Properties of related lattices

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Abstract

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1. Introduction

The connection between lattice theory and several parts of mathematics is based on the fact that mathematical structures are accompanied by related structures, which are lattices in several cases. The elements of related lattices are certain objects connected with the escorted structures, mostly subsets or relations. In our case the related lattices will consist of sublattices, congruences, submodules, quasiorders and subspaces.

2. Sublattice lattices

For a lattice L, let $\mathrm{Sub}(L)$ denote the sublattice lattice of L. The first results concerning the isomorphism of sublattice lattices are due to Filippov [Fil66]. He provided necessary and sufficient conditions for $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$, however, these conditions are rather complicated. Grätzer [Gra78] posed several problems concerning sublattice lattices. In this chapter we consider the following three problems:

I. Find conditions under which $\mathrm{Sub}(L)$ determines L, that is, $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ implies $L \cong K$ or $L \cong K^d$? ([Gra78], Problem I.4.)

II. Which lattice properties Φ are preserved under the isomorphism of sublattice lattices, that is, $\Phi(L)$ and $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ imply $\Phi(K)$? ([Gra78], Problem I.8.)

III. Which lattice varieties $\mathcal{V}$ are closed under the isomorphism of sublattice lattices, that is, $L \in \mathcal{V}$ and $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ imply $K \in \mathcal{V}$? ([Gra78], Problem I.5.)

The third problem is clearly a special case of the second one, namely when $\Phi(L)$ means $L \in \mathcal{V}$.

Let H be a sublattice of L. H is said to be nontrivial if $H \neq L$ and $|H| > 1$. H is called *prime* if $L \setminus H$ is also a sublattice and *homogeneous* if any element of H is either comparable or incomparable with every element of $L \setminus H$. The following theorem is a partial answer to the first problem in the language of homogeneous prime sublattices.

THEOREM 2.1. ([Fil66]) *If the lattice L does not contain any nontrivial homogeneous prime sublattices then* $\mathrm{Sub}(L)$ *determines L.*

Making use of this theorem we proved the following partial answers to the first problem.

THEOREM 2.2. ([Tak97], [Tak98], [Taka], [Fil66]) *If the lattice L fulfills one of the following properties then L is determined by its sublattice lattice:*

(1) L is an ordinal sum indecomposable modular lattice;

(2) L is an ordinal sum indecomposable, weakly complemented lattice;

(3) L is relatively complemented;

(4) L is an ordinal sum indecomposable, semimodular, strongly atomic lattice;

(5) L is simple;

(6) L is directly reducible;

(7) L is a selfdual subdirectly irreducible lattice of locally finite length.

Note that (1) and (3) are due to Filippov. On the other hand, in the proof of (7) we used the characterization of the isomorphism of sublattice lattices rather than Theorem 2.1.

Concerning the second problem we proved the following "dependence" result.

THEOREM 2.3. ([Takb]) *If L is a lattice of finite length then* $\mathrm{Sub}(L)$ *determines* $\mathrm{Con}(L)$, *that is,* $\mathrm{Sub}(L) \cong \mathrm{Sub}(K)$ *implies* $\mathrm{Con}(L) \cong \mathrm{Con}(K)$.

It contrasts with the fact that by Baranskiĭ [Bar79] and Urquhart [Urq78], the congruence lattice and the automorphism group of a finite lattice are independent. A simple example shows that the sublattice lattice does not determines the automorphism group in general.

The third problem was studied more generally: we considered quasivarieties, i.e. lattice classes defined by Horn sentences. By a Horn sentence we mean a universally quantified lattice sentence

$$\psi : \qquad \bigwedge_{i=1}^{n} p_i = q_i \implies p = q,$$

where p_i, q_i, p, q are lattice terms. It is clear that no non-selfdual lattice quasivariety is closed under the isomorphism of sublattice lattices. Using Theorem 2.2. (1) we proved the following.

