Theses of Ph.D. Dissertation

Modular and semimodular lattices

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2013

Szeged

Introduction

Modularity. The concept of modular lattices is as old as that of lattices itself, and both are due to Dedekind. A lattice is said to be *modular* if it satisfies the following identity:

$$x \vee \big(y \wedge (x \vee z)\big) = (x \vee y) \wedge (x \vee z).$$

Dedekind showed around 1900 that the submodules of a module form a modular lattice with respect to set inclusion. Many other algebraic structures are closely related to modular lattices: both normal subgroups of groups and ideals of rings form modular lattices; distributive lattices (thus also Boolean algebras) are special modular lattices. Later, it turned out that, in addition to algebra, modular lattices appear in other areas of mathematics as well, such as geometry and combinatorics.

Chapter 1 of the dissertation deals with von Neumann normalized frames, *frames* shortly, wich are due to von Neumann [41]. Although he worked in lattice theory just for two years between 1935 and 1937, many lattice theorists, including Grätzer [27, p. 292], say that his results belong to the deepest part of lattice theory. For instance, Birkhoff, the founder and pioneer of universal algebra and lattice theory, wrote in a paper [5] about him: "John von Neumann's brilliant mind blazed over lattice theory like a meteor." Also, it was Birkhoff who turned von Neumann's attention to lattice theory. Then von Neumann began to think that he could probably use lattice theory as a tool. At that time he was trying to find an appropriate concept of space for modern physics. In contrast to the usual concept of dimension, where the dimension function has a discrete range $(0, 1, 2, \dots)$, he was looking for a dimension function with a continuous range. In full extent, his work was published much later, see [41]. It is centered around the concept of *continuous geometries*, which are special complemented modular lattices.

On his way to continuous geometries, von Neumann introduced the concept of frames, and he used them to extend the classical Veblen-Young coordinatization theorem of projective spaces [49, 50] to arbitrary complemented modular lattices with frames. Note that the best known method for the classical coordinatization is "von Staudt's algebra of throws", cf. Grätzer [28, p. 384]. As a first step, von Neumann associated a ring, the so-called *coordinate ring*, with each frame. It turned out that in case of a complemented modular lattice with a frame, the coordinate ring satisfies some additional property, which is nowadays called *von Neumann regularity*.

It is worth mentioning that this property has proved to be particularly useful. The theory of von Neumann regular rings has become an independent discipline later.

Since there is a one to one correspondence between modular geometric lattices and projective spaces, many properties, including Desargues' theorem, can be formulated in the language of lattices, see, e.g., Grätzer [28, Section V.5]. Von Neumann's work exemplifies that lattice theory can be helpful to handle geometric problems in a more elegant and compact way. Analogous applications of lattice theory were given later by others. For example, Jónsson [36] provided lattice identities that hold in a modular geometric lattice if and only if Desargues' theorem holds in the associated projective space. Note that there exists a similar characterization of Pappus' theorem, see Day [20].

Although von Neumann considered a *complemented* modular lattice L of length $n \geq 4$, his construction of the coordinate ring (without coordinatization) extends to arbitrary modular lattices without complementation, see Artmann [1] and Freese [23], and even to $n = 3$ if L is Arguesian, see Day and Pickering [21].

A concept equivalent to frames is that of *Huhn diamonds*, see Huhn [32]. Since distributive lattices played a central role already in the beginning of lattice theory, cf. Grätzer [28, p. xix], Huhn's original purpose was to generalize the distributive law. He also wanted to find generalizations for many well known theorems and applications of distributive lattices. His new identity, called *n-distributivity*, proved to be a particularly fruitful generalization of distributivity. While distributive lattices among modular lattices are characterized by excluding M_3's (Birkhoff's criteria), in case of n-distributive lattices, M_3's are replaced by Huhn diamonds. Huhn diamonds are connected to many interesting theorems, for instance, Huhn proved with them that the automorphism group of a finitely presented modular lattice can be infinite, see [33].

Frames and Huhn diamonds are used in the proof of several deep results showing how complicated modular lattices are, only to mention Freese [23], Huhn [33], and Hutchinson [34]. Frames or Huhn diamonds were also used in the theory of congruence varieties, see Hutchinson and Czédli [35], Czédli [13], and Freese, Herrmann and Huhn [24]; and in commutator theory, see Freese and McKenzie [25, Chapter XIII].

