity.*

**Proof** Follows from Corollary 1.2.6 and the proofs in [É99]. □

The following fact is known from [BE93].

**Proposition 1.2.8** *Except for the commutative and the group identities, all of the identities from 1. to 12. hold in all partial Conway theories.*

As shown in [Elg75] and [BE93], unique solutions to fixed point equations in algebraic theories give rise to partial iteration theories. We give the details of this result below.

**Definition 1.2.9** *A* partial iterative theory *is a theory $T$ with a distinguished ideal $D(T)$ such that for each $f : n \to n + p$ in $D(T)$, the fixed point equation $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ associated with $f$ has a unique solution in $T$.*

A *morphism* $T \to T'$ of partial iterative theories is a theory morphism $\varphi$ with $D(T)\varphi \subseteq D(T')$. Every partial iterative theory $T$ is a partial dagger theory with dagger operation that maps a morphism $f : n \to n + p$ in $D(T)$ to the unique morphism $f^\dagger : n \to p$ with $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$. It is clear that any morphism of partial iterative theories is a partial dagger theory morphism.

The following fact follows from the results in [BGR77].

**Remark 1.2.10** Suppose that $T$ is a theory with distinguished ideal $D(T)$. If the equation $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution for each scalar morphism $f : 1 \to 1 + p$ in $D(T)$, where $\xi$ ranges over the set of morphisms $1 \to p$, then $T$ is a partial iterative theory.

By the following result from [BE93], the prime examples of partial iteration theories are the partial iterative theories.

**Theorem 1.2.11** *Every partial iterative theory is a partial iteration theory.*

**Definition 1.2.12** *A* dagger theory *is a partial dagger theory $T$ with $D(T) = T$. A* Conway theory *(*iteration theory*) is a dagger theory which is a partial Conway theory (partial iteration theory). A morphism of these theories is a partial dagger theory morphism.*

Note that a partial dagger theory $T$ is a dagger theory iff $D(T)$ contains $\mathbf{1}_1$, or at least one distinguished morphism. The axioms of Conway theories may be simplified. The following fact is known, cf. [BE93], Chapter 6.

**Proposition 1.2.13** *Suppose that $T$ is a dagger theory. Then the following are equivalent:*

- *$T$ satisfies the left zero, right zero, pairing and permutation identities.*
- *$T$ satisfies the base parameter, composition and double dagger identities.*
- *$T$ satisfies the fixed point, base parameter, simplified composition and double dagger identities.*
- *$T$ satisfies the scalar versions of the parameter, composition and double dagger identities, and the pairing identity for $m = 1$.*
- *$T$ satisfies the scalar versions of the fixed point, parameter, simplified composition and double dagger identities, and the pairing identity for $m = 1$.*

**Example 1.2.14** Examples of iteration theories are the theories of continuous or monotone functions over complete partial orders equipped with the *least fixed point operation* as dagger. See [BE93] for details.

**Remark 1.2.15** Iteration theories are exactly the dagger theories satisfying all identities true of continuous theories, or equivalently, the tree theories $\Sigma\mathrm{TR}$ or $\Sigma\mathrm{tr}$ with a total dagger operation as in Example 1.2.16. The free iteration theories may be described as the theories $\Sigma\mathrm{tr}$ of regular trees. The free Conway theories have been described in [BE98]. It is decidable in polynomial time whether an identity holds in all iteration theories, whereas the equational theory of Conway theories is **PSPACE**-complete, cf. [BE98].

**Example 1.2.16** Let $\Sigma$ be a ranked alphabet and let $T$ be the theory $\Sigma\mathrm{TR}$, or the theory $\Sigma\,\mathrm{tr}$. Let the ideal $D(T)$ consist of those morphisms $f : n \to p$ in $T$ whose components $i_n \cdot f$, $i \in [n]$, are proper trees. It is known that for each $f : n \to n + p$ in $D(T)$, the equation

$$\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle \tag{1.2}$$

has a unique solution in the variable $\xi : n \to p$. Denoting this unique solution by $f^\dagger$, $T$ becomes a partial iterative theory. Moreover, if $\Sigma_0$ is not empty, so that there is at least one morphism in $T(1, 0)$, then for any choice of a morphism $\perp : 1 \to 0$ the partial dagger operation can be *uniquely* extended to a totally defined dagger operation such that $T$ becomes an iteration theory. See [BEW80a] and [É82], or [BE93].

**Example 1.2.17** Suppose that $\Sigma$ contains a single letter $\perp$ that has rank 0. Then any scalar morphism in $\Sigma$TR is either a distinguished morphism, or a morphism $\perp_{1,p} = \perp \cdot 0_p : 1 \to p$. Given $f : n \to n+p$, it holds that $f^\dagger = f^n \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle$, where $\perp_{n,p} = \langle \perp_{1,p}, \dots, \perp_{1,p} \rangle : n \to p$, $f^0 = \mathbf{1}_n \oplus 0_p$ and $f^{k+1} = f \cdot \langle f^k, 0_n \oplus \mathbf{1}_p \rangle$. Let $\perp$TR denote this Conway theory. It is known that $\perp$TR is an initial Conway theory (and an initial iteration theory).

**Example 1.2.18** Let $\Theta'$ be the theory whose morphisms $n \to p$ are the partial functions $[n] \to [p]$. Composition is function composition and the distinguished morphisms are defined as in the theory $\Theta$. For each $n, p \geq 0$, let $\perp_{n,p}$ denote the totally undefined partial function $[n] \to [p]$. For each $\rho : n \to n+p$, define

$$\rho^\dagger = \rho^n \cdot \langle \perp_{n,p} \mathbf{1}_p \rangle.$$

Then $\Theta'$ is an iteration theory isomorphic to $\perp$TR. Thus, $\Theta'$ is also an initial iteration theory. It is known that $\Theta'$ is also an initial Conway theory.

**Example 1.2.19** Each nontrivial Conway theory $T$ contains a subtheory isomorphic to $\Theta'$. Given a partial function $\rho : [n] \to [p]$, let us define the corresponding *partial base morphism* in $T$ as the morphism $\langle f_1, \dots, f_n \rangle$, where for each $i \in [n]$, $f_i = j_p$ if $i\rho = j$ is defined, and $f_i = \perp \cdot 0_p = \perp_{1,p}$, if $i\rho$ is not defined. Here, $\perp = \mathbf{1}_1{}^\dagger$.

### 1.2.1 Partial Conway and iteration semirings

Recall Example 1.1.2. Let $T = \mathbf{Mat}_S$ be a matrix theory. If $T'$ is a subtheory of $T$, $T'$ is not necessarily a matrix theory. But when $T'$ contains the zero matrices $0_{n,p}$ and is closed under sum, then $T'$ is a matrix theory, called a *sub matrix theory* of $T$. It is easy to see that in this case $S_0 := T'(1,1)$ is a subsemiring of $S$, and a matrix is in $T'$ iff each of its entries belongs to $S_0$, so that $T'$ is $\mathbf{Mat}_{S_0}$. Conversely, every subsemiring of $S$ determines a sub matrix theory of $T$.

Suppose now that $I = (I(n,p))_{n,p}$ is a collection of morphisms containing the zero morphisms $0_{n,p}$ closed under sum and left and right composition with any morphism in $T$. Then we call $I$ a *two-sided ideal* of $T$. Each two-sided ideal of $T$ determines and is determined by a two-sided ideal of the semiring $S$ (cf. [Gol99]), since if $I$ is a two-sided ideal of $T$ then $I(1,1)$ is a two-sided ideal of $S$, and if $I_0$ is a two-sided ideal of $S$ then the collection of those matrices all of whose entries belong to $I_0$ is a two-sided ideal of $T$. Note that every two-sided ideal of a matrix theory $T$ is an ideal of $T$ as defined at the beginning of Section 1.2.

**Definition 1.2.20** *A* partial Conway matrix theory *is a matrix theory* $T = \mathbf{Mat}_S$ *which is a partial Conway theory such that* $D(T)$ *is a two-sided ideal. A* partial iteration matrix theory *is a partial Conway matrix theory that is a partial iteration theory. A* Conway matrix theory *(*iteration matrix theory*) is matrix theory that is a Conway theory (iteration theory).*

When $\mathbf{Mat}_S$ is a Conway matrix theory, the dagger operation determines a *star operation* mapping a matrix $A : n \to n$ to a matrix $A^* : n \to n$ by

$$A^* = \begin{pmatrix} A & \mathbf{1}_n \end{pmatrix}^\dagger .$$

In particular, $S$ is equipped with a star operation $^* : S \to S$. The equational properties of the dagger operation are then reflected by corresponding properties of the star operation. For example, the fixed point identity corresponds to the identity

$$A^* = AA^* + \mathbf{1}_n \tag{1.3}$$

where $A : n \to n$. Moreover, the simplified double dagger identity corresponds to the identity

$$(A + B)^* = A^*(BA^*)^*$$

where $A, B : n \to n$, and the composition identity corresponds to

$$(AB)^* = 1 + A(BA)^*B$$

where $A : n \to m$, $B : m \to n$.

The star operation in turn gives rise to a dagger operation:

$$\begin{pmatrix} A & B \end{pmatrix}^\dagger = A^*B \tag{1.4}$$

where $A$ is an $n \times n$ matrix and $B$ is an $n \times p$ matrix over $I$.
Similar facts hold for *partial Conway matrix theories*.

Following [BEK08], we define a *partial Conway semiring* to be a semiring $S$ equipped with a distinguished two-sided ideal $I$ and a star operation $^* : I \to S$ such that

$$(a + b)^* = a^*(ba^*)^*, \quad a, b \in I, \tag{1.5}$$

$$(ab)^* = 1 + a(ba)^*b, \quad a \in I \text{ or } b \in I. \tag{1.6}$$

The star operation can be extended to square matrices over $I$ using the following well-known matrix formula (which corresponds to the pairing identity as explained in [BE93]):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} A^* + A^*B(D + CA^*B)^*CA^* & A^*B(D + CA^*B)^* \\ (D + CA^*B)^*CA^* & (D + CA^*B)^* \end{pmatrix}$$

where $A$ and $D$ are square matrices. (There are several equivalent formulas, see [BE93].) A partial Conway semiring $S$ is a *Conway semiring* iff $I = S$.

A partial Conway semiring is a *partial iteration semiring* [É99, BEK08] iff for each finite group $G = \{1, \ldots, n\}$ it satisfies the *group identity associated with* $G$ :

$$e_1 M_G^* u_n = (a_1 + \ldots + a_n)^*$$

where $a_1, \ldots, a_n$ are arbitrary elements in the distinguished two-sided ideal of $S$, and $M_G$ is the $n \times n$ matrix whose $(i,j)$th entry is $a_{i^{-1}j}$, for all $i, j \in G$, and $e_1$ is the $1 \times n$ matrix whose first entry is 1 and whose other entries are 0. Finally, $u_n$ is the $n \times 1$ matrix all of whose entries are 1. A partial iteration semiring $S$ is an *iteration semiring* iff $I = S$.

Suppose that $S$ is a partial Conway semiring. Let $T = \mathbf{Mat}_S$ and denote by $D(T)$ the ideal of those matrices all of whose entries are in $I$. Then $T$, equipped with the dagger operation defined in (1.4) on the morphisms $n \to n + p$ in $D(T)$, $n, p \geq 0$, is a partial Conway matrix theory. When $I = S$, $T$ is a Conway matrix theory.

Let $S$, $S'$ be partial Conway semirings. A semiring morphism $\varphi : S \to S'$ is a *(partial) Conway semiring morphism* iff it preserves the distinguished ideal in $S$ and the star operation, i.e. when $D(S)\varphi \subseteq D(S')$ and $(a^*)\varphi = (a\varphi)^*$, for each $a \in D(S)$. A *(partial) iteration semiring morphism* is a (partial) Conway semiring morphism that happens to go between (partial) iteration semirings. The partial iteration semiring $S$ is a *partial subiteration semiring* of the partial iteration semiring $S'$ iff $S$ is a subsemiring of $S'$ with $D(S') \subseteq D(S)$ and the star operation of $S$ is the restriction of the star operation of $S'$ to $D(S)$.

Let $S$ be a partial Conway semiring. A *star congruence* on $S$ is a semiring-congruence which preserves the partially defined star operation, i.e. for every $a, b \in D(S)$, whenever $a$ is equivalent to $b$ then $a^*$ is equivalent to $b^*$. With this definition the kernels of the star semiring morphisms are exactly the star congruences. The quotient of the *partial Conway semiring* $S$ with the star congruence $\theta$ will be denoted $S/\theta$. By definition $D(S/\theta)$ contains the equivalence classes $s/\theta$ such that $s \in D(S)$.

The *category of (partial) iteration semirings* is the following category. The objects are the (partial) iteration semirings and the morphisms are the (partial) iteration semiring morphisms.

Recall that $\mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$ denotes the semiring of rational power series over the semiring $\mathbb{N}$ of the nonnegative integers. $\mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$ is an example of a partial iteration semiring with the usual definition of star. More can be said.

Let $\eta : \Delta \to \mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$ be the injection of $\Delta$ into the corresponding series in $\mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$. The following theorem is from [BE09].

**Theorem 1.2.21** *The partial iteration semiring* $\mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$ *of rational power series over the semiring of the nonnegative integers is freely generated by* $\Delta$ *in the category of partial iteration semirings. More precisely, for each partial iteration semiring* $S$ *and for any mapping* $h : \Delta \to D(S)$*, there is a unique partial iteration semiring morphism* $h^\# : \mathbb{N}^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle \to S$ *such that* $\eta \cdot h^\# = h$.

In Section 5 we will need the description of $S_0$, the initial iteration semiring. This is from [BE93]. We define a linear order $\leqslant$ on $S_0$ by enumerating it's elements:

$$0, 1, 2, \ldots, k, \ldots, 1^*, (1^*)^2, \ldots, (1^*)^k, \ldots, 1^{**}. \tag{1.7}$$

The sum and product of integers are defined as usual, the sum and the product on the remaining elements are defined as follows.

$$\begin{gathered} x + y = \max\{x, y\}, \text{ if } 1^* \leqslant x \text{ or } 1^* \leqslant y, \\ (1^*)^n (1^*)^p = (1^*)^{n+p}, \\ x1^{**} = 1^{**} = 1^{**}x, \text{ for } x \neq 0. \end{gathered}$$

Moreover, we define $0^* = 1$, $x^* = 1^*$, for $x = 1$ and $x^* = 1^{**}$ for each $x \in S_0$ such that $2 \leqslant x$.

It is known [BE93] that $S_0$ is an initial iteration semiring.

There are several computationally interesting quotients of $S_0$. Among these are the Boolean semiring $\mathbb{B}$ and the semiring $\mathbb{N}_\infty$, which can be obtained from the semiring $\mathbb{N}$ by adding a new element $\infty$ and extending the semiring operations, as usual. Here $\mathbb{N}$ denotes the usual semiring of nonnegative integers and $\mathbb{B}$ denotes the usual Boolean semiring. In $\mathbb{N}_\infty$, $0^* = 1$ and $a^* = \infty$ for each nonzero $a \in \mathbb{N}$. In $\mathbb{B}$, $0^* = 1^* = 1$. Thus, $\mathbb{N}_\infty$ can be obtained by taking the quotient of $S_0$ with the least star congruence containing the all pairs $((1^*)^k, 1^{**})$, for all $k \geqslant 1$. Alternatively, it can be seen that $\mathbb{N}_\infty$ is the quotient of $S_0$ with respect to the smallest star congruence identifying $1^*$ and $1^{**}$, or $1^*1^*$. Moreover, $\mathbb{B}$ can be obtained by taking the quotient of $S_0$ with respect to the star congruence that identifies all the nonzero elements of $S_0$ but keeps zero intact. Equivalently, $\mathbb{B}$ is the quotient of $S_0$ with respect to the congruence generated by the pair $(1, 1^*)$.

Clearly, $\mathbb{N}_\infty$ is an iteration semiring, which is initial in the variety of iteration semirings satisfying $1^* = 1^{**}$. Moreover, for each $\Delta$, $\mathbb{N}_\infty^{\mathrm{rat}}\langle\langle\Delta^*\rangle\rangle$ is an iteration semiring, a free iteration semiring generated by $\Delta$ in the variety of iteration semirings satisfying the following identities: [2]

[2]The first identity could be replaced by $1^* = 1^{**}$, cf. [BE09]. The last identity could be replaced by $(1 + a)^* = 1^*a^*$.

$$1^*1^* = 1^*,$$
$$1^*a = a1^*,$$
$$1^*a^* = 1^*(1^*a)^*.$$

It is known that continuous, $\omega$-continuous, complete and countably complete semirings equipped with the star operation defined as $x^* = \sum_{n\geqslant 0} x^n$ satisfy exactly the identities of this variety. For the missing definitions see Remark 47 from [BE09], additional references can be found there.

For example, the semiring $\mathbb{Q}_{\infty,\geqslant 0}$ is countably complete, thus, it is an iteration semiring. Here $\mathbb{Q}_{\infty,\geqslant 0}$ is obtained from the usual semiring of non-negative rational numbers by adding a new element $\infty$ and extending the semiring operations as usual. The star operation is defined as follows. If $x \geqslant 1$, then $x^* = \infty$. If $x = 0$, then $x^* = 1$. Lastly, if $0 < x < 1$, then $x^* = \sum_{n\geqslant 0} x^n = \frac{1}{1-x}$, which is a positive rational number.

Moreover, it is known that $\mathbb{B}^{\mathrm{rat}}\langle\langle \Delta^* \rangle\rangle$ is an iteration semiring, which is free in the category of iteration semirings satisfying $1^* = 1$. These results were shown in [Kro91, BE93].

### 1.2.2 Partial Conway (and iteration) matricial theories

Recall Example 1.1.3. When a matricial theory $\mathbf{Matr}_{S,V}$ is a Conway theory, then the dagger operation determines a star operation on the underlying matrix theory as well as an *omega operation* mapping a matrix $A : n \to n$ in $\mathbf{Mat}_S$ to a vector in $V^n$, and thus also an star operation on $S$ and a omega operation $S \to V$. For details, see [BE93].

In [É11], a *partial Conway semiring-semimodule pair* is defined as a semiring-semimodule pair $(S, V)$ equipped with a two-sided ideal $I \subseteq S$ and star and *omega power* operations $^* : I \to S$ and $^\omega : I \to V$ such that $S$ is a partial Conway semiring and

$$\begin{aligned}(a+b)^\omega &= (a^*b)^\omega + (a^*b)^*a^\omega, \quad a, b \in I,\\ (ab)^\omega &= a(ba)^\omega, \quad a \in I \text{ or } b \in I.\end{aligned}$$

A *Conway semiring-semimodule pair* [BE93] is a partial Conway semiring-semimodule pair $(S, V)$ with $S$ as distinguished ideal.

Suppose that $(S, V)$ is a partial Conway semiring-semimodule pair with distinguished two-sided ideal $I$. Then, as shown in [BE93, É11], the omega power operation can be extended to square matrices over $I$ in the following way.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\omega = \begin{pmatrix} A^*B(D+CA^*B)^\omega + A^*B(D+CA^*B)^*CA^\omega + A^\omega \\ (D+CA^*B)^\omega + (D+CA^*B)^*CA^\omega \end{pmatrix}.$$

Here $A : n \to n$, $B : n \to m$, $C : m \to n$ and $D : m \to m$ are matrices over $I$, and $n, m \geqslant 1$. Using star and omega power, we can define dagger by

$$\left( \begin{array}{cc} A & B \end{array} ; \ v \right)^\dagger = \left( A^*B \ ; \ A^\omega + A^*v \right)$$

where $A \in I^{n\times n}$, $B \in I^{n\times p}$ and $v \in V^n$. Note that the dagger operation in turn determines both the star and the omega power operations, since $A^\omega = (A; 0)^\dagger$, for all square matrices $A$.

It is shown in [BE93] that when $(S, V)$ is a Conway semiring-semimodule pair, then $\mathbf{Matr}_{S,V}$ is a Conway theory, called a *Conway matricial theory.* The same argument proves that when $(S, V)$ is a partial Conway semiring-semimodule pair, with distinguished two-sided ideal $I$, then $\mathbf{Matr}_{S,V}$ is a *partial Conway matricial theory* with distinguished ideal the set of those morphisms $(A; v) : n \to p$ such that $A$ is a matrix over $I$. The concepts of *partial iteration matricial theory* and *partial iteration semiring-semimodule pair* were also investigated [BE93, É11], but we won't use them in the thesis.

### 1.2.3 Partial Conway and iteration grove theories

**Definition 1.2.22** *A* grove theory *[BE93] is a theory equipped with the constants* $+ : 1 \to 2$ *and* $\# : 1 \to 0$ *satisfying the following equations:*

$$\begin{aligned} 1_2 + 2_2 &= 2_2 + 1_2, \\ (1_3 + 2_3) + 3_3 &= 1_3 + (2_3 + 3_3), \\ 1_1 + 0_{1,1} &= 1_1. \end{aligned}$$

*The equations above are understood in the following way.*

*Suppose that* $f, g : 1 \to p$ *are morphisms in a grove theory. We define*

$$f + g = + \cdot \langle f, g\rangle.$$

*Moreover, for arbitrary* $f = \langle f_1, \ldots, f_n\rangle, g = \langle g_1, \ldots, g_n\rangle : n \to p$ *we define*

$$f + g = \langle f_1 + g_1, \ldots, f_n + g_n\rangle.$$

*A* grove theory morphism $\varphi : T \to T'$ *between the grove theories* $T$ *and* $T'$ *is a theory morphism preserving the constants, i.e.* $+\varphi = +$ *and* $\#\varphi = \#$*. We say that the grove theory* $T$ *is a* subgrove theory *of the grove theory* $T'$ *if* $T$ *is a subtheory of* $T'$ *with the same constants* $+$ *and* $\#$*.*

It follows that for each $n, p \geq 0$, $(T(n, p), +, 0_{n,p})$ is a commutative monoid. Moreover,

$$\begin{aligned} (f + g) \cdot h &= (f \cdot h) + (g \cdot h), \\ 0_{m,n} \cdot f &= 0_{m,p}, \end{aligned}$$

for all $f, g : n \to p$ and $h : p \to q$. Note that distributivity on the left need not hold. If $T$ and $T'$ are grove theories and $\varphi : T \to T'$ is a grove theory morphism, then $(f + g)\varphi = f\varphi + g\varphi$ and $0_{n,p}\varphi = 0_{n,p}$ for all $n, p \geq 0$ and $f, g : n \to p$. Similarly, if $T'$ is a subgrove theory of $T$, then $T'$ contains the zero morphisms $0_{n,p}$ of $T$ and is closed under the sum operation of $T$.

Examples of grove theories include all matrix theories $\mathbf{Mat}_S$ and all matricial theories $\mathbf{Matr}_{S,V}$, see Examples 1.1.2 and 1.1.3.

In $\mathbf{Mat}_S$, the morphism $+$ is the matrix

$$\begin{pmatrix} 1 & 1 \end{pmatrix} : 1 \to 2$$

and $\#$ is the unique matrix $1 \to 0$.

In $\mathbf{Matr}_{S,V}$

$$+ = \left( \begin{pmatrix} 1 & 1 \end{pmatrix} ;\ \ 0 \right),$$

$$\# = \left( \ ;\ \ 0 \right).$$

A grove theory which is a (partial) Conway theory is a *(partial) Conway grove theory.* A grove theory which is a (partial) iteration theory is a *(partial) iteration grove theory.* A morphism of (partial) Conway or iteration grove theories is both a grove theory morphism and a (partial) Conway or iteration theory morphism, respectively. A *(partial) sub-Conway grove theory* $T'$ of a (partial) Conway grove theory $T$ is a subgrove theory that is a (partial) sub-Conway theory of $T$.

In the next chapter we will need the following observations.

**Remark 1.2.23** In a grove theory, for every base morphism $\rho : m \to n$ and every pair of morphisms $f, g : n \to p$ it holds that $\rho \cdot (f + g) = (\rho \cdot f) + (\rho \cdot g)$, moreover $\rho \cdot 0_{n,p} = 0_{m,p}$, for all $n, m, p \geqslant 0$.

**Example 1.2.24** Suppose that $L$ is a complete lattice with least element $\perp$. Thus, each direct power $L^n$ of $L$ is also a complete lattice. Recall that a function $L^p \to L^n$ is *continuous* [Sco72, BE93] if it preserves the suprema of (nonempty) directed sets. Let $\mathbf{Cont}_L$ denote the theory of all continuous functions over $L$. Thus, $\mathbf{Cont}_L$ is the subtheory of $\mathbf{Fun}_L$ determined by the continuous functions.

Let $+$ denote the function $L^2 \to L$, $(x, y) \mapsto x \vee y$, the supremum of the set $\{x, y\}$. It follows that for any $f, g : 1 \to p$, $f + g$ is the function $L^p \to L$ mapping $x \in L^p$ to $f(x) \vee g(x)$. Moreover, let $\#$ denote the least element $\perp$ considered as a function $L^0 \to L$. Then $\mathbf{Fun}_L$ and $\mathbf{Cont}_L$ are grove theories. Note that for each $n, p$, the morphism $0_{n,p}$ is the function $L^p \to L^n$ which maps each $z \in L^p$ to $\perp_n$, the least element of $L^n$.

**Example 1.2.25** Let $\Sigma$ be a ranked alphabet. The theory $\mathbf{Lang}_\Sigma$ has morphisms $1 \to p$ the $\Sigma$-tree languages $L \subseteq T_\Sigma(X_p)$. The morphisms $n \to p$ are the $n$-tuples of morphisms $1 \to p$. Let $L : 1 \to p$ and $L' = (L'_1, \ldots, L'_p) : p \to q$. Then $L \cdot L'$ is the collection of all trees in $T_\Sigma(X_q)$ that can be obtained by OI-substitution [ES77, ES78], i.e., the set of those trees $t$ such that there is a tree $s \in L$ such that $t$ can be constructed from $s$ by replacing each leaf labeled $x_i$ for $i \in [p]$ by some tree in $L'_i$ so that different occurrences of $x_i$ may be replaced by different trees. The distinguished morphism $i_n$ is the set $\{x_i\}$, and the morphisms $+$ and $\#$ are the sets $\{x_1, x_2\}$ and $\varnothing$, respectively. It then follows that addition is (component-wise) set union, and each component of any $0_{n,p}$ is $\varnothing$. The theory $\mathbf{Lang}_\Sigma$ is a grove theory.

# Chapter 2

# Generalized Star

In Section 1.2.1 we have seen that in matrix theories the dagger operation determines and is uniquely determined by a star operation. Every matrix theory is a grove theory, but there are grove theories that are not matrix theories. In this chapter, we consider grove theories equipped with a dagger operation and grove theories equipped with a generalized star operation, and under some natural assumptions we establish a correspondence between them in terms of a categorical isomorphism. We then use this isomorphism to relate equational properties of the dagger operation to equational properties of the generalized star operation. Sections 2.1, 2.2 and 2.3 illustrate that our generalization of the correspondence between the dagger and star operations in matrix theories is well behaved and natural. The contents of this chapter were published in [EH09].

