THESES OF Ph. D. DISSERTATION

ZEROS OF ORTHOGONAL POLYNOMIALS AND CHRISTOFFEL FUNCTIONS

Tamás Varga

Supervisor:
Prof. Vilmos Totik
full member of MTA

Doctoral School in Mathematics and Computer Science
University of Szeged
Bolyai Institute

Szeged, 2013

1 Introduction

When Stieltjes introduced the notion of orthogonal polynomials associated to an arbitrary measure at the end of the 19th century, it began a general theory. Thenceforth the research of the general properties of orthogonal polynomials (asymptotic behavior, zero spacing, distribution, etc.) became as important as the classical polynomial families (Jacobi-polynomials, Hermite-polynomials, etc.). The results of this thesis relates to the former approach.

Our starting-point is the article [6] of Mastroianni and Totik published in 2010 which deals with the zero spacing of orthogonal polynomials associated with a doubling measure on the interval $[-1, 1]$.

The results are divided into two groups. In the first one we have worked on the real line. The central question is the zero spacing of orthogonal polynomials on an interval where the associated measure possesses the doubling property. In this part we have succeeded in generalizing the results of Mastroianni and Totik showing that those are actually a consequence of a local property of the measure. In the proof the Christoffel functions and fast decreasing polynomials play an important role.

In the second part the Christoffel functions has came to the fore. We have given matching estimates on curves and arcs, respectively, if the associated measure is doubling. The significance of our results is that, contrary to the preceding results which assumed smoothness, we have required only quasismoothness which allows the existence of corners. As applications we have given matching estimates for orthogonal polynomials and we have established Nikolskii-type inequalities. During the proofs, in addition to fast decreasing polynomials, we have made use the tools of conformal mappings inspired by Andrievskii's recently published articles [1, 2, 3].

The results are based on the following papers of the author:

- Vilmos Totik és Tamás Varga, Non-symmetric fast decreasing polynomials and applications, *J. Math. Anal. Appl.*, **394** (2012), 378–390. [9]

- Tamás Varga, Uniform spacing of zeros of orthogonal polynomials for locally doubling measures, *Analysis (Munich)*, **33** (2013), 1-12. [11]

- Tamás Varga, Christoffel functions for doubling measures on quasismooth curves and arcs, *Acta Math. Hungar.*, **141** (2013), 161-184. [12]

2 Preliminaries

Let μ be a measure on the complex plane which has a compact and infinite support. Then there exists a unique polynomial sequences $\{\pi_n\}_{n\in\mathbb{N}}$ such that $\deg(\pi_n) = n$, the leading coefficient of π_n is positive and

$$\int \pi_m \pi_n \, d\mu = \begin{cases} 1 & \text{ha } m = n \\ 0 & \text{ha } m \neq n. \end{cases}$$

The members of this sequence are called *orthogonal (orthonormal) polynomials* associated to the measure μ.

In our research we have worked with doubling measures.

Definition. *Let μ be a measure with the support $[-1, 1]$. We say that μ is a doubling measure if there is a constant L such that*

$$L\mu\Big([x-\delta, x+\delta]\Big) \geq \mu\Big([x-2\delta, x+2\delta]\Big)$$

for every $x \in [-1, 1]$ and $\delta > 0$.

The n-th Christoffel function associated to the measure μ is defined as

$$\lambda_n(\mu, p, x) = \inf_{\substack{q(x)=1 \\ \deg(q)\leq n}} \int |q|^p \, d\mu,$$

where the infimum is taken for all polynomials of degree at most n with the property $P_n(x) = 1$.

Introducing the following notation:

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2},$$

we cite Mastroianni and Totik's theorem on zero spacing.

Theorem [6, Theorem 1]. *Let μ be a doubling measure on the interval $[-1, 1]$, and let $\{\pi_n\}_{n\in\mathbb{N}}$ be the corresponding orthogonal polynomials. If $x_{n,0} := -1 < x_{n,1} < \cdots < x_{n,n} < x_{n,n+1} := 1$ denote the zeros of π_n then there is a constant A such that for every $n \in \mathbb{N}$ and for every $0 \leq k \leq n$ we have*

$$\frac{1}{A}\Delta_n(x_{n,k}) \leq x_{n,k+1} - x_{n,k} \leq A\Delta_n(x_{n,k}). \tag{$\diamond$}$$

We emphasize that A is independent of both n and k. Consider the magnitude of $\Delta_n(x)$. We can see that, according to the theorem, the distances between the consecutive zeros far from the endpoints are of $1/n$ order of magnitude which gradually changes into $1/n^2$ as x moves towards the endpoints. This regular spacing can be seen best if we project the zeros onto the unit circle in the upper half plane. Then the arc-length distances between the consecutive zeros are of $1/n$

order of magnitude.

