UNIVERSITY OF SZEGED

FACULTY OF SCIENCE AND INFORMATICS

DEPARTMENT OF THEORETICAL PHYSICS

DOCTORAL SCHOOL OF PHYSICS

Investigation of integrable many-particle systems by Hamiltonian reduction

Theses of Ph.D. dissertation

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2013

1 Introduction

The study of integrable, exactly solvable systems is an important branch of mathematical physics. Many examples of integrable systems arise in non-linear optics, particle physics and general relativity. One of the reasons for their importance is that these models may provide suitable starting points for the analysis of more complicated problems. Realistic models can be often investigated as perturbations of integrable systems. Integrable systems can be used also for testing the accuracy of numerical methods.

It is easier to illustrate integrability with examples than to define it precisely. The Korteweg–de Vries (KdV) equation is a well-known example for integrable systems in classical field theory [5, 14]. The KdV equation describes one-dimensional water waves in a shallow channel. Such wave was first observed by Scott Russell in 1834 [5]. The wave was localized and travelled with constant velocity, while maintaining its shape. The solutions with these properties are called solitons. In the KdV equation the non-linear term cancels the effect of the dispersive term, so the wave packet does not spread. Another remarkable integrable model with soliton solutions is the sine-Gordon equation, which first appeared in studies of pseudospherical surfaces. A mechanical model of the sine-Gordon equation can be constructed using a series of pendulums connected to an elastic rubber band. The many-soliton solutions possess the interesting property that the solitons can "pass through" each other, after the interaction their phases change but their shapes remain the same.

The two best-known examples of integrable classical mechanical systems are the harmonic oscillator and the Kepler problem. The Toda chain, the Calogero–Sutherland and Ruijsenaars–Schneider type systems are important examples of integrable many-particle systems [3, 9]. In several cases these systems can be derived by symmetry reduction. The Calogero–Sutherland type systems are finite dimensional dynamical systems that are integrable both at the classical and quantum mechanical level [3, 9, 15]. These models describe the pair interaction of arbitrary ($n \geq 2$) number of point particles moving on the line or on the circle. In the most important cases the pair potential is proportional to a particular rational, hyperbolic, trigonometric or elliptic function of the particle coordinate differences. The integrability of the n-particle rational Calogero system was first proved in the quantum mechanical framework by Calogero [2], later Moser proved the classical integrability of the system [7]. The Ruijsenaars-Schneider type integrable systems also describe the pair interaction of n point particles moving in one-dimensional space [3, 13]. In the most important cases their generalized pair potentials are again proportional to a rational, hyperbolic, trigonometric or elliptic function of the particle coordinate differences. The defining Hamiltonians of the Ruijsenaars–Schneider systems are one-parameter deformations of the corresponding Calogero–Sutherland Hamiltonians. The extra parameter is often interpreted as the speed of light. The "relativistic" Ruijsenaars–Schneider systems posses a translation and a boost generator, which together with the Hamiltonian generate the Poincaré algebra in $1 + 1$ dimension through the Poisson bracket. In the non-relativistic limit the Ruijsenaars–Schneider Hamiltonians become equal to the corresponding Calogero–Sutherland Hamiltonians. Some variants of the systems mentioned above are suitable for analysing solutions of soliton equations. The motion of the "Calogero particles" is equivalent to the time evolution of the poles and zeros of certain KdV solutions [4]. The Ruijsenaars–Schneider systems also have interesting applications to integrable partial differential equations. The n-particle hyperbolic Ruijsenaars–Schneider system describes the n-soliton solutions of the sine-Gordon equation [11, 13].

2 Research aims and methods

The principal aim of my work was the investigation of some interesting aspects of classical integrable many-particle systems. I focused on certain variants of the Calogero–Sutherland and Ruijsenaars–Schneider type integrable systems. I concentrated on three research topics, namely: the application of symmetry reduction, superintegrability and duality in concrete systems.

I relied on Hamiltonian reduction to investigate the many-particle systems. The relevant systems were all obtained by reduction of some higher dimensional "free systems". Starting from an integrable system with rich symmetries one may obtain integrable reduced systems under suitable conditions. Since the method of Hamiltonian reduction plays a central role in the thesis, I briefly comment on the relevant concept of symmetry. In addition to its mathematical beauty, symmetries are also very important theoretical physics. In many cases symmetries lead to conservation laws and combined with the reduction method are often useful for simplifying complicated problems. We here use the word "symmetry" in the context of Hamiltonian dynamical systems, i.e., assume that the Hamiltonian and the Poisson brackets of the system are invariant with respect to the action of some Lie group on the phase space. Then the system can be projected to a lower dimensional phase space by fixing the values of the constants of motion (that generate the symmetry) and implementing the Marsden–Weinstein reduction procedure [1]. In essence, one obtains a lower dimensional reduced system by factorization with the symmetry, which permits elimination of some degrees of freedom.