Generalized star operations in grove theories were first studied in [BE93]. Natural sources of the generalized star operation include theories of continuous (or monotone) functions on complete lattices [BE93, É00, Ési13] where the additive structure is given by the binary supremum operation, theories of tree languages where the additive structure is given by set union [Ési98, Ési10], theories of formal tree series over semirings with pointwise addition, and theories of synchronization trees and theories of synchronization trees with respect to various behavioral equivalences, see e.g. [É00, É02, EK03, EH09].

**Definition 2.0.26** *Suppose that $T$ is a theory. We say that $T$ is a* generalized star theory *if $T$ is* grove theory *equipped with a* (generalized) star *operation*

$$^{\otimes} : T(n, n+p) \to T(n, n+p), \quad n, p \geq 0$$

*which need not satisfy any particular properties. Morphisms of generalized star theories also preserve the generalized star operation.*

**Notation** In any grove theory $T$, for any morphism $f : n \to n+p$ let $f^\tau$ denote the following morphism:[1]

$$f^\tau = f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p) : n \to n+n+p.$$

Thus, when $T = \mathbf{Fun}_A$, then $f^\tau$ is the function $A^{n+n+p} \to A^n$

$$(x, y, z) \mapsto f(x, z) + y$$

for every $x, y \in A^n$, $z \in A^p$.

Suppose that $T$ is a grove theory which is dagger theory. Then we define a generalized star operation by

$$f^\otimes = (f^\tau)^\dagger : n \to n+p \tag{2.1}$$

for all $f : n \to n+p$. We denote by $T_\otimes$ the resulting generalized star theory. Conversely, suppose now that $S$ is a generalized star theory. Then we define a dagger operation on $S$ by

$$f^\dagger = f^\otimes \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle : n \to p \tag{2.2}$$

for all $f : n \to n+p$. Let $S_\dagger$ denote the resulting dagger theory which is also a grove theory.

**Proposition 2.0.27** *The category of grove theories which are dagger theories and for each $f : n \to n+p$ satisfy the equation*

$$f^\dagger = (f^\tau)^\dagger \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \tag{2.3}$$

*is isomorphic to the category of those grove theories which are generalized star theories and satisfy*

$$f^\otimes = (f^\tau)^\otimes \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle \tag{2.4}$$

*for all $f : n \to n+p$.*

**Proof** The equations (2.3) and (2.4) yield

$$(T_\otimes)_\dagger = T \quad \text{and} \quad (S_\dagger)_\otimes = S$$

for any grove theories $T, S$ such that $T$ is a dagger theory and $S$ is a generalized star theory. Moreover, for any grove theories $T, T'$ which are dagger theories, it holds that any grove theory morphism $T \to T'$ preserving dagger

[1] Here and from now on we assume that composition has higher precedence than sum.

preserves the generalized star operation and is thus a morphism $T_\otimes \to T'_\otimes$. Conversely, if $S, S'$ are grove theories which are generalized star theories, and if $\varphi$ is a grove theory morphism $S \to S'$ preserving the generalized star operation, then $\varphi$ is also a morphism $S_\dagger \to S'_\dagger$. □

An isomorphism maps a dagger theory $T$ which is a grove theory to $T_\otimes$ and a generalized star theory $S$ to $S_\dagger$.

By the above proposition, when a grove theory is both a dagger theory and a generalized star theory and the two operations are related by (2.1) and (2.2), then properties of the dagger operation are reflected by certain properties of the generalized star operation and vice versa.

Below we will provide equivalent forms of the iteration theory identities in grove theories that use the generalized star operation instead of dagger, provided that the two operations are related by (2.1) and (2.2). By Proposition 2.0.27, such a translation is always possible, but we might get rather complicated equations as the result of a direct application of Proposition 2.0.27. Some of the equivalences proved below assume the parameter identity. This is no problem for the applications, since any well-behaved dagger operation does satisfy this identity. See also Proposition 2.0.30.

**Proposition 2.0.28** *Suppose that $T$ is a grove theory which is both a dagger theory and a generalized star theory. Suppose that the dagger and generalized star operations are related by (2.1) and (2.2). Then the following equivalences hold in $T$:*

**(a)** *the fixed point identity holds iff the* generalized star fixed point identity *holds:*

$$f^\otimes = f \cdot \langle f^\otimes, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p)$$

*where $f : n \to n + p$,*

**(b)** *the parameter identity holds iff the* generalized star parameter identity *holds:*

$$f^\otimes \cdot (\mathbf{1}_n \oplus g) = (f \cdot (\mathbf{1}_n \oplus g))^\otimes$$

*where $f : n \to n + p$ and $g : p \to q$,*

**(c)** *the left zero identity holds iff the* generalized star left zero identity *holds:*

$$(0_n \oplus f)^\otimes = (0_n \oplus f) + (\mathbf{1}_n \oplus 0_p)$$

*where $f : n \to p$,*

**(d)** *the right zero identity holds iff the* generalized star right zero identity *holds:*

$$(f \oplus 0_q)^{\otimes} = f^{\otimes} \oplus 0_q$$

*where* $f : n \to n + p$.

*Moreover if the* parameter identity *holds in* $T$, *then the following equivalences are valid:*

**(e)** *the simplified form of the double dagger identity holds iff the* generalized double star identity *holds:*

$$(f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p))^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle \quad = \quad (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^{\otimes}$$

*where* $f : n \to n + n + p$ *and* $\pi = \langle 0_n \oplus \mathbf{1}_n, \mathbf{1}_n \oplus 0_n \rangle$,

**(f)** *the composition identity holds iff the* generalized star composition identity *holds:*

$$(f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^{\otimes} = f^{\tau} \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p \rangle)^{\otimes}, 0_m \oplus \mathbf{1}_{p'} \rangle \cdot \langle 0_{m,p'}, \mathbf{1}_{p'} \rangle$$

*for every* $f : n \to m + p$ *and* $g : m \to n + p$, *where* $p' = n + p$, *and* $f^{\tau} = f \cdot (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p) + (0_m \oplus \mathbf{1}_n \oplus 0_p)$,

**(g)** *the permutation identity holds iff the* generalized star permutation identity *holds:*

$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^{\otimes} \quad = \quad \pi \cdot f^{\otimes} \cdot (\pi^{-1} \oplus \mathbf{1}_p)$$

*for all* $f : n \to n + p$ *and base permutation* $\pi : n \to n$ *with inverse* $\pi^{-1}$,

**(h)** *the pairing identity holds iff the* generalized star pairing identity *holds:*

$$\langle f, g \rangle^{\otimes} \quad = \quad \langle f^{\otimes} \cdot \langle \mathbf{1}_n \oplus 0_{m+p}, k^{\otimes} \cdot (\pi^{-1} \oplus \mathbf{1}_p), 0_{n+m} \oplus \mathbf{1}_p \rangle, k^{\otimes} \cdot (\pi^{-1} \oplus \mathbf{1}_p) \rangle$$

*for all*

$$f : n \to n + m + p$$

*and*

$$g : m \to n + m + p$$

*where*

$$\pi = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n \rangle : n + m \to m + n$$

*with inverse*

$$\pi^{-1} = \langle 0_n \oplus \mathbf{1}_m, \mathbf{1}_n \oplus 0_m \rangle : m + n \to n + m$$

*and*

$$k = g \cdot \langle f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p \rangle : m \to m + n + p.$$

**Proof**

**(a)** Suppose that the fixed point identity holds. Then

$$\begin{array}{rcl} f^{\otimes} & = & (f^{\tau})^{\dagger} \\ & =_1 & f^{\tau} \cdot \langle (f^{\tau})^{\dagger}, \mathbf{1}_{n+p} \rangle \\ & = & (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle \\ & = & f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle + (0_n \oplus \mathbf{1}_n \oplus 0_p) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle \\ & = & f \cdot \langle f^{\otimes}, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p) \end{array}$$

for all $f : n \to n+p$. Suppose now that the generalized star fixed point identity holds. Then

$$\begin{array}{rcl} f^{\dagger} & = & f^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ & = & (f \cdot \langle f^{\otimes}, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p)) \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ & = & f \cdot \langle f^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle, (0_n \oplus \mathbf{1}_p) \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \rangle + (\mathbf{1}_n \oplus 0_p) \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ & = & f \cdot \langle f^{\dagger}, \mathbf{1}_p \rangle + 0_{n,p} \\ & = & f \cdot \langle f^{\dagger}, \mathbf{1}_p \rangle \end{array}$$

for all $f : n \to n+p$. We used the generalized star fixed point identity in the second equation.

**(b)** Let $f : n \to n+p$ and $g : p \to q$. Suppose that the parameter identity holds. Then

$$\begin{array}{rcl} f^{\otimes} \cdot (\mathbf{1}_n \oplus g) & = & (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \cdot (\mathbf{1}_n \oplus g) \\ & =_4 & ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_{2n} \oplus g))^{\dagger} \\ & = & (f \cdot (\mathbf{1}_n \oplus 0_n \oplus g) + (0_n \oplus \mathbf{1}_n \oplus 0_q))^{\dagger} \\ & = & (f \cdot (\mathbf{1}_n \oplus g) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_q) + (0_n \oplus \mathbf{1}_n \oplus 0_q))^{\dagger} \\ & = & (f \cdot (\mathbf{1}_n \oplus g))^{\otimes}. \end{array}$$

Suppose now that the generalized star parameter identity holds. Then

$$\begin{array}{rcl} f^{\dagger} \cdot g & = & f^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \cdot g \\ & = & f^{\otimes} \cdot \langle 0_{n,q}, g \rangle \\ & = & f^{\otimes} \cdot (\mathbf{1}_n \oplus g) \cdot \langle 0_{n,q}, \mathbf{1}_q \rangle \\ & = & (f \cdot (\mathbf{1}_n \oplus g))^{\otimes} \cdot \langle 0_{n,q}, \mathbf{1}_q \rangle \\ & = & (f \cdot (\mathbf{1}_n \oplus g))^{\dagger}. \end{array}$$

We used the generalized star parameter identity in the fourth equation.

**(c)** Suppose that the left zero identity holds. Then

$$\begin{aligned}(0_n \oplus f)^{\otimes} &= ((0_n \oplus f) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\ &= ((0_{2n} \oplus f) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\ &= (0_n \oplus ((0_n \oplus f) + (\mathbf{1}_n \oplus 0_p)))^{\dagger} \\ &=_2 (0_n \oplus f) + (\mathbf{1}_n \oplus 0_p)\end{aligned}$$

for all $f : n \to p$. Suppose now that the generalized star left zero identity holds. Then

$$\begin{aligned}(0_n \oplus f)^{\dagger} &= (0_n \oplus f)^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ &= ((0_n \oplus f) + (\mathbf{1}_n \oplus 0_p)) \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ &= f + 0_{n,p} \\ &= f\end{aligned}$$

for all $f : n \to p$. We used the generalized star left zero identity in the second equation.

**(d)** Suppose that the right zero identity holds. Then

$$\begin{aligned}(f \oplus 0_q)^{\otimes} &= ((f \oplus 0_q) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_{p+q}) + (0_n \oplus \mathbf{1}_n \oplus 0_{p+q}))^{\dagger} \\ &= (((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p)) \oplus 0_q) + (0_n \oplus \mathbf{1}_n \oplus 0_{p+q}))^{\dagger} \\ &= ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \oplus 0_q)^{\dagger} \\ &=_3 (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \oplus 0_q \\ &= f^{\otimes} \oplus 0_q\end{aligned}$$

for all $f : n \to n + p$. The other direction also holds: assuming the generalized star right zero identity, we have

$$\begin{aligned}(f \oplus 0_q)^{\dagger} &= (f \oplus 0_q)^{\otimes} \cdot \langle 0_{n,p+q}, \mathbf{1}_{p+q} \rangle \\ &= f^{\otimes} \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus 0_q) \cdot \langle 0_{n,p+q}, \mathbf{1}_{p+q} \rangle \\ &= f^{\otimes} \cdot \langle 0_{n,p+q}, \mathbf{1}_p \oplus 0_q \rangle \\ &= (f^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle) \oplus 0_q \\ &= f^{\dagger} \oplus 0_q\end{aligned}$$

for all $f : n \to n + p$. We used the generalized star right zero identity in the second equation.

**(e)** Let $f : n \to n + n + p$ and let $\pi = \langle 0_n \oplus \mathbf{1}_n, \mathbf{1}_n \oplus 0_n \rangle$. Suppose that the parameter identity and the simplified form of the double dagger identity holds.

$$\begin{array}{rl}(f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p))^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p}\rangle = & \\ = & ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_{n+p}) + (0_n \oplus \mathbf{1}_n \oplus 0_{n+p}))^{\dagger} \cdot (\pi \oplus \mathbf{1}_p))^{\dagger} \\ =_4 & ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_{n+p}) + (0_n \oplus \mathbf{1}_n \oplus 0_{n+p})) \cdot (\mathbf{1}_n \oplus \pi \oplus \mathbf{1}_p))^{\dagger\dagger} \\ = & (f \cdot (\mathbf{1}_{2n} \oplus 0_n \oplus \mathbf{1}_p) + (0_{2n} \oplus \mathbf{1}_n \oplus 0_p))^{\dagger\dagger} \\ =_{12} & (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n\rangle \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\ = & (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n\rangle \oplus \mathbf{1}_p))^{\otimes}.\end{array}$$

Suppose now that the parameter and generalized double star identities hold. Then:

$$\begin{array}{rcl}(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n\rangle \oplus \mathbf{1}_p))^{\dagger} & = & (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n\rangle \oplus \mathbf{1}_p))^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p\rangle \\ & = & (f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p))^{\otimes} \cdot \langle 0_{n,n+p} \oplus \mathbf{1}_{n+p}\rangle \cdot \langle 0_{n,p}, \mathbf{1}_p\rangle \\ & = & (f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p))^{\dagger} \cdot \langle 0_{n,p}, \mathbf{1}_p\rangle \\ & =_4 & (f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \langle 0_{n,p}, \mathbf{1}_p\rangle))^{\dagger} \\ & = & (f^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p}\rangle)^{\dagger} \\ & = & f^{\dagger\dagger}.\end{array}$$

We used the generalized double star identity in the second equation.

**(f)** We now show that the composition identity holds iff the generalized star composition identity holds. Suppose first that the composition identity holds. Then

$$\begin{array}{rl}(f \cdot \langle g, 0_n \oplus \mathbf{1}_p\rangle)^{\otimes} = & \\ = & (f \cdot \langle g, 0_n \oplus \mathbf{1}_p\rangle \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\ = & (f \cdot \langle g \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p), 0_{2n} \oplus \mathbf{1}_p\rangle + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\ = & (h \cdot \langle t, 0_n \oplus \mathbf{1}_{p'}\rangle)^{\dagger}\end{array}$$

where

$$h = f^{\tau} = f \cdot (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p) + (0_m \oplus \mathbf{1}_n \oplus 0_p) : n \to m + p'$$

and

$$t = g \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) : m \to n + p'.$$

Then, by the composition identity we have

$$\begin{array}{rl}(h \cdot \langle t, 0_n \oplus \mathbf{1}_{p'}\rangle)^{\dagger} = & \\ =_8 & h \cdot \langle (t \cdot \langle h, 0_m \oplus \mathbf{1}_{p'}\rangle)^{\dagger}, \mathbf{1}_{p'}\rangle \\ = & f^{\tau} \cdot \langle (g \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \cdot \langle f^{\tau}, 0_m \oplus \mathbf{1}_{p'}\rangle)^{\dagger}, \mathbf{1}_{p'}\rangle \\ = & f^{\tau} \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p\rangle)^{\dagger}, \mathbf{1}_{p'}\rangle \\ = & f^{\tau} \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p\rangle)^{\otimes}, 0_m \oplus \mathbf{1}_{p'}\rangle \cdot \langle 0_{m,p'}, \mathbf{1}_{p'}\rangle\end{array}$$

Suppose now that the generalized star composition identity holds. Then

$$(f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle = (f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^{\dagger}$$

and also

$$\begin{aligned}
& f^{\tau} \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p \rangle)^{\dagger}, \mathbf{1}_{p'} \rangle \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle = \\
= \;& f^{\tau} \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle))^{\dagger}, 0_{n,p}, \mathbf{1}_p \rangle \\
= \;& f \cdot (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p) \cdot \\
& \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle))^{\dagger}, 0_{n,p}, \mathbf{1}_p \rangle \\
= \;& f \cdot \langle (g \cdot \langle f^{\tau}, 0_{m+n} \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle))^{\dagger}, \mathbf{1}_p \rangle \\
= \;& f \cdot \langle (g \cdot \langle f^{\tau} \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle), 0_m \oplus \mathbf{1}_p \rangle)^{\dagger}, \mathbf{1}_p \rangle \\
= \;& f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^{\dagger}, \mathbf{1}_p \rangle
\end{aligned}$$

since

$$\begin{aligned}
& f^{\tau} \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle) = \\
= \;& (f \cdot (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p) + (0_m \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_m \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle) \\
= \;& f + 0_{n,n+p} \\
= \;& f.
\end{aligned}$$

From an application of the generalized star composition identity the composition identity follows.

**(g)** Let $f : n \to n + p$ and assume that $\pi : n \to n$ is a base permutation with inverse $\pi^{-1}$. Suppose that the parameter and the permutation identities hold.

$$\begin{aligned}
& (\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^{\otimes} = \\
= \;& (\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \\
= \;& (\pi \cdot f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \cdot (\pi^{-1} \oplus \mathbf{1}_n \oplus \mathbf{1}_p) + \\
& + (0_n \oplus \mathbf{1}_n \oplus 0_p) \cdot (\pi^{-1} \oplus \mathbf{1}_n \oplus \mathbf{1}_p))^{\dagger} \\
= \;& ((\pi \cdot f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\pi^{-1} \oplus \mathbf{1}_n \oplus \mathbf{1}_p))^{\dagger} \\
= \;& ((\pi \cdot f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + \pi \cdot (0_n \oplus \pi^{-1} \oplus 0_p)) \cdot (\pi^{-1} \oplus \mathbf{1}_n \oplus \mathbf{1}_p))^{\dagger} \\
= \;& (\pi \cdot (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \pi^{-1} \oplus 0_p)) \cdot (\pi^{-1} \oplus \mathbf{1}_{n+p}))^{\dagger} \\
=_5 \;& \pi \cdot (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \pi^{-1} \oplus 0_p))^{\dagger} \\
= \;& \pi \cdot ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_n \oplus \pi^{-1} \oplus \mathbf{1}_p))^{\dagger} \\
=_4 \;& \pi \cdot (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \cdot (\pi^{-1} \oplus \mathbf{1}_p) \\
= \;& \pi \cdot f^{\otimes} \cdot (\pi^{-1} \oplus \mathbf{1}_p)
\end{aligned}$$

Supposing that the generalized star permutation identity holds, we have

$$\begin{array}{rcl}(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger & = & (\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\otimes \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ & = & \pi \cdot f^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \\ & = & \pi \cdot f^\dagger.\end{array}$$

We have used the generalized star permutation identity in the second equation.

**(h)** Assume that $f : n \to n+m+p$ and $g : m \to n+m+p$. Let $\pi$ denote the base permutation

$$\pi = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n \rangle : n+m \to m+n$$

with inverse

$$\pi^{-1} = \langle 0_n \oplus \mathbf{1}_m, \mathbf{1}_n \oplus 0_m \rangle : m+n \to n+m.$$

Define

$$k = g \cdot \langle f^\otimes \cdot (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p \rangle : m \to m+n+p.$$

We show that when the parameter identity holds, then the pairing identity holds iff the following generalized star pairing identity holds:

$$\langle f, g \rangle^\otimes = \langle f^\otimes \cdot \langle \mathbf{1}_n \oplus 0_{m+p}, k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p), 0_{n+m} \oplus \mathbf{1}_p \rangle, k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p) \rangle.$$

Assume first that the parameter identity and the pairing identity hold. Then

$$\begin{array}{rcl}\langle f, g \rangle^\otimes & = & (\langle f, g \rangle^\tau)^\dagger \\ & = & (\langle f, g \rangle \cdot (\mathbf{1}_{n+m} \oplus 0_{n+m} \oplus \mathbf{1}_p) + (0_{n+m} \oplus \mathbf{1}_{n+m} \oplus 0_p))^\dagger \\ & = & \langle f \cdot (\mathbf{1}_{n+m} \oplus 0_{n+m} \oplus \mathbf{1}_p) + (0_{n+m} \oplus \mathbf{1}_n \oplus 0_{m+p}), \\ & & g \cdot (\mathbf{1}_{n+m} \oplus 0_{n+m} \oplus \mathbf{1}_p) + (0_{n+m+n} \oplus \mathbf{1}_m \oplus 0_p) \rangle^\dagger \\ & = & \langle \overline{f}, \overline{g} \rangle^\dagger.\end{array}$$

Thus, by the pairing identity,

$$\langle f, g \rangle^\otimes \quad = \quad \langle \overline{f}^\dagger \cdot \langle h^\dagger, \mathbf{1}_{n+m+p} \rangle, h^\dagger \rangle$$

where

$$h \quad = \quad \overline{g} \cdot \langle \overline{f}^\dagger, \mathbf{1}_{m+n+m+p} \rangle.$$

Now

$$\begin{array}{rcl}
\overline{f}^\dagger & = & (f \cdot (\mathbf{1}_{n+m} \oplus 0_{n+m} \oplus \mathbf{1}_p) + (0_{n+m} \oplus \mathbf{1}_n \oplus 0_{m+p}))^\dagger \\
& = & ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_{m+p}) + (0_n \oplus \mathbf{1}_n \oplus 0_{m+p})) \cdot (\mathbf{1}_n \oplus \pi \oplus 0_m \oplus \mathbf{1}_p))^\dagger \\
& = & (f^\tau \cdot (\mathbf{1}_n \oplus \pi \oplus 0_m \oplus \mathbf{1}_p))^\dagger \\
& =_4 & f^\otimes \cdot (\pi \oplus 0_m \oplus \mathbf{1}_p).
\end{array}$$

Thus,

$$\begin{array}{rcl}
h & = & \overline{g} \cdot \langle f^\otimes \cdot (\pi \oplus 0_m \oplus \mathbf{1}_p), \mathbf{1}_{m+n+m+p} \rangle \\
& = & (g \cdot (\mathbf{1}_{n+m} \oplus 0_{n+m} \oplus \mathbf{1}_p) + (0_{n+m+n} \oplus \mathbf{1}_m \oplus 0_p)) \cdot \\
& & \cdot \langle f^\otimes \cdot (\pi \oplus 0_m \oplus \mathbf{1}_p), \mathbf{1}_{m+n+m+p} \rangle \\
& = & g \cdot \langle f^\otimes \cdot (\pi \oplus 0_m \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_{n+m} \oplus \mathbf{1}_p \rangle + (0_{m+n} \oplus \mathbf{1}_m \oplus 0_p) \\
& = & (g \cdot \langle f^\otimes \cdot (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_m \oplus 0_m \oplus \mathbf{1}_{n+p}) + \\
& & + (0_m \oplus \mathbf{1}_m \oplus 0_{n+p})) \cdot (\mathbf{1}_m \oplus \pi^{-1} \oplus \mathbf{1}_p) \\
& = & (g \cdot \langle f^\otimes \cdot (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p \rangle)^\tau \cdot (\mathbf{1}_m \oplus \pi^{-1} \oplus \mathbf{1}_p) \\
& = & k^\tau \cdot (\mathbf{1}_m \oplus \pi^{-1} \oplus \mathbf{1}_p)
\end{array}$$

where

$$k \quad = \quad g \cdot \langle f^\otimes \cdot (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p \rangle.$$

Thus, an application of the parameter identity yields

$$h^\dagger \quad = \quad k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p).$$

Using this,

$$\begin{array}{rcl}
\langle f, g \rangle^\otimes = & & \\
= & \langle f^\otimes \cdot (\pi \oplus 0_m \oplus \mathbf{1}_p) \cdot \langle k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p), \mathbf{1}_{n+m+p} \rangle, k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p) \rangle \\
= & \langle f^\otimes \cdot \langle \mathbf{1}_n \oplus 0_{m+p}, k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p), 0_{n+m} \oplus \mathbf{1}_p \rangle, k^\otimes \cdot (\pi^{-1} \oplus \mathbf{1}_p) \rangle.
\end{array}$$

Suppose now that the parameter identity and the generalized star pairing identity hold. Let $f, g$ be as above. We want to show that

$$\langle f, g \rangle^\dagger \quad = \quad \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle$$

where

$$h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle.$$

We have

$$
\begin{aligned}
\langle f,g\rangle^{\dagger} \;&=\; \langle f,g\rangle^{\otimes}\cdot\langle 0_{n+m,p},\mathbf{1}_p\rangle =\\
=\;& \langle f^{\otimes}\cdot\langle \mathbf{1}_n\oplus 0_{m+p}, k^{\otimes}\cdot(\pi^{-1}\oplus\mathbf{1}_p), 0_{n+m}\oplus\mathbf{1}_p\rangle, k^{\otimes}\cdot(\pi^{-1}\oplus\mathbf{1}_p)\rangle\cdot\\
&\cdot\langle 0_{n+m,p},\mathbf{1}_p\rangle
\end{aligned}
$$

where $k$ was defined above. We used the generalized star pairing identity in the second equation. First we show that

$$k^{\otimes}\cdot(\pi^{-1}\oplus\mathbf{1}_p)\cdot\langle 0_{n+m,p},\mathbf{1}_p\rangle \;=\; h^{\dagger}.$$

Indeed,

$$
\begin{aligned}
&k^{\otimes}\cdot(\pi^{-1}\oplus\mathbf{1}_p)\cdot\langle 0_{n+m,p},\mathbf{1}_p\rangle =\\
&\quad=\; k^{\otimes}\cdot\langle 0_{m+n,p},\mathbf{1}_p\rangle\\
&\quad=\; k^{\otimes}\cdot\langle 0_{m,n+p},\mathbf{1}_{n+p}\rangle\cdot\langle 0_{n,p},\mathbf{1}_p\rangle\\
&\quad=\; k^{\dagger}\cdot\langle 0_{n,p},\mathbf{1}_p\rangle\\
&\quad=_4\; (k\cdot(\mathbf{1}_m\oplus\langle 0_{n,p},\mathbf{1}_p\rangle))^{\dagger}\\
&\quad=\; (g\cdot\langle f^{\otimes}\cdot(\pi\oplus\mathbf{1}_p),\mathbf{1}_m\oplus 0_n\oplus\mathbf{1}_p\rangle\cdot(\mathbf{1}_m\oplus\langle 0_{n,p},\mathbf{1}_p\rangle))^{\dagger}\\
&\quad=\; (g\cdot\langle f^{\otimes}\cdot\langle 0_{n,m+p},\mathbf{1}_{m+p}\rangle,\mathbf{1}_{m+p}\rangle)^{\dagger}\\
&\quad=\; (g\cdot\langle f^{\dagger},\mathbf{1}_{m+p}\rangle)^{\dagger}\\
&\quad=\; h^{\dagger}.
\end{aligned}
$$

Using this,

$$
\begin{aligned}
f^{\otimes}\cdot\langle \mathbf{1}_n\oplus 0_{m+p}, k^{\otimes}\cdot(\pi^{-1}\oplus\mathbf{1}_p), 0_{n+m}\oplus\mathbf{1}_p\rangle\cdot\langle 0_{n+m,p},\mathbf{1}_p\rangle &=\\
=\; f^{\otimes}\cdot\langle 0_{n,p},h^{\dagger},\mathbf{1}_p\rangle&\\
=\; f^{\otimes}\cdot\langle 0_{n,m+p},\mathbf{1}_{m+p}\rangle\cdot\langle h^{\dagger},\mathbf{1}_p\rangle&\\
=\; f^{\dagger}\cdot\langle h^{\dagger},\mathbf{1}_p\rangle&
\end{aligned}
$$

completing the proof.

