Dynamical r-matrices and their appearance in Calogero–Moser models

PhD Thesis

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2003

Contents

Introduction

The theory of integrable systems is a very rapidly developing branch of modern mathematical physics. From a physical point of view the importance of this subject is quite clear. A great amount of knowledge about nature is based on the use of special, exactly solvable models. The harmonic oscillator and the Kepler problem play a central role in classical mechanics as well as in quantum theory. The KdV and KP equations help us to understand the wave motion of shallow water and indicate the behaviour of 'solitons' in general. The nonlinear Schrödinger equation found applications in the theory of optical fibres. The sine-Gordon equation serves as a theoretical laboratory for particle physicists. These selected examples are enough to convince us about the importance of integrable systems, even in their own right. Furthermore, it is a well known fact that beside numerical simulations it is perturbation theory that allows us to get an insight into the details of physical phenomena. It is worth keeping in mind that all perturbative calculations rely heavily on exactly solved problems. From a mathematical point of view it is much simpler and much more obvious to give grounds for the investigations of these models. First, they are challenging mathematical problems. Second, almost all branches of mathematics can be used during calculations. Not only the classical parts of mathematics are applicable, but the newest methods play a role, too. In conclusion, we can state that the study of integrable systems is highly motivated.

The outline of the present work is the following. In Chapter 1 we give a short overview of the theory of integrability [1, 2] . After recalling Liouville's definition of a finite dimensional integrable hamiltonian system, we introduce the main concepts of classical integrability, i.e. Lax pairs and classical r-matrices. We touch upon the classical Yang–Baxter equation and the classical dynamical Yang–Baxter equation, too. These concepts are the main players in the subsequent chapters.

In Chapter 2 we begin to present our own results [3, 4]. After recalling the definition of the degenerate Calogero–Moser models [5, 6, 7], we present a complete description of the non-dynamical r-matrices of these models based on gl_n. First the most general momentum independent r-matrices are given for the standard Lax representation of these systems and those r-matrices whose coordinate dependence can be gauged away are selected. Then the constant r-matrices resulting from gauge transformation are determined and are related to well-known r-matrices. In the hyperbolic/trigonometric case a non-dynamical r-matrix equivalent to a real/imaginary multiple of the Cremmer–Gervais [9] classical r-matrix is found. In the rational case the constant r-matrix corresponds to the antisymmetric solution of the classical Yang–Baxter equation associated with the Frobenius subalgebra of gl_n consisting of the matrices with vanishing last row. These claims are consistent with previous results of Hasegawa [10] and others, which imply that Belavin's [11] elliptic r-matrix and its degenerations appear in the Calogero–Moser models. The advantages of our analysis are that it is elementary and also clarifies the extent to which the constant r-matrix is unique in the degenerate cases.

In Chapter 3 we start the analysis of the classical dynamical r-matrices. It is well known [12, 13] that a classical dynamical r-matrix can be associated with every finite-dimensional self-dual Lie algebra $\mathcal{G}$ by the definition $R(\omega) := f(\mathrm{ad}\omega)$, where $\omega \in \mathcal{G}$ and f is the holomorphic function given by $f(z) = \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z}$ for $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. We present a new, direct proof of the statement that this 'canonical' r-matrix satisfies the modified classical dynamical Yang–Baxter equation on $\mathcal{G}$ [14].

In Chapter 4 we continue the study of the classical dynamical r-matrices. We associate a dynamical r-matrix with every self-dual Lie algebra $\mathcal{A}$ which is graded by finite-dimensional subspaces as $\mathcal{A} = \oplus_{n\in\mathcal{Z}}\mathcal{A}_n$, where $\mathcal{A}_n$ is dual to $\mathcal{A}_{-n}$ with respect to the invariant scalar product on $\mathcal{A}$, and $\mathcal{A}_0$ admits a nonempty open subset $\check{\mathcal{A}}_0$ for which $\mathrm{ad}\,\kappa$ is invertible on $\mathcal{A}_n$ if $n \neq 0$ and $\kappa \in \check{\mathcal{A}}_0$. Examples are furnished by taking $\mathcal{A}$ to be an affine Lie algebra obtained from the central extension of a twisted loop algebra $\ell(\mathcal{G}, \mu)$ of a finite-dimensional self-dual Lie algebra $\mathcal{G}$. These r-matrices, $R : \check{\mathcal{A}}_0 \to \mathrm{End}(\mathcal{A})$, yield generalizations of the basic trigonometric dynamical r-matrices that, according to Etingof and Varchenko [15], are associated with the Coxeter automorphisms of the simple Lie algebras, and are related to Felder's [16] elliptic r-matrices by evaluation homomorphisms of $\ell(\mathcal{G}, \mu)$ into $\mathcal{G}$. The spectral-parameter-dependent dynamical r-matrix that corresponds analogously to an arbitrary scalar-product-preserving finite order automorphism of a self-dual Lie algebra is calculated explicitly [17, 18].