THEOREM 2.4. ([Tak98]) *A modular lattice quasivariety is closed under the isomorphism of sublattice lattices iff it is selfdual.*

For non-modular quasivarieties this is no longer true: we constructed a non-closed selfdual variety. On the other hand, as a consequence of Theorem 2.2. (7) we have

COROLLARY 2.5. *Selfdual splitting varieties are closed under the isomorphism of sublattice lattices.*

3. Duality of submodule lattices

In this chapter an elementary proof is given for Hutchinson's duality theorem. Given a ring R with unit element $1 = 1_R$, the class of left modules over R is denoted by R–**Mod**. Let $T(R)$ denote the set of all lattice identities that hold in the submodule lattices of all R-modules, i.e., in the class of $\{\mathrm{Sub}(M) : M \in R\text{–}\mathbf{Mod}\}$. Using the heavy machinery of abelian category theory and Theorem 4 from [HC78], Hutchinson [Hut73,HC78] has proved the following duality result.

THEOREM 3.1. (Hutchinson [Hut73,HC78]) *For every ring R, $T(R)$ is a selfdual set of lattice identities. In other words, a lattice identity λ holds in $\{\mathrm{Sub}(M) : M \in R\text{–}\mathbf{Mod}\}$ iff so does the dual of λ.*

The goal of the present chapter is to give an easy new proof of this theorem, based on [CT]. Our elementary approach does not resort to category theory and uses much less from [HC78] than the original one.

4. Generation of quasiorder lattices

Given a set A, let Quord(A) denote the (complete) lattice of all quasiorders (i.e., reflexive and transitive relations) on A. Similarly, the lattice of equivalences on A will be denoted by Equ(A). By a (complete) involution lattice we mean a (complete) lattice equipped with an extra unary operation * such that * is an involutory automorphism of the lattice reduct. Typical example is Quord(A) where $\alpha^* = \{(x,y) \in A^2 : (y,x) \in \alpha\}$ for $\alpha \in$ Quord(A). In this chapter Quord(A) is always considered a complete involution lattice.

By Strietz [Str75] and Zádori [Zad86], the (complete) lattice Equ(A), $4 < |A| < \infty$ has a four element generating set, but cannot be generated by three elements. Modifying Zádori's method, Chajda and Czédli [CC96] have shown that Quord(A) has a three element generating set for all finite and some infinite A. This result gave rise to studying infinite equivalence lattices: Czédli [Cze96] proved that Equ(A) is four-generated if there is no inaccessible cardinal m such that $m \leq |A|$.

Combining the methods of the above papers, we proved the following theorem.

THEOREM 4.1. ([Tak96a]) *Let A be a set with at least three elements and assume that there is no inaccessible cardinal* $\leq |A|$. *Then the complete involution lattice Quord(A) has a three element generating set. In fact, it can be generated by three partial orders.*

By Kuratowski [Kur25] (cf. also Levy [Lev79]), ZFC has a model without inaccessible cardinals, therefore this theorem holds for all sets in an appropriate model of set theory.

By Chajda and Czédli [CC96], it is sufficient to prove the result only for infinite sets. The proof is an induction on $|A|$, and a so called box structure was built on A. In order to make the proofs shorter, we defined the notion of semibox in [Tak96a], which was utilized later in [Cze].

5. Lattice identities for projective geometries

In this chapter we considered projective geometrical configurational conditions (Schliessungssätze) and translated them to the language of lattice theory. That is, given a configurational condition T, we present a lattice identity λ_T such that a projective geometry satisfies T iff λ_T holds in its subspace lattice. In some cases we should restrict the investigations to projective planes.

The most known lattice identity motivated by projective geometry is Jónsson's [Jon53] Arguesian identity which corresponds to Desargues' Theorem. The Arguesian identity plays a prominent role in the investigation of modular lattices. It was used first to present a modular lattice having no representation with permuting equivalences. Another lattice identities motivated by projective geometry are the Pappian identity (Day [Day81]), the Fano identity and the Moufang identity (Wille [Wil73]), and an identity in Pálfy and Szabó [PS95] that was created to distinguish the congruence variety of Abelian groups and that of all groups.