□

Thus, when the generalized star fixed point identity holds, then for each $f : n \to n+p$, $f^{\otimes}$ solves the fixed point equation

$$\xi = f\cdot\langle \xi, 0_n\oplus\mathbf{1}_p\rangle + (\mathbf{1}_n\oplus 0_p)$$

in the variable $\xi : n \to n+p$. When $p = 0$ this becomes

$$\xi = f\cdot\xi + \mathbf{1}_n \tag{2.5}$$

Compare (2.5) to (1.3). Note also that if the pairing identity holds, then the dagger operation is completely determined by its restriction to the scalar morphisms $1 \to 1+p$, $p \geq 0$. Similarly, if the generalized star pairing identity holds, then the generalized star operation is completely determined by its restriction to the scalar morphisms $1 \to 1+p$.

**Proposition 2.0.29** *Let $T$ be a grove theory equipped with a dagger and a generalized star operation, which are related by (2.1) and (2.2). Suppose that the parameter identity holds in $T$. Then for a group $S$ of order $n$ the identity $C(S)$ holds iff the following identity $C_\otimes(S)$ holds:*

$$g_S^\otimes \cdot (\tau_n \oplus \mathbf{1}_p) = \tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^\otimes \tag{2.6}$$

*where $g : 1 \to n+p$.*

**Proof** Since

$$g_S^\otimes \cdot (\tau_n \oplus \mathbf{1}_p) \cdot \langle 0_{1,p}, \mathbf{1}_p \rangle = g_S^\otimes \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle = g_S^\dagger$$

and

$$\tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^\otimes \cdot \langle 0_{1,p}, \mathbf{1}_p \rangle = \tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger$$

if $C_\otimes(S)$ holds, then so does $C(S)$.

In the calculations below, in order to save space, we will only indicate the generic $i$th component of a tuple. We will use the following equation:

$$\begin{aligned} &\langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p), \ldots \rangle = \\ &= \langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p), \ldots \rangle + (0_n \oplus \tau_n \oplus 0_p). \end{aligned}$$

Indeed, since

$$\begin{aligned} &i_n \cdot \langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p), \ldots \rangle = \\ &= g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p) \\ &= i_n \cdot \langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p), \ldots \rangle + i_n \cdot (0_n \oplus \tau_n \oplus 0_p) \\ &= i_n \cdot (\langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p), \ldots \rangle + (0_n \oplus \tau_n \oplus 0_p)) \end{aligned}$$

for all $i \in [n]$.

Suppose now that $C(S)$ holds, then

$$\begin{array}{rl}
\tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^{\otimes} = & \\
= & \tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p) \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) + (0_1 \oplus \mathbf{1}_1 \oplus 0_p))^{\dagger} \\
= & \tau_n \cdot ((g \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p)) \cdot (\tau_n \oplus \mathbf{1}_1 \oplus \mathbf{1}_p))^{\dagger} \\
=_{10} & (g \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p))^{\dagger}_S \\
= & \langle \ldots, (g \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_1 \oplus 0_p)) \cdot (\rho_i^S \oplus \mathbf{1}_{1+p}), \ldots \rangle^{\dagger} \\
= & (\langle \ldots, g \cdot (\rho_i^S \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus 0_1 \oplus \mathbf{1}_p), \ldots \rangle + (0_n \oplus \tau_n \oplus 0_p))^{\dagger} \\
= & ((g_S)^{\tau} \cdot (\mathbf{1}_n \oplus \tau_n \oplus \mathbf{1}_p))^{\dagger} \\
=_4 & g_S^{\otimes} \cdot (\tau_n \oplus \mathbf{1}_p).
\end{array}$$

Note that we have used the group identity $C(S)$ in the third equation. $\square$

The parameter identity and the generalized star parameter identity are of special importance due to the following fact.

**Proposition 2.0.30** *Suppose that $T$ is a grove theory. If $T$ is a dagger theory satisfying the parameter identity, then (2.3) holds. If $T$ is a generalized star theory satisfying the generalized star left zero identity and the generalized double star identity, then (2.4) holds.*

**Proof** Assume first that $T$ is a dagger theory satisfying the parameter identity. Then,

$$\begin{array}{rl}
(f^{\tau})^{\dagger} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle = & \\
=_4 & (f^{\tau} \cdot (\mathbf{1}_n \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle))^{\dagger} \\
= & ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_n \oplus \langle 0_{n,p}, \mathbf{1}_p \rangle))^{\dagger} \\
= & (f + 0_{n,n+p})^{\dagger} \\
= & f^{\dagger}
\end{array}$$

for all $f : n \to n + p$.

Suppose now that $T$ is a generalized star theory satisfying the generalized star left zero identity and the generalized double star identity. In order to prove that (2.4) holds, suppose that $g : n \to n + p$ in $T$, and define $f = 0_n \oplus g : n \to n + n + p$. Let $\pi = \langle 0_n \oplus \mathbf{1}_n, \mathbf{1}_n \oplus 0_n \rangle$. Then, using the generalized star left zero identity,

$$\begin{aligned}
f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p) &= ((0_n \oplus g) + (\mathbf{1}_n \oplus 0_{n+p})) \cdot (\pi \oplus \mathbf{1}_p) \\
&= (0_n \oplus g) \cdot (\pi \oplus \mathbf{1}_p) + (\mathbf{1}_n \oplus 0_{n+p}) \cdot (\pi \oplus \mathbf{1}_p) \\
&= (0_n \oplus g) \cdot \langle 0_n \oplus \mathbf{1}_n \oplus 0_p, \mathbf{1}_n \oplus 0_{n+p}, 0_{2n} \oplus \mathbf{1}_p \rangle \\
&\quad + (\mathbf{1}_n \oplus 0_{n+p}) \cdot \langle 0_n \oplus \mathbf{1}_n \oplus 0_p, \mathbf{1}_n \oplus 0_{n+p}, 0_{2n} \oplus \mathbf{1}_p \rangle \\
&= g \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p) \\
&= g^{\tau}.
\end{aligned}$$

Thus,

$$(f^{\otimes} \cdot (\pi \oplus \mathbf{1}_p))^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle = (g^{\tau})^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle.$$

Also,

$$\begin{aligned}
(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^{\otimes} &= ((0_n \oplus g) \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^{\otimes} \\
&= ((0_n \oplus g) \cdot \langle \mathbf{1}_n \oplus 0_p, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p \rangle)^{\otimes} \\
&= g^{\otimes}.
\end{aligned}$$

Thus, if the generalized double star identity holds, then

$$g^{\otimes} = (g^{\tau})^{\otimes} \cdot \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle,$$

proving (2.4). □

## 2.1 Conway and iteration star theories

**Definition 2.1.1** *A* Conway star theory *is a generalized star theory satisfying the generalized star left zero, right zero, pairing and permutation identities. A morphism of Conway star theories is a generalized star theory morphism.*

**Corollary 2.1.2** *A generalized star theory is a Conway star theory iff it satisfies the generalized star parameter, double star and star composition identities; or the scalar versions of the generalized star parameter, double star, star composition and star pairing identities.*

The scalar versions are defined in the same way as for the dagger identities. By Propositions 2.0.27, 2.0.28 and 2.0.30 we have:

**Corollary 2.1.3** *The category of Conway grove theories is isomorphic to the category of Conway star theories.*

For the definition of identity $C_\otimes(G)$ see (2.6).

**Definition 2.1.4** *An* iteration star theory *is a Conway star theory satisfying the identities $C_\otimes(G)$ for all finite groups $G$. A morphism of iteration star theories is a Conway star theory morphism.*

**Corollary 2.1.5** *All identities $C_\otimes(S)$ hold in all iteration star theories, where $S$ is any finite group.*

**Corollary 2.1.6** *The category of iteration grove theories is isomorphic to the category of iteration star theories.*

In Conway theories, the group identities $C(S)$ are implied by a simple implication. Let $T$ be a dagger theory and $C$ a subset of the morphisms of $T$. Following [BE93], we say that $T$ satisfies the *functorial dagger implication* for $C$ if for all $f : n \to n+p$, $g : m \to m+p$ in $T$ and for all $h : n \to m$ in $C$,

$$f \cdot (h \oplus \mathbf{1}_p) = h \cdot g \quad \Rightarrow \quad f^\dagger = h \cdot g^\dagger.$$

It is known that any Conway theory satisfies the functorial dagger implication for injective base morphisms, and that a Conway theory satisfies the functorial dagger implication for all base morphisms iff it satisfies the functorial dagger implication for the set of base morphisms $n \to 1$, $n \geq 2$.

**Definition 2.1.7** *Let $T$ be a generalized star theory and $C$ a subset of the set of morphisms of $T$. We define two versions of the* generalized functorial star implication *for $C$. We say that $T$ satisfies the first version if for all $f : n \to n+p$, $g : m \to m+p$ in $T$ and $h : n \to m$ in $C$,*

$$f^\tau \cdot (h \oplus \langle 0_{n,p}, \mathbf{1}_p\rangle) = h \cdot g \quad \Rightarrow \quad f^\otimes \cdot \langle 0_{n,p}, \mathbf{1}_p\rangle = h \cdot g^\otimes \cdot \langle 0_{m,p}, \mathbf{1}_p\rangle. \quad (2.7)$$

*Moreover, we say that $T$ satisfies the second version if for all $f, g, h$ as above,*

$$f^\tau \cdot (h \oplus h \oplus \mathbf{1}_p) = h \cdot g^\tau \quad \Rightarrow \quad f^\otimes \cdot (h \oplus \mathbf{1}_p) = h \cdot g^\otimes. \quad (2.8)$$

**Proposition 2.1.8** *Let $T$ be a grove theory which is both a dagger and a generalized star theory in which the dagger and generalized star operations are related by (2.1) and (2.2). Moreover, suppose that $T$ satisfies the parameter identity.*

*Then for an arbitrary set $C$ of $T$-morphisms the functorial dagger implication holds for $C$ iff the first version of the generalized functorial star implication holds. Also, the functorial dagger implication holds for some set $C$ of base morphisms iff the second version of the generalized functorial star implication holds wrt. $C$.*

**Proof** To prove the first claim, let $C$ be an arbitrary set of morphisms. Then by $f^\tau \cdot (h \oplus \langle 0_{n,p}, \mathbf{1}_p\rangle) = f \cdot (h \oplus \mathbf{1}_p)$ and $f^\otimes \cdot \langle 0_{n,p}, \mathbf{1}_p\rangle = f^\dagger$, $h \cdot g^\otimes \cdot \langle 0_{m,p}, \mathbf{1}_p\rangle = h \cdot g^\dagger$, the functorial dagger implication holds for $C$ iff (2.7) holds.

Now let $C$ be a set of base morphisms and assume that (2.8) holds. Let $f : n \to n + p$, $g : m \to m + p$ and $h : n \to m$ with $h \in C$. Assume that $f \cdot (h \oplus \mathbf{1}_p) = h \cdot g$. Then, using Remark 1.2.23,

$$\begin{aligned} f^\tau \cdot (h \oplus h \oplus \mathbf{1}_p) &= f \cdot (h \oplus 0_m \oplus \mathbf{1}_p) + (0_m \oplus h \oplus 0_p) \\ &= f \cdot (h \oplus \mathbf{1}_p) \cdot (\mathbf{1}_m \oplus 0_m \oplus \mathbf{1}_p) + (0_m \oplus h \oplus 0_p) \\ &= h \cdot g \cdot (\mathbf{1}_m \oplus 0_m \oplus \mathbf{1}_p) + (0_m \oplus h \oplus 0_p) \\ &= h \cdot g^\tau. \end{aligned}$$

Thus, by (2.8),

$$f^\otimes \cdot (h \oplus \mathbf{1}_p) = h \cdot g^\otimes,$$

so that

$$\begin{aligned} f^\dagger &= f^\otimes \cdot \langle 0_{m,p}, \mathbf{1}_p\rangle \\ &= f^\otimes \cdot (h \oplus \mathbf{1}_p) \cdot \langle 0_{m,p}, \mathbf{1}_p\rangle \\ &= h \cdot g^\otimes \cdot \langle 0_{m,p}, \mathbf{1}_p\rangle \\ &= h \cdot g^\dagger. \end{aligned}$$

We have thus proved that the functorial dagger implication holds.

Suppose now that the functorial dagger implication holds for $C$. We show that (2.8) also holds for $C$. For this reason, let $f$, $g$ and $h$ be as above, and assume that $f^\tau \cdot (h \oplus h \oplus \mathbf{1}_p) = h \cdot g^\tau$. Let

$$\overline{f} = f^\tau \cdot (\mathbf{1}_n \oplus h \oplus \mathbf{1}_p) = f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) + (0_n \oplus h \oplus 0_p) : n \to n + m + p.$$

Then,

$$\begin{aligned} \overline{f} \cdot (h \oplus \mathbf{1}_{m+p}) &= f \cdot (h \oplus 0_m \oplus \mathbf{1}_p) + (0_m \oplus h \oplus 0_p) \\ &= f^\tau \cdot (h \oplus h \oplus \mathbf{1}_p) \\ &= h \cdot g^\tau. \end{aligned}$$

Thus, by the functorial dagger implication, also

$$\overline{f}^\dagger = h \cdot (g^\tau)^\dagger.$$

Using this fact and the parameter identity,

$$\begin{aligned} f^\otimes \cdot (h \oplus \mathbf{1}_p) &= (f^\tau)^\dagger \cdot (h \oplus \mathbf{1}_p) \\ &=_4 (f^\tau \cdot (\mathbf{1}_n \oplus h \oplus \mathbf{1}_p))^\dagger \\ &= \overline{f}^\dagger \\ &= h \cdot (g^\tau)^\dagger \\ &= h \cdot g^\otimes. \end{aligned}$$

□

**Corollary 2.1.9** *Let $T$ be a Conway star theory. Then the generalized functorial star implications hold for the set of injective base morphisms. Moreover, the generalized functorial star implications hold for the set of all base morphisms iff they hold for the set of base morphisms $n \to 1$ for all $n \geq 2$.*

*Any Conway star theory satisfying one of the two versions of the functorial star implication for base morphisms is an iteration star theory.*

## 2.2 Ordered iteration grove theories

An *ordered theory* is a theory $T$ equipped with a partial order $\leq$ on each hom-set $T(n,p)$ which is preserved by the composition and tupling operations. More precisely, we require that for all $f, f' : n \to p$ and $g, g' : p \to q$ in $T$ if $f \leq f'$ and $g \leq g'$ then $f \cdot g \leq f' \cdot g'$. Moreover, for all $f, f' : n \to p$ and $h, h' : m \to p$ in $T$ if $f \leq f'$ and $h \leq h'$ then $\langle f, h\rangle \leq \langle f', h'\rangle$. A morphism of ordered theories is a theory morphism which preserves the partial order.

**Proposition 2.2.1** *Suppose that $T$ is both a grove theory and an ordered theory. Then the sum operation is monotone:*

$$f \leq f' \;\&\; g \leq g' \quad \Rightarrow \quad f + g \leq f' + g', \qquad f, f', g, g' : n \to p.$$

**Definition 2.2.2** *An* ordered grove theory *is a grove theory $T$ which is an ordered theory such that for each $p$, $0_{1,p}$ is the least morphism $1 \to p$. A morphism of ordered grove theories is a grove theory morphism which is an ordered theory morphism.*

It then follows that for any pair $n, p$ of nonnegative integers, $0_{n,p}$ is least in $T(n,p)$.

**Example 2.2.3** Suppose that $T$ is a grove theory such that $+ \cdot \langle \mathbf{1}_1, \mathbf{1}_1\rangle = \mathbf{1}_1$. It then follows that the sum operation is idempotent: $f + f = f$ for all $f : n \to p$, and we call $T$ an *idempotent grove theory.*

When $T$ is idempotent, there is a unique partial order $\leq$ turning $T$ into an ordered grove theory: We have $f \leq g$ for $f, g : n \to p$ iff $f + g = g$ iff there is some $h : n \to p$ with $f + h = g$. It follows that any grove theory morphism between idempotent grove theories preserves this order and is thus an ordered grove theory morphism.

**Definition 2.2.4** *An* ordered dagger theory *is an ordered theory which is a dagger theory such that the dagger operation is monotone and for each $n, p$, $\perp_{n,p} = (\mathbf{1}_n \oplus 0_p)^\dagger$ is the least morphism $n \to p$. An* ordered iteration theory *is an ordered dagger theory which is an iteration theory. An* ordered generalized star theory *is an ordered grove theory which is a generalized star theory such that the generalized star operation is monotone. An* ordered iteration star theory *is an ordered generalized star theory which is an iteration star theory. Morphisms of these structures also preserve the order.*

Note that by Definitions 2.2.2 and 2.2.4, $\perp_{n,p} = 0_{n,p}$ holds for all $n, p \geq 0$ in an ordered iteration star theory.

The theories $\mathbf{Cont}_L$ defined in Example 1.2.24 satisfy the following *fixed point induction rule*, cf. [Par69, É97]:

$$f \cdot \langle g, \mathbf{1}_p \rangle \leq g \quad \Rightarrow \quad f^\dagger \leq g$$

for all $f : n \to n + p$ and $g : n \to p$.

**Proposition 2.2.5** *Suppose that $T$ is an ordered grove theory which is both a dagger theory and a generalized star theory. Suppose that the dagger and generalized star operations are related by (2.1) and (2.2). If $T$ satisfies the fixed point induction rule and the parameter identity, then $T$ satisfies the following* generalized star fixed point induction rule*:*

$$f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle + h \leq g \quad \Rightarrow \quad f^\otimes \cdot \langle h, 0_n \oplus \mathbf{1}_p \rangle \leq g,$$

*for all $f, g, h : n \to n + p$. Moreover, if $T$ satisfies the generalized star parameter identity and the generalized star fixed point induction rule then $T$ satisfies the fixed point induction rule.*

**Proof** To prove the first claim, suppose that $T$ satisfies the parameter identity and the fixed point induction rule. Assume that $f, g, h : n \to n + p$ with $f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle + h \leq g$. Then

$$\begin{aligned}
&f^\tau \cdot (\mathbf{1}_n \oplus \langle h, 0_n \oplus \mathbf{1}_p \rangle) \cdot \langle g, \mathbf{1}_{n+p} \rangle = \\
&\quad = \quad f^\tau \cdot \langle g, h, 0_n \oplus \mathbf{1}_p \rangle \\
&\quad = \quad (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot \langle g, h, 0_n \oplus \mathbf{1}_p \rangle \\
&\quad = \quad f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle + h \\
&\quad \leq \quad g.
\end{aligned}$$

Thus, by the fixed point induction rule and the parameter identity,

$$\begin{aligned}
f^\otimes \cdot \langle h, 0_n \oplus \mathbf{1}_p \rangle \quad &= \quad (f^\tau)^\dagger \cdot \langle h, 0_n \oplus \mathbf{1}_p \rangle \\
&=_4 \quad (f^\tau \cdot (\mathbf{1}_n \oplus \langle h, 0_n \oplus \mathbf{1}_p \rangle))^\dagger \\
&\leq \quad g.
\end{aligned}$$

Suppose now that the generalized star fixed point induction rule holds. Let $f : n \to n + p$ and $g : n \to p$ with $f \cdot \langle g, \mathbf{1}_p \rangle \leq g$. Then

$$f \cdot \langle 0_n \oplus g, 0_n \oplus \mathbf{1}_p \rangle + 0_{n,n+p} \leq 0_n \oplus g,$$

so that

$$f^{\otimes} \cdot \langle 0_{n,n+p}, 0_n \oplus \mathbf{1}_p \rangle \leq 0_n \oplus g.$$

Composing both sides with $\langle 0_{n,p}, \mathbf{1}_p \rangle$ on the right, this gives

$$f^{\dagger} = f^{\otimes} \cdot \langle 0_{n,p}, \mathbf{1}_p \rangle \leq g.$$

□

## 2.3 Applications

In this section, we present some applications of the results of the previous sections. Below, by a *dagger term* we will mean any term built in the usual way from symbols representing morphisms in dagger grove theories and the distinguished morphisms by composition, the cartesian operations, sum and dagger. *Star terms* are defined in an analogous way. Note that each dagger or star term has a source $n$ and a target $p$, and under each evaluation of the morphism variables, the term evaluates to a morphism $n \to p$ in any dagger theory or generalized star theory. An equation $t = t'$, or inequation $t \leq t'$ between dagger or star terms is a formal (in)equality between terms $t, t' : n \to p$. The validity or satisfaction of an (in)equation in a dagger grove theory or a generalized star theory is defined as usual.

**Example 2.3.1** Let $L$ be a complete lattice. We define a dagger and a generalized star operation on $\mathbf{Cont}_L$. Let $f : L^{n+p} \to L^n$ be a continuous function. By the Knaster-Tarski fixed point theorem, for each $z \in L^p$, the endofunction $L^n \to L^n$, $x \mapsto f(x, z)$ has a least (pre-)fixed point. We define $f^{\dagger}(z)$ as this least pre-fixed point. An easy argument shows that $f^{\dagger}$ is also continuous.

Note that when $f : L^{n+p} \to L^n$ is continuous, then so is the function $f^{\tau} : L^{n+n+p} \to L^n$, defined by $(x, y, z) \mapsto f(x, z) \vee y$. We define $f^{\otimes} := (f^{\tau})^{\dagger}$.

By definition, (2.1) holds. Since $f^{\tau}(x, \perp_n, z) = f(x, z)$, it follows that (2.2) also holds. Moreover, since equipped with the dagger operation, $\mathbf{Cont}_L$ is a iteration grove theory, cf. [BE93, É00], it is also an iteration star theory, in fact an idempotent iteration grove theory and an idempotent iteration star theory.

**Proposition 2.3.2** *Suppose that $f : L^{n+p} \to L^n$ in $\mathbf{Cont}_L$. Then for each $y \in L^n$ and $z \in L^p$, $f^{\otimes}(y, z)$ is the least pre-fixed point of the endofunction $f_z : L^n \to L^n$, $x \mapsto f(x, z)$ which is greater than or equal to $y$.*

**Proof** Since $\mathbf{Cont}_L$ is an iteration star theory, the generalized star fixed point identity holds. Thus, for any $y$ and $z$, $f^{\otimes}(y, z) = f(f^{\otimes}(y, z), z) \vee y$ so that $f(f^{\otimes}(y, z), z) \leq f^{\otimes}(y, z)$ and $y \leq f^{\otimes}(y, z)$. Suppose now that $x \in L^n$ satisfies $f(x, z) \leq x$ and $y \leq x$. Then $f^{\tau}(x, y, z) = f(x, z) \vee y \leq x$ and thus $f^{\otimes}(y, z) = (f^{\tau})^{\dagger}(y, z) \leq x$ by the definition of dagger. $\Box$

The following result was proved in [É00]. (For dagger terms without + see also [BE93].)

**Theorem 2.3.3** *An (in)equation between dagger terms holds in all theories $\mathbf{Cont}_L$, where $L$ is any complete lattice iff it holds in all ordered iteration grove theories satisfying $+^{\dagger} = \mathbf{1}_1$.*

The last equation can also be written as $(1_2 + 2_2)^{\dagger} = \mathbf{1}_1$. As a corollary of this result we obtain:

**Corollary 2.3.4** *An equation between star terms holds in all theories $\mathbf{Cont}_L$, where $L$ is any complete lattice iff it holds in all ordered iteration star theories satisfying $\mathbf{1}_1{}^{\otimes} = \mathbf{1}_1$.*

The following result is a reformulation of a result of [É00].

**Theorem 2.3.5** *An equation between dagger terms holds in all theories $\mathbf{Cont}_L$ iff it holds in all ordered idempotent grove theories which are dagger theories satisfying the (scalar versions) of the fixed point identity, the parameter identity, and the fixed point induction rule.*

**Corollary 2.3.6** *An equation between star terms holds in all theories $\mathbf{Cont}_L$ iff it holds in all ordered idempotent generalized star theories satisfying the (scalar versions of the) generalized star fixed point identity, the generalized star parameter identity, and the generalized star fixed point induction rule.*

The last two results also hold for the broader class of monotone functions.

The free theories in the corresponding equational class can be described as theories of regular synchronization trees modulo simulation equivalence. See [É00].

**Example 2.3.7** Suppose that $S$ is a *continuous monoid*, i.e., a commutative monoid $S = (S, +, 0)$ equipped with a partial order $\leq$ such that $(S, \leq)$ is a cpo with least element 0, so that the supremum of each nonempty directed set exists, and the sum operation preserves such suprema (and is thus monotone).

Let $\mathbf{Cont}_S$ denote the theory of continuous functions over $S$. It is an iteration grove theory in the same way as the theory $\mathbf{Cont}_L$, where $L$ is a complete lattice. But unless the monoid $S$ is idempotent, $\mathbf{Cont}_S$ is not necessarily idempotent. Note that unlike in [Boz99] or [Kui00], we do not require here any linearity conditions for the functions themselves.

The following results were proved in [É02].

**Theorem 2.3.8** *An (in)equation between dagger terms holds in all theories $\mathbf{Cont}_S$, where $S$ is any continuous monoid iff it holds in all ordered iteration grove theories satisfying $(1_3+2_3+3_3)^{\dagger\dagger} = (1_2+2_2)^\dagger$ and $(1_2+2_2)^\dagger \cdot (f+g) = ((1_2+2_2)^\dagger \cdot f) + ((1_2+2_2)^\dagger \cdot g)$.*

**Theorem 2.3.9** *An (in)equation between dagger terms holds in all theories $\mathbf{Cont}_S$ iff it holds in all ordered dagger grove theories satisfying the (scalar versions) of the fixed point identity, the parameter identity and the fixed point induction rule, together with the equations $(1_3+2_3+3_3)^{\dagger\dagger} = (1_2+2_2)^\dagger$ and $(1_2+2_2)^\dagger \cdot (f+g) = ((1_2+2_2)^\dagger \cdot f) + ((1_2+2_2)^\dagger \cdot g)$.*

**Corollary 2.3.10** *An (in)equation between star terms holds in all theories $\mathbf{Cont}_S$, where $S$ is any continuous monoid iff it holds in all ordered iteration star theories satisfying $\mathbf{1}_1{}^{\otimes\otimes} = \mathbf{1}_1{}^{\otimes}$ and $\mathbf{1}_1{}^{\otimes} \cdot (f+g) = (\mathbf{1}_1{}^{\otimes} \cdot f) + (\mathbf{1}_1{}^{\otimes} \cdot g)$, or when it holds in all ordered generalized star theories satisfying the star forms the (scalar versions) of fixed point identity, the parameter identity, the fixed point induction rule, together with the equations $\mathbf{1}_1{}^{\otimes\otimes} = \mathbf{1}_1{}^{\otimes}$ and $\mathbf{1}_1{}^{\otimes} \cdot (f+g) = (\mathbf{1}_1{}^{\otimes} \cdot f) + (\mathbf{1}_1{}^{\otimes} \cdot g)$.*

The free theories in the corresponding equational class can be described as theories of regular synchronization trees modulo injective simulation equivalence. See [É02].