Chapter 1

Overview of the theory of integrable systems

Integrable systems play important role in the area of dynamical systems. The main feature of an integrable system is the existence of global first integrals of motion. Liouville's theorem gives a precise connection between the existence of conserved quantities and solvability. Practically all systems for which the equation of motion has been solved explicitly are integrable in the sense of Liouville. Let us consider a list of some classical examples:

- Any system with only one degree of freedom is integrable.
- The motion of a point particle in a central potential (Newton).
- The motion of a point particle in the gravitational field of two fixed centers (Euler).
- Free motion of a particle on the ellipsoid (Jacobi).
- Motion of a particle on the sphere under the influence of linear force (K. Neumann).
- The motion of three particles in one dimensional space, with two-body interactions inversely proportional to the square of the distance (Jacobi).
- The spinning top, i.e. a solid body rotating around one fixed point, in the special cases of Euler (no external force), Lagrange (in the presence of a gravitational field, but when the top has a rotational symmetry axis passing through the fixed point), and Kowalewski.

The investigation of these systems was an important line of study in the 19th century. Early in the 20th century, however, the work of H. Poinaré made it clear that global integrals of motion for hamiltonian systems exist only in exceptional cases, and the interest in integrable systems declined. Further progress in this field was made only a short time ago. In 1967 Gardner, Greene, Kruskal, and Miura [19] discovered the inverse scattering method. This approach was cast in algebraic form by Peter Lax [20]. Their pioneering work shed a new light on integrable systems. The inverse scattering method, also called the isospectral method, was originally applied to nonlinear partial differential equations, such as the Korteweg-de Vries equation, the nonlinear Schrödinger equation and the sine-Gordon equation. However, this method is applicable not only in infinite dimensional cases, but in the realm of many-particle systems, too. The most famous integrable many-particle systems are the Toda chains and the Calogero–Moser models.

The content of this chapter is based on the papers [1, 2]. After this introduction, the chapter consists of two sections. The concept of Liouville integrability and Lax pairs is described in section 1.1. Section 1.2 is devoted to the definition of the r-matrices and the classical Yang–Baxter equations.

1.1 Liouville integrability and Lax pairs

Let us consider a finite dimensional hamiltonian system $(\mathcal{M}, \{,\}, h)$. The phase space $\mathcal{M}$ is a $2n$ dimensional differentiable manifold, $\{,\}$ is a Poisson bracket and h is a hamiltonian. The system is said to be integrable *in the sense of Liouville*, if it possesses n independent integrals of motion $F_i\,(i = 1, \ldots, n)$ in involution, i.e.

$$\{h, F_i\} = 0, \qquad \{F_i, F_j\} = 0, \qquad (\forall i, j \in \{1, \ldots, n\}) \tag{1.1}$$

where h is not independent of the F's. Under these assumptions the following theorem, the so-called Liouville theorem holds.

Theorem 1.1 *The solution of the equations of motion of a Liouville integrable system is obtained by quadrature.*

Proofs can be found in [2, 21]. Somewhat informally, the essence of this theorem is that the existence of first integrals in involution is a good indication for the problem being exactly solvable.

The use of Lax pairs proves extremely useful to produce integrable systems. A Lax pair L, M consists of two functions on the phase space $\mathcal{M}$ with the values in some Lie algebra $\mathcal{G}$, such that the

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] \tag{1.2}$$

equation holds along every solution of the hamiltonian evolution equations. We will denote by G a connected Lie group having $\mathcal{G}$ as a Lie algebra. The solution of the *Lax equation* (1.2) is of the form

$$L(t) = g(t)^{-1} L(0)\, g(t), \tag{1.3}$$

where $g(t) \in G$ is determined by the equation

$$M(t) = g(t)^{-1} \frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{1.4}$$

Recalling that the adjoint action of G on $\mathcal{G}$ is given by $Ad_g(X) = gXg^{-1} \quad (\forall g \in G, \forall X \in \mathcal{G})$, the following proposition is a trivial consequence of (1.3).