The values of the Christoffel functions at the zeros of the corresponding orthogonal polynomials are called *Cotes* or *Christoffel* numbers.

With the help of the previous theorem it turns out that the quotient of the Cotes numbers corresponding to consecutive zeros are uniformly bounded away from both 0 and ∞.

Corollary [6, Theorem 2]. *If μ is a doubling measure on the interval $[-1, 1]$ then there is a constant B such that for every $n \in \mathbb{N}$ and for every $0 \le k \le n$ we have*

$$\frac{1}{B} \le \frac{\lambda_n(x_{n,k+1})}{\lambda_n(x_{n,k})} \le B. \qquad (\diamond\,\diamond)$$

The previous two theorems together have a converse.

Theorem [6, Theorem 3]. *If μ is a measure supported on the interval $[-1, 1]$ and if ($\diamond$) and ($\diamond\,\diamond$) hold for every $n \in \mathbb{N}$ and $1 \le k < n$ with some constants A and B then μ is doubling on* $[-1, 1]$.

The first part of our results state that if the measure possesses the doubling property only on some subinterval of its support rather then on the whole one, then the regular spacing and the matching estimate for the quotient of the consecutive Cotes numbers hold on that subinterval. Our proofs follow Mastroianni and Totik's argument in which Christoffel functions play an important role. In connection with this we cite the following result:

Theorem [7, (7.14)]. *If μ is doubling on the interval $[-1, 1]$ then there exits a constant c such that the following matching estimate is true for every $x \in [-1, 1]$:*

$$\frac{1}{c}\mu\Big([x - \Delta_n(x), x + \Delta_n(x)]\Big) \le \lambda_n(\mu, p, x) \le c\mu\Big([x - \Delta_n(x), x + \Delta_n(x)]\Big). \qquad (\circ)$$

In the second part of the thesis the generalization of this theorem is given in the sense that the interval is replaced by a quasismooth curve or arc.

3 Zero spacing on doubling intervals

In this part it is supposed that the support of the measure is a compact subset of the real line. If $I = [\tau - \delta, \tau + \delta]$ then $2I$ denotes the twice enlarged interval with the same midpoint as I, namely, the interval $[\tau - 2\delta, \tau + 2\delta]$. The length (Lebesgue measure) of I is denoted by $|I|$.

3.1 Local zero spacing

Definition. *The measure μ is doubling on the interval $[a, b]$ with the (doubling) constant $L = L(\mu, [a, b])$ if $\mu([a, b]) > 0$ and*

$$\mu(2I) \leq L\mu(I)$$

whenever $2I \subset [a, b]$.

We are ready to claim the local variant of [6, Theorem 1].

Theorem 3.2. [1] *Let μ be a measure with compact support on the real line and with the doubling property on $[a, b]$. Then for every $\delta > 0$ there exists a constant A independent of n such that*

$$\frac{1}{An} \leq x_{n,k+1} - x_{n,k} \leq \frac{A}{n}, \qquad k = j, j+1, \ldots, l-1, \tag{*}$$

where $x_{n,j} < x_{n,j+1} < \ldots < x_{n,l}$ are the zeros of the n-th orthogonal polynomial in $[a + \delta, b - \delta]$.

Remark that the statement is not necessarily true if $[a + \delta, b - \delta]$ is replaced by (a, b). For example, consider some doubling weight on $[a, b] := [-1, 1]$. Then $(\diamond)$ shows that the distances are of $1/n^2$ order of magnitude near to the endpoints.