In our last example, we consider tree languages. Recall Example 1.2.25.

**Example 2.3.11** Let $\Sigma$ be a ranked set and consider the ordered idempotent theory $\mathbf{Lang}_\Sigma$. Each hom-set is a complete lattice and the theory and sum operations are continuous. It follows that for each morphism $L = (L_1, \ldots, L_n) : n \to n + p$ the fixed point equation

$$X = L \cdot \langle X, \mathbf{1}_p \rangle$$

has a least solution in the variable $X = (X_1, \ldots, X_n)$ over $\mathbf{Lang}_\Sigma(n, p)$, denoted $L^\dagger$. Given the dagger operation, we can also define a generalized star operation in the usual way. For each $L$ as above, $L^{\otimes}$ provides a least solution to the equation

$$Y = L \cdot \langle Y, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p)$$

in the variable $Y = (Y_1, \ldots, Y_n)$ ranging over the set of morphisms $n \to p$. (When $p = 0$, the above equation reads as $Y = L \cdot Y + \mathbf{1}_n$.) Equipped with the dagger operation, $\mathbf{Lang}_\Sigma$ is an ordered iteration grove theory, and equipped with the generalized star operation, $\mathbf{Lang}_\Sigma$ is an ordered iteration star theory, containing the theory of regular tree languages as a sub iteration grove (resp, star) theory denoted $\mathbf{Reg}_\Sigma$. $\mathbf{Reg}_\Sigma$ has exactly those morphisms of $\mathbf{Lang}_\Sigma$ that are tuples of regular tree languages. Note also that on morphisms $L : 1 \to 1 + p$ (i.e., tree languages $L \subseteq T_\Sigma(X_{1+p})$), $L^{\otimes}$ is the familiar $x_1$-*iterate* of $L$, cf. [GS84].

# Chapter 3

# The partial and the total

## 3.1 Dagger Extension Theorem

The dagger operation is necessarily a partial operation in (nontrivial) iterative theories. In [BEW80b, É82] (and [BE93], Theorem 6.4.5) it was shown that any iterative theory with at least one morphism $1 \to 0$ can be turned into an iteration theory that has a total dagger operation.

In this chapter, our aim is to provide a generalization of this construction that is applicable to partial iterative theories. In the main result we give a sufficient condition ensuring that the dagger operation of a partial iterative theory be extendible to a total dagger operation such that the resulting theory becomes a Conway theory or an iteration theory. Our "Dagger Extension Theorem" (Theorem 3.1.4) is a generalization of the Matrix Extension Theorem found in [BE93], on pages 323-335, and the extension theorem concerning grove theories, found in [BE03]. Possible applications of these extension theorems include Process Algebra, where one deals with unique fixed points of guarded fixed point equations (cf. [Fok07]). The contents of this chapter were published in [EH11a].

In what follows, we will make heavy use of the following notation. In any theory, we will denote the base morphism $\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s \rangle : s + n \to n + s$ by $\pi_{n,s}$, for all $n, s \geqslant 0$. Note that $\pi_{n,s}$ corresponds to the permutation $[s + n] \to [n + s]$ mapping each $i \in [s]$ to $n + i$ and each $j \in [s + n]$ with $j > s$ to $j - s$.

Let $T$ be a partial dagger theory with dagger operation ${}^{\dagger_D}$ defined on the morphisms $f : n \to n + p$ in $D(T)$ and let $T_0$ be a subtheory of $T$. Suppose that $T_0$ is a dagger theory with dagger operation ${}^{\dagger_0} : T_0(n, n+p) \to T_0(n, p)$,

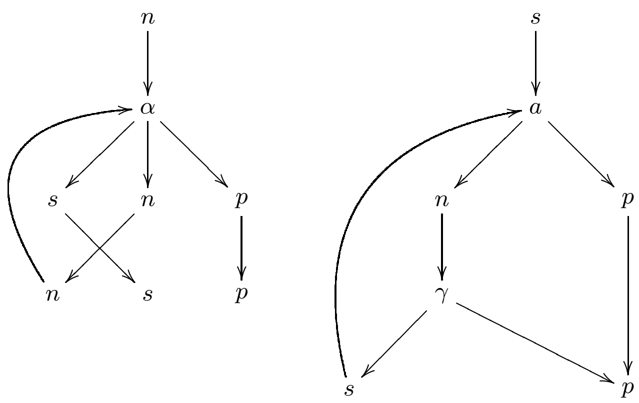

Figure 3.1: $\gamma$ is on the left and $c$ is on the right.

$n, p \geq 0$.

**Definition 3.1.1** *A* description $(\alpha, a) : n \to q$ *of weight $s$ consists of a morphism $\alpha : n \to s + q$ in $T_0$ and a morphism $a : s \to q$ in $D(T)$. We write $|(\alpha, a)|$ for the morphism $\alpha \cdot \langle a, \mathbf{1}_q \rangle$ in $T$, and call this morphism the* behavior *of the description $(\alpha, a)$. Moreover, for a description $(\alpha, a) : n \to n + p$ of weight $s$, we define $(\alpha, a)^\wedge$ to be the description $(\gamma, c) : n \to p$ of weight $s$, where*

$$\begin{aligned} \gamma &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s + p, \\ c &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to p, \end{aligned}$$

*see Fig. 3.1.*

**Example 3.1.2** Consider $\{\perp, \sigma, \tau\}$TR, where the arities of $\perp$, $\sigma$, $\tau$ are $0, 1, 1$, respectively. Let $T_0$ be the subtheory $\perp$TR of $\{\perp, \sigma, \tau\}$TR and let $D(\{\perp, \sigma, \tau\}\mathrm{TR})$ be the ideal containing those $(t_1, \ldots, t_n) : n \to p$ in $\{\perp, \sigma, \tau\}$TR such that the root of each $t_i$ is labeled by $\sigma$.

Then $(\alpha, a)$ is a description $3 \to 5$ of weight 3, where

$$\alpha = \langle 2_5, \perp, 1_5 \rangle = (x_2, \perp, x_1)$$

is $3 \to 8$ and

$$a = (\sigma(\tau(x_1)), \sigma(x_2), \sigma(x_2))$$

is $3 \to 5$. Then

$$|(\alpha, a)| = \alpha \cdot \langle a, \mathbf{1}_5 \rangle = (\sigma(x_2), \perp, \sigma(\tau(x_1))).$$

Now let

$$(\gamma, c) = (\alpha, a)^\wedge.$$

Then

$$\gamma = (x_5, \perp, x_4)$$

and

$$c = (\sigma(\tau(x_2)), \sigma(\perp), \sigma(\perp)).$$

**Proposition 3.1.3** *If $T$ is a Conway theory with dagger operation $^\dagger$, and $^{\dagger_0}$ is the restriction of $^\dagger$ to $T_0$, then for each description $(\alpha, a) : n \to n+p$ of weight $s$ we have $|(\alpha, a)^\wedge| = |(\alpha, a)|^\dagger$.*

**Proof**

$$\begin{array}{rcl}
|(\alpha, a)|^\dagger & = & (\alpha \cdot \langle a, \mathbf{1}_{n+p}\rangle)^\dagger \\
& = & (\alpha \cdot \langle a, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
& = & (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_n \oplus 0_p, a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
& =_{12} & (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \langle a, 0_n \oplus \mathbf{1}_p\rangle))^{\dagger\dagger} \\
& =_4 & ((\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^\dagger \cdot \langle a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
& = & (\gamma \cdot \langle a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
& =_8 & \gamma \cdot \langle (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
& = & \gamma \cdot \langle c, \mathbf{1}_p\rangle = |(\gamma, c)| = |(\alpha, a)^\wedge|
\end{array}$$

where $\gamma = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^\dagger$ and $c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p\rangle)^\dagger$. □

Now we give a sufficient condition ensuring that a partially defined dagger operation on the theory $T$ be uniquely extendible to a totally defined dagger operation such that $T$ becomes a Conway theory. Recall that each partial iterative theory $T$ yields a partial dagger theory with a dagger operation that provides unique solutions to fixed point equations $\xi = f \cdot \langle \xi, \mathbf{1}_p\rangle$, for all $f : n \to n+p$ in $D(T)$. Below we will denote this operation by $^{\dagger_D}$.

**Theorem 3.1.4** *Let $T$ be a partial iterative theory so that $T$ is also a partial dagger theory with the operation $^{\dagger_D}$ defined on the morphisms $f : n \to n+p$ in $D(T)$. Suppose that the following hold:*

*3.1.4.1. $T_0$ is a subtheory of $T$ and a Conway theory with the operation*

$$^{\dagger_0} : T_0(n, n+p) \to T_0(n, p)$$

$n, p \geq 0.$

*3.1.4.2.* *Each morphism $n \to p$ in $T$ can be written as $\alpha \cdot \langle a, \mathbf{1}_p \rangle$, where $\alpha : n \to s + p$ is in $T_0$ and $a : s \to p$ is in $D(T)$.*

*3.1.4.3.* *For all $\alpha : n \to s + n + p$, $\alpha' : n \to r + n + p$ in $T_0$ and $a : s \to n + p$, $a' : r \to n + p$ in $D(T)$ the following holds:*

$$|(\alpha, a)| = |(\alpha', a')| \quad \Longrightarrow \quad |(\alpha, a)^\wedge| = |(\alpha', a')^\wedge|$$

*i.e.,*

$$\alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle = \alpha' \cdot \langle a', \mathbf{1}_{n+p} \rangle \quad \Longrightarrow \quad \gamma \cdot \langle c, \mathbf{1}_p \rangle = \gamma' \cdot \langle c', \mathbf{1}_p \rangle$$

*where*

$$(\gamma, c) = (\alpha, a)^\wedge \quad \text{and} \quad (\gamma', c') = (\alpha', a')^\wedge.$$

*Then the operations ${}^{\dagger_0}$ and ${}^{\dagger_D}$ can be uniquely extended to a totally defined operation ${}^\dagger : T(n, n + p) \to T(n, p)$ such that $T$ equipped with ${}^\dagger$ is a Conway theory. Moreover, if $T_0$ is an iteration theory then $T$ is an iteration theory.*

**Proof** First, note that for morphisms in both $T_0$ and $D(T)$ the operations ${}^{\dagger_0}$ and ${}^{\dagger_D}$ coincide, since the fixed point identity holds in $T_0$. For a morphism $f : n \to n + p$ in $T$, we define $f^\dagger = |(\alpha, a)^\wedge|$, where $(\alpha, a)$ is a description such that $f = |(\alpha, a)|$. By 3.1.4.2 such a description exists and by 3.1.4.3 the operation ${}^\dagger$ is well defined.

Our aim is to show that $T$, equipped with the dagger operation ${}^\dagger$ is a Conway theory. To this end, we will prove that $T$ satisfies the following identities:

a) the fixed point identity,

b) the base parameter identity,

c) the double dagger identity,

d) the simplified composition identity.

Proposition 1.2.13 yields the desired result. We will use the following facts. Let $f : n \to n + p$ be an arbitrary morphism in $T$ with $f = |(\alpha, a)|$, where $\alpha : n \to s + n + p$ in $T_0$ and $a : s \to n + p$ in $D(T)$. Moreover, let $(\alpha, a)^\wedge = (\gamma, c)$. Then

$$\gamma \;=\; \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle. \tag{3.1}$$

Indeed,

$$\begin{aligned} \gamma &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &=_1 \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot \langle \gamma, \mathbf{1}_{s+p} \rangle \\ &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Moreover,

$$c = a \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \tag{3.2}$$

since we have

$$\begin{aligned} c &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &=_1 (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle) \cdot \langle c, \mathbf{1}_p \rangle \\ &= a \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle. \end{aligned}$$

Theorem 1.2.11 implies the fixed point identity used in this calculation. Now we proceed to prove that the fixed point, base parameter, double dagger and the simplified composition identities hold in $T$.

a) First, we show that $T$ satisfies the fixed point identity. Let $f : n \to n+p$ be an arbitrary morphism in $T$, and let $(\alpha, a)$ be a description of weight $s$ with behavior $f$.

   We calculate as follows:

$$\begin{aligned} f^\dagger &= |(\alpha, a)^\wedge| = \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle c, \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= f \cdot \langle f^\dagger, \mathbf{1}_p \rangle. \end{aligned}$$

   Here we have applied (3.1) in the third equation and (3.2) in the fifth. Thus, we have proved that the fixed point identity holds in $T$ for the operation $^\dagger$.

b) Now we show that the base parameter identity holds in $T$. Let $f : n \to n+p$ be an arbitrary morphism in $T$ and $g : p \to q$ a base morphism. Let $(\alpha, a)$ be a description of weight $s$, with behavior $f$. Let $(\gamma, c) = (\alpha, a)^\wedge$, thus

$$\gamma = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s + p,$$
$$c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to p.$$

Then $f \cdot (\mathbf{1}_n \oplus g) = \alpha \cdot (\mathbf{1}_{s+n} \oplus g) \cdot \langle a \cdot (\mathbf{1}_n \oplus g), \mathbf{1}_{n+q} \rangle$, and since $g$ is a base morphism, $\alpha \cdot (\mathbf{1}_{s+n} \oplus g)$ is in $T_0$. Thus

$$|(\alpha \cdot (\mathbf{1}_{s+n} \oplus g), a \cdot (\mathbf{1}_n \oplus g))| \quad = \quad f \cdot (\mathbf{1}_n \oplus g).$$

Let us define $(\delta, d) = (\alpha \cdot (\mathbf{1}_{s+n} \oplus g), a \cdot (\mathbf{1}_n \oplus g))^\wedge$, then

$$\delta = \gamma \cdot (\mathbf{1}_s \oplus g), \tag{3.3}$$

$$d = c \cdot g. \tag{3.4}$$

Indeed,

$$\begin{aligned}
\delta &= (\alpha \cdot (\mathbf{1}_{s+n} \oplus g) \cdot (\pi_{n,s} \oplus \mathbf{1}_q))^{\dagger_0} \\
&= (\alpha \cdot (\pi_{n,s} \oplus g))^{\dagger_0} \\
&= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_{n+s} \oplus g))^{\dagger_0} \\
&=_4 \gamma \cdot (\mathbf{1}_s \oplus g)
\end{aligned}$$

and

$$\begin{aligned}
d &= (a \cdot (\mathbf{1}_n \oplus g) \cdot \langle \delta, 0_s \oplus \mathbf{1}_q \rangle)^{\dagger_D} \\
&= (a \cdot \langle \gamma \cdot (\mathbf{1}_s \oplus g), 0_s \oplus g \rangle)^{\dagger_D} \\
&= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_s \oplus g))^{\dagger_D} \\
&=_4 (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \cdot g \\
&= c \cdot g.
\end{aligned}$$

Using these facts we conclude with the following calculation:

$$\begin{aligned}
(f \cdot (\mathbf{1}_n \oplus g))^\dagger &= |(\delta, d)| \\
&= \delta \cdot \langle d, \mathbf{1}_q \rangle \\
&= \gamma \cdot (\mathbf{1}_s \oplus g) \cdot \langle d, \mathbf{1}_q \rangle \\
&= \gamma \cdot \langle d, g \rangle \\
&= \gamma \cdot \langle c \cdot g, g \rangle \\
&= \gamma \cdot \langle c, \mathbf{1}_p \rangle \cdot g \\
&= |(\gamma, c)| \cdot g \\
&= f^\dagger \cdot g.
\end{aligned}$$

Here we used (3.3) in the third step of this calculation and (3.4) in the fifth. With this we have proved that the base parameter identity holds in $T$.

c) We will now prove that the (simplified) double dagger identity holds in $T$. Let $f : n \to n + n + p$ be an arbitrary morphism in $T$, and let $(\alpha, a)$ be a description of weight $s$ with behavior $f$. Let $(\beta, b) = (\alpha, a)^\wedge$ and $(\gamma, c) = (\beta, b)^\wedge$. Then, by definition,

$$\begin{aligned} \beta &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_{n+p}))^{\dagger_0} : n \to s + n + p, \\ b &= (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle)^{\dagger_D} : s \to n + p. \end{aligned}$$

This yields

$$\gamma = (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0}. \tag{3.5}$$

Indeed,

$$\begin{aligned} \gamma &= (\beta \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &=_4 (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_{n+p}) \cdot (\mathbf{1}_n \oplus \pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0 \dagger_0} \\ &= (\alpha \cdot (\langle 0_{n+n} \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_{n+s}, 0_n \oplus \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0 \dagger_0} \\ &=_{12} (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0}. \end{aligned}$$

Moreover,

$$\begin{aligned} c &= (b \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= ((a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle)^{\dagger_D} \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &=_4 (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle \cdot (\mathbf{1}_s \oplus \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle))^{\dagger_D \dagger_D} \\ &=_{12} (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \end{aligned}$$

and finally, using (3.1) we conclude that

$$c = (a \cdot \langle \gamma, \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D}. \tag{3.6}$$

On the other hand,

$$f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) = |(\delta, d)| \tag{3.7}$$

where $\delta = \alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)$ and $d = a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)$. Since

$$\begin{aligned} f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) &= \alpha \cdot \langle a, \mathbf{1}_{n+n+p} \rangle \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \\ &= \alpha \cdot \langle (a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)), \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p \rangle \\ &= \alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot \langle d, \mathbf{1}_{n+p} \rangle \\ &= \delta \cdot \langle d, \mathbf{1}_{n+p} \rangle \\ &= |(\delta, d)|. \end{aligned}$$

We define $(\eta, h) = (\delta, d)^\wedge$. Then

$$\begin{aligned}
\eta &= (\delta \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0} \\
&= \gamma.
\end{aligned}$$

The last step uses (3.5). Moreover,

$$\begin{aligned}
h &= (d \cdot \langle \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= (a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot \langle \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= (a \cdot \langle \eta, \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= c
\end{aligned}$$

where the last step is due to (3.6). We conclude the proof of the claim as follows:

$$\begin{aligned}
f^{\dagger\dagger} &= |(\alpha, a)^\wedge|^\dagger \\
&= |(\beta, b)|^\dagger \\
&= |(\beta, b)^\wedge| \\
&= |(\gamma, c)| \\
&= |(\eta, h)| \\
&= |(\delta, d)^\wedge| \\
&= |(\delta, d)|^\dagger \\
&= (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger.
\end{aligned}$$

We used Proposition 3.1.3 in the third and seventh equations and (3.7) in the last one.

d) We show that the simplified composition identity holds in $T$. Let $f : n \to m$ and $g : m \to n + p$ be arbitrary morphisms in $T$. Then there exist descriptions $(\alpha, a), (\beta, b)$ of weight $s$ and $r$, say, with behaviors $f$ and $g$, respectively.

Consider $(\gamma, c) : n \to n + p$ of weight $s + r$, where $\gamma = \alpha \cdot (\mathbf{1}_s \oplus \beta)$ and $c = \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle, b \rangle$. Then $|(\gamma, c)| = |(\alpha, a)| \cdot |(\beta, b)| = f \cdot g$. Let $(\delta, d) = (\gamma, c)^\wedge : n \to p$. Then

$$\delta = (\gamma \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s + r + p,$$
$$d = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s + r \to p.$$

By the definition of $^\dagger$, we have $|(\delta, d)| = (f \cdot g)^\dagger$. Note that

$$f \oplus \mathbf{1}_p = (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle$$

and

$$g \cdot (f \oplus \mathbf{1}_p) = \beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle x, \mathbf{1}_{m+p} \rangle$$

where $x = \langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle, a \oplus 0_p \rangle$. By definition, we have $|(\eta, h)| = (g \cdot (f \oplus \mathbf{1}_p))^\dagger$, where

$$\eta = (\beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p))^{\dagger_0} : m \to r + s + p$$
$$h = (x \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} : r + s \to p$$

so that $(\eta, h) : m \to p$ is of weight $r + s$. Moreover,

$$|(\tau, t)| = |(\alpha, a)| \cdot |(\eta, h)| = f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger$$

where $\tau = \alpha \cdot (\mathbf{1}_s \oplus \eta)$ and $t = \langle a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle, h \rangle$, so that $(\tau, t) : n \to p$ is of weight $s + r + s$. We want to prove that

$$|(\tau, t)| = \tau \cdot \langle t, \mathbf{1}_p \rangle = \delta \cdot \langle d, \mathbf{1}_p \rangle = |(\delta, d)|. \tag{3.8}$$

It follows that the simplified composition identity holds in $T$.

First, we prove that

$$\delta \quad = \quad \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p). \tag{3.9}$$

We calculate as follows:

$$\begin{aligned}
\delta &= (\alpha \cdot (\mathbf{1}_s \oplus \beta) \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p))^{\dagger_0} \\
&=_9 \alpha \cdot ((\mathbf{1}_s \oplus \beta) \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p) \cdot (\alpha \oplus \mathbf{1}_{s+r+p}))^{\dagger_0} \\
&= \alpha \cdot \langle 0_{s+m} \oplus \mathbf{1}_s \oplus 0_{r+p}, \beta \cdot (\langle 0_{s+m+s} \oplus \mathbf{1}_r, \alpha \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p) \rangle^{\dagger_0} \\
&= \alpha \cdot \langle f', g' \rangle^{\dagger_0} \\
&=_6 \alpha \cdot \langle f'^{\dagger_0} \cdot \langle h'^{\dagger_0}, \mathbf{1}_{s+r+p} \rangle, h'^{\dagger_0} \rangle \\
&= \alpha \cdot \langle \mathbf{1}_s \oplus 0_{r+p}, h'^{\dagger_0} \rangle
\end{aligned}$$

where

$$\begin{aligned}
f'^{\dagger_0} &= (0_{s+m} \oplus \mathbf{1}_s \oplus 0_{r+p})^{\dagger_0} \\
&=_2 0_m \oplus \mathbf{1}_s \oplus 0_{r+p}
\end{aligned}$$

and

$$\begin{aligned} h' &= g' \cdot \langle f'^{\dagger_0}, \mathbf{1}_{m+s+r+p} \rangle \\ &= \beta \cdot (\langle 0_{s+m+s} \oplus \mathbf{1}_r, \alpha \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p) \cdot \langle 0_m \oplus \mathbf{1}_s \oplus 0_{r+p}, \mathbf{1}_{m+s+r+p} \rangle \\ &= \beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot (\langle 0_{m+s} \oplus \mathbf{1}_r, 0_m \oplus \mathbf{1}_s \oplus 0_r, \mathbf{1}_m \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p). \end{aligned}$$

From this we have that

$$\begin{aligned} \delta \cdot (\pi_{r,s} \oplus \mathbf{1}_p) &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_{r+p}, h'^{\dagger_0} \rangle \cdot (\pi_{r,s} \oplus \mathbf{1}_p) \\ &= \alpha \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, h'^{\dagger_0} \cdot (\pi_{r,s} \oplus \mathbf{1}_p) \rangle \\ &=_4 \alpha \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, \eta \rangle \\ &= \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, \mathbf{1}_{r+s} \oplus 0_p, 0_{r+s} \oplus \mathbf{1}_p \rangle \\ &= \tau \cdot (\langle 0_r \oplus \mathbf{1}_s, \mathbf{1}_{r+s} \rangle \oplus \mathbf{1}_p) \end{aligned}$$

yielding

$$\begin{aligned} \delta &= \tau \cdot (\langle 0_r \oplus \mathbf{1}_s, \mathbf{1}_{r+s} \rangle \oplus \mathbf{1}_p) \cdot (\pi_{s,r} \oplus \mathbf{1}_p) \\ &= \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p). \end{aligned}$$

Thus, we have proved that (3.9) holds. Therefore, in order to prove (3.8), it is enough to show that

$$t = \langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \cdot d.$$

This holds if we can show that

d.1) $(0_r \oplus \mathbf{1}_s) \cdot h = a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle$,

d.2) $d = \pi_{r,s} \cdot h$.

Let us note that

$$h = \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D} \rangle \tag{3.10}$$

where

$$e = b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle$$

and

$$y = a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle.$$

Indeed,

$$\begin{aligned} h &= (x \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= (\langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle, a \oplus 0_p \rangle \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= \langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle, a \cdot \eta \rangle^{\dagger_D} \\ &=_6 \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D} \rangle. \end{aligned}$$

Now we prove d.1) and d.2).

d.1) Since $(0_r \oplus \mathbf{1}_s) \cdot h = y^{\dagger_D}$ we have to prove that

$$y^{\dagger_D} = a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle.$$

Indeed,

$$\begin{aligned} a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle &= a \cdot \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= y \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ &=_1 y^{\dagger_D} \end{aligned}$$

proving d.1).

d.2) Since $d = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle)^{\dagger_D}$ and $T$ is a partial iterative theory, it is enough to show that $\pi_{r,s} \cdot h$ is the unique solution of

$$\xi = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle) \cdot \langle \xi, \mathbf{1}_p \rangle$$

i.e. that

$$\begin{aligned} &\langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \rangle = \\ &= (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \end{aligned} \tag{3.11}$$

where we have used (3.10). First, note that

$$\begin{aligned} &\langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \\ &= \langle \delta \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \end{aligned}$$

which can be written alternatively as

$$\begin{aligned} &\langle \delta \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \\ &= \langle \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p) \cdot \\ &\quad \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \langle \tau \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle. \end{aligned}$$

In the above calculation we have used (3.9).
Now, let us define

$$u = \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle.$$

We conclude that the right-hand side of (3.11) equals

$$\begin{aligned} c \cdot u &= \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle, b \rangle \cdot u \\ &= \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u, b \cdot u \rangle. \end{aligned}$$

Thus, by the coproduct property we have that (3.11) holds iff the following hold:

d.2.1) $e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle = b \cdot u$,
d.2.2) $y^{\dagger_D} = a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u$.

Proof of d.2.1). We calculate as follows:

$$\begin{aligned} & e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle = \\ =_4 \quad & (b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_r \oplus \langle y^{\dagger_D}, \mathbf{1}_p \rangle))^{\dagger_D} \\ = \quad & (b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot z, z, 0_r \oplus \mathbf{1}_p \rangle)^{\dagger_D} \end{aligned}$$

where

$$z = \eta \cdot (\mathbf{1}_r \oplus \langle y^{\dagger_D}, \mathbf{1}_p \rangle).$$

Thus, using that

$$a \cdot z \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = y \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle = y^{\dagger_D}$$

and

$$z \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle$$

as a corollary of Theorem 1.2.11 we have

$$\begin{aligned} & e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle =_1 \\ =_1 \quad & b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot z, z, 0_r \oplus \mathbf{1}_p \rangle \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ = \quad & b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle y^{\dagger_D}, \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ = \quad & b \cdot \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ = \quad & b \cdot u. \end{aligned}$$

Proof of d.2.2). First, note that

$$\begin{aligned} \eta \quad = \quad & (\beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p))^{\dagger_0} \\ =_1 \quad & \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p) \cdot \langle \eta, \mathbf{1}_{r+s+p} \rangle \\ = \quad & \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{r+s} \oplus 0_p, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Now, using Theorem 1.2.11 we calculate as follows:

$$\begin{aligned} y^{\dagger_D} \quad =_1 \quad & (a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle) \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ = \quad & a \cdot \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle \\ = \quad & a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{r+s} \oplus 0_p, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle \cdot t \\ = \quad & a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \eta \cdot t, \mathbf{1}_p \rangle \\ = \quad & a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle b \cdot u, y^{\dagger_D}, \eta \cdot t, \mathbf{1}_p \rangle \\ = \quad & a \cdot \beta \cdot \langle b \cdot u, \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, t \rangle, \mathbf{1}_p \rangle \\ = \quad & a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u \end{aligned}$$

where $t = \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle$. Note that we have used d.2.1) in the fifth equation.