Dealing with quasi-fractal generated non-distributive modular lattice varieties, Czédli [14] introduced the concept of *product frames*. Chapter 1 of the dissertation is based on a joint work with Czédli [18]. We show that product frames are closely related to matrices. Namely, the coordinate ring of the so-called *outer frame* of a product frame is a matrix ring over the coordinate ring of the so-called *inner frame* of the product frame, see Theorem 1.

Semimodularity. One of the most fruitful generalization of modularity is the so-called semimodularity. A lattice is said to be (upper) *semimodular* if it satisfies the following Horn formula

$$x \prec y \Rightarrow x \vee z \preceq y \vee z.$$

In contrast to modular lattices, the class of semimodular lattices cannot be characterized by identities. In the preface of his book titled Semimodular Lattices, Manfred Stern [47] attributes the abstract concept of semimodularity to Birkhoff [6]. He also mentions that classically semimodular lattices came from closure operators that satisfies the nowadays usually called Steinitz-Mac Lane Exchange Property, cf. [47, page ix, 2 and 40]. One of the most important class of semimodular lattices that was systematically studied at first is the class of *geometric lattices*, which are semimodular, atomistic algebraic lattices, cf. Birkhoff [6, Chapter IV] and Crawley and Dilworth [9, Chapter 14]. Since one can think of finite geometric lattices as (simple) *matroids*, it is not surprising that the theory of semimodular lattices has been developing simultaneously with matroid theory since the beginning, cf. the Preface of Stern [47].

Chapter 2 of the dissertation deals with lattice embeddings into geometric lattices, which also have nice consequences for semimodular lattices. Lattice embeddings have been heavily studied since the beginning of lattice theory. The first important result was published by Birkhoff [4] in 1935. He proved that *every partition lattice is embeddable into the lattice of subgroups of some group.* Later, in 1946, Whitman [51] showed that *every lattice is embeddable into a partition lattice.* These two results together imply that *every lattice is embeddable into the lattice of subgroups of some group.* These embeddings have considerable consequences; for example, there is no nontrivial lattice identity that holds in all partition lattices or in all subgroup lattices.

Perhaps the best-known proof for Whitman's theorem is due to Jónsson [36]. However, both in Whitman's and Jónsson's proofs, the constructed partition lattices are much bigger than the original ones, for instance, they are infinite even for finite lattices. The question whether a finite lattice is embeddable into a finite partition lattice arose already in Whitman [51]. He conjectured that this question had a positive answer.

Partition lattices belong to the class of geometric lattices. A *finite geometric* lattice is an atomistic semimodular lattice. The first step towards

Whitman's conjecture was a result of Finkbeiner [22]. He proved that *every finite lattice can be embedded into a finite semimodular lattice.* His construction is based on two steps. On the one hand, he showed that every finite lattice that has a so-called *pseudo rank function* can be embedded into a finite semimodular lattice. On the other hand, he pointed out that every finite lattice has a pseudo rank function. His embedding "preserves" the pseudorank function; that is, if L is embedded into S, say $L \leq S$, and p denotes the pseudorank function of L, and h denotes the height function of S then p and h coincide on L. Note that Finkbeiner credits his proof as an unpublished result of Dilworth. The second step towards Whitman's conjecture was a result of Dilworth, which was published later in Crawley and Dilworth [9]. He showed that *every finite lattice can be embedded into a finite geometric lattice.* The last step was made by Pudlák and Tůma [43, 44], who showed in 1977 that Whitman's conjecture is true.

Although Finkbeiner did not manage to prove Whitman's conjecture, his proof drew attention to embeddings that preserve pseudo rank functions. Such embeddings are called *isometrical.* In 1986, blending the results of Finkbeiner and Dilworth, Grätzer and Kiss [29] showed that *every finite lattice with a pseudorank function has an isometrical embedding into a finite geometric lattice.* The question whether a finite lattice with a pseudorank function has an isometrical embedding into a partition lattice is still open. Grätzer and Kiss' theorem has a straightforward corollary for semimodular lattices. Given a finite semimodular lattice, its height function is a pseudorank function, and an isometrical embedding (with respect to the height function) is an embedding that preserves the height of each element. It is equivalent to the condition that the embedding preserves the covering relation. Such embeddings are called *cover-preserving.* Now, Grätzer and Kiss' theorem implies that *every finite semimodular lattice has a cover-preserving embedding into a finite geometric lattice.* Note that this corollary together with Finkbeiner's result imply Grätzer and Kiss' theorem.