The claim in the theorem can be formulated in the following way. If $x_{n,k(n)}$ denotes a sequence of zeros such that $x_{n,k(n)} \in [a + \delta, b - \delta]$ then

$$0 < \frac{1}{A} \leq \liminf_{n\to\infty} n(x_{n,k(n)+1} - x_{n,k(n)}) \leq \limsup_{n\to\infty} n(x_{n,k(n)+1} - x_{n,k(n)}) \leq A < \infty.$$

From this form it immediately follows that Theorem 3.2. generalizes Y. Last and B. Simon's following two theorems:

Corollary 3.4 [5, Theorem 8.5]. *Suppose $\mathrm{d}\mu = w(x)\mathrm{d}(x)$ is purely absolutely continuous in a neighborhood of the point E_0, and for some $q > 0$,*

$$0 < \liminf_{x\to E_0} \frac{w(x)}{|x - E_0|^q} \leq \limsup_{x\to E_0} \frac{w(x)}{|x - E_0|^q} < \infty. \tag{3.1}$$

Then

$$\limsup_{n\to\infty} n|x_n^{(1)}(E_0) - x_n^{(-1)}(E_0)| < \infty,$$

where $x_n^{(1)}(E_0)$ is the smallest zero and $x_n^{(-1)}(E_0)$ is the largest zero of p_n for which $x_n^{(-1)}(E_0) \leq E_0 < x_n^{(1)}(E_0)$.

Corollary 3.5 [5, Theorem 9.3]. *Suppose $\mathrm{d}\mu = w\,\mathrm{d}x + \mathrm{d}\mu_s$, where, for the singular part,*

[1] The numbering corresponds to that of the dissertation.

$\mu_s([x_0-\delta, x_0+\delta]) = 0$ *and, for the absolutely continuous part,*

$$0 < \inf_{|y-x_0|\le\delta} w(x) \le \sup_{|y-x_0|\le\delta} w(x) < \infty. \tag{3.2}$$

Then for any $\epsilon < \delta$,

$$\inf_{|y-x_0|<\epsilon} \liminf_{n\to\infty} n|x_n^{(1)}(y) - x_n^{(-1)}(y)| > 0.$$

Mastroianni and Totik's matching estimate with respect to the quotient of consecutive Cotes numbers and the converse also remains true.

Corollary 3.6. *If* μ *is a measure with compact support on the real line and with the doubling property on* $[a,b]$*, then for every* $\delta > 0$ *there exists a constant* $B = B_\delta$ *such that*

$$\frac{1}{B} \le \frac{\lambda_{n,k}}{\lambda_{n,k+1}} \le B, \tag{**}$$

whenever $x_{n,k}$ *and* $x_{n,k+1} \in [a+\delta, b-\delta]$.

Theorem 3.7. *Let* μ *be a measure with compact support. If (*) and (**) hold on every interval* $[a+\delta, b-\delta] \subset \mathrm{supp}(\mu)$*,* $\delta > 0$ *(with some A and B in (*) and (**) that may depend on* δ*), then* μ *has the doubling property on every such interval.*

In the foregoing the zero spacing is investigated far from the endpoints. However, one can not generally give an estimate: as we have seen, for a doubling measure on $[-1,1]$ the distances are of $1/n^2$ order of magnitude near the endpoints; whereas if a measure is doubling not only on $[a,b]$ but also on $[a-\varepsilon, b+\varepsilon]$, then the order of magnitude of the distances continue to be $1/n$. This shows one should be cautious at the endpoints because there the zero spacing is influenced by the properties of the measure outside the interval. We have studied the case when the measure vanishes on a small interval on the left hand side of a left-endpoint. We have verified that, in this case, the pattern follows Mastroianni and Totik's purely doubling situation [6], that is the $1/n$ order of magnitude gradually changes into $1/n^2$ as we move towards the endpoint. One difference is possible, namely, the distance between the endpoint and the nearest zero. We have shown by an example that in the case of a local endpoint this distance can be much smaller (e.g. e^{-n}) than $1/n^2$.

Denote $\mathrm{supp}(\mu)$ the support of μ.

Definition. *The point* $\mathbf{b}$ *is called a left-endpoint of the support of* μ*, if for some* $\alpha > 0$ *we have* $\mu([\mathbf{b}-\alpha, \mathbf{b})) = 0$ *but* $\mu([\mathbf{b}, \mathbf{b}+\beta)) > 0$ *for all* $\beta > 0$.

We can now claim the endpoint variants of Theorem 3.2, Corollary 3.6 and Theorem 3.7.