With this we have proved that the simplified composition identity holds in $T$.

It follows that $T$ is a Conway theory with the dagger operation $^\dagger$. Now we prove that whenever $T_0$ is an iteration theory then $T$ equipped with $^\dagger$ is also an iteration theory. Suppose that $T_0$ is an iteration theory, so that the group identities hold in $T_0$. We prove that the group identities hold in $T$.

We will use the following fact. For each morphism $f : n \to n+p$ in $D(T)$ and $h : n \to m$, $g : m \to m+p$ in $T$, the functorial dagger implication:

$$f \cdot (h \oplus \mathbf{1}_p) = h \cdot g \implies f^{\dagger_D} = h \cdot g^\dagger \tag{3.12}$$

is valid. Indeed,

$$\begin{aligned} h \cdot g^\dagger &= h \cdot g \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot (h \oplus \mathbf{1}_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot \langle h \cdot g^\dagger, \mathbf{1}_p \rangle \end{aligned}$$

thus, since $T$ is a partial iterative theory, we obtain $f^{\dagger_D} = h \cdot g^\dagger$.

Let $f : 1 \to n+p$ be an arbitrary morphism in $T$ and let $(\alpha, a)$ be a description of weight $s$, with behavior $f$. For each base morphism $\rho : n \to k$ we define the description

$$D_\rho = (\alpha \cdot (\mathbf{1}_s \oplus \rho \oplus \mathbf{1}_p), a \cdot (\rho \oplus \mathbf{1}_p)) : 1 \to k+p.$$

Then $|D_\rho| = f \cdot (\rho \oplus \mathbf{1}_p)$. Let us consider the unique base morphism $\tau_n : n \to 1$ and the description $D^\wedge_{\tau_n} = (\beta, b) : 1 \to p$ of weight $s$,

$$\begin{gathered} \beta = (\alpha \cdot (\mathbf{1}_s \oplus \tau_n \oplus \mathbf{1}_p) \cdot (\pi_{1,s} \oplus \mathbf{1}_p))^{\dagger_0} = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}))^{\dagger_0}, \\ b = (a \cdot (\tau_n \oplus \mathbf{1}_p) \cdot \langle \beta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} = (a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D}. \end{gathered}$$

Then

$$|D^\wedge_{\tau_n}| = (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger.$$

Let $\rho_i^S : n \to n$ for $i \in [n]$ be the base morphisms associated with the group $S$ of order $n$ (see Definition 1.2.2) and let us define the description $(\gamma, c) : n \to n+p$ of weight $n \cdot s$, where

$$\begin{gathered} \gamma = \langle \alpha \cdot (\mathbf{1}_s \oplus \rho_1^S \oplus \mathbf{1}_p) \cdot (\lambda_1 \oplus \mathbf{1}_{n+p}), \ldots, \alpha \cdot (\mathbf{1}_s \oplus \rho_n^S \oplus \mathbf{1}_p) \cdot (\lambda_n \oplus \mathbf{1}_{n+p}) \rangle, \\ c = \langle a \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, a \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle \end{gathered}$$

where $\lambda_i = (0_{(i-1)\cdot s} \oplus \mathbf{1}_s \oplus 0_{(n-i)\cdot s})$, with $(i-1) \cdot s$ denoting the usual multiplication of the integers $(i-1)$ and $s$. We have

$$|(\gamma, c)| = \langle |D_{\rho_1^S}|, \ldots, |D_{\rho_n^S}| \rangle = \langle f \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, f \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle = f_S : n \to n+p.$$

Let $(\gamma, c)^\wedge = (\delta, d) : n \to p$ of weight $n \cdot s$. We have

$$\begin{aligned}
\delta &= (\gamma \cdot (\pi_{n,n\cdot s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= \langle \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\rho_1^S \oplus \lambda_1 \oplus \mathbf{1}_p), \ldots, \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\rho_n^S \oplus \lambda_n \oplus \mathbf{1}_p) \rangle^{\dagger_0}
\end{aligned}$$

and

$$\begin{aligned}
d &= (c \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= \langle a \cdot (\rho_1^S \oplus \mathbf{1}_p) \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle, \ldots, a \cdot (\rho_n^S \oplus \mathbf{1}_p) \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle \rangle^{\dagger_D}.
\end{aligned}$$

Moreover

$$|(\delta, d)| = f_S^\dagger = \langle f \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, f \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle^\dagger.$$

Now let $(\eta, h) : n \to p$ of weight $n \cdot s$, where

$$\eta = \langle (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_1 \oplus \mathbf{1}_p))^{\dagger_0}, \ldots, (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_n \oplus \mathbf{1}_p))^{\dagger_0} \rangle,$$
$$h = \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot b.$$

For each $i \in [n]$ we have

$$\begin{aligned}
&(\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_i \oplus \mathbf{1}_p))^{\dagger_0} \cdot \langle h, \mathbf{1}_p \rangle =_4 \\
&=_4 \ (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_i \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \langle h, \mathbf{1}_p \rangle))^{\dagger_0} \\
&= \ (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}) \cdot (\mathbf{1}_n \oplus \langle b, \mathbf{1}_p \rangle))^{\dagger_0} \\
&=_4 \ (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}))^{\dagger_0} \cdot \langle b, \mathbf{1}_p \rangle \\
&= \ \beta \cdot \langle b, \mathbf{1}_p \rangle \\
&= \ |D_{\tau_n}^\wedge|.
\end{aligned}$$

Thus,

$$\begin{aligned}
|(\eta, h)| &= \langle |D_{\tau_n}^\wedge|, \ldots, |D_{\tau_n}^\wedge| \rangle && (3.13) \\
&= \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger. && (3.14)
\end{aligned}$$

Now note that

$$\delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) =_{4,10} \tau_n \cdot \beta =_4 \eta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p). \qquad (3.15)$$

Also, we see that

$$\begin{aligned}
&(c \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle) \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) = \\
&= \langle a \cdot \langle \rho_1^S \cdot \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p), 0_s \oplus \mathbf{1}_p \rangle, \ldots \\
&\quad \ldots, a \cdot \langle \rho_n^S \cdot \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p), 0_s \oplus \mathbf{1}_p \rangle \rangle \\
&= \langle a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle, \ldots, a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle \rangle \\
&= \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle
\end{aligned}$$

from which the functorial dagger implication, i.e. (3.12) yields

$$d = \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot b.$$

Finally, using (3.15) we have

$$\begin{aligned}
f_S^\dagger &= \delta \cdot \langle d, \mathbf{1}_p \rangle \\
&= \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\
&= \eta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\
&= \eta \cdot \langle h, \mathbf{1}_p \rangle \\
&= |(\eta, h)| \\
&= \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger
\end{aligned}$$

where in the last step we used (3.13) and (3.14).

We have shown that whenever $T_0$ is an iteration theory then $T$ equipped with the dagger operation $^\dagger$ is an iteration theory. It remains to show that $^\dagger$ extends $^{\dagger_0}$ and $^{\dagger_D}$.

First we show that $^\dagger$ extends $^{\dagger_0}$. Let $(\alpha, 0_{n+p}) : n \to n+p$ be of weight 0, where $\alpha$ is an arbitrary morphism in $T_0$. Then $|(\alpha, 0_{n+p})| = \alpha$. Observe that from

$$(\alpha, 0_{n+p})^\wedge = (\alpha^{\dagger_0}, 0_p)$$

we get

$$\alpha^\dagger = |(\alpha, 0_{n+p})^\wedge| = \alpha^{\dagger_0}.$$

Thus, $^\dagger$ extends $^{\dagger_0}$. The fact that $^\dagger$ extends $^{\dagger_D}$ follows from the fact that $T$ is a partial iterative theory and that $T$, equipped with $^\dagger$ satisfies the fixed point identity, see above.

Finally, as a corollary of Proposition 3.1.3 we have that the definition of $^\dagger$ is forced. Thus, there is at most one desired extension of the operations $^{\dagger_0}$ and $^{\dagger_D}$. We have seen that such an extension exists, the proof is complete. $\square$

## 3.2 Corollaries of the Dagger Extension Theorem

We now consider some corollaries of the Dagger Extension Theorem. In the first corollary, we replace Assumption 3.1.4.3 of the Dagger Extension Theorem by a condition based on the notion of simulation [BE93] that is useful in many applications.

For the rest of this subsection, suppose that $T$ is a partial iterative theory with a subtheory $T_0$ which is a Conway theory with dagger operation ${}^{\dagger_0}$. Since $T$ is a partial iterative theory, there is a dagger operation ${}^{\dagger_D}$ defined on the morphisms $n \to n+p$ in $D(T)$ that provides unique solutions to fixed point equations $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ with $f : n \to n+p$.

**Definition 3.2.1** *Let $D = (\alpha, a)$ and $E = (\beta, b)$ be descriptions $n \to p$ of weight $s$ and $r$, respectively, and let $\rho : s \to r$ be a morphism in $T_0$. We call $\rho$ a* simulation *$D \to E$ and write*

$$D \to^{\rho} E$$

*when*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) = \beta$$

*and*

$$a = \rho \cdot b.$$

*We define $D \to E$ if there is a morphism $\rho : s \to r$ in $T_0$ such that $D \to^{\rho} E$, and we let $\leftrightarrow^*$ denote the least equivalence relation containing $\to$. The relation $\leftrightarrow^*$ is called* simulation equivalence*.*

In addition, we write $D \equiv E$ when $|D| = |E|$, i.e., when $D$ and $E$ are (behaviorally) *equivalent*.

**Lemma 3.2.2**

*a) Let $D$, $E$ be descriptions $n \to p$. If $D \to^{\rho} E$, then $D \equiv E$.*

*b) Let $D$, $E$ be descriptions $n \to n+p$. If $D \to^{\rho} E$, then $D^{\wedge} \to^{\rho} E^{\wedge}$.*

**Proof**

a) Let $D = (\alpha, a)$, $E = (\beta, b)$ be descriptions $n \to p$, of weight $s$ and $r$. Assume that $D \to^{\rho} E$ holds for a $T_0$-morphism $\rho : s \to r$. Then

$$\begin{aligned} |D| &= \alpha \cdot \langle a, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \rho \cdot b, \mathbf{1}_p \rangle \\ &= \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\ &= \beta \cdot \langle b, \mathbf{1}_p \rangle \\ &= |E|. \end{aligned}$$

b) First recall that the functorial dagger implication, i.e. (3.12) holds for all morphisms $f : n \to n+p$ and $g : m \to m+p$ in $D(T)$ and any morphism $h : n \to m$. Assume that $D \to^\rho E$, where $\rho : s \to r$ is in $T_0$ and $D = (\alpha, a)$ and $E = (\beta, b)$ are descriptions $n \to n+p$ of weight $s$ and $r$, respectively. Let $D^\wedge = (\gamma, c)$ and $E^\wedge = (\delta, d)$. Then

$$\begin{aligned}
\gamma \cdot (\rho \oplus \mathbf{1}_p) &=_4 (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \rho \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\alpha \cdot (\rho \oplus \mathbf{1}_{n+p}) \cdot (\pi_{n,r} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\beta \cdot (\pi_{n,r} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= \delta.
\end{aligned}$$

Thus, from $a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot b \cdot \langle \delta, 0_r \oplus \mathbf{1}_p \rangle$ an application of (3.12) yields

$$\begin{aligned}
c &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= \rho \cdot (b \cdot \langle \delta, 0_r \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= \rho \cdot d.
\end{aligned}$$

We conclude that $D^\wedge \to^\rho E^\wedge$.

□

We have obtained the following corollary of the Dagger Extension Theorem.

**Corollary 3.2.3** *Let $T$ be a partial iterative theory. Suppose that the following hold:*

*3.2.3.1. $T_0$ is a subtheory of $T$ which is a Conway theory with the operation ${}^{\dagger_0} : T_0(n, n+p) \to T_0(n, p)$, $n, p \geq 0$.*

*3.2.3.2. Each morphism $n \to p$ in $T$ can be written as $\alpha \cdot \langle a, \mathbf{1}_p \rangle$, where $\alpha : n \to s+p$ is in $T_0$ and $a : s \to p$ is in $D(T)$,*

*3.2.3.3. For all descriptions $D, E : n \to p$, $n, p \geq 0$, if $D \equiv E$ then $D \leftrightarrow^* E$.*

*Then ${}^{\dagger_0}$ can be uniquely extended to a totally defined operation*

$${}^\dagger : T(n, n+p) \to T(n, p) \qquad n, p \geq 0$$

*such that $T$ equipped with ${}^\dagger$ becomes a Conway theory. Moreover, if $T_0$ is an iteration theory, then so is $T$.*

**Proof** Let $D, E$ be equivalent descriptions $n \to n+p$. From 3.2.3.3 we have $D \leftrightarrow^* E$. Moreover, from $D \leftrightarrow^* E$ it follows that $D^\wedge \leftrightarrow^* E^\wedge$, as a corollary of Lemma 3.2.2 b). Lastly, from $D^\wedge \leftrightarrow^* E^\wedge$ Lemma 3.2.2 a) yields $D^\wedge \equiv E^\wedge$, and we conclude that 3.1.4.3 holds. □

**Corollary 3.2.4** *Let $T$ be a Conway theory with dagger operation $^\dagger$ and let $D(T)$ be an ideal in $T$. Suppose that the following hold:*

*3.2.4.1. $T_0$ is a sub-Conway theory of $T$ which is an iteration theory.*

*3.2.4.2. For each $f : n \to n+p$ in $D(T)$, $f^\dagger$ is the unique morphism $n \to p$ in $T$ such that*

$$f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$

*3.2.4.3. Each morphism $n \to p$ in $T$ can be written as $\alpha \cdot \langle a, \mathbf{1}_p \rangle$ for some $\alpha : n \to s+p$ in $T_0$ and $a : s \to p$ in $D(T)$.*

*Then $T$ is an iteration theory.*

**Proof** 3.2.4.1 and 3.2.4.3 obviously imply 3.1.4.1 and 3.1.4.2, respectively. As a corollary of 3.2.4.2, $T$ is a partial iterative theory with the distinguished ideal $D(T)$. Suppose that $|(\alpha, a)| = |(\alpha', a')|$, where $\alpha, \alpha'$ and $a, a'$ are as in 3.1.4.3. Then using Proposition 3.1.3 in the first and last equations we obtain

$$|(\alpha, a)^\wedge| = |(\alpha, a)|^\dagger = |(\alpha', a')|^\dagger = |(\alpha', a')^\wedge|.$$

This establishes 3.1.4.3. The result follows by Theorem 3.1.4. □

## 3.3 Applications

### 3.3.1 Pointed iterative theories

Call a morphism $f : n \to p$ of a nontrivial theory $T$ *ideal* if none of the components $i_n \cdot f$ for $i \in [n]$ is a distinguished morphism. An *iterative theory* [Elg75] is a nontrivial partial iterative theory $T$ such that $D(T)$ is the collection of all ideal morphisms. Thus, each iterative theory comes with a partial dagger operation defined on the ideal morphisms $n \to n+p$. Below we denote this operation by $^\dagger$.

In this section we show that the following result from [BEW80b] and [É82] is an instance of the Dagger Extension Theorem.

**Theorem 3.3.1** *Suppose that $T$ is an iterative theory and $\perp : 1 \to 0$. Then there is a unique way of defining a dagger operation on $T$ such that $T$ becomes a Conway theory with $\mathbf{1}_1{}^\dagger = \perp$. Moreover, equipped with this dagger operation, $T$ is an iteration theory.*

**Proof** We may assume that $T$ is nontrivial, since otherwise the claim is clear. Since $T$ is an iterative theory, it is also a partial iterative theory where the distinguished ideal $D(T)$ is the collection of all ideal morphisms. Any scalar morphism in $T$ is either a distinguished morphism or an ideal morphism in $D(T)$.

Consider the least subtheory $T_0$ of $T$ containing the morphism $\perp$. Then $T_0$ is isomorphic to the theory $\Theta'$ of partial functions (see Example 1.2.18) which is a Conway theory (in fact an iteration theory) in a unique way. Thus, $T_0$ may be turned into a Conway theory in a unique way. The dagger operation on $T_0$, denoted $^{\dagger_0}$, is defined by $f^{\dagger_0} = f^n \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle$, for all $f : n \to n + p$ in $T_0$, where $\perp_{n,p} = \langle \perp \cdot 0_p, \dots, \perp \cdot 0_p \rangle$. We clearly have $\mathbf{1}_1{}^\dagger = \perp$.

To complete the proof, we need to verify that the partial iterative theory $T$ and the iteration (sub)theory $T_0$ satisfy the assumptions of Corollary 3.2.3. The only nontrivial fact is that the assumption 3.2.3.3 holds. This follows from Lemma 3.2.2 and Remark 3.3.5 below. □

In the lemmas below, we will consider descriptions $(\alpha, a) : n \to p$, where $\alpha : n \to s + p$ is in $T_0$, isomorphic to the theory $\Theta'$.

**Lemma 3.3.2** *For any description $(\alpha, a) : n \to p$ of weight $s$ there is a description $(\beta, b) : n \to p$ of weight $s+1$ such that $\beta$ is a base morphism and there is a simulation $(\beta, b) \to (\alpha, a)$.*

**Proof** We identify $\alpha$ with a partial function $[n] \to [s + p]$. Let $b = \langle a, \perp_{1,p} \rangle$ and let $\beta$ be the base morphism corresponding to the function $[n] \to [s+1+p]$ defined by

$$i\beta = \begin{cases} i\alpha & \text{if } i\alpha \in [s] \text{ is defined,} \\ i+1 & \text{if } i\alpha > s \text{ is defined,} \\ s+1 & \text{if } i\alpha \text{ is undefined.} \end{cases}$$

It is clear that the partial function $\rho : [s + 1] \to [s]$ which is undefined on $s+1$ and maps any integer in $[s]$ to itself defines a simulation $(\beta, b) \to (\alpha, a)$. □

Now we will consider descriptions $(\alpha, a)$ where $\alpha$ is a base morphism. Let $D = (\alpha, a)$ be such a description $n \to p$ of weight $s$. We call $D$ *accessible*, if for all $j \in [s]$ there is an $i \in [n]$ such that $i_n \cdot \alpha = j_{s+p}$. Moreover, we call $D$ *reduced*, if for all $i, j \in [s]$, $i_s \cdot a = j_s \cdot a$ implies $i = j$. With these definitions it is clear that the following lemmas hold.

**Lemma 3.3.3** *For every description $D = (\alpha, a) : n \to p$ such that $\alpha$ is a base morphism there exist an accessible description $D' = (\alpha', a') : n \to p$ and an accessible and reduced description $D'' = (\alpha'', a'') : n \to p$ such that $\alpha'$ and $\alpha''$ are base morphisms and $D' \to^\rho D$ and $D' \to^\tau D''$ hold for some injective base morphism $\rho$ and surjective base morphism $\tau$.*

**Lemma 3.3.4** *Let $D = (\alpha, a)$ and $F = (\beta, b)$ be accessible descriptions such that $D \equiv F$. If $F$ is reduced, then there is a surjective base morphism $\rho$ such that $D \to^\rho F$.*

**Proof** Let $s$ and $r$ denote the weights of $D$ and $F$, respectively. Since $D \equiv F$, $D$ is accessible and $F$ is reduced, for each $i \in [s]$ there is a unique $j = i\rho \in [r]$ with $i_s \cdot a = j_r \cdot b$. Since $F$ is accessible, $\rho$ is surjective. It is routine matter to verify that $\rho$ defines a simulation. □

**Remark 3.3.5** Note that the above lemmas yield that whenever $(\alpha, a), (\beta, b)$ are descriptions such that $(\alpha, a) \equiv (\beta, b)$ then $(\alpha, a) \leftrightarrow^* (\beta, b)$.

Indeed, as a corollary of Lemmas 3.3.2 and 3.2.2 a) it suffices to consider the case when $\alpha$ and $\beta$ are base morphisms. Then Lemma 3.3.3 yields that there are descriptions $(\alpha', a')$, $(\alpha'', a'')$ and $(\beta', b')$ such that

$$(\alpha'', a'') \leftarrow (\alpha', a') \to (\alpha, a) \tag{3.16}$$

and

$$(\beta', b') \to (\beta, b) \tag{3.17}$$

where $(\alpha', a')$, $(\beta', b')$ are accessible and $(\alpha'', a'')$ is accessible and reduced.

Supposing that $(\alpha, a) \equiv (\beta, b)$ we have

$$(\alpha'', a'') \equiv (\alpha', a') \equiv (\alpha, a) \equiv (\beta, b) \equiv (\beta', b').$$

where the first two and the last equivalences are due to applications of Lemma 3.2.2 a) to (3.16) and (3.17), respectively. Then Lemma 3.3.4 yields $(\beta', b') \to (\alpha'', a'')$, therefore $(\alpha, a) \leftrightarrow^* (\beta, b)$.

### 3.3.2 Iteration semirings

Recall Section 1.2.1. In this section our aim is to show that the Dagger Extension Theorem (Theorem 3.1.4) generalizes the Matrix Extension Theorem of [BE93]. The Matrix Extension Theorem [BE93] comes in two versions.

**Theorem 3.3.6** *Let $T = \mathbf{Mat}_S$ be a matrix theory with a distinguished two-sided ideal $I$. Suppose that the following hold:*

*3.3.6.1.* $T_0 = \mathbf{Mat}_{S_0}$ *is a sub matrix theory of* $T$ *which is a Conway matrix theory equipped with a star operation mapping a morphism* $\alpha : n \to n$ *in* $T_0$ *to a morphism* $\alpha^{*_0} : n \to n$ *in* $T_0$.

*3.3.6.2. For each* $a : n \to n$ *in* $I$ *and* $b : n \to p$ *in* $T$, *there is a unique morphism that solves the equation* $\xi = a\xi + b$ *in the variable* $\xi : n \to p$ *in* $T$.

*3.3.6.3. Each morphism* $f : n \to p$ *can be written as* $\alpha + a$, *where* $\alpha : n \to p$ *is in* $T_0$ *and* $a : n \to p$ *is in* $I$.

*3.3.6.4. If* $\alpha, \alpha' : n \to p$ *in* $T_0$ *and* $a, a' : n \to p$ *in* $I$ *with* $\alpha + a = \alpha' + a'$, *then* $\alpha = \alpha'$ *and* $a = a'$.

*Then the operation* $^{*_0}$ *can be extended in a unique way to a star operation* $f \mapsto f^*$ *defined on all morphisms* $f : n \to n$ *in* $T$ *such that* $T$ *becomes a Conway matrix theory. Moreover, if* $T_0$ *is an iteration matrix theory, then* $T$ *also becomes an iteration matrix theory.*

The second version uses semirings.

**Theorem 3.3.7** *Let* $S$ *be a semiring with a distinguished two-sided ideal* $I_0$. *Suppose the following:*

*3.3.7.1.* $S_0$ *is a subsemiring of* $S$ *that is a Conway semiring with star operation* $^{*_0}$.

*3.3.7.2. For each* $a \in I_0$ *and* $b \in S$, *the equation* $x = ax + b$ *has a unique solution in* $S$.

*3.3.7.3. Each* $s \in S$ *can be written as* $s = x + a$ *for some* $x \in S_0$ *and* $a \in I_0$.

*3.3.7.4. For all* $x, x' \in S_0$ *and* $a, a' \in I_0$, *if* $x + a = x' + a'$ *then* $x = x'$ *and* $a = a'$.

*Then the operation* $^{*_0}$ *can be extended in a unique way to a star operation* $^* : S \to S$ *such that* $S$ *becomes a Conway semiring. Moreover, if* $S_0$ *is an iteration semiring, then* $S$ *also becomes an iteration semiring.*

We prove the first version by showing that the assumptions of this theorem imply the assumptions of the Dagger Extension Theorem with $D(T) := I$ and the following definition of $^{\dagger_D}$. Let $f : n \to n + p$ in $D(T)$. Since $T$ is a matrix theory, we may write $f = \begin{pmatrix} a & b \end{pmatrix}$, where $a : n \to n$ and $b : n \to p$ in $D(T)$. By 3.3.6.2, there is a unique $\xi : n \to n$ in $T$ with $\xi = a\xi + \mathbf{1}_n$. Let $a^{*_D}$ denote this unique solution. Then $a^{*_D}b$ is the unique solution of the equation $\xi = a\xi + b$ and of $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$, where $\xi$ now ranges over the morphisms $n \to p$ in $T$. In other words $T$ is a partial iterative theory. So we define $f^{\dagger_D} := a^{*_D} b$. Now if $f : n \to p$ in $T$, then by 3.3.6.3, $f = \alpha + a$ for some $\alpha : n \to p$ in $T_0$ and $a : n \to p$ in $D(T)$. So we may write $f$ as $((\mathbf{1}_n \oplus 0_p) + (0_n \oplus \alpha)) \cdot \langle a, \mathbf{1}_p \rangle$, proving 3.1.4.2.