Proposition 1.2 *If I is an Ad-invariant function on $\mathcal{G}$ then*

$$\frac{\mathrm{d}I(L(t))}{\mathrm{d}t} = 0. \tag{1.5}$$

The message of this statement is that by means of the *Ad*-invariant functions numerous conserved quantities become available. If L and M are taken in some representation of $\mathcal{G}$, the invariants are essentially the eigenvalues of L. This is why the time evolution of equation (1.2) is referred to as *isospectral deformation*.

Now we explain [1] that every finite dimensional Liouville integrable system admits a Lax pair. It is a well known fact that the Liouville theorem relies heavily on the existence of action-angle variables. Given a Liouville integrable system described at the beginning of this section there exists a system of conjugate coordinates $I_i, \Theta_i\,(i = 1, \ldots, n)$, where I_j are functions of the F_i's only and the equations of motion take the very simple form

$$\frac{\mathrm{d}I_j}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\Theta_j}{\mathrm{d}t} = \frac{\partial h}{\partial I_j}. \tag{1.6}$$

To prove the existence of a Lax pair, it is enough to show one such pair. This is straightforward in the action-angle coordinate system. Introduce the Lie algebra $\mathcal{G}$ generated by $H_i, E_i\,(i = 1, \ldots, n)$ with the relations

$$[H_i, H_j] = 0, \qquad [H_i, E_j] = 2\delta_{ij} E_j, \qquad [E_i, E_j] = 0. \tag{1.7}$$

Set

$$L := \sum_{j=1}^{n} \left(I_j H_j + 2 I_j \Theta_j E_j \right), \qquad M := \sum_{j=1}^{n} \frac{\partial h}{\partial I_j} E_j. \tag{1.8}$$

It can be easily seen that the equation

$$\dot{L} = [L, M] \tag{1.9}$$

is equivalent to (1.6).

Before turning out attention to the theory of r-matrices, let us finish this section with two concluding remarks. First, Lax pairs are not unique: even the Lie algebra $\mathcal{G}$ may be changed. Second, there is a natural gauge transformation group acting on the Lax pair:

$$L \mapsto L' = gLg^{-1}, \qquad M \mapsto M' = gMg^{-1} - \frac{\mathrm{d}g}{\mathrm{d}t} g^{-1}, \tag{1.10}$$

where g is an arbitrary smooth G-valued function on phase space $\mathcal{M}$. Simply, L', M' also serves as a Lax pair for the given system.

1.2 Poisson structure and r-matrices

A Lax pair provides us with integrals of motion without referring to a Poisson structure. The notion of Liouville integrability requires the knowledge of the involution property of the conserved quantities as well. Suppose we are given a Lax pair L, M in some matrix representation of some Lie algebra $\mathcal{G}$. Assuming that the L matrix is diagonalizable, its eigenvalues are first integrals as we have already mentioned in the previous section. Babelon and Viallet [1] gave an algebraic characterization of the involution property of the eigenvalues.

Before formulating their result we need some notations. Let T_μ be a basis of the Lie algebra $\mathcal{G}$. We can write

$$L = \sum_{\mu} L^\mu T_\mu, \tag{1.11}$$

where L^μ are functions on phase space $\mathcal{M}$. We may evaluate their Poisson brackets $\{L^\mu, L^\nu\}$ and gather the results as follows. Set

$$L_1 := L \otimes 1, \qquad L_2 := 1 \otimes L, \qquad \{L_1, L_2\} := \sum_{\mu\nu} \{L^\mu, L^\nu\} T_\mu \otimes T_\nu, \tag{1.12}$$

and if $\alpha \in \mathcal{G} \otimes \mathcal{G}$, denote

$$\alpha = \alpha_{12} = \sum_{\mu\nu} \alpha^{\mu\nu} T_\mu \otimes T_\nu, \qquad \alpha_{21} := \sum_{\mu\nu} \alpha^{\mu\nu} T_\nu \otimes T_\mu. \tag{1.13}$$

Proposition 1.3 *The involution property of the eigenvalues of L is equivalent to the existence of functions $a, b\colon \mathcal{M} \mapsto \mathcal{G} \otimes \mathcal{G}$ such that*

$$\{L_1, L_2\} = [a_{12}, L_1] + [b_{12}, L_2] \, . \tag{1.14}$$

Using the antisymmetry of the Poisson bracket we can write

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] \, , \tag{1.15}$$

where $r_{12} = \frac{1}{2}(a_{12} - b_{21})$. It is customary to call r_{12} a *classical r-matrix*. A classical r-matrix is *non-dynamical* if it does not depend on the dynamical variables, i.e., constant over the phase space $\mathcal{M}$, and *dynamical* otherwise. It is wort mentioning that the (1.15) form of the Poisson bracket is preserved by the (1.10) gauge transformations. Namely, if (1.15) holds and $L' = g^{-1}Lg$, then

$$\{L'_1, L'_2\} = [r'_{12}, L'_1] - [r'_{21}, L'_2], \tag{1.16}$$

where

$$r'_{12} = g_1^{-1} g_2^{-1} \left(r_{12} - \{g_1, L_2\} + \frac{1}{2} \left[\{g_1, g_2\} \, g_1^{-1} g_2^{-1}, L_2 \right] \right) g_1 g_2. \tag{1.17}$$