Theorem 3.10. *If* $\mathbf{b}$ *is a left-endpoint of the support of* μ *and* μ *is doubling on the interval*

$[\mathbf{b}, \mathbf{b}+\beta]$ *for some* $\beta > 0$ *then for every* $0 < \gamma < \beta$ *there is a constant* A_γ *such that*

$$\frac{1}{A_\gamma}\left(\frac{\sqrt{x_{n,k}-\mathbf{b}}}{n}+\frac{1}{n^2}\right) \leq x_{n,k+1}-x_{n,k} \leq A_\gamma\left(\frac{\sqrt{x_{n,k}-\mathbf{b}}}{n}+\frac{1}{n^2}\right) \qquad (\star)$$

holds, whenever $x_{n,k}$ *and* $x_{n,k+1}$ *denote two consecutive zeros of the n-th orthogonal polynomial on* $[\mathbf{b}, \mathbf{b}+\gamma]$.

Corollary 3.12. *If* $\mathbf{b}$ *is a left-endpoint of the support of* μ *and* μ *is doubling on the interval* $[\mathbf{b}, \mathbf{b}+\beta]$ *for some* $\beta > 0$ *then for every* $\gamma < \beta$*, there is a constant* B_γ *such that*

$$\frac{1}{B_\gamma} \leq \frac{\lambda_{n,k}}{\lambda_{n,k+1}} \leq B_\gamma, \qquad (\star\ \star)$$

whenever $x_{n,k}, x_{n,k+1} \in [\mathbf{b}, \mathbf{b}+\gamma]$.

The properties $(\star)$ and $(\star\ \star)$ again implies the doubling property.

Theorem 3.13. *Assume that* μ *vanishes on* $[\mathbf{b}-\alpha, \mathbf{b}]$*, and that* $(\star)$ *and* $(\star\ \star)$ *hold on every interval* $[\mathbf{b}, \mathbf{b}+\gamma]$*,* $\gamma < \beta$*. Then* μ *has the doubling property on every such interval.*

Remark. Of course, the notion of right-endpoint can be defined in the same way as the left-endpoint and the previous theorems are valid with right-endpoints too.

The following statement gives an example which shows that one should indeed take care of the distance between a local endpoint and the nearest zero.

Statement 3.25. *There is a* $d \in (0,1)$ *such that if* μ *is the restriction of the Lebesgue-measure onto* $[-2,-1] \cup [0,d]$*, then for infinitely many n we have for the smallest positive zero* x_{n,j_0} *of the n-th orthogonal polynomial the inequality*

$$\frac{1}{2}e^{-n} \leq x_{n,j_0} \leq 2e^{-n}.$$

In the proof of both Theorem 3.2 and Theorem 3.10 the upper estimate proves more difficult. The following two theorems concerning Christoffel functions play an important role in the verification.

Lemma 3.15. *Let* μ *be a measure with compact support on the real line and with the doubling property on* $[a,b]$*. Then for every* $\delta > 0$ *there is a constant D such that for* $n > \frac{2}{\delta}$

$$\frac{1}{D}\mu\left(\left[\xi-\tfrac{1}{n}, \xi+\tfrac{1}{n}\right]\right) \leq \lambda_n(\mu, p, \xi) \leq D\mu\left(\left[\xi-\tfrac{1}{n}, \xi+\tfrac{1}{n}\right]\right),$$

whenever $\xi \in [a+\delta, b-\delta]$.

Lemma 3.20. *Let* $\mathbf{b}$ *be a left-endpoint of the support of* μ. *Assume that* μ *is a doubling measure on some interval* $[\mathbf{b}, \mathbf{b}+\beta]$, *and let* $\gamma < \beta$. *Then uniformly in* $a \in [\mathbf{b}, \mathbf{b}+\gamma]$ *we have*

$$\frac{1}{C_\gamma}\mu([a-\Delta_n(a), a+\Delta_n(a)]) \le \lambda_n(\mu, p, a) \le C_\gamma \mu([a-\Delta_n(a), x+\Delta_n(a)]),$$

where

$$\Delta_n(a) = \frac{\sqrt{a-A}}{n} + \frac{1}{n^2}.$$

To prove the two preceding lemmas fast decreasing polynomials have been needed.

Let ψ be a nonnegative and increasing function on $[0,\infty)$, such that $\psi(0+) = 0$ and $\psi(x) \le M\psi(x/2)$ with some constant M independent of n.