To complete the proof, we show that 3.3.6.2 and 3.3.6.4 imply 3.1.4.3. Let $(\alpha, a) : n \to n + p$ be a description of weight $s$. Since $T$ is a matrix theory, we can write $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix}$ and $a = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$, where $\alpha_0 : n \to s$, $\alpha_1 : n \to n$, $\alpha_2 : n \to p$ are in $T_0$ and $a_1 : s \to n$ and $a_2 : s \to p$ are in $D(T)$. Thus,

$$\begin{aligned} |(\alpha, a)| &= \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle \\ &= \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_p \end{pmatrix} \\ &= \begin{pmatrix} \alpha_0 a_1 + \alpha_1 & \alpha_0 a_2 + \alpha_2 \end{pmatrix}. \end{aligned}$$

Let $(\gamma, c) = (\alpha, a)^\wedge$. Thus, $\gamma = \begin{pmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{pmatrix}^{\dagger_0} = \begin{pmatrix} \alpha_1^{*_0} \alpha_0 & \alpha_1^{*_0} \alpha_2 \end{pmatrix}$, moreover,

$$\begin{aligned} c &= (\begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \alpha_1^{*_0} \alpha_0 & \alpha_1^{*_0} \alpha_2 \\ 0 & \mathbf{1}_p \end{pmatrix})^{\dagger_D} \\ &= \begin{pmatrix} a_1 \alpha_1^{*_0} \alpha_0 & a_1 \alpha_1^{*_0} \alpha_2 + a_2 \end{pmatrix}^{\dagger_D} \\ &= (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} (a_1 \alpha_1^{*_0} \alpha_2 + a_2). \end{aligned}$$

Therefore,

$$\begin{aligned} |(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \begin{pmatrix} \alpha_1^{*_0} \alpha_0 & \alpha_1^{*_0} \alpha_2 \end{pmatrix} \begin{pmatrix} (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} (a_1 \alpha_1^{*_0} \alpha_2 + a_2) \\ \mathbf{1}_p \end{pmatrix} \\ &= \alpha_1^{*_0} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} (a_1 \alpha_1^{*_0} \alpha_2 + a_2) + \alpha_1^{*_0} \alpha_2 \\ &= \alpha_1^{*_0} (\alpha_0 a_1 \alpha_1^{*_0})^{*_D} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_2 + a_2) + \alpha_1^{*_0} \alpha_2. \end{aligned}$$

The last equation is shown as follows:

$$\begin{aligned} \alpha_0 a_1 \alpha_1^{*_0} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} + \alpha_0 &= \alpha_0 (a_1 \alpha_1^{*_0} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} + \mathbf{1}_p) \\ &= \alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} \end{aligned}$$

where we have used 3.3.6.2. Note that $\alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D}$ solves the equation $(\alpha_0 a_1 \alpha_1^{*_0}) \xi + \alpha_0 = \xi$, therefore 3.3.6.2 yields

$$\alpha_0 (a_1 \alpha_1^{*_0} \alpha_0)^{*_D} = (\alpha_0 a_1 \alpha_1^{*_0})^{*_D} \alpha_0.$$

Now suppose that $(\beta, b) : n \to n+p$ is a description of weight $r$, where $\beta = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$ with $\beta_0 : n \to r$, $\beta_1 : n \to n$, $\beta_2 : n \to p$ in $T_0$ and $b_1 : r \to n$, $b_2 : r \to p$ in $D(T)$.

Suppose that $|(\alpha, a)| = |(\beta, b)|$. Then

$$|(\alpha, a)| = \begin{pmatrix} \alpha_0 a_1 + \alpha_1 & \alpha_0 a_2 + \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta_0 b_1 + \beta_1 & \beta_0 b_2 + \beta_2 \end{pmatrix} = |(\beta, b)|.$$

From this, 3.3.6.4 yields

$$\begin{aligned} \alpha_0 a_1 &= \beta_0 b_1, && (3.18)\\ \alpha_1 &= \beta_1, && (3.19)\\ \alpha_0 a_2 &= \beta_0 b_2, && (3.20)\\ \alpha_2 &= \beta_2. && (3.21) \end{aligned}$$

Now let $(\delta, d) = (\beta, b)^\wedge$. Then

$$\begin{aligned} |(\delta, d)| = \delta \cdot \langle d, \mathbf{1}_p \rangle &= \beta_1^{*_0}(\beta_0 b_1 \beta_1^{*_0})^{*_D} \beta_0 (b_1 \beta_1^{*_0} \beta_2 + b_2) + \beta_1^{*_0} \beta_2 \\ &= \alpha_1^{*_0}(\alpha_0 a_1 \alpha_1^{*_0})^{*_D} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_2 + a_2) + \alpha_1^{*_0} \alpha_2 \\ &= \gamma \cdot \langle c, \mathbf{1}_p \rangle = |(\gamma, c)| \end{aligned}$$

as a corollary of (3.18), (3.19), (3.20) and (3.21).

We have shown that 3.3.6.4 implies 3.1.4.3. Thus, the Matrix Extension Theorem, Theorem 3.3.6, follows from the Dagger Extension Theorem.

Applications of Theorem 3.3.6 were given in [BE93] and [BE09]. Here we only mention the following result from [BE93].

**Corollary 3.3.8** *If $S$ is an iteration semiring, then $S\langle\langle \Delta^* \rangle\rangle$, the semiring of formal power series over an alphabet $\Delta$ with coefficients in $S$ is an iteration semiring. The same holds for $S^{rat}\langle\langle \Delta^* \rangle\rangle$, the semiring of rational power series over $\Delta$ with coefficients in $S$.*

### 3.3.3 Iteration grove theories

Recall Section 1.2.3. In this section we will show that the Dagger Extension Theorem proves an extension theorem for grove theories found in [BE03].

Suppose that $T$ is a grove theory and $T_0$ is a sub-grove theory of $T$. Moreover, suppose that $T_0$ is a matrix theory. Note that if an ideal $D(T)$ is closed under composition with arbitrary morphisms from $T_0$ on the left, then for all $f, g : n \to p$ in $D(T)$ we have $f + g \in D(T)$ and $0_{n,p} \in D(T)$. We call an ideal $D(T)$ a $T_0$-*ideal*, if it is closed under composition with arbitrary morphisms from $T_0$ on the left.

The Grove Extension Theorem [BE03] is as follows.

**Theorem 3.3.9** *Let $T$ be a grove theory and $T_0$ a sub-grove theory of $T$ that is a matrix theory. Further, assume that the following hold:*

*3.3.9.1. $D(T)$ is a $T_0$-ideal.*

*3.3.9.2. Every morphism in $T$ can be written uniquely as $\alpha + a$, for some $\alpha$ in $T_0$ and $a$ in $D(T)$.*

*3.3.9.3. For all $\alpha : n \to p$ in $T_0$ and $f, g : p \to q$ in $T$ we have*

$$\alpha \cdot (f + g) = (\alpha \cdot f) + (\alpha \cdot g).$$

*3.3.9.4. $T_0$ is a Conway theory with dagger operation ${}^{\dagger_0} : T_0(n, n+p) \to T_0(n, p)$, $n, p \geq 0$.*

*3.3.9.5. For every $\alpha : n \to p$ in $T_0$ and $a : n \to n + p$ in $D(T)$, the fixed point equation $\xi = ((0_n \oplus \alpha) + a) \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution.*

*Then there is a unique way to define a total dagger operation ${}^\dagger$ on $T$ extending ${}^{\dagger_0}$ such that $T$ becomes a Conway theory. Further, if $T_0$ is an iteration theory, so is $T$.*

**Proof** For every $\alpha : n \to p$ in $T_0$ and $a : n \to n + p$ in $D(T)$, we let $((0_n \oplus \alpha) + a)^{\dagger_D}$ denote the unique morphism $n \to p$ in $T$ with

$$((0_n \oplus \alpha) + a)^{\dagger_D} = ((0_n \oplus \alpha) + a) \cdot \langle ((0_n \oplus \alpha) + a)^{\dagger_D}, \mathbf{1}_p \rangle.$$

We show that the assumptions in this theorem imply the assumptions of the Dagger Extension Theorem, with the choice of $T_0$ and $D(T)$ as the notation suggests.

Assumption 3.3.9.4 is the same as 3.1.4.1. It also holds that $T$, equipped with the distinguished ideal $D(T)$ is a partial iterative theory. Moreover, 3.3.9.2 implies that every $n \to p$ morphism in $T$ can be written as

$$\alpha + a = ((0_n \oplus \alpha) + (\mathbf{1}_n \oplus 0_p)) \cdot \langle a, \mathbf{1}_p \rangle$$

for some $\alpha : n \to p$ in $T_0$ and $a : n \to p$ in $D(T)$. Thus, 3.3.9.2 implies 3.1.4.2.

Now we show that 3.3.9.2, 3.3.9.4 and 3.3.9.5 imply 3.1.4.3. Let $(\alpha, a) : n \to n + p$ be a description of weight $s$. Since $T_0$ is a matrix theory, we can write $\alpha$ uniquely as $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix}$, where $\alpha_0 : n \to s$, $\alpha_1 : n \to n$ and $\alpha_2 : n \to p$ in $T_0$. Thus,

$$\begin{aligned} |(\alpha, a)| &= \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle \\ &= \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix} \cdot \langle a, \mathbf{1}_{n+p} \rangle \\ &= \alpha_0 a + \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} : n \to n + p. \end{aligned}$$

Let $(\gamma, c) = (\alpha, a)^\wedge$. Then

$$\begin{aligned}\gamma &= \begin{pmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{pmatrix}^{\dagger_0} \\ &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} : n \to s + p\end{aligned}$$

where ${}^{*_0}$ is the partial star operation determined by ${}^{\dagger_0}$, see Section 1.2.1. Moreover,

$$c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to s + p.$$

Let us define

$$x = a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle : s \to p.$$

Note that this definition makes sense by 3.3.9.5.

We prove that $c = x$. Since

$$a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle x, \mathbf{1}_p \rangle = a \cdot \langle \gamma \cdot \langle x, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$$

and

$$\begin{aligned}\gamma \cdot \langle x, \mathbf{1}_p \rangle &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} \cdot \langle a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0}\alpha_0 a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \alpha_1^{*}\alpha_2 \\ &= (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2))) \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle \\ &= (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}\end{aligned}$$

we obtain $a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle x, \mathbf{1}_p \rangle = x$. Thus, 3.3.9.5 yields $c = x$.

Moreover, we have

$$\begin{aligned}|(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0}\alpha_0 c + \alpha_1^{*_0}\alpha_2 \\ &= \alpha_1^{*_0}\alpha_0 x + \alpha_1^{*_0}\alpha_2.\end{aligned}$$

Suppose that $(\beta, b) : n \to n + p$ is a description of weight $r$, where $\beta = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$ with $\beta_1 : n \to r$, $\beta_2 : n \to n$, $\beta_3 : n \to p$ in $T_0$ and $b : r \to r + p$ in $D(T)$. Moreover, suppose that $|(\alpha, a)| = |(\beta, b)|$. Then

$$\begin{aligned}|(\alpha, a)| &= \alpha_0 a + \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \\ &= \beta_0 b + \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \\ &= |(\beta, b)|.\end{aligned}$$

thus, 3.3.9.2 yields

$$\begin{aligned} \alpha_0 a &= \beta_0 b, & (3.22)\\ \alpha_1 &= \beta_1, & (3.23)\\ \alpha_2 &= \beta_2. & (3.24) \end{aligned}$$

Let $(\delta, d) = (\beta, b)^\wedge$. Using (3.22), (3.23) and (3.24) we calculate as follows:

$$\begin{aligned} |(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0} \alpha_0 x + \alpha_1^{*_0} \alpha_2 \\ &= \alpha_1^{*_0} \alpha_0 a \cdot \langle (\alpha_1^{*_0} \alpha_0 a + (0_n \oplus (\alpha_1^{*_0} \alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \alpha_1^{*_0} \alpha_2 \\ &= \beta_1^{*_0} \beta_0 b \cdot \langle (\beta_1^{*_0} \beta_0 b + (0_n \oplus (\beta_1^{*_0} \beta_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \beta_1^{*_0} \beta_2 \\ &= \delta \cdot \langle d, \mathbf{1}_p \rangle \\ &= |(\delta, d)|. \end{aligned}$$

Thus, 3.1.4.3 holds. □

A corollary of the Grove Extension Theorem is the following, see [BE03] and Section 1.2.1 for the missing definitions. Recall that a formal tree series $1 \to p$ with coefficients in $S$ is a mapping $T_\Sigma(X_p) \to S$ from the set of $\Sigma$-trees to a semiring $S$, cf. [EK03].

**Corollary 3.3.10** *The formal tree series over a ranked alphabet with coefficients in a Conway semiring $S$ form a Conway grove theory containing the rational tree series as a sub-Conway grove theory. When $S$ is an iteration semiring, both theories are iteration grove theories.*

# Chapter 4

# Kleene Theorem for Partial Conway theories

## 4.1 Kleene Theorem

Iteration theories can be axiomatized by the Conway theory identities and an additional set of identities, one per each finite group, cf. [É99]. Whereas the group identities are needed for completeness, several constructions in automata theory and other areas of computer science only require the Conway identities.

In [BE93], a Kleene type theorem was proved for all Conway theories extending Kleene's classical theorem, [Kle56]. However, in many models of interest, the dagger operation is only partially defined.

In this chapter we give a Kleene-type theorem for partial Conway theories and discuss several applications of this result. The contents of this chapter were published in [EH11b].

Let $T$ be a partial dagger theory, $T_0$ a subtheory of $T$, and let $A$ be a set of scalar morphisms in $D(T)$. We write $A(T_0)$ for the set of morphisms $\langle f_1, \ldots, f_n\rangle : n \to p$, $n, p \geq 0$ such that each $f_i$ is the composition of a morphism in $A$ with a morphism in $T_0$. In particular, $0_p \in A(T_0)$ for all $p \geq 0$. Note that if $T_0$ is $T$ then $A(T_0)$ is the least ideal in $T$ containing the morphisms in $A$, and if $A$ is the set of scalar morphisms in $D(T)$, then $A(T_0) = D(T)$ for every subtheory $T_0$ of $T$.

We say that $(T_0, A)$ is *dagger compatible*, if for each $\alpha : n \to s + n + p$ in $T_0$ and $a : s \to s + n + p$ in $A(T_0)$, $s, n, p \geq 0$,

$$\alpha \cdot \langle a^\dagger, \mathbf{1}_{n+p}\rangle \in D(T) \Longrightarrow \alpha \cdot \langle a, 0_s \oplus \mathbf{1}_{n+p}\rangle \in A(T_0).$$

This condition is clearly fulfilled in a partial dagger theory $T$ if $(T_0, A)$ is *strongly dagger compatible*:

1. For all $\alpha : n \to p \in T_0$ and $a : p \to g \in A(T_0)$, it holds $\alpha \cdot a \in A(T_0)$, i.e., when $A(T_0)$ is closed under left composition with $T_0$-morphisms.
2. If $\alpha \cdot \langle f, \mathbf{1}_p \rangle \in D(T)$ for some $\alpha : n \to m+p \in T_0$ and $f : m \to p \in D(T)$, then $\alpha = \beta \oplus 0_p$ for some $\beta : n \to m$ in $T_0$.

Indeed, if these conditions hold and $\alpha \cdot \langle a^\dagger, \mathbf{1}_{n+p} \rangle \in D(T)$ for some $\alpha : n \to s + n + p$ in $T_0$ and $a : s \to s + n + p$ in $A(T_0)$, then there exists $\beta : n \to s$ in $T_0$ with $\alpha = \beta \oplus 0_{n+p}$. Thus, $\alpha \cdot \langle a, 0_s \oplus \mathbf{1}_{n+p} \rangle = \beta \cdot a$ is in $A(T_0)$.

Of course, it suffices to require the above conditions when the morphism $\alpha$ is scalar.

**Remark 4.1.1** When $(T_0, A)$ is dagger compatible, then for every $\alpha : n \to n + p$ in $T_0$, if $\alpha \cdot \langle 0^\dagger_{n+p}, \mathbf{1}_{n+p} \rangle = \alpha \cdot \langle 0_{n+p}, \mathbf{1}_{n+p} \rangle = \alpha$ is in $D(T)$, then it is in $A(T_0)$.

Below, when we write that $(T_0, A)$ *is a basis*, we will mean that $T_0$ is a subtheory of $T$ and $A$ is a set of scalar morphisms in $D(T)$.

**Definition 4.1.2** *A* presentation $n \to p$ *of dimension* $s$ *over a basis* $(T_0, A)$ *is an ordered pair:*

$$D = (\alpha, a) : n \to p$$

*where* $\alpha : n \to s + p$ *is in* $T_0$ *and* $a : s \to s + p$ *is in* $A(T_0)$.

*The* behavior *of* $D$ *is the following morphism in* $T$*:*

$$|D| = \alpha \cdot \langle a^\dagger, \mathbf{1}_p \rangle : n \to p.$$

**Definition 4.1.3**

a) *Let* $D = (\alpha, a) : n \to p$ *and* $E = (\beta, b) : m \to p$ *be presentations of dimension* $s$ *and* $r$*, respectively. We define*

$$\langle D, E \rangle = (\gamma, c) : n + m \to p$$

*as the presentation of dimension* $s + r$*, where*

$$\begin{aligned} \gamma &= \langle \alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p), 0_s \oplus \beta \rangle, \\ c &= \langle a \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p), 0_s \oplus b \rangle. \end{aligned}$$

b) *Let $D = (\alpha, a) : n \to p$ and $E = (\beta, b) : p \to q$ be presentations of dimension $s$ and $r$, respectively. Let us define*

$$D \cdot E = (\gamma, c) : n \to q$$

*as the presentation of dimension $s + r$, where*

$$\begin{aligned} \gamma &= \alpha \cdot (\mathbf{1}_s \oplus \beta), \\ c &= \langle a \cdot (\mathbf{1}_s \oplus \beta), 0_s \oplus b \rangle. \end{aligned}$$

c) *Let $D = (\alpha, a) : n \to n + p$ be a presentation of dimension $s$ with $|D| \in D(T)$. Suppose that $(T_0, A)$ is dagger compatible. Then we define*

$$D^\dagger = (\beta, b) : n \to p$$

*as the presentation of dimension $s + n$, where*

$$\begin{aligned} \beta &= (0_s \oplus \mathbf{1}_n \oplus 0_p), \\ b &= \langle a, \alpha \cdot \langle a, 0_s \oplus \mathbf{1}_{n+p} \rangle \rangle. \end{aligned}$$

*And if $T_0 \subseteq D(T)$ is closed under dagger, then we define*

$$D^\dagger = (\beta, b) : n \to p$$

*as the presentation of dimension $s$, where*

$$\begin{aligned} \beta &= (\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^\dagger, \\ b &= a \cdot \langle \mathbf{1}_s \oplus 0_p, \beta, 0_s \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Note that if $T_0 \subseteq D(T)$ then $T = D(T)$, so that $T$ is a Conway theory.

**Lemma 4.1.4** *Let $T$ be a partial Conway theory with basis $(T_0, A)$. Then*

a) *for each presentations $D : n \to p$, $E : p \to q$ we have $|D| \cdot |E| = |D \cdot E|$,*

b) *for each presentations $D : n \to p$ and $E : m \to p$ we have $\langle |D|, |E| \rangle = |\langle D, E \rangle|$, and*

c) *if $(T_0, A)$ is dagger compatible, or when $T_0 \subseteq D(T)$ is closed under dagger, then for each presentation $D : n \to n + p$ such that $|D|$ is in $D(T)$, we have $|D|^\dagger = |D^\dagger|$.*

**Proof** Although the proofs of a) and b) are the same as the proofs of the corresponding facts on pages 450–452 in [BE93], we include them for the reader's convenience.

Proof of a).

$$\begin{aligned}
|\langle D, E\rangle| &= \gamma \cdot \langle c^\dagger, \mathbf{1}_p\rangle \\
&=_{6,4} \gamma \cdot \langle a^\dagger, b^\dagger, \mathbf{1}_p\rangle \\
&= \langle \alpha \cdot \langle a^\dagger, \mathbf{1}_p\rangle, \beta \cdot \langle b^\dagger, \mathbf{1}_p\rangle\rangle \\
&= \langle |D|, |E|\rangle.
\end{aligned}$$

In the second line, we used the pairing and parameter identities.

Proof of b).

$$\begin{aligned}
|D \cdot E| &= \gamma \cdot \langle c^\dagger, \mathbf{1}_q\rangle \\
&=_6 \gamma \cdot \langle (a \cdot (\mathbf{1}_s \oplus \beta))^\dagger \cdot \langle b^\dagger, \mathbf{1}_q\rangle, b^\dagger, \mathbf{1}_q\rangle \\
&=_4 \alpha \cdot (\mathbf{1}_s \oplus \beta) \cdot \langle a^\dagger \cdot \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle, b^\dagger, \mathbf{1}_q\rangle \\
&= \alpha \cdot \langle a^\dagger, \mathbf{1}_p\rangle \cdot \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle \\
&= |D| \cdot |E|.
\end{aligned}$$

Proof of c). First suppose that $(T_0, A)$ is dagger compatible. Using the first definition of $D^\dagger$ we have:

$$\begin{aligned}
|D^\dagger| &= \beta \cdot \langle b^\dagger, \mathbf{1}_p\rangle \\
&= (0_s \oplus \mathbf{1}_n \oplus 0_p) \cdot \langle b^\dagger, \mathbf{1}_p\rangle \\
&= (0_s \oplus \mathbf{1}_n) \cdot b^\dagger \\
&= (0_s \oplus \mathbf{1}_n) \cdot \langle a, \alpha \cdot \langle a, 0_s \oplus \mathbf{1}_{n+p}\rangle\rangle^\dagger \\
&=_6 (\alpha \cdot \langle a, 0_s \oplus \mathbf{1}_{n+p}\rangle \cdot \langle a^\dagger, \mathbf{1}_{n+p}\rangle)^\dagger \\
&=_1 (\alpha \cdot \langle a^\dagger, \mathbf{1}_{n+p}\rangle)^\dagger \\
&= |D|^\dagger.
\end{aligned}$$

Next, suppose that $T_0 \subseteq D(T)$ is closed under dagger. Then, using the second definition of $D^\dagger$,

$$\begin{aligned}
|D^\dagger| &= \beta \cdot \langle b^\dagger, \mathbf{1}_p\rangle \\
&= \beta \cdot \langle (a \cdot \langle \mathbf{1}_s \oplus 0_p, \beta, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
&= \beta \cdot \langle (a^\dagger \cdot \langle \beta, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
&=_8 (\beta \cdot \langle a^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= ((\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s\rangle \oplus \mathbf{1}_p))^\dagger \cdot \langle a^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s\rangle \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_n \oplus 0_p, a^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (\alpha \cdot \langle a^\dagger, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (\alpha \cdot \langle a^\dagger, \mathbf{1}_{n+p}\rangle)^\dagger \\
&= |D|^\dagger.
\end{aligned}$$

In the third and sixth equation we used

$$(f^\dagger \cdot \langle g, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \quad = \quad (f \cdot \langle \mathbf{1}_n \oplus 0_p, g, 0_n \oplus \mathbf{1}_p)^\dagger$$

$f : n \to n + m + p$, $g : m \to n + p$ that holds in all Conway theories as a corollary of the parameter and double dagger identities. A proof of this fact can be found in [BE93] page 452, (1) from Chapter 11. The above calculation with the second definition of $D^\dagger$ can also be found on the same page in [BE93]. □

**Lemma 4.1.5** *Let $T$ be a partial Conway theory with basis $(T_0, A)$. Then every $T_0$-morphism and every morphism in $A$ is the behavior of some presentation.*

**Proof** Indeed, when $\alpha : n \to p$ is in $T_0$, then $\alpha = |D_\alpha|$ for the presentation $D_\alpha = (\alpha, 0_p)$, and when $a : 1 \to p$ is in $A$, then $a = |D_a|$ where

$$D_a = (1_{1+p}, 0_1 \oplus a).$$

The latter fact requires the left zero identity. □

Using the previous lemmas we obtain the following Kleene type theorem for partial Conway theories.

**Theorem 4.1.6** *Let $T$ be a partial Conway theory with basis $(T_0, A)$. Suppose that either $(T_0, A)$ is dagger compatible or $T_0 \subseteq D(T)$ is closed under dagger. Then a morphism $f$ belongs to the least partial sub-Conway theory of $T$ containing $T_0$ and $A$ iff $f$ is the behavior of some presentation over $(T_0, A)$.*

**Proof** The necessity of our claim follows from the previous two lemmas. Suppose now that $f : n \to p$ is the behavior of a presentation $(\alpha, a) : n \to p$ over $(T_0, A)$. Let $T'$ denote the least partial sub-Conway theory of $T$ containing $T_0$ and $A$, so that $D(T') = T' \cap D(T)$. Since $a \in D(T')$, $a^\dagger \in T'$. Since $T'$ is a subtheory containing $T_0$, it follows that $f = \alpha \cdot \langle a^\dagger, \mathbf{1}_p \rangle \in T'$. □

**Corollary 4.1.7** *Let $T$ be a partial Conway theory with basis $(T_0, A)$. Suppose that either $(T_0, A)$ is dagger compatible or $T_0 \subseteq D(T)$ is closed under dagger. Then the following are equivalent:*

a) *$T_0 \cup A$ generates $T$, so that every morphism can be constructed from the morphisms in $T_0 \cup A$ by the theory operations and dagger.*

b) *For each morphism $f$ in $T$ there is a presentation over $(T_0, A)$ whose behavior is $f$.*

c) *For each scalar morphism $f$ in $T$ there is a presentation over $(T_0, A)$ whose behavior is $f$.*

**Remark 4.1.8** Let $T$ be a partial Conway theory with basis $(T_0, A)$ such that every morphism in $T$ is the behavior of a presentation over $(T_0, A)$. Suppose that $D(T)$ is closed with respect to left composition by $T_0$-morphisms and if $\alpha \cdot \langle f, \mathbf{1}_p \rangle$ is in $D(T)$, where $\alpha : n \to s + p \in T_0$ and $f : s \to p \in D(T)$, then $\alpha = \beta \oplus 0_p$ for some $\beta : n \to s$. Then $D(T)$ is closed with respect to left composition by every $T$-morphism.

Indeed, suppose that $f : n \to p$ and $g : p \to q$ in $T$ with $g \in D(T)$. By assumption, there exist presentations $(\alpha, a)$ and $(\beta, b)$ with $|(\alpha, a)| = f$ and $|(\beta, b)| = g$. Since $g \in D(T)$, there exists some $\gamma$ in $T_0$ with $\beta = \gamma \oplus 0_p$. Thus, $g = \gamma \cdot b^\dagger$, and we conclude that

$$f \cdot g = \alpha \cdot \langle a^\dagger, \mathbf{1}_p \rangle \cdot \gamma \cdot b^\dagger = \alpha \cdot \langle a^\dagger \cdot \gamma \cdot b^\dagger, \gamma \cdot b^\dagger \rangle$$

is a morphism in $D(T)$.

The above assumptions hold if $T$ is generated by $T_0 \cup A$ and $(T_0, A)$ is strongly dagger compatible.

Indeed, suppose that $(T_0, A)$ is strongly dagger compatible. Then $D(T)$ is closed with respect to left composition with $T_0$-morphisms. To prove this, suppose that $f : n \to p$ in $D(T)$. Since $(T_0, A)$ is strongly dagger compatible, there exist $\alpha : n \to s \in T_0$ and $a : s \to s + p \in A(T_0)$ with $f = \alpha \cdot a^\dagger$. Let $\beta : m \to n$ be a $T_0$-morphism. Then

$$\beta \cdot f = \beta \cdot (\alpha \cdot a^\dagger) = (\beta \cdot \alpha \cdot a) \cdot \langle a^\dagger, \mathbf{1}_p \rangle$$

is in $D(T)$.

## 4.2 Corollary for grove theories

Recall Section 1.2.3.

**Proposition 4.2.1** *Suppose that $T$ is a partial Conway grove theory with basis $(T_0, A)$. If $T_0$ is a subgrove theory, then the set of behaviors of presentations over $(T_0, A)$ contains the morphisms $0_{n,p}$ and is closed under the sum operation.*

**Proof** Every morphism in $T_0$ is the behavior of some presentation over $(T_0, A)$. Since $\#$ and $+$ are in $T_0$, this fact applies to these morphisms. $\square$

An important special case of Theorem 4.1.6 concerns partial Conway grove theories $T$ with a basis $(T_0, A)$ such that $T_0$ is a matrix theory.