We took two more configurational conditions; the first one claims that the fourth point of a harmonic set is determined by the other three. That is why we call the geometries satisfying the following "theorem" (i.e., condition) harmonic geometries.

THE THEOREM OF COMPLETE QUADRANGLES. For any line l and points a, b, c, p, q, p', q' such that $a, b, c \leq l$, $a \neq c$, $p, q, p', q' \not\leq l$, $p \neq q$, $p' \neq q'$ and $b \leq p+q, p'+q'$, we have $(r+s)(r'+s') \leq l$, where $s = (a+p)(c+q), r = (a+q)(c+p), s' = (a+p')(c+q'), r' = (a+q')(c+p')$.

We translated this property into a lattice identity as follows.

THEOREM 5.1. ([Tak96b]) *A projective geometry is harmonic iff the identity*

$$(a+c)(p+q)(p'+q') \leq a + (r+s)(r'+s') \tag{1}$$

holds in its subspace lattice, where $s = (a+p)(c+q), r = (a+q)(c+p), s' = (a+p')(c+q'), r' = (a+q')(c+p')$.

The other configurational condition to be latticized is the perspective case of Pappus' theorem.

PERSPECTIVE PAPPUS' THEOREM (Kerékjártó [Ker63]). Let a and b two distinct lines, and let a_0, a_1, a_2, resp. b_0, b_1, b_2 points on the line a resp. b such that $|\{ab, a_0, a_1, a_2, b_0, b_1, b_2\}| = 7$. If the lines a_0+b_0, a_1+b_1 and a_2+b_2 are concurrent, then the points $(a_0+b_1)(a_1+b_0)$, $(a_0+b_2)(a_2+b_0)$ and $(a_1+b_2)(a_2+b_1)$ are collinear.

To eliminate the stipulation "four points in general position" in the perspective Pappian identity, we need the dual of Day's [Day81] line pair configuration.

DEFINITION 5.2. ([Tak96b]) *Let L be a modular lattice. A quadruple* $(a_0, a_1, b_0, b_1) \in L^4$ *is called a dual line pair if*

$$a_0 + b_0b_1 = a_1 + b_0b_1 = b_0 + a_0a_1 = b_1 + a_0a_1.$$

THEOREM 5.3. ([Tak96b]) *A projective plane is perspective Pappian iff the following "identity" holds in its sublattice lattice: $LP^d(x, y, a, c)$ and $ac \leq b$ imply $B \leq J$, where*

$$\begin{aligned} B &= (ax+cy)(ay+cx)(bx+ay+cy)(by+ax+cx), \\ J &= (ax+by)(ay+bx)+(cx+by)(cy+bx). \end{aligned}$$

Due to the projectivity of the dual line pair configuration, the above Horn sentence is equivalent to a real identity.

An advantage of a lattice identity over the original geometric notion is that while the latter involves several stipulations (like non-collinear, x is a line, of general position, etc.), the former does not. When a geometric property (e.g. Pappian) implies another (e.g. arguesian) then the proof of this fact has to handle many ramifying cases, generally. But passing from geometric properties to corresponding lattice identities one can expect an easier lattice theoretic proof.

As an illustration, we proved that the Fano identity implies the harmonic identity for modular lattices, and the Arguesian identity implies the harmonic identity for complemented modular lattices. Day's [Day80] question that whether the frame-Pappian identity implies the Arguesian identity is answered in the negative.

6. Summary of methods and applicability of results

We used standard algebraic methods in the investigations, generally. The identities of Chapter 5 was tested with the help of computer programs developed by Czédli[1]. The modification of the algorithm of Czédli [Cze91] facilitated the proof of geometrical implications.

The notion of semibox in Chapter 4 and the lemmas relating semibox extensions was already utilized by Czédli [Cze]. The method that was used in the proof of Theorem 2.2. seems general enough to prove another answers to the first problem in Chapter 1.

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[1]The programs can be found at the web site of Czédli, namely at `http://www.math.u-szeged.hu/tagok/czedli`.

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