Theorem 3.19. *Suppose that*

$$\int_1^\infty \frac{\psi(x)}{x^2}\,\mathrm{d}x < \infty.$$

Then there are constants $C, c > 0$ *such that for all* $a \in [0, 1/2]$ *and for all n there are polynomials* $P_n = P_{n,a}$ *of degree at most n with the properties that* $P_n(0) = 1$, $|P_n(x)| \le 2$, $x \in [-a, 1]$, *and*

$$P_n(x) \le C\exp\left(-c\psi\left(\frac{n|x|}{\sqrt{|x|}+\sqrt{a}}\right)\right), \qquad x \in [-a, 1].$$

4 Christoffel functions on curves and arcs

4.1 Matching estimate for Christoffel functions

In this part the generalization of ($\circ$) is given in the sense that the interval can be replaced by quasismooth curves or arcs.

Our methods relies on V. Andriewskii's work [1, 2, 3] who extended some results of [7] to quasismooth curves or arcs.

In what follows the Jordan curves (homeomorphic images of the unit circle) and arcs (homeomorphic images of $[-1,1]$) are assumed to be rectifiable. Let L be a curve or an arc and let z_1 and z_2 be two points on L. We denote by $L(z_1, z_2)$ the (shorter) subarc of L that joins $z_1 \in L$ and $z_2 \in L$, and by $|L(z_1, z_2)|$ the arc length of $L(z_1, z_2)$.

Definition. *The Jordan curve or arc L is quasismooth (in the sense of Lavrentiev), if there is a*

(Lavrentiev) constant Λ_L such that

$$|L(z_1, z_2)| \le \Lambda_L |z_1 - z_2|$$

holds for arbitrary $z_1, z_2 \in L$.

In the treatment of L the Riemann mapping plays an important role. Let $\mathbb{C}_\infty$ be the extended complex plane and denote Ω the exterior of L (the connected component of $\mathbb{C}_\infty \setminus L$ containing ∞). Then there exists a unique function Φ that conformally maps Ω onto the exterior of the unit disk $\{z \in \mathbb{C}_\infty : |z| > 1\} =: \mathbb{D}^*$ with the normalization $\Phi(\infty) = \infty$ and $\Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. [8, Chapter 4.4]

For $\delta > 0$ let

$$L_\delta := \{\zeta \in \Omega : |\Phi(\zeta)| = 1 + \delta\}$$

be the $(1 + \delta)$-level line of Φ and

$$\rho_\delta(z) := \mathrm{d}(L_\delta, z) = \inf_{\zeta \in L_\delta} |z - \zeta|$$

the distance from z to this level line.

The notion of the doubling property for a measure with support on a Jordan curve or arc is a natural extension of the interval-case. Let

$$B(z, \delta) = \{\zeta : |\zeta - z| \le \delta\}$$

denote the closed disk of radius δ about the point z.

Definition. *Let μ be a measure on the complex plane the support of which is a quasismooth curve or arc L. μ is called a doubling measure on L, if there is a constant c_μ such that*

$$c_\mu \, \mu(B(z, \delta)) \ge \mu(B(z, 2\delta))$$

holds for any $z \in L$ and $\delta > 0$.

Take an orientation of L (say, from ζ_1 to ζ_2 in the arc-case, if ζ_1 and ζ_2 denote the two endpoints of L; counterclockwise in the curve-case), and let $\delta > 0$ be such that $\sup_{z\in L} \rho_\delta(z) < |L|/2$. For a point $z \in L$ let $z_{-\delta}$ be the point followed by z and z_δ that follows z such that $|L(z_{-\delta}, z)| = \rho_\delta(z)/2$ and $|L(z, z_\delta)| = \rho_\delta(z)/2$ respectively. If there is no such point (in the arc case) then set $z_{-\delta} := \zeta_1$ and $z_\delta := \zeta_2$, respectively. With these we set

$$l_\delta(z) := L(z_{-\delta}, z_\delta),$$

and we introduce the following function

$$\upsilon_\delta(z) := \mu(l_\delta(z)).$$

On quasismooth curves and arcs we introduce a further parameter in the definition of the Christoffel function:

Definition. *Let L be a quasismooth curve or arc, μ a doubling measure on L, $p \in [1, \infty)$ and $t \in \mathbb{R}$. Then the function*

$$\lambda_n(\mu, p, t, z) := \inf_{\substack{\deg(p_n)\leq n \\ p_n(z)=1}} \int \rho_{\frac{1}{n}}(\zeta)^t |p_n(\zeta)|^p \,\mathrm{d}\mu(\zeta)$$

is called the n-th Christoffel function associated to μ with parameter (p, t).