**Corollary 4.2.2** *Suppose that $T$ is a partial Conway grove theory with basis $(T_0, A)$ such that $T_0$ is a matrix theory. Suppose that one of the following two conditions holds:*

1. *For all $x : 1 \to p$ in $T_0$ and $f : 1 \to p \in D(T)$, if $x + f \in D(T)$ then $x = 0_{1,p}$. Moreover, for all $x : 1 \to 1 \in T_0$ and $a, b : 1 \to p \in A(T_0)$, $x \cdot a \in A(T_0)$ and $a + b \in A(T_0)$.*

2. *For every $x : 1 \to 1 \in T_0$, $x^*$ is defined and belongs to $T_0$.*

*Then a morphism $n \to p$ belongs to the least partial sub-Conway grove theory of $T$ containing $T_0$ and $A$ iff it is the behavior of some presentation over $(T_0, A)$.*

**Proof** Suppose that for all $x : 1 \to p$ in $T_0$ and $f : 1 \to p \in D(T)$, if $x + f \in D(T)$ then $x = 0_{1,p}$. Then if $\alpha \cdot \langle a, \mathbf{1}_p \rangle$ belongs to $D(T)$, for some $\alpha : n \to s+p$ in $T_0$ and $a : s \to p \in D(T)$, then $\alpha = \beta \oplus 0_p$ for some $\beta : n \to s$. Since $T_0$ is a matrix theory, $A(T_0)$ is closed under left composition with $T_0$-morphisms iff for each $p \geq 0$, the set of morphisms $1 \to p$ in $T_0$ is closed under sum and left composition with morphisms $1 \to 1$ in $T_0$. The second condition is equivalent to requiring that $T_0 \subseteq D(T)$ and $T_0$ is closed under dagger, or to the condition that $\alpha^*$ exists and is in $T_0$ for all $\alpha : n \to n$ in $T_0$. $\square$

Note that the condition that for all $x : 1 \to p \in T_0$ and $f : 1 \to p$ in $A(T_0)$, if $x + f \in D(T)$ then $x = 0_{1,p}$ holds whenever each (scalar) morphism of $T$ can be written in at most one way as the sum of a (scalar) $T_0$-morphism and a (scalar) morphism in $D(T)$.

## 4.3 Applications

In this section we present several applications of Theorem 4.1.6.

### 4.3.1 Trees

Suppose that $\Sigma$ is a ranked alphabet and consider the theory $T = \Sigma\mathrm{TR}$. Equipped with the ideal $D(T)$ determined by the proper trees, $\Sigma\mathrm{TR}$ is a partial iterative theory and thus a partial Conway theory. Let $T_0$ be the subtheory determined by those trees not containing any vertex labeled by a letter of $\Sigma$, and let $A$ be the collection of all atomic trees corresponding to the letters of $\Sigma$. Every tree in $T_0$ may be considered as a function in the initial theory $\Theta$. Then $A(T_0)$ is the ideal of all proper trees and $(T_0, A(T_0))$ is a strongly dagger compatible basis. A presentation $(\alpha, a) : n \to p$ of dimension $s$ is nothing but a *flowchart scheme* $n \to p$ over $\Sigma$, cf. [BE93]. We can write $a : s \to s + p$ in a unique way

$$a \quad = \quad \langle \sigma_1 \cdot \rho_1, \ldots, \sigma_s \cdot \rho_s \rangle \tag{4.1}$$

where each $\sigma_i$ is in $\Sigma_{n_i}$ for some $n_i \geq 0$, and each base morphism $\rho_i$ corresponds to some function $[n_i] \to [s + p]$, also denoted $\rho_i$. Such a scheme is a finite, directed, ordered graph whose vertices are the integers in the set $[s + p]$. A vertex $i \in [s]$ is labeled $\sigma_i$ and has $n_i$ linearly ordered outgoing edges so that the $j$th edge leads to the vertex $j\rho_i$. Each vertex $s + i$ with $i \in [p]$ is labeled $x_i$, the $i$th variable in the set $\{x_1, \ldots, x_p\}$. The base morphism $\alpha : n \to s + p$ corresponds to a function, also denoted $\alpha$, that picks the $i$th begin vertex $i\alpha$ for each $i \in [n]$. The behavior of $(\alpha, a)$ is the tree $t = (t_1, \ldots, t_n) : n \to p$ obtained by unfolding the flowchart scheme. It is known that a tree is the unfolding of a flowchart scheme iff it is regular. Thus the Kleene theorem asserts that a tree can be constructed from the atomic trees corresponding to the letters in $\Sigma$ by the theory operations and (scalar) dagger iff it is the behavior of a scheme.

When $\Sigma_0$ contains the letter $\perp$, $T = \Sigma\mathrm{TR}$ can be turned in a unique way into a Conway theory with a totally defined dagger operation such that $\mathbf{1}_1{}^\dagger = \perp$, see Example 1.2.17. Thus, $D(T) = T$. Accordingly, we may choose $T_0$ to be the subtheory of all trees not having any vertex labeled by a symbol in $\Sigma$ other than $\perp$. Then $T_0$ is closed under dagger, in fact $T_0$ is uniquely isomorphic to the theory $\Theta'$ of Example 1.2.18. The isomorphism $\Theta' \to T_0$ maps a partial function $\rho : [n] \to [p]$ to the tree $n \to p$ whose $i$th component is the variable $x_{i\rho}$ if $i\rho$ is defined and the tree $\perp$ otherwise, for all $i \in [n]$. Consider a presentation $(\alpha, a) : n \to p$ of dimension $s$ over $(T_0, A)$, where $A$ is the collection of all atomic trees corresponding to the letters of $\Sigma$ other than

$\perp$. Here, $\alpha$ may be viewed as a partial function $[n] \to [s+p]$ and $a$ is given as in (4.1), where now each $\rho_i$ corresponds to a partial function $[n_i] \to [s+p]$. When $j\rho_i$ is undefined, for some $j \in [n_i]$, then the $j$th outgoing edge of vertex $i$ leads to the extra vertex labeled $\perp$. Similarly, when $j\alpha$ is undefined for some $j \in [n]$, then this means that the $j$th begin vertex is the vertex labeled $\perp$. The behavior of the scheme is again the unfolding of the scheme. Since these unfoldings are again the regular trees, the Kleene theorem asserts that a tree can be constructed from the letters in $\Sigma$ (other than $\perp$) by the theory operations and the total (scalar) dagger operation iff it is regular.

### 4.3.2 Synchronization trees

Suppose that $\Sigma$ is an alphabet. A *synchronization tree* $t : 1 \to p$ over $\Sigma$ is an at most countable directed tree whose edges are labeled by the letters in the set $\Sigma \cup \{\mathsf{ex}_1, \ldots, \mathsf{ex}_p\}$, where the $\mathsf{ex}_i$ are referred to as the *exit symbols*. It is required that whenever an edge is labeled $\mathsf{ex}_i$, for some $i$, then its target is a leaf. A morphism between trees $1 \to p$ preserves the root, the edges and the labeling. We identify isomorphic trees. A synchronization tree $n \to p$ over $\Sigma$ is an $n$-tuple $(t_1, \ldots, t_n)$ of synchronization trees $1 \to p$ over $\Sigma$.

Synchronization trees over $\Sigma$ form a category $\mathrm{ST}_\Sigma$ with composition defined in the following way. Suppose that $t : 1 \to p$ and $t' = (t'_1, \ldots, t'_p) : p \to q$. Then $t \cdot t'$ is the synchronization tree obtained from $t$ by replacing each edge labeled $\mathsf{ex}_i$ for some $i \in [p]$ by a copy of $t_i$. When $t = (t_1, \ldots, t_n) : n \to p$, $t \cdot t'$ is defined as the tree $(t_1 \cdot t', \ldots, t_n \cdot t')$. With the trees $1 \to n$, $n \geq 0$, having a single edge labeled by an exit symbol as distinguished morphisms, $\mathrm{ST}_\Sigma$ is a theory. Let $+$ denote the tree $1 \to 2$ with two edges, an edge labeled $\mathsf{ex}_1$, and an edge labeled $\mathsf{ex}_2$, and let $\# : 1 \to 0$ be the empty tree having a single vertex and no edges. Equipped with these constants, $\mathrm{ST}_\Sigma$ is a grove theory. For each $n, p$, each component of $0_{n,p}$ is an empty tree. When $t, t' : 1 \to p$, $t + t'$ is the tree $1 \to p$ obtained by taking the disjoint union of $t$ and $t'$ and merging the roots. When $t, t' : n \to p$, $i_n \cdot (t + t') = i_n \cdot t + i_n \cdot t'$, for all $i \in [n]$. For more details, we refer to [BE93].

Let $T = \mathrm{ST}_\Sigma$ and define $D(T)$ to be the ideal determined by the *guarded trees* having no exit edge originating in the root. It is known that for each $t : n \to n + p$, the fixed point equation $\xi = t \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution, denoted $t^\dagger$. When each component of $t$ is *finitely branching*, then the same holds for $t^\dagger$. Thus, $T$ equipped with the ideal $D(T)$ is a partial iterative theory and hence a partial Conway grove theory.

Let $T_0$ denote the subtheory determined by the finitely branching synchronization trees with no edge labeled in $\Sigma$. Then $T_0$ is isomorphic to the matrix theory $\mathbf{Mat}_{\mathbb{N}}$ for the semiring $\mathbb{N}$ of natural numbers. Let $A$ denote the collection of all trees $1 \to 1$ corresponding to letters $\sigma$ in $\Sigma$ that have a single path consisting of two edges, labeled $\sigma$ and $\mathsf{ex}_1$, respectively. Then a presentation $D : 1 \to p$ over $(T_0, A)$ of dimension $s$ is an ordered pair $(\alpha, a)$ consisting of a row matrix $\alpha$ of dimension $s + p$ over $\mathbb{N}$ and a morphism $a = \langle a_1 \cdot \rho_1, \ldots, a_s \cdot \rho_s \rangle : s \to s + p$ where each $a_i$ is in $\Sigma$ and each $\rho_i$ is a row matrix over $\mathbb{N}$ of dimension $s + p$:

$$\begin{aligned} \alpha &= (\alpha_j)_{j \in [s+p]}, \\ \rho_i &= (\rho_{i,j})_{j \in [s+p]}. \end{aligned}$$

Such a presentation $(\alpha, a) : 1 \to p$ of dimension $s$ determines and is determined by a finitely branching *transition system* whose set of states is $[s + p]$ together with an *external exit state* $\mathsf{ex}$ and a *begin state* $b$. For a pair of states $(i, j) \in [s] \times [s + p]$, there are $\rho_{i,j}$ transitions labeled $a_j$ from state $i$ to state $j$. In addition, there are $\rho_{i,s+j}$ edges labeled $\mathsf{ex}_j$ from state $i$ to the external exit state $\mathsf{ex}$, for all $j \in [p]$. Finally, for each $j \in [s]$ there are $\alpha_j$ edges labeled $a_j$ from $b$ to state $j$, and for each $j \in [p]$, there are $\alpha_{s+j}$ edges labeled $\mathsf{ex}_j$ from $b$ to the external exit state $\mathsf{ex}$. The behavior of $(\alpha, a)$ is the unfolding of this transition system from the begin state $b$.

Now $(T_0, A)$ is a strongly dagger compatible basis and in this setting the Kleene theorem is the assertion that a tree $1 \to p$ can be constructed from the trees corresponding to the letters in $\Sigma$ and the empty tree by the theory operations, sum, and (scalar) dagger applied to guarded trees iff it is the unfolding of a finitely branching transition system $1 \to p$. These trees are exactly the finitely branching regular trees having a finite number of subtrees.

When $T_0$ is the subtheory of all synchronization trees not having any edge labeled in $\Sigma$, then $T_0$ is isomorphic to $\mathbf{Mat}_{\mathbb{N}_\infty}$ for the semiring $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ obtained from $\mathbb{N}$ by adding a point at infinity with the usual operations. The dagger operation defined on the guarded trees can be (uniquely) extended to all trees $t : n \to n + p$ in such a way that $\mathrm{ST}_\Sigma$ becomes a Conway grove theory with $+^\dagger = \infty \cdot \mathbf{1}_1$ being the tree $1 \to 1$ that has a countably infinite number of edges leaving the root, each labeled $\mathsf{ex}_1$, cf. [BE93]. Now $T_0$ is closed under dagger, and corresponds to the star operation defined on $\mathbb{N}_\infty$ by $0^* = 1$ and $n^* = \infty$ for all $n \in \mathbb{N}_\infty$, $n \neq 0$. Let $A$ denote the collection of all trees corresponding to the letters in $\Sigma$. Then a presentation corresponds to a transition system as before, but now $\alpha$ and the $\rho_i$ are row matrices over $\mathbb{N}_\infty$. The behavior is obtained in the same way. Using the second part of Theorem 4.1.6, we conclude that a tree $1 \to p$ can be constructed from the trees corresponding to the letters in $\Sigma$ by the theory operations, sum and (scalar) dagger iff it is the unfolding of a transition system. These are exactly the regular synchronization trees.

### 4.3.3 Bisimulation

Let $\Sigma$ be an alphabet and consider the Conway grove theory $\mathrm{ST}_\Sigma$. For $t, t' : 1 \to p$, define $t \sim t'$ iff $t$ and $t'$ are *bisimilar*, i.e., when there is a bisimulation between them [Par69, Mil89]. For trees $t = (t_1, \ldots, t_n) : n \to p$ and $t' = (t'_1, \ldots, t'_n) : n \to p$, let $t \sim t'$ iff $t_i \sim t'_i$ for all $i$. Then the relation $\sim$ is an equivalence relation on each hom-set $T(n, p)$ preserved by all operations including dagger. Thus, we can form the quotient Conway grove theory of bisimilarity equivalence classes.

Suppose now that $T$ is the quotient partial Conway theory of $\mathrm{ST}_\Sigma$ with respect to the relation $\sim$. We identify each letter in $\Sigma$ with the bisimilarity equivalence class of the corresponding tree. Let $A$ denote the collection of all these equivalence classes.

Let $T_0$ be the subtheory determined by the equivalence classes of those trees having no edge labeled in $\Sigma$, so that $T_0$ may be identified with the theory $\mathbf{Mat}_{\mathbb{B}}$ of matrices over the boolean semiring $\mathbb{B}$. The transition system corresponding to a presentation $1 \to p$ over $(T_0, A)$ is defined in the same way as in Subsection 4.3.2 but without any parallel edges labeled by the same symbol. The behavior of the presentation is the bisimulation equivalence class of its unfolding.

The Kleene theorem asserts that a bisimilarity equivalence class of a tree $1 \to p$ can be constructed from the equivalence classes corresponding to the letters in $\Sigma$ by the theory operations, sum and (scalar) dagger iff it is the behavior of a transition system. It is known that these behaviors are the bisimilarity equivalence classes of the regular synchronization tees. For more details, see [BE93].

### 4.3.4 Weighted tree automata

Suppose that $S$ is a semiring and $\Sigma$ is a ranked alphabet. A function $s : T_\Sigma(X_p) \to S$ is called a (finite) *tree series* [BR82, BLB83, EK03] with coefficients in $S$, sometimes denoted as a formal sum

$$\sum_{t \in T_\Sigma(X_p)} (s, t)t.$$

The *support* of $s$ is the set of all trees mapped to a non-zero element of $S$. Let $S\langle\langle T_\Sigma(X_p)\rangle\rangle$ stand for the set of all such series. Note that each $\sigma \in \Sigma_p$ has a corresponding series in $S\langle\langle T_\Sigma(X_p)\rangle\rangle$ that maps the atomic tree corresponding to $\sigma$ to 1 and all other trees to 0.

We can form a theory $S\langle\langle \Sigma\mathrm{Term}\rangle\rangle$ whose morphisms $n \to p$ are all $n$-tuples of series in $S\langle\langle T_\Sigma(X_p)\rangle\rangle$. Composition is defined in the following way, cf. [BE03].

Let $s : 1 \to p$ and $r = (r_1, \ldots, r_p) : p \to q$, and consider a tree $u \in T_\Sigma(X_q)$. Write $u$ in all possible ways as

$$u \quad = \quad \hat{u} \cdot \langle u_1, \ldots, u_k \rangle \tag{4.2}$$

where $\hat{u} \in T_\Sigma(X_k)$ has exactly one leaf labeled $x_i$ for each $i \in [k]$ and the label sequence (from left to right) of these leaves is $x_1 \ldots x_k$, and where $u_1, \ldots, u_k \in T_\Sigma(X_q)$. Note that there are a finite number of such decomposition. Now for each possible decomposition (4.2), and for each base morphism $\rho : k \to p$, consider the product

$$(s, \hat{u} \cdot \rho)(r_{1\rho}, u_1) \cdots (r_{k\rho}, u_k)$$

where we have identified $\rho$ with the corresponding function $[k] \to [p]$ as usual. Finally, $(s \cdot r, u)$ is the sum of all these products over all possible decompositions of $u$ and all possible choices of $\rho$. When $s = (s_1, \ldots, s_n) : n \to p$, define $s \cdot r = (s_1 \cdot r, \ldots, s_n \cdot r)$. For each $i \in [n]$, $n \geq 0$, the distinguished morphism $i_n : 1 \to n$ is the series which maps $x_i$ to 1 and all other trees in $T_\Sigma(X_n)$ to 0. Let $+$ denote the series $1 \to 2$ that maps $x_1$ and $x_2$ to 1 and all other trees in $T_\Sigma(X_2)$ to 0, and let $\#$ stand for the series $1 \to 0$ that maps all trees in $T_\Sigma(X_0) = T_\Sigma(\emptyset)$ to 0. Equipped with these constants, $S\langle\langle\Sigma\mathrm{Term}\rangle\rangle$ is a grove theory. The sum operation determined by the constant $+$ is the pointwise sum, so that

$$(s + s', t) \quad = \quad (s, t) + (s', t)$$

for all $s, s' : 1 \to p$ and $t \in T_\Sigma(X_p)$, and

$$s + s' \quad = \quad (s_1 + s'_1, \ldots, s_n + s'_n)$$

for all $s = (s_1, \ldots, s_n) : n \to p$ and $s' = (s'_1, \ldots, s'_n) : n \to p$.

Consider the theory $T = S\langle\langle\Sigma\mathrm{Term}\rangle\rangle$. Call $s : 1 \to p$ *proper* if $(s, x_i) = 0$ for all $x_i \in X_p$. Moreover, call $s : n \to p$ proper if $i_n \cdot s$ is proper for all $i \in [n]$. The proper morphisms form an ideal $D(T)$, and for every proper $s : n \to n + p$, the equation $\xi = s \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution in the set of morphisms $n \to p$. Thus we have a partial iterative grove theory and a partial Conway grove theory.

Now let $T_0$ denote the subtheory determined by those series that map every proper tree to 0. Clearly, $T_0$ may be identified with the theory $\mathbf{Mat}_S$. In particular, each element $s$ of $S$ may be identified with the series $1 \to 1$ that maps $x_1$ to $s$ and all other trees to 0. Let $A$ denote the collection of all series whose support is finite and includes only trees of the form $\sigma \cdot \rho$, where $\sigma \in \Sigma$ and $\rho$ is base. Note that $A(T_0) = A$ and $(T_0, A)$ is a strongly dagger compatible basis. A presentation $D = (\alpha, a) : 1 \to p$ of weight $s$ over $(T_0, A)$ may be viewed as a (variant of a) *weighted tree automaton*, see [BR82, EK03]. Indeed, each component $a_i$ of $a$ is a series $1 \to s+p$ in $A$, and $\alpha$ is a row matrix over $S$ of dimension $s + p$. The corresponding weighted tree automaton has $[s + p]$ as its set of states, with $s + 1, \ldots, s + p$ being the initial states corresponding to the variables $x_1, \ldots, x_p$. For a letter $\sigma \in \Sigma_k$ and states $i_1, \ldots, i_k$ and $i$, there is a transition from $(i_1, \ldots, i_k)$ to $i$ labeled $\sigma$ and having weight $(a_i, \sigma(x_{i_1}, \ldots, x_{i_k}))$ if this value is not 0. The row matrix $\alpha$ determines the final weight of each state. The initial weights of the states $s+1, \ldots, s+p$ are all 1, whereas the initial weight of any state in $[s]$ is 0. The behavior of $D$ is the tree series recognized by the corresponding weighted tree automaton. Thus, a tree series is recognizable iff it can be constructed from the series corresponding to the letters of $\Sigma$ and the series corresponding to the elements of $S$ using the theory operations, sum and dagger. (The dagger operation may be replaced by a generalized star operation, see Chapter 2. )

### 4.3.5 Partial Conway semirings

Recall Section 1.2.1. Let $S$ be a partial Conway semiring where $I$ denotes the distinguished two-sided ideal of $S$. Let $A \subseteq I$ and $S_0$ be a subsemiring of $S$. An *automaton* over $(S_0, A)$ is a triple $(\alpha, M, \beta)$ where $\alpha \in S_0^{1\times n}$, $M \in (S_0A)^{n\times n}$ and $\beta \in S_0^{n\times 1}$, where $S_0A$ is the set of all finite linear combinations of elements of $A$ with coefficients in $S_0$. The behavior of $(\alpha, M, \beta)$ is $\alpha M^*\beta$. Corollary 4.2.2 gives the following result, cf. [BE93, BEK08]:

**Theorem 4.3.1** *Suppose that either $S_0 \subseteq I$ is closed under star, or that whenever $x + a \in I$ for some $x \in S_0$ and $a \in D(T)$, then $x = 0$. Then an element of $S$ is the behavior of some automaton over $(S_0, A)$ iff $s$ can be generated from $S_0 \cup A$ by the rational operations of $+, \cdot$ and star.*

We note that if $S_0 \subseteq I$ then $1 \in I$ and $I = S$, so that $S$ is a Conway semiring. When $S$ is the power series semiring $S_0\langle\langle\Sigma^*\rangle\rangle$, for some alphabet $\Sigma$, $I$ is the ideal of proper series and $A$ is the collection of all series associated with the letters in $\Sigma$, this is Schützenberger's theorem, see [Sch61, Sch62] or [KS85]. If in addition $S_0$ is $\mathbb{B}$ with star operation $0^* = 1^* = 1$, then $S_0\langle\langle\Sigma^*\rangle\rangle$ may be identified with the usual Conway semiring of all subsets of $\Sigma^*$, and we have Kleene's classical theorem [Kle56].

### 4.3.6 Partial Conway semiring-semimodule pairs

Recall Section 1.2.2. Suppose that $(S, V)$ is a partial Conway semiring-semimodule pair with distinguished two-sided ideal $I$. Let $S_0$ be a subsemiring of $S$ and $A$ a subset of $I$. Then a *Büchi automaton* over $(S_0, A)$ is a triple $(\alpha, M, k)$, where $\alpha \in S_0^{1\times n}$, $M \in (S_0 A)^{n\times n}$ and $k \le n$. The behavior of $(\alpha, M, k)$ is $\alpha M^{\omega_k}$, where if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $A$ is $k \times k$ and $D$ is $(n-k) \times (n-k)$, then

$$M^{\omega_k} = \begin{pmatrix} (A + BD^*C)^\omega \\ D^*C(A + BD^*C)^\omega \end{pmatrix}.$$

Using Corollary 4.2.2, we have the following result, see [ÉK05, É11]:

**Theorem 4.3.2** *Suppose that $(S, V)$ is a partial Conway semiring-semimodule pair with distinguished two-sided ideal $I$. Let $S_0$ be a subsemiring of $S$ and $A \subseteq I$. Suppose that either $S_0 \subseteq I$ is closed under star, or that $x + a \in I$ with $x \in S_0$ and $a \in I$ implies that $x = 0$. Then $v \in V$ is the behavior of a Büchi-automaton over $(S_0, I)$ iff $v$ can be generated from $S_0 \cup A$ by the rational operations of $+, \cdot$, star and omega power.*

The above result extends Büchi's classic theorem, cf. [Bü62].

# Chapter 5

# Partial and total iteration semirings

Recall Section 1.2.1. The aim of this chapter is to give a description of the free iteration semirings using a simple congruence. This chapter is based on [EH14].

Recall the following theorem, which can be found in [DEK13]. Notice that this is an improvement of Theorem 3.3.7.

**Theorem 5.0.3** *Let $S$ be a semiring with a distinguished two-sided ideal $I_0$. Suppose the following:*

*5.0.3.1 $S_0$ is a subsemiring of $S$ that is a Conway semiring with star operation ${}^{*_0}$.*

*5.0.3.2 Each $s \in S$ can be written as $s = x + a$ for some $x \in S_0$ and $a \in I_0$.*

*5.0.3.3 For all $x, x' \in S_0$ and $a, a' \in I_0$, if $x + a = x' + a'$ then $x = x'$ and $a = a'$.*

*Then the operation ${}^{*_0}$ can be extended in a unique way to the a star operation ${}^* : S \to S$ such that $S$ becomes a Conway semiring. Moreover, if $S_0$ is an iteration semiring, then $S$ also becomes an iteration semiring.*

The assumptions imply that $S_0 \cap I = \{0\}$, moreover, $0^* = 0^{*_0} = 1$. In fact, as shown in [DEK13], the assumption that $S$ is the direct sum of $S_0$ and $I$ may be weakened, since it suffices to suppose that for all $x, y \in S_0$ and $a, b \in I$, if $x + a = y + b$, then $(x^{*_0}a)^*x^{*_0} = (y^{*_0}b)^*y^{*_0}$.

For later use we note:

**Lemma 5.0.4** *Suppose that $S$ and $S'$ are Conway semirings, $X$ is a subset of $S$ and $I$ is an ideal of $S$. Suppose that $h : S \to S'$ is a semiring morphism which preserves the star operation on $X$ and $I$, so that $h(x^*) = (h(x))^*$ and $h(a^*) = (h(a))^*$ for all $x \in X$ and $a \in I$. If each $s \in S$ may be written as $s = x + a$ for some $x \in X$ and $a \in I$, then $h$ preserves the star operation, so that $h$ is a Conway semiring morphism.*

**Proof** Suppose that $s \in S$ and write $s$ as $s = x + a$ where $x \in X$ and $a \in I$. By assumption, $h(x^*) = (h(x))^*$ and $h((x^*a)^*) = (h(x^*a))^* = (h(x^*)h(a))^* = ((h(x))^*h(a))^*$. Thus,

$$\begin{aligned} h(s^*) &= h((x^*a)^*x^*) \\ &= h((x^*a)^*)h(x^*) \\ &= ((h(x))^*h(a))^*(h(x))^* \\ &= (h(x) + h(a))^* \\ &= h(s)^*. \end{aligned}$$

□

Of course, the same fact holds for iteration semirings.

### About the equational structure of the free iteration semirings

In this section, we provide a characterization of the free iteration semirings using rational power series. To this end, let $A$ be a set, $A_\perp = A \cup \{\perp\}$ where $\perp \notin A$, and consider the partial iteration semiring $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ with the set of all proper rational series as distinguished ideal.