3 Studied topics

1. A Liouville integrable system is called superintegrable if it admits more time-independent constants of motions than the maximal number that can be Poisson commuting [16]. Probably the best-known superintegrable system is the Kepler problem. Another frequently cited example is the rational Calogero system [17]. I investigated the superintegrability of the rational Ruijsenaars–Schneider system, which can be viewed as a relativistic deformation of the rational Calogero system. The analysis is based on the derivation of the Ruijsenaars–Schneider system by Hamiltonian reduction [6]. I considered two ways for demonstrating its superintegrability, one relies on an explicit construction and the other on the so-called global action-angle map of maximally non-compact type.

2. There exist generalizations of the Sutherland system that describe "charged" particles, where the different charges attract and the identical charges repulse each other in a special manner. The first Sutherland type system with charged particles was introduced by Calogero

[3] by the trick of shifting $m < n$ coordinates in the usual n-particle Sutherland system by $(\mathrm{i}\,\frac{\pi}{2})$. In this model the interaction potential is attractive between the particles with different charges and it is proportional to the $\cosh^{-2}$ function of particle position differences, while the repulsive interaction potential between the particles with identical charges is proportional to the $\sinh^{-2}$ function. Later this system was derived by Olshanetsky and Rogov by reduction [8]. In the dissertation I derived a generalized Sutherland system with the aid of Hamiltonian reduction, which describes charged particles and contains three independent coupling constants.

3. In the impressive series of papers [10, 11, 12] Ruijsenaars investigated the dynamics and the duality relations of various Calogero type integrable many-particle systems. The phase spaces of dual pairs of integrable many-particle systems are related by a symplectomorphism that identifies the action variables of the "first" system as the particle positions of the "second" system, and vice versa. This symplectomorphism is also known as "duality transformation". The n-particle trigonometric Sutherland system admits three different physical interpretations depending on the choice of the domain of the position variables. It can be regarded as a system of n indistinguishable particles moving on the circle, or as systems of distinguishable particles moving either on the circle or one the line. The following configuration spaces correspond to these choices:

$$Q(n), \quad U(1) \times SQ(n), \quad \mathbb{R} \times SQ(n).$$

By using a direct method, Ruijsenaars constructed canonical transformations between the phase spaces of the three variants of the trigonometric Sutherland system and their duals, and also constructed covering maps between the dual pairs associated with the above alternative configuration spaces [12]. In my work I examined the group theoretic interpretation of the web of duality transformations and covering maps.

4 New results

Next I summarize my new results, which were mainly obtained by Hamiltonian reduction. The results are arranged in three paragraphs according to the topics.

1. I presented an explicit construction of the extra constants of motion of the rational Ruijsenaars–Schneider system that are responsible for its maximal superintegrability [A1]. The construction is based on the following Poisson bracket algebra

$$\{I_k, I_j\}_M = 0\,, \quad \{I_k^1, I_j^1\}_M = (j-k)I_{k+j}^1\,, \quad \{I_k^1, I_j\}_M = jI_{j+k},$$

which generalizes a similar algebra exhibited by Wojciechowski for the rational Calogero system [17]. I explained how can the above Poisson algebra be used to construct additional constants of motion for the Hamiltonians depending only on the Poisson commuting I_k functions. In [A1], I gave a new realization of this Poisson bracket algebra utilizing the derivation of the rational Ruijsenaars–Schneider system in the symplectic reduction framework based on the reduction of the $T^*GL(n,\mathbb{C})$ cotangent bundle [6]. The claimed Poisson bracket relations were proved by studying suitable invariant functions. Based on [A1,A2], I explained how does superintegrability follow from the existence of a global action-angle map of maximally non-compact type, and described the connection between duality and superintegrability in the case of those Calogero–Sutherland and Ruijsenaars–Schneider systems that possess purely scattering motions.