Finkbeiner, Grätzer and Kiss focused on finite lattices. The question arises naturally whether their results can be generalized for infinite lattices. Czédli and Schmidt [17] proved that the corollary of Grätzer and Kiss' theorem can be extended for semimodular lattices of finite length. Chapter 2 of the dissertation is based on [46]. We show that Grätzer and Kiss' theorem can also be extended for lattices of finite length, moreover, it can be extended for a larger class of lattices that we called *finite height generated lattices*, see Theorem 2.

Chapter 3 of the dissertation deals with Mal'cev conditions. The classic theorem of Mal'cev [39] states that the congruences of any algebra of a variety $\mathcal{V}$ permute if and only if there is a ternary term p such that $\mathcal{V}$ satisfies the following identities:

$$p(x, y, y) = x \text{ and } p(x, x, y) = y.$$

Jónsson [37] and Day [19] proved similar results for distributivity and modularity. These results led to the concept of Mal'cev(-type) conditions, see Grätzer [26]. Using Grätzer's concept, Jónsson's resp. Day's result says that the class of congruence distributive resp. congruence modular varieties can be defined by a Mal'cev condition. Later, beside the concept of Mal'cev condition, two similar concepts appeared, the strong and weak Mal'cev conditions, cf. Taylor [48].

After Mal'cev's, Jónsson's and Day's results, many classes of varieties have proved to be definable by (strong/weak) Mal'cev conditions. Both permutability and distributivity have some generalizations, the so-called n-permutability and n-distributivity. Hagemann and Mitschke [31] characterized n-permutability ($n \geq 2$) by a strong Mal'cev condition. On the other hand, n-distributivity, which was introduced by Huhn [32], turned out to be equivalent with distributivity in congruence varieties, cf. Nation [40]. Thus Jónsson's result [37] also characterizes congruence n-distributivity by a Mal'cev condition. Let us mention here that distributivity and n-distributivity are not equivalent in general. Distributivity implies n-distributivity, but, e.g., M_3 is an n-distributive lattice that is not distributive (if $n > 2$).

As for congruence modularity, Gumm [30] improved Day's result and found a Mal'cev condition for congruence modularity that contains ternary terms, see also Lakser, Taylor and Tschantz [38]. Then Czédli and Horváth [15] proved that every lattice identity that implies modularity in congruence varieties can be characterized by a Mal'cev condition. Their proof is heavily based on one of their former paper with Radeleczki [16]. Note that it is still an open problem whether all congruence lattice identities can be characterized by a Mal'cev condition. On the other hand, Wille [52] and Pixley [42] showed that every congruence lattice identity can be characterized by a weak Mal'cev condition.

In connection with Mal'cev conditions, we consider important to mention that Csákány was the first person from Szeged, who dealt with Mal'cev condition, for example, one of his results is the characterization of regular varieties by a Mal'cev condition [10]. He also wrote his thesis for the doctor of science degree about Mal'cev conditions and their applications [11].

Nowadays, Mal'cev conditions, especially Jónsson's, Day's and Gumm's terms, are frequently used in universal algebra and related areas such as CSP, cf., e.g., Barto and Kozik [2, 3].

Observe that, in case of groups, rings and modules, congruences are determined by normal subgroups, ideals and submodules. Although one congruence class does not usually determine the whole congruence, these examples show that given an algebra with a constant operation symbol c, the congruence class that contains c can play a special role. To recall a related concept from Chajda [7], let $\lambda : p(x_1, \dots, x_n) \le q(x_1, \dots, x_n)$ be a lattice identity, and let $\mathcal{V}$ be a variety with a constant operation symbol 0 in its type. We say that λ holds for the congruences of $\mathcal{V}$ at 0 if for every $\mathbf{A} \in \mathcal{V}$ and for all congruences $\alpha_1, \dots, \alpha_n$ of $\mathbf{A}$, we have $[0]p(\alpha_1, \dots, \alpha_n) \subseteq [0]q(\alpha_1, \dots, \alpha_n)$. In particular, if λ is $\alpha_1 \wedge (\alpha_2 \vee \alpha_3) \le (\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3)$ resp. $(\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_3) \le \alpha_1 \vee (\alpha_2 \wedge (\alpha_1 \vee \alpha_3))$, then we say that $\mathcal{V}$ is *congruence distributive* resp. *congruence modular at* 0.