Note that by setting $t = 0$ we obtain the classical L^p Christoffel functions of μ.

Now we are ready to extend [7, (7.14)].

Theorem 4.4. *Let L be a quasismooth curve or arc and μ a doubling measure on L. If $p \in [1, \infty)$, $t \in \mathbb{R}$ then there is a constant $c = c(L, c_\mu, p, t)$ such that*

$$\frac{1}{c}\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z) \leq \lambda_n(\mu, p, t, z) \leq c\,\rho_{\frac{1}{n}}(z)^t \upsilon_{\frac{1}{n}}(z)$$

is true for any $z \in L$ and $n \in \mathbb{N}$.

Corollary 4.5. *The same result holds if L is a finite union of quasismooth curves or arcs lying exterior to one another, and μ is a doubling measure on L remarking that $\rho_{\frac{1}{n}}(z)$ with respect to L means the function that is equal to $\rho_{\frac{1}{n}}(z)$ with respect to the connected component of L containing z.*

The case $t = 0$ may be the most interesting one. It shows that the magnitude of the n-th Christoffel function at a point $z \in L$ is about as large as the μ-measure of $l_{\frac{1}{n}}(z)$.

4.2 Corollaries

4.2.1 Estimate for orthonormal polynomials

Using the fact that

$$\lambda_n(\mu, 2, 0, z) = \frac{1}{\sum_{k=0}^{n} |\pi_k(z)|^2},$$

where π_k is the k-th orthonormal polynomial associated to μ (see e.g. [10, Theorem 1.4]) we immediately obtain the following corollaries.

Corollary 4.6. *Let μ be a doubling measure on a quasismooth curve or arc L. Then*

$$|\pi_n(z)| \leq \frac{\sqrt{c}}{\sqrt{\upsilon_{\frac{1}{n}}(z)}}$$

holds for every $z \in L$.

Corollary 4.7. *Let μ be a doubling measure on a quasismooth curve or arc L. Then*

$$\max_{0 \le k \le n} |\pi_k(z)| \ge \frac{1}{\sqrt{c}\,\sqrt{n}\,\sqrt{\upsilon_{\frac{1}{n}}(z)}}.$$

Moreover, if we know that $n \cdot \upsilon_{\frac{1}{n}}(z) \to 0$ then here we can delete “max”:

Corollary 4.8. *Let μ be a doubling measure on a quasismooth curve or arc L. Then for any $z \in L$ for which $n \cdot \upsilon_{\frac{1}{n}}(z) \to 0$ holds there is an infinite subset $\mathbb{M} = \mathbb{M}(z)$ of $\mathbb{N}$ such that for all $n \in \mathbb{M}$*

$$|\pi_n(z)| \ge \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\upsilon_{\frac{1}{n}}(z)}}.$$

Without the assumption $n \cdot \upsilon_{1/n}(z) \to 0$ we can only prove a weaker corollary.

Corollary 4.10. *Let μ be a doubling measure on a quasismooth curve or arc L. Then for any $z \in L$ and $\varepsilon > 0$ there is an infinite subset $\mathbb{M} = \mathbb{M}(z, \varepsilon)$ of $\mathbb{N}$ such that for all $n \in \mathbb{M}$*

$$|\pi_n(z)| \ge \frac{1}{n^{1/2+\varepsilon}} \frac{1}{\sqrt{\upsilon_{\frac{1}{n}}(z)}}.$$

4.2.2 Christoffel functions on Dini-smooth curves and arcs

In the subsequent two corollaries we have some further assumptions about the smoothness of L. We suppose that the considered curve or arc is Dini-smooth or has Dini-smooth corner at some point. Then we have an explicit form for the magnitude of $\rho_{\frac{1}{n}}(z)$.

Definition. *A Jordan curve or arc L is Dini-smooth, if it has a parametrization $\gamma(t)$ with non-zero and Dini-continuous derivative, that is*

$$\int_0^{\pi} \frac{\omega(t)}{t}\,\mathrm{d}t < \infty$$

holds for the modulus of continuity of the derivative γ'

$$\omega(\delta) := \sup_{\substack{t_1, t_2 \in [0, 2\pi] \\ |t_1 - t_2| < \delta}} |\gamma'(t_1) - \gamma'(t_2)|.$$

We say that L has a corner at $\gamma(t_0) =: \zeta$ if the half-tangents exist at $\gamma(t_0)$. If we speak of a corner with angle $\beta\pi$ then we always consider the angular domain (determined by the half-tangents) which falls in the exterior of the curve in the curve-case and we always consider the angular domain with greater angle in the arc-case.