2. In [A3], I studied a generalized Sutherland system that describes "charged" particles and possesses three independent coupling constants. I obtained the system by reducing the free geodesic motion on the group $G = SU(n,n)$. Two commuting involutions were introduced on the group G having corresponding fixed-point groups G_+ and G^+. The reduction of the T^*G cotangent bundle was based on the symmetry group $G_+ \times G^+$, where G_+ is the maximal compact subgroup of G, and it was analyzed using a generalized Cartan decomposition of G. The Hamiltonian of the reduced system describes attractive-repulsive interactions of n charged particles moving on the half-line, which are influenced also by their mirror images and a positive charge fixed at the origin. The attractive interaction of the particles with different charges is defined by the $\cosh^{-2}$ function and the repulsive interaction between identical charges is governed by the $\sinh^{-2}$ function. I have shown that the Liouville integrability of the system is a direct consequence of the Hamiltonian reduction. By utilizing the geometric picture and the "free flows", I gave a linear-algebraic method for constructing the time evolution of the particle positions and their canonical conjugates.

3. In [A4], I investigated the dual pairs associated with three different physical interpretation of the trigonometric Sutherland system from a group theoretic viewpoint, and described the connection between the symplectic covering maps of Ruijsenaars [12] and the group theoretic covering homomorphisms

$$G_2 := \mathbb{R} \times SU(n) \longrightarrow G_1 := U(1) \times SU(n) \longrightarrow G := U(n)$$

I derived the dual pairs by symplectic reduction of the phase spaces T^*G, T^*G_1, T^*G_2 using the symmetry group $\bar{G} = G/\mathbb{Z}_G \simeq G_1/\mathbb{Z}_{G_1} \simeq G_2/\mathbb{Z}_{G_2}$ (where $\mathbb{Z}_G$ is the center of G). My main result is the group theoretic interpretation of the following commutative diagram:

$$\begin{tikzcd}
T^*\mathbb{R} \times T^*SQ(n) \arrow[r, "\mathrm{id}_2 \times \mathcal{R}_0"] \arrow[d, "\psi_2^{\mathrm{I}}"'] & T^*\mathbb{R} \times \mathbb{C}^{n-1} \arrow[d, "\psi_2^{\mathrm{II}}"] \\
T^*U(1) \times T^*SQ(n) \arrow[r, "\mathrm{id}_1 \times \mathcal{R}_0"] \arrow[d, "\psi_1^{\mathrm{I}}"'] & T^*U(1) \times \mathbb{C}^{n-1} \arrow[d, "\psi_1^{\mathrm{II}}"] \\
P = T^*Q(n) \arrow[r, "\mathcal{R}"] & \hat{P}_c = \mathbb{C}^{n-1} \times \mathbb{C}^\times
\end{tikzcd}$$

On the left side the alternative Sutherland phase spaces can be seen, on the right the corresponding Ruijsenaars–Schneider ones. The vertical lines denote the symplectic covering maps and the horizontal lines denote duality symplectomorphisms. The dual pairs in the three horizontal lines of the diagram involve the three physically different versions of the trigonometric Sutherland model. The first line corresponds to distinguishable particles moving on the line, the second to distinguishable particles moving on the circle, and the third to indistinguishable particles moving on the circle. The diagram was first constructed by Ruijsenaars [12] with the aid of direct methods. I described the group theoretic–geometric interpretation of this web of dualities and coverings, which permitted to significantly simplify the proof of the Poisson bracket preserving property of the duality transformations [A4].

5 Publications

My results reported in the thesis are based on the following publications:

[A1] V. Ayadi, L. Fehér, *On the superintegrability of the rational Ruijsenaars-Schneider model,* Phys. Lett. A **374**, 1913 (2010)

[A2] V. Ayadi, L. Fehér, T.F. Görbe, *Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals,* J. Geom. Symmetry Phys. **27**, 27 (2012)

[A3] V. Ayadi, L. Fehér, *An integrable BC(n) Sutherland model with two types of particles*, J. Math. Phys. **52**, 103506 (2011)

[A4] L. Fehér, V. Ayadi, *Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction,* J. Math. Phys. **51**, 103511 (2010)

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[16] P. Tempesta, P. Winternitz *et al* (Editors), *Superintegrability in Classical and Quantum Systems, CRM Proceedings and Lecture Notes*, **37**, Amer. Math. Soc., Providence, 2004

[17] S. Wojciechowski, *Superintegrability of the Calogero–Moser system*, Phys. Lett. A **95**, 279 (1983)