This concept is not as trivial as it may seem. For example, while the variety $\mathcal{S}$ of meet semilattices with 0 is congruence distributive at 0, the dual of the distributive law does not hold for congruences of $\mathcal{S}$ at 0.

Returning to Mal'cev conditions, Chajda [7] has given a Mal'cev condition characterizing congruence distributivity at 0, and Czédli [12] has pointed out that the satisfaction of λ for congruences at 0 can always be characterized by a *weak* Mal'cev condition. (This is particularly useful when each congruence α is determined by $[0]\alpha$, see the comment following Prop. 2 in Czédli [12].) Later, Chajda and Halaš [8] took some steps towards characterizing congruence modularity at 0. Then we gave a Mal'cev condition in [45] that characterizes congruence modularity at 0, see Theorem 4. Note that Jónsson's and Day's characterization of congruence distributivity and congruence modularity follows from the characterization of congruence distributivity and congruence modularity at 0.

Major results

We recall the major results of the dissertation chapter by chapter.

Von Neumann frames

For definition, let $2 \le m$, let L be a nontrivial modular lattice with 0 and 1, and let $\vec{a} = (a_1, \dots, a_m) \in L^m$ and $\vec{c} = (c_{12}, \dots, c_{1m}) \in L^{m-1}$. We say that $(\vec{a}, \vec{c}) = (a_1, \dots, a_m,\ c_{12}, \dots, c_{1m})$ is a *spanning m-frame* (or a frame of *order m*) of L, if $a_1 \neq a_2$ and the following equations hold for all $j \le m$ and $2 \le k \le m$:

$$\sum_{i \le m} a_i = 1, \qquad a_j \sum_{i \le m,\, i \neq j} a_i = 0,$$

$$a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, \qquad a_1 c_{1k} = a_k c_{1k} = 0.$$

Let us mention here that in coordinatization theory, the lattice operations join and meet are traditionally denoted by + and $\cdot$ (mostly juxtaposition) such that meets take precedence over joins.

To understand the concept of von Neumann frames better, let us consider the following example. Let K be a ring with 1. Let v_i denote the vector $(0, \dots, 0, 1, 0, \dots, 0) \in K^m$ (1 at the ith position). Letting $a_i = K v_i$ and $c_{1j} = K(v_1 - v_j)$, we obtain a spanning m-frame of the submodule lattice $\mathrm{Sub}(K^m)$, where K^m is, say, a left module over K in the usual way. This frame is called the *canonical m-frame* of $\mathrm{Sub}(K^m)$.

We also need the concept of a coordinate ring. If $m \ge 4$ and $(\vec{a}, \vec{c}) = (a_1, \dots, a_m,\ c_{12}, \dots, c_{1m})$ is a spanning m-frame of L then one can define addition and multiplication on the set $R\langle 1, 2\rangle = \{x \in L : x + a_2 = a_1 + a_2,\ x a_2 = 0\}$ such that $R\langle 1, 2\rangle$ forms a ring with a unit. This ring is called the *coordinate ring* of $(\vec{a}, \vec{c})$. Note that the ring construction also works if $m = 3$ and L is Arguesian.

Now, we are in position to formulate the main result of the first chapter.

Theorem 1 ([18, Theorem 1.1]).

(a) *Let L be a lattice with $0, 1 \in L$, and let $m, n \in \mathbb{N}$ with $n \ge 2$. Assume that*

$$L \text{ is modular and } m \ge 4. \tag{a1}$$

Let $(\vec{a}, \vec{c}) = (a_1, \dots, a_m, c_{12}, \dots, c_{1m})$ be a spanning von Neumann m-frame of L and $(\vec{u}, \vec{v}) = (u_1, \dots, u_n, v_{12}, \dots, v_{1n})$ be a spanning von Neumann n-frame of the interval $[0, a_1]$. Let R^* denote the coordinate *ring of $(\vec{a}, \vec{c})$. Then there is a ring S^* such that R^* is isomorphic to the ring of all $n \times n$ matrices over S^*. If*

$$n \geq 4, \tag{a2}$$

then we can choose S^* as the coordinate ring of $(\vec{u}, \vec{v})$.