Let $\Theta$ denote the smallest congruence relation of $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ such that

$$(\perp + \perp)\ \Theta\ \perp \quad \text{and} \quad (\perp + 1)\ \Theta\ \perp \tag{5.1}$$

hold. For each $r$ in $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ we define

$$r^+ = rr^*.$$

**Lemma 5.0.5** *The following hold:*

$$\perp^n \ \Theta\ (\perp^n + \perp^m), \quad n \geq m,\ n > 0 \tag{5.2}$$

$$\perp^* \ \Theta\ (\perp^* + \perp^n), \quad n \geq 0 \tag{5.3}$$

$$\perp^* \ \Theta\ (\perp^* \cdot \perp^n), \quad n \geq 0 \tag{5.4}$$

$$\perp^* \ \Theta\ (\perp^* + \perp^*) \tag{5.5}$$

$$\perp^* \ \Theta\ (\perp^n)^*, \quad n > 0 \tag{5.6}$$

$$\perp^* \ \Theta\ (\perp^* \cdot \perp^*) \tag{5.7}$$

$$\perp^* \ \Theta\ (\perp^+)^* \tag{5.8}$$

**Proof** *Proof of (5.2).* First note that $(\perp^n + \perp^n) \,\Theta \perp^n$ and $(\perp^n + \perp^{n-1}) \,\Theta\, \perp^n$ for all $n > 0$ as an immediate consequence of (5.1). It follows that

$$\perp^n \,\Theta\, \perp^n + \perp^{n-1} + \cdots + 1$$

for all $n > 0$. Thus, when $n \geq m$ and $n > 0$, then

$$\begin{array}{rcl} \perp^n & \Theta & \perp^n + \perp^{n-1} + \cdots + 1 \\ & \Theta & \perp^n + \perp^{n-1} + \cdots + 1 + \perp^m + \perp^{m-1} + \cdots + 1 \\ & \Theta & \perp^n + \perp^m. \end{array}$$

*Proof of (5.3).* Since $\perp^* = \perp\perp^* + 1$, a simple inductive argument proves $\perp^* = \perp^{n+2}\perp^* + \perp^{n+1} + \perp^n + \cdots + 1$ for all $n \geq 0$. But

$$(\perp^{n+1} + \perp^n + \cdots + 1) + \perp^n \,\Theta\, \perp^{n+1} + \perp^n + \cdots + 1$$

for all $n \geq 0$ as shown above. Thus, $(\perp^* + \perp^n) \,\Theta\, \perp^*$ for all $n \geq 0$.

*Proof of (5.4).* We have

$$\perp^*\perp = (\perp^+ + 1)\perp = \perp^+\perp + \perp \,\Theta\, \perp^+\perp + \perp + 1 = \perp^+ + 1 = \perp^*.$$

It follows now by a straightforward induction on $n$ that $\perp^*\perp^n \,\Theta\, \perp^*$ for all $n \geq 0$. In the same way, also $\perp^n\perp^* \,\Theta\, \perp^*$ for all $n \geq 0$. Observe that in the case when $n = 1$, we have established $\perp^* \,\Theta\, \perp^+$.

*Proof of (5.5).* $\perp^* + \perp^* \,\Theta\, \perp^*\perp + \perp^*\perp = \perp^*(\perp + \perp) \,\Theta\, \perp^*\perp \,\Theta\, \perp^*$, where we have used (5.4) twice.

*Proof of (5.6).* We argue by induction on $n$. When $n = 1$, our claim is obvious. Suppose now that $n > 1$ and the claim holds for $n - 1$. Then, using (1.5), (5.2), (5.4) and the induction hypothesis,

$$\begin{array}{rcl} (\perp^n)^* & \Theta & (\perp^{n-1} + \perp^n)^* \\ & = & ((\perp^{n-1})^*\perp^n)^*(\perp^{n-1})^* \\ & \Theta & (\perp^*\perp^n)^*\perp^* \\ & \Theta & (\perp^*\perp)^*\perp^* \\ & \Theta & (\perp + \perp)^* \\ & \Theta & \perp^*. \end{array}$$

*Proof of (5.8).* Note that by $(\perp + \perp) \,\Theta\, \perp$ and (1.5), (5.4), also

$$(\perp^+)^*\perp^* \,\Theta\, (\perp^*\perp)^*\perp^* = (\perp + \perp)^* \,\Theta\, \perp^*.$$

But $\perp^+ \,\Theta\, \perp^*$, thus $(\perp^+)^*\perp^+ \,\Theta\, \perp^+$, i.e., $(\perp^+)^+ \,\Theta\, \perp^+$. By adding 1 to both sides, we conclude that $(\perp^+)^* \,\Theta\, \perp^*$.

*Proof of (5.7).*

$$\begin{array}{rcl} \perp^*\perp^* & \Theta & \perp^+\perp^* \\ & = & \perp^+\perp^+ + \perp^+ \\ & \Theta & \perp^+(\perp^+)^+ + \perp^+ \\ & = & (\perp^+)^+ \\ & \Theta & \perp^*. \end{array}$$

□

**Lemma 5.0.6** *For each integer $n \geq 0$, let us also denote by $n$ the $n$-fold sum of the series 1 with itself. Then the series $n$, $\perp^m$ and $\perp^*$, where $n \geq 0$ and $m > 0$, are pairwise inequivalent with respect to $\Theta$.*

**Proof** This follows from the fact that by Theorem 1.2.21, there is a partial iteration semiring morphism $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle \to S_0$ which maps all elements of $A_\perp$ to $1^*$, and that this morphism is surjective. Moreover, the kernel of this morphism collapses $\perp + \perp$, $\perp + 1$ and $\perp$. □

**Lemma 5.0.7** *Suppose that $r, s \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ with $\mathrm{supp}(r) \subseteq \{\perp\}^*$. If $r\,\Theta\,s$ then also $\mathrm{supp}(s) \subseteq \{\perp\}^*$. Moreover, $r\,\Theta\,s$ iff $r = s$ and $\mathrm{supp}(r) \subseteq \{\epsilon\}$, or there is some integer $n > 0$ such that both $\perp^n \in \mathrm{supp}(r) \cap \mathrm{supp}(s)$ and $\mathrm{supp}(r) \cup \mathrm{supp}(s) \subseteq \{\perp^n, \perp^{n-1}, \ldots, 1\}$ hold, or both $\mathrm{supp}(r)$ and $\mathrm{supp}(s)$ are infinite (and thus both contain an infinite number of powers of $\perp$).*

**Proof** Since $\Theta$ is the congruence generated by the pairs $(\perp + \perp, \perp)$ and $(\perp + 1, \perp)$, we have $r\,\Theta\,s$ iff $(r, s)$ can be derived from these two pairs by the following rules:

- $r_1 = r_2 \vdash (r_1, r_2)$.
- $(r_1, r_2) \vdash (r_2, r_1)$.
- $(r_1, r_2), (r_2, r_3) \vdash (r_1, r_3)$.
- $(r_1, r_2) \vdash (r_1 + r_3, r_2 + r_3)$.
- $(r_1, r_2) \vdash (r_1 r_3, r_2 r_3)$.
- $(r_1, r_2) \vdash (r_3 r_1, r_3 r_2)$.
- $(r_1, r_2) \vdash (r_1^*, r_2^*)$, provided that $r_1, r_2$ are proper.

It follows by an easy induction that whenever $(r,s)$ is derivable and no word in $\mathrm{supp}(r)$ contains a letter in $A$, then the same holds for $s$. Thus, if $\mathrm{supp}(r) \subseteq \{\perp\}^*$ then also $\mathrm{supp}(s) \subseteq \{\perp\}^*$. The second fact follows from the previous lemmas. One proves that for all $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ with $\mathrm{supp}(r) \subseteq \{\perp\}^*$, if $\mathrm{supp}(r) \subseteq \{\epsilon\}$, then $r = n$ for some $n \geq 0$, where $n$ also denotes the $n$-fold sum of 1 with itself. Moreover, if $\mathrm{supp}(r)$ is a finite subset of $\{\perp\}^*$ containing at least one nonempty word, then $r \;\Theta\; \perp^n$ (or equivalently, $r \;\Theta\; (\perp^n + \cdots + 1)$) for the largest integer $n > 0$ with $\perp^n \in \mathrm{supp}(r)$, and if $\mathrm{supp}(r)$ is an infinite subset of $\{\perp\}^*$, then $r \;\Theta\; \perp^*$. Since $r$ is rational and $\mathrm{supp}(r) \subseteq \{\perp\}^*$, $r$ can be constructed from $0, 1$ and $\perp$ by the sum, product and star operations (the latter applied only to proper series). One argues using Lemma 5.0.5 by induction on the least number of operations needed to obtain $r$ from $\{0, 1, \perp\}$. $\square$

**Corollary 5.0.8** *Suppose that* $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$*. Then* $\mathrm{supp}(r) \subseteq \{\perp\}^*$ *iff* $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle \{\perp\}^* \rangle\rangle$*.*[1] *Moreover, in this case either $r \;\Theta\; n$ or $r \;\Theta\; \perp^m$ for some $n \geq 0$, $m \geq 1$, where $n$ also denotes the $n$-fold sum of the series* 1 *with itself, or $r \;\Theta\; \perp^*$.*

Consider now the quotient partial iteration semiring $F_A = \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle / \Theta$. For each rational series $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$, let $[r]$ denote the $\Theta$-equivalence class of $r$. As shown above, when $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ with $\mathrm{supp}(r) \subseteq \{\perp\}^*$, then $\mathrm{supp}(s) \subseteq \{\perp\}^*$ for all $s \in [r]$. Let $S_0'$ be the set of all such $\Theta$-equivalence classes, so that by Corollary 5.0.8 the elements of $S_0'$ are the equivalence classes $[n]$, $[\perp^m]$ and $[\perp^*]$, for $n \geq 0$ and $m > 0$.

**Lemma 5.0.9** *$S_0'$ is a semiring isomorphic to the underlying semiring of the initial iteration semiring $S_0$.*

**Proof** Map each equivalence class $[n]$ to $n$, each equivalence class $[\perp^m]$ to $(1^*)^m$, map $[\perp^*]$ to $1^{**}$, and use Lemma 5.0.5. $\square$

The star operation of $F_A$ is defined on the ideal $I$ of all equivalence classes containing at least one proper rational series in $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$. Our next aim is to show that the star operation of $F_A$ may be turned into a totally defined operation such that $F_A$ becomes an iteration semiring. To this end, we will make use of Theorem 5.0.3. In order to apply this result, first turn $S_0'$ into an iteration semiring isomorphic to $S_0$. Thus, we define $[0]^* = [1]$, $[1]^* = [\perp]$ and $[r]^* = [\perp^*]$ for all $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle \{\perp\}^* \rangle\rangle$, $r \neq 0, 1$. (Note that on the elements $[\perp^m]$ and $[\perp^*]$ of $S_0'$, this star operation agrees with the one inherited from $F_A$ as a quotient of $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$. ) The elements $[r]$ with $\mathrm{supp}(r) \subseteq A_\perp^* A A_\perp^*$

[1] When $r \in \mathbb{N}\langle\langle A_\perp^* \rangle\rangle$ with $\mathrm{supp}(r) \subseteq \{\perp\}^*$, we may view $r$ as a series in $\mathbb{N}\langle\langle \{\perp\}^* \rangle\rangle$.

form another ideal $J$ (included in $I$), moreover $F_A$, equipped with this ideal and the partial star operation defined on this ideal inherited from $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^*\rangle\rangle$, is a partial iteration semiring. Each element of $F_A$ can be written as the sum of an element of $S_0'$ with an element of $J$. If we can prove that this decomposition is unique, then by Theorem 5.0.3 there is a unique extension of the star operations on $S_0'$ and $J$ to a star operation on $F_A$ such that $F_A$ becomes an iteration semiring. The extension is given as follows. Given $[s] \in F_A$, write $[s]$ as a sum $[s] = [x] + [r]$ with $[x] \in S_0'$ and $[r] \in J$. Then define $[s]^* = ([x]^*[r])^*[x]^*$.

**Lemma 5.0.10** *Let* $x, x'r, r' \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^*\rangle\rangle$ *such that* $\mathrm{supp}(x) \cup \mathrm{supp}(x') \subseteq \{\perp\}^*$ *and* $\mathrm{supp}(r) \cup \mathrm{supp}(r') \subseteq A_\perp^* A A_\perp^*$. *If* $(x + r)\ \Theta\ (x' + r')$ *then* $x\,\Theta\,x'$ *and* $r\,\Theta\,r'$.

**Proof** Below $x, x', x_1, \ldots$ always denote series in $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^*\rangle\rangle$ whose support is included in $\{\perp\}^*$, while $r, r', r_1, \ldots$ denote series in $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^*\rangle\rangle$ whose support is a subset of $A_\perp^* A A_\perp^*$.

Our claim is clear when both $x + r$ and $x' + r'$ are in the set $\{\perp, \perp + \perp\}$, or in the set $\{\perp, \perp + 1\}$. To complete the proof, suppose that the claim holds for $x_i + r_i$ and $x_i' + r_i'$, where $(x_i + r_i)\ \Theta\ (x_i' + r_i')$, $i = 1, 2$. Then, by assumption, $x_i\,\Theta\,x_i'$ and $r_i\,\Theta\,r_i'$, for $i = 1, 2$. Since $\Theta$ is the smallest congruence relation with (5.1), it suffices to prove the following facts (see the proof of Lemma 5.0.7).

- Let $x = x_1 + x_2$, $x' = x_1' + x_2'$, $r = r_1 + r_2$ and $r' = r_1' + r_2'$, so that

  $$(x + r) = (x_1 + x_2) + (r_1 + r_2)\ \Theta\ (x_1' + x_2') + (r_1' + r_2') = (x' + r').$$

  Then, since $x_i\,\Theta\,x_i'$ and $r_i\,\Theta\,r_i'$ for $i = 1, 2$, $x = (x_1 + x_2)\ \Theta\ (x_1' + x_2') = x'$ and $r = (r_1 + r_2)\ \Theta\ (r_1' + r_2') = r'$.

- Let $x = x_1 x_2$, $x' = x_1' x_2'$, $r = x_2 r_1 + x_1 r_2 + r_1 r_2$ and $r' = x_2' r_1' + x_1' r_2' + r_1' r_2'$, so that

  $$(x + r) = (x_1 + r_1)(x_2 + r_2)\ \Theta\ (x_1' + r_1')(x_2' + r_2') = (x' + r').$$

  Then $x = (x_1 x_2)\ \Theta\ (x_1' x_2') = x'$ and $r = (x_2 r_1 + x_1 r_2 + r_1 r_2)\ \Theta\ (x_2' r_1' + x_1' r_2' + r_1' r_2') = r'$.

- Finally, suppose that $x = (x_1)^*$, $x' = (x_1')^*$, $r = ((x_1)^* r_1)^+ (x_1)^*$ and $r' = ((x_1')^* r_1')^+ (x_1')^*$, so that

  $$(x + r) = (x_1 + r_1)^*\ \Theta\ (x_1' + r_1')^* = (x' + r').$$

Then

$$x = (x_1)^* \;\Theta\; (x_1')^* = x'$$

and

$$r = ((x_1)^* r_1)^+ (x_1)^* \;\Theta\; ((x_1')^* r_1')^+ (x_1')^* = r'.$$

$\square$

We now prove the final result of this chapter:

**Theorem 5.0.11** *$F_A$ is the free iteration semiring, freely generated by $A$.*

**Proof** We already know that $F_A$ is an iteration semiring. It is also clear that $F_A$ is generated by the image of $A$ with respect to the embedding $A \hookrightarrow F_A$, $a \mapsto [a]$. To complete the proof, suppose that $S$ is an iteration semiring and $h : A \to S$. We need to show that there is a morphism $h^\sharp : F_A \to S$ of iteration semirings extending $h$. (The extension is unique since $F_A$ is generated by the elements $[a]$ for $a \in A$.)

Let us extend $h$ to a function $A_\perp \to S$, also denoted $h$, by defining $h(\perp) = 1^*$. Since by Theorem 1.2.21 $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$ is a free partial iteration semiring freely generated by $A_\perp$, there is a (unique) partial iteration semiring morphism $\overline{h} : \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle \to S$ extending $h$. We have $\overline{h}(\perp + \perp) = \overline{h}(\perp) + \overline{h}(\perp) = 1^* + 1^* = 1^* = \overline{h}(\perp)$. Similarly, $\overline{h}(\perp + 1) = \overline{h}(\perp)$. Thus, the kernel of $\overline{h}$ is included in $\Theta$, so that $\overline{h}$ factors through the quotient map $\mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle \to F_A$ giving rise to the extension $h^\sharp : F_A \to S$. Clearly, $h^\sharp$ is a semiring morphism. Since $\overline{h}(r^*) = (\overline{h}(r))^*$ for all proper series $r \in \mathbb{N}^{\mathrm{rat}}\langle\langle A_\perp^* \rangle\rangle$, we have $h^\sharp([r]^*) = (h^\sharp([r]))^*$ for all $[r] \in I$ and thus for all $[r] \in J$ and all $[r] \in S_0'$ such that $[r] \notin \{[n] : n > 0\}$. But $h^\sharp$ also preserves the star of the elements of the sort $[n]$, where $n > 0$. Indeed,

$$h^\sharp([1]^*) = h^\sharp([\perp]) = \overline{h}(\perp) = 1^* = (h^\sharp([1]))^*,$$

and if $n > 1$, then

$$h^\sharp([n]^*) = h^\sharp([\perp^*]) = \overline{h}(\perp^*) = (\overline{h}(\perp))^* = 1^{**}$$

and

$$(h^\sharp([n]))^* = (\overline{h}(n))^* = 1^{**}.$$

Since each element of $F_A$ can be written as a sum $[x] + [r]$ with $[x] \in S_0'$ and $[r] \in J$, it follows by Lemma 5.0.4 that $h^\sharp$ preserves the star operation. $\square$

Theorem 5.0.11 does not immediately provide a decision procedure for the equational theory of iteration semirings. If there is a canonical way of selecting a unique representative of each $\Theta$-equivalence class, then one might obtain a decision procedure.

# Chapter 6

## 6.1 Summary

This thesis is concerned with partial Conway and iteration theories and their relation to Conway and iteration theories.

In **Chapter 1** we introduced the basic concepts used in the thesis. In a nutshell, a *theory* is a category whose objects are the nonnegative integers such that each object $n$ is the $n$-fold coproduct of object 1 with itself. A (partial) Conway theory is a theory equipped with a (partially defined) dagger operation taking a morphism $n \to n+p$ to a morphism $n \to p$. Moreover, we require that each (partial) Conway theory satisfies a certain set of identities. A (partial) iteration theory is a (partial) Conway theory subject to one more set of identities, one identity for each finite group. An iterative theory is a theory equipped with a partially defined dagger operation that takes a morphism $f : n \to n + p$ to a morphism $f^\dagger : n \to p$ which is the unique solution to the fixed point equation associated with $f$ :

$$\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$$

in the variable $\xi : n \to p$. Each iterative theory is a partial iteration theory. More can be said: the equational properties of the dagger operation in iterative theories is axiomatized by the iteration theory identities. These concepts were investigated in [BE93], where additional references can be found.

As was shown in [BE93], whether one is interested in least fixed points, in unique fixed points, or initial fixed points, the fixed point operation satisfies the iteration theory identities. Thus, results about abstract iteration theories have as corollaries facts about:

- ordered theories, e.g. continuous functions on complete posets;
- continuous 2-theories, such as $\omega$-functors and natural transformations on $\omega$-categories;

- matrix theories, e.g. matrices of regular sets over some alphabet;
- synchronization trees, and other theories of trees;
- partial correctness logic.

It is known [BE93], that in several subvarieties of iteration theories the free theories can be described using bisimulation equivalence classes of certain state-transition systems, or as input-output behaviors of these state-transition systems.

If a theory is equipped with an additional structure, such as an additive structure, then the dagger operation is usually related to some "Kleenean operations". For example, the theory of matrices over a semiring $S$ has an additive structure. Under a natural condition, cf. [BE93], any dagger operation over a matrix theory determines and is determined by a star operation mapping an $n \times n$ square matrix $A$ (i.e., a morphism $A : n \to n$) to an $n \times n$ square matrix $A^*$. Properties of the dagger operation are then reflected by corresponding properties of the star operation. In **Chapter 2** we have shown that this correspondence between the dagger and star operations can be naturally generalized to arbitrary grove theories.

In [BEW80b, É82] (see also [BE93], Theorem 6.4.5) it was shown that any iterative theory with at least one "constant" (i.e., morphism $1 \to 0$) can be turned into an iteration theory that has a total dagger operation. Moreover, the extension of the dagger operation to a total operation only depends on the choice of the constant that serves as the canonical solution of the fixed point equation associated with the identity morphism $1 \to 1$.

In **Chapter 3**, we have given a generalization of this construction that is applicable to partial iterative theories. We have given a sufficient condition ensuring that a partially defined dagger operation of a partial iterative theory can be extended to a total operation so that the resulting theory becomes an iteration theory. We have shown that this general result can be instantiated to prove that every iterative theory with at least one constant can be extended to an iteration theory. We also applied our results to theories equipped with an additive structure. We have shown that our result implies the Matrix Extension Theorem of [BE93] and the Grove Extension Theorem of [BE03].

In [BE93], a general Kleene type theorem was proved for all Conway theories. However, in many models of interest, the dagger operation is only partially defined. In **Chapter 4**, we have shown a Kleene theorem for partial Conway theories. We have also discussed several application of this generic result.

In **Chapter 5** of this thesis we have given a description of the free iteration semirings using a simple congruence. At the time of the writing of this thesis we do not have a concrete description, yet. Chapter 5 is based on [EH14], a yet unpublished manuscript.

## 6.2 Összefoglaló

Ez a tézis parciális Conway és iterációs elméletekkel foglalkozik és a Conway és iterációs elméletekkel való kapcsolatukkal.

**1-es fejezetben** bevezettük az alapvető fogalmakat melyeket használunk a tézis során. Dióhéjban, *elmélet* alatt egy olyan kategóriát értünk, melynek objektumai a nemnegatív egészek és minden $n$ objektum az 1 objektumnak önmagával vett $n$-szeres co-szorzata. (Parciális) Conway elmélet alatt egy olyan elméletet értünk amely egy (parciálisan definiált) iterációs művelettel van ellátva, mely egy $n \to n+p$ morfizmust egy $n \to p$ morfizmusba visz. Továbbá, megköveteljük, hogy minden (parciális) Conway elmélet kielégítse azonosságok egy bizonyos halmazát. (Parciális) iterációs elmélet alatt olyan (parciális) Conway elméletet értünk, mely azonosságok még egy halmazának tesz eleget, véges (egyszerű) csoportonként egy azonosságnak. Iteratív elmélet alatt olyan elméletet értünk, mely egy parciálisan definiált iterációs művelettel van ellátva, mely egy $f : n \to n+p$ morfizmust egy $f^\dagger : n \to p$ morfizmusba visz, mely az egyértelmű megoldása az $f$ által meghatározott fixpont egyenletnek:

$$\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$$

a $\xi : n \to p$ változóban. Minden iteratív elmélet parciális iterációs elmélet is egyben. Többet is el lehet mondani: a parciális iterációs elmélet fogalmát definiáló azonosságok axiomatizálják az iterációs művelet ekvacionális tulajdonságait iteratív elméletekben. Ezen fogalmak vizsgálatra kerültek a [BE93] könyvben, melyben további hivatkozások találhatóak.

Amint a [BE93] könyvben meg lett mutatva, attól függetlenül, hogy valaki legkisebb fixpontokkal, egyértelmű fixpontokkal, vagy iniciális fixpontokkal dolgozik, a fixpont művelet eleget tesz az iterációs elmélet azonosságoknak. Ily módon megmutatásra került, hogy az absztrakt iterációs elméletekre vonatkozó eredményeknek vonatkozásai vannak a következő fogalmakra nézve:

- rendezett elméletek, mint például a folytonos függvények elmélete egy teljes háló felett;
- folytonos 2-elméletek, mint például $\omega$-funktorok és természetes transzformációk $\omega$-kategóriák felett;
- mátrix elméletek, mint például valamely ábécé feletti reguláris nyelvek mátrixainak elmélete;
- szinkronizációs fák elmélete, és további fák által alkotott elméletek;

- parciális helyesség logika.

Ismert [BE93], hogy iterációs elméletek bizonyos részvarietásaiban a szabad iterációs elméletek leírhatóak bizonyos állapot - átmeneti rendszerek biszimulációra vett ekvivalenciaosztályait használva, vagy ezen állapot-átmeneti rendszerek bemenet-kimeneti viselkedését használva.

Ha egy elmélet fel van ruházva némi plusz struktúrával, mondjuk egy additív struktúrával, akkor az iterációs művelet általában kapcsolódik valamilyen „Kleene-féle„ művelethez. Például az $S$ félgyűrű feletti mátrixok elmélete felruházható egy additív struktúrával. Egy bizonyos [BE93] egyszerű feltétel teljesülése esetén bármely mátrix elmélet feletti iterációs művelet meghatároz egy csillag műveletet, mely egy $n \times n$-es négyzetes $A$ mátrixot (azaz egy $A : n \to n$ morfizmust) egy $n \times n$-es négyzetes $A^*$ mátrixba képez. Továbbá, az iterációs művelet meghatározott egy ilyen csillag művelet által. Az iterációs művelet tulajdonságai ekkor a csillag művelet megfelelő tulajdonságaiban tükröződnek. A **2-es fejezetben** megmutattuk, hogy ez a kapcsolat az iterációs művelet és a csillag művelet között természetes módon általánosítható tetszőleges grove elméletekre.

A [BEW80b, É82] cikkekben (lásd még [BE93], 6.4.5-ös tétel) meg lett mutatva, hogy bármely iteratív elmélet amely rendelkezik legalább egy darab „konstanssal" (azaz egy $1 \to 0$ morfizmussal) iterációs elméletté alakítható, amely totálisan definiált iterációs művelettel van ellátva. Továbbá, az iterációs művelet totális műveletté való kiterjesztése csak a konstans megválasztásától függ, mely konstans az $1 \to 1$ identitásmorfizmus által meghatározott fixpont egyenlet megoldását adja.

A **3-as fejezetben** ezen konstrukció egy általánosítását mutattuk meg, mely parciális iteratív elméletekre alkalmazható. Adtunk egy elegendő feltételt arra vonatkozóan, hogy a parciálisan definiált iterációs művelet egy parciális iteratív elméletben kiterjeszthető legyen egy totális műveletté úgy, hogy az eredményül kapott elmélet egy iterációs elmélet legyen. Megmutattuk, hogy ennek az általános eredménynek következménye az, hogy minden, legalább egy konstanssal rendelkező iteratív elmélet kiterjeszthető iterációs elméletté. Eredményünket olyan elméletekre is alkalmaztuk, melyek bizonyos extra additív struktúrával vannak ellátva. Megmutattuk, hogy eredményünkből következik a Mátrix Kiterjesztési Tétel a [BE93] könyvből és a Grove Kiterjesztési Tétel a [BE03] cikkből.

A [BE93] könyvben egy általános Kleene típusú tétel került igazolásra, mely tetszőleges Conway elméletre vonatkozik. Azonban sok érdekelt modellben az iterációs művelet parciálisan definiált. A **4-es fejezetben** egy Kleene típusú tételt adtunk parciális Conway elméletekre. Ezen kívül néhány alkalmazását tárgyaltuk ennek a tételnek.

Ezen tézis **5-ös fejezetében** megadtuk a szabad iterációs félgyűrűk egy leírását egy egyszerű kongruencia segítségével. Ezen fejezet a [EH14] cikkre alapszik, mely a tézis megírásának időpontjában még publikálatlan.

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