The remaining question is the connection between the Jacoby identity of the Poisson bracket and the r-matrix. In the approach of classical integrable hamiltonian systems developed by the St Petersburg School of L.D. Faddeev and collaborators, the key equation is

$$\{L_1, L_2\} = [r_{12}, L_1 + L_2] \, , \tag{1.18}$$

which is a trivial consequence of (1.15) when r is antisymmetric, i.e., $r_{12} + r_{21} = 0$. Since the left hand side of (1.18) is a Poisson bracket, it must satisfy the antisymmetry property and the Jacoby identity. This yields constraints on the r-matrix. In the very special case when r is non-dynamical, the following statement is valid.

Proposition 1.4 *The antisymmetry property of the Poisson bracket and the Jacoby identity are equivalent to the equations:*

$$[r_{12} + r_{21}, L_1 + L_2] \;=\; 0, \tag{1.19}$$

$$[\varphi, L_1 + L_2 + L_3] \;=\; 0, \tag{1.20}$$

with

$$\varphi = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] \, . \tag{1.21}$$

Here the standard tensorial notations are used, $L_1 = L \otimes 1 \otimes 1, r_{12} = r \otimes 1, r_{23} = 1 \otimes r$ etc. If r is antisymmetric, one obtains the better-known and much studied form of (1.21):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \varphi. \tag{1.22}$$

If r obeys (1.22) with $\varphi = 0$ we say that r is a solution of the *classical Yang–Baxter equation* (CYBE). If r satisfies (1.22) with $\varphi \neq 0$, where φ is some constant $\mathcal{G}$-invariant element of $\mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$, we say that r is a solution of the *modified classical Yang–Baxter equation* (mCYBE).

Dynamical generalizations of the Yang–Baxter equations and the associated algebraic structures are in the focus of current interest due to their applications in the theory of integrable systems and other areas of mathematical physics and pure mathematics (see [22] for a review). Let us recall that dynamical r-matrices in the sense of Etingof-Varchenko [15] are associated with any subalgebra $\mathcal{H}$ of any (complex or real) Lie algebra $\mathcal{G}$. By definition, a dynamical r-matrix is a (holomorphic or smooth) $\mathcal{G} \otimes \mathcal{G}$-valued function on an open subset $\check{\mathcal{H}}^*$ of the dual space $\mathcal{H}^*$ of $\mathcal{H}$ subject to the following three conditions. First, r must satisfy the modified classical dynamical Yang–Baxter equation (mCDYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] + T^1_j \frac{\partial r_{23}}{\partial \omega_j} - T^2_j \frac{\partial r_{13}}{\partial \omega_j} + T^3_j \frac{\partial r_{12}}{\partial \omega_j} = \varphi, \tag{1.23}$$

where φ is some constant, $\mathcal{G}$-invariant element of $\mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$. The ω_j are coordinates on $\mathcal{H}^*$ with respect to a basis $\{T_j\}$ of $\mathcal{H}$, and the usual tensorial notations as well as the summation convention are used. The second condition is that $(r + r^T)$, where $(X_a \otimes Y^a)^T = Y^a \otimes X_a$, is a $\mathcal{G}$-invariant constant. The third condition requires the map $r : \check{\mathcal{H}}^* \to \mathcal{G} \otimes \mathcal{G}$ to be equivariant with respect to the (coadjoint and adjoint) infinitesimal actions of $\mathcal{H}$ on the corresponding spaces. The mCDYBE becomes the CDYBE for $\varphi = 0$.

Chapter 2

Degenerate Calogero–Moser models

The purpose of this chapter is to provide a complete description of the non-dynamical, constant r-matrices of the standard Calogero–Moser models [5, 6] associated with degenerate potential functions, which can be obtained by gauge transformations of their usual Lax representation.