(b) *The previous part of the theorem remains valid if* (a1) *and* (a2) *are replaced by*

$$L \text{ is Arguesian and } m \geq 3 \tag{b1}$$

and

$$n \geq 3, \tag{b2}$$

respectively.

We could formulate the theorem without recalling the concepts of a product frame and the corresponding outer and inner frames. However, it is worth mentioning here that S^* is the coordinate ring associated to the product frame that occurs in the proof of the theorem. While $(\vec{a}, \vec{c})$ and $(\vec{u}, \vec{v})$ are the corresponding outer and inner frames, respectively.

Isometrical embeddings

Given a lattice L with a lower bound 0, a function $p\colon L \to \mathbb{N}_\infty = \{0, 1, \ldots, \infty\}$ is called a *pseudorank function* if it has the following properties:

(i) $p(0) = 0$;
(ii) $a \leq b$ implies $p(a) \leq p(b)$ for all $a, b \in L$;
(iii) $a < b$ implies $p(a) < p(b)$ for all $a, b \in L$ of finite height;
(iv) $p(a \wedge b) + p(a \vee b) \leq p(a) + p(b)$ for all $a, b \in L$;
(v) $p(a) < \infty$ iff a is of finite height.

In case of finite lattices, this definition coincide that of Finkbeiner [22] and Stern [47]. It is an easy consequence of the Jordan-Hölder Chain Condition that the height function of any semimodular lattice is a pseudorank function.

Consider a lattice L with a lower bound 0, a pseudorank function $p\colon L \to \mathbb{N}_\infty$ and a geometric lattice G whose height function is denoted by h. Then L is embeddable *isometrically* into G iff there is a lattice embedding $\varphi\colon L \to G$ such that $p = h \circ \varphi$, cf. Grätzer and Kiss [29].

We need one more concept in order to formulate the main result of this chapter, which generalizes a result of Grätzer and Kiss [29]. A lattice is said to be *finite height generated* iff it is complete and every element is the join of some elements of finite height. Note that lattices of finite length are finite height generated. To show a finite height generated lattice that is not of finite length, consider, for instance, $\mathbb{N}_\infty$ with the usual ordering.

Theorem 2 ([46, Theorem 1]). *Every finite height generated algebraic lattice with a pseudorank function can be embedded isometrically into a geometric lattice.*

This theorem has a straidforward corollary for semimodular lattices. A lattice embedding is said to be *cover-preserving* iff it preserves the covering relation.

Corollary 3 ([46, Corollary 2]). *Every finite height generated semimodular algebraic lattice has a cover-preserving embedding into a geometric lattice.*

Mal'cev conditions

Let $\mathcal{V}$ be a variety that has a constant operation symbol 0 in its type. We say that $\mathcal{V}$ is congruence modular at 0 iff for every algebra $\mathbf{A} \in \mathcal{V}$ and for all congruences α, β and γ of $\mathbf{A}$, we have $[0]\alpha\vee(\beta\wedge(\alpha\vee\gamma)) = [0](\alpha\vee\beta)\wedge(\alpha\vee\gamma)$, cf. Chajda [7] and Chajda and Halaš [8]. Notice that congruence modularity implies congruence modularity at 0, for instance, any group or ring variety is congruence modular at 0, since it is congruence modular. However, the converse is not true.

The main result of the third chapter characterizes congruence modularity at 0 by a Mal'cev condition. A similar result for congruence modularity was published by Day [19]. Note that our proof is heavily based on that of Day.

Theorem 4 ([45, Theorem 1]). *For a variety $\mathcal{V}$ of algebras with a constant* 0*, the following conditions are equivalent:*

(i) $\operatorname{Con}\mathbf{A}$ *is modular at* 0 *for all* $\mathbf{A} \in \mathcal{V}$*;*

(ii) *there is a natural number n and there are ternary terms m_i ($i = 0, \dots, n$) such that $\mathcal{V}$ satisfies the following identities:*

$$m_0(x,y,z) = 0 \text{ and } m_n(x,y,z) = z; \tag{m1}$$

$$m_i(x,x,0) = 0 \qquad \text{for all } i; \tag{m2}$$

$$m_i(x,x,z) = m_{i+1}(x,x,z) \qquad \text{for } i \text{ odd}; \tag{m3}$$

$$m_i(0,z,z) = m_{i+1}(0,z,z) \qquad \text{for } i \text{ even}. \tag{m4}$$

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