The *corner at $\gamma(t_0)$ is Dini-smooth* if there are two subarcs of L ending and lying on the opposite sides of $\gamma(t_0)$ which are Dini-smooth, and, similarly, the *endpoint ζ_i is Dini-smooth* if it is an endpoint of a Dini-smooth subarc of L. Note that if L is Dini-smooth then at any point (except for the endpoints) there is a Dini-smooth straight angle, so the following statements include the corner-free case, as well.

We introduce the following function at a Dini-smooth corner ζ with angle β which replaces $\rho_{\frac{1}{n}}(z)$ in Theorem 4.4. We set

$$\Delta_n(z) := \begin{cases} \frac{1}{n^\beta} & \text{if } z \in l_{\frac{1}{n}}(\zeta) \\ \frac{|z-\zeta|^{1-\frac{1}{\beta}}}{n} & \text{if } z \in L \setminus l_{\frac{1}{n}}(\zeta). \end{cases}$$

Theorem 4.12. *Let L be a quasismooth curve or arc which has a Dini-smooth corner/endpoint at ζ with angle $\beta\pi$ ($0 < \beta < 2$ in the curve-case, $1 \le \beta < 2$ in the arc-case and $\beta := 2$ if ζ is an endpoint). Then there are an $\varepsilon = \varepsilon(L, \zeta) > 0$ and a constant $c_7 = c_7(L, \zeta, \varepsilon, \beta)$ such that for all points $z \in L$ with $|z - \zeta| \le \varepsilon$ the inequalities*

$$\frac{1}{c_7}\Delta_n(z) \le \rho_{\frac{1}{n}}(z) \le c_7\Delta_n(z)$$

hold.

Remark. The previous lemma can also be claimed at endpoints. Then it is required that the endpoint is also an endpoint of a Dini-smooth subarc of L while β is replaced by 2 as if there were a corner at the endpoint with angle 2π.

Combining this lemma with Theorem 4.4 we immediately obtain

Corollary 4.14. *Let L be a quasismooth curve or arc which has a Dini-smooth corner/endpoint at ζ with angle $\beta\pi$ ($0 < \beta < 2$ in the curve-case, $1 \le \beta < 2$ in the arc-case and $\beta := 2$ if ζ is an endpoint). Then there are an $\varepsilon = \varepsilon(L, \zeta) > 0$ and a constant $c = c(L, c_\mu, p, t, \zeta, \varepsilon, \beta)$ such that for all points $z \in L$ with $|z - \zeta| \le \varepsilon$ the inequalities*

$$\frac{1}{c}\,\Delta_n(z)^t \upsilon_{\frac{1}{n}}(z) \le \lambda_n(\mu, p, t, z) \le c\,\Delta_n(z)^t \upsilon_{\frac{1}{n}}(z)$$

hold.

In particular, if the curve or arc is piecewise Dini-smooth, then the previous corollaries are globally valid. We first give an appropriate form for $\Delta_n(z)$. Let $\zeta_1, \ldots, \zeta_n$ be the corners of L

with angles different from π where in the arc-case ζ_1, ζ_n continue denoting the endpoints of L. Let $\beta_1\pi,\dots,\beta_n\pi$ be the corresponding angles ($0<\beta_i<2$ in the curve-case; $\beta_i:=2$, if $i=1$ or n and $1<\beta_i<2$, if $1<i<n$ in the arc-case). With

$$\Delta_n(z) := \begin{cases} \frac{1}{n^{\beta_i}} & \text{if } z\in l_{\frac{1}{n}}(\zeta_i)\ \ (i=1,2,\dots,n) \\ \frac{\prod_{i=1}^n |z-\zeta_i|^{1-\frac{1}{\beta_i}}}{n} & \text{if } z\in L\setminus\left(\cup_{i=1}^n l_{\frac{1}{n}}(\zeta_i)\right) \end{cases}$$

we get the global variant of Theorem 4.12:

Theorem 4.15. *If L is a piecewise Dini-smooth curve or arc, then there exists a constant $c_8=c_8(L)$ such that*

$$\frac{1}{c_8}\Delta_n(z)\le \rho_{\frac{1}{n}}(z)\le c_8\Delta_n(z)$$

for all $z\in L$.

Corollary 4.16. *If L is a piecewise Dini-smooth curve or arc, then there exists a constant $c=c(L,c_\mu,p,t)$ such that*

$$\frac{1}{c}\Delta_n(z)^t\upsilon_{\frac{1}{n}}(z)\le\lambda_n(\mu,p,t,z)\le c\,\Delta_n(z)^t\upsilon_{\frac{1}{n}}(z)$$

holds for all $z\in L$.

Consider that ($\circ$) corresponds to the choice $L=[-1,1]$ and $t=0$.

4.2.3 Nikolskii-type inequalities

If $1\le p<q$, then, by Hölder's inequality, we can estimate the L^p-norm by the L^q-norm from above. In the opposite direction the so-called Nikolskii-type inequalities are used for polynomials. With the help of Theorem 4.4 we can create such inequalities for doubling measures supported on a quasismooth curve or arc. So these results partly overlap with [2, Theorem 6] in which Andrievskii proved an unweighted Nikolskii-type inequality over rectifiable curve in another way.

We introduce the following notations:

$$M_n := \sup_{\zeta\in L}\frac{1}{\upsilon_{\frac{1}{n}}(\zeta)},$$

and for a function f on L let

$$\|f\|_\infty := \sup_{\zeta\in L}|f(\zeta)|,\qquad \|f\|_{\mu,p} := \left(\int_L |f(\zeta)|^p\,\mathrm{d}\mu(\zeta)\right)^{\frac{1}{p}}.$$

With these we have the following Nikolskii-type inequalities.

Corollary 4.17. *Let L be a quasismooth curve or arc and μ a doubling measure on L. If $1 \le p < q$, then there is a constant $c = c(p, q)$ independent of n such that*

$$\|p_n\|_\infty \le c M_n^{\frac{1}{p}} \|p_n\|_{\mu,p},$$

as well as

$$\|p_n\|_{\mu,q} \le M_n^{\frac{1}{p}-\frac{1}{q}} \|p_n\|_{\mu,p} \tag{4.1}$$

for every polynomial p_n of degree at most n.

Remark. In the unweighted case it can be shown that the magnitude of M_n is n if L is a Dini-smooth curve, and n^2 if L is an arc . So our corollary includes the classical results for the unit circle as well as for $[-1, 1]$ (up to constants).

References

[1] V. V. Andrievskii, Weighted L_p Bernstein-type inequalities on a quasismooth curve in the complex plane, *Acta Math. Hungar.*, **135 (1–2)** (2011), 8–23.

[2] V. V. Andrievskii, Weighted Polynomial Inequalities in the Complex Plane, *J. Approx. Theory*, **164** (2012), 1165–1183.

[3] V. V. Andrievskii, Weighted Polynomial Inequalities with Doubling Weights on a Quasismooth Arc, *Constr. Approx.*, **36** (2011), 117–143.

[4] K. G. Ivanov; V. Totik, Fast decreasing polynomials, *Constr. Approx.* **6** (1990), no. 1, 1–20.

[5] Y. Last; B. Simon, Fine structure of the zeros of orthogonal polynomials IV. A priori bounds and clock behavior, *Comm. Pure Appl. Math.* **61** (2008), no. 4, 486–538.

[6] G. Mastroianni; V. Totik, Uniform spacing of zeros of orthogonal polynomials, *Constr. Approx.* **32** (2010), no. 2, 181–192.

[7] G. Mastroianni; V. Totik, Weighted polynomial inequalities with doubling and A_∞ weights, *Constr. Approx.* **16** (2000), no. 1, 37–71.

[8] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[9] V. Totik and T. Varga, Non-symmetric fast decreasing polynomials and applications, *J. Math. Anal. Appl.*, **394** (2012), 378-390.

[10] W. Van Assche, Orthogonal Polynomials in the Complex Plane and on the Real Line, *Fields Inst. Commun.Amer. Math. Soc* (1997), 211-245.

[11] T. Varga, Uniform spacing of zeros of orthogonal polynomials for locally doubling measures, *Analysis (Munich)*, **33** (2013), 1-12.

[12] T. Varga, Christoffel functions for doubling measures on quasismooth curves and arcs, *Acta Math. Hungar.*, **141** (2013), 161-184.