The Calogero–Moser type many particle systems (for a review, see [7]) have been much studied recently due to their fascinating mathematics and applications [8] ranging from solid state physics to Seiberg-Witten theory. The definition of these models involves a root system and a potential function depending on the inter-particle 'distance'. The potential is given either by the Weierstrass $\mathcal{P}$-function or one of its (hyperbolic, trigonometric or rational) degenerations. The classical equations of motion of the models admit Lax representations (1.2),

$$\dot{L} = [L, M], \tag{2.1}$$

which underlie their integrability. A Lax representation of the Calogero–Moser models based on the root systems of the classical Lie algebras was found by Olshanetsky and Perelomov [23] using symmetric spaces. Recently new Lax representations for these systems as well as their exceptional Lie algebraic analogues and twisted versions have been constructed [24, 25].

As we have already seen in section 1.2, Liouville integrability can be understood as a consequence of the Poisson brackets of the Lax matrix having the r-matrix form (1.15),

$$\{L_1, L_2\} = \{L^\mu, L^\nu\} T_\mu \otimes T_\nu = [r_{12}, L_1] - [r_{21}, L_2]. \tag{2.2}$$

Of course, L and r may also depend on a spectral parameter in general, but this does not occur for the systems of our interest, and thus is suppressed in (2.2). When the r-matrix really does depend on the phase space variables, one says that it is 'dynamical'.

The classical r-matrix has been calculated first for the standard Lax representation of the gl_n Calogero–Moser systems associated with degenerate potentials [26], and then for Krichever's [27] spectral parameter dependent Lax matrix in the elliptic case [28, 29]. The r-matrices found in these papers are dynamical, but depend only on the coordinates of the particles. These r-matrices have been re-derived by means of Hamiltonian reduction in [30, 31], and in a recent paper [32] they have been generalized explicitly for the BC_n system as well as for all classical Lie algebras. In the physically most interesting gl_n case, dynamical r-matrices have also been found [33, 34, 35] for the relativistic deformations of the Calogero–Moser models introduced by Ruijsenaars and Schneider [36]. Then the quantization of the non-relativistic [37] and the relativistic models [38, 39, 40] has been investigated in a new framework based on quantum dynamical R-matrices.

The above developments have close connections with the new theory of dynamical r-matrices and associated quantized structures reviewed in [22]. However, since the present understanding of most integrable systems involves constant (i.e. 'non-dynamical') r-matrices, which form a direct link to Poisson-Lie groups and quantum groups [41], it is natural to ask if the Lax representation of the Calogero–Moser models can be chosen in such a way to exhibit non-dynamical r-matrices. The obvious way to search for new Lax representations with this property is to perform gauge transformations on the usual Lax representations. In the elliptic case of the standard gl_n models a new Lax representation associated with Belavin's [11] constant elliptic r-matrix has recently been found in this way [42]. To be more precise, the results of [42] are already contained in a somewhat less explicit form in the seminal paper by Hasegawa [10], where the commuting Ruijsenaars operators [43] have been interpreted as commuting transfer matrices based on a realization of the $RLL = LLR$ relation with Belavin's elliptic R-matrix and certain difference L-operators. In fact, the dynamical twisting and the classical and non-relativistic limits of the L-operator leading to Krichever's Lax matrix for the elliptic Calogero–Moser model are indicated in [10] (see also [40]). Then in the paper [44] some delicate limit procedures have been considered, whereby non-dynamical R-matrices can be obtained for the trigonometric degenerations of the Ruijsenaars-Schneider and Calogero–Moser models. The resulting R-matrix was found to be non-unique, one possibility [44] being the spectral parameter independent Cremmer–Gervais R-matrix discovered in a different context in [9].

It is clear from the above that Lax representations for the degenerate Calogero–Moser models with non-dynamical r-matrices can be obtained by taking limits of Hasegawa's $RLL = LLR$ relation. However, the details of the admissible limiting procedures appear rather complicated and the starting point requires familiarity with quite advanced results. In this circumstance, it might be worthwhile to understand the possible non-dynamical r-matrices also from an elementary viewpoint. This is the objective of the present chapter, where we aim to perform a self-contained, systematic analysis of the gauge transformations of the usual Lax representation of the degenerate Calogero–Moser models that lead to constant r-matrices.

The organization and the main results of this chapter are as follows. First, we describe the most general momentum independent dynamical r-matrices for the standard Lax representation in section 2.1. This amounts to a slight but necessary generalization of the Avan-Talon [26] r-matrix as given by Theorem 2.1. Second, we select those dynamical r-matrices that become constant by a gauge transformation (defined by eq. (2.18)) and determine the corresponding 'gauge potentials' $A_k(q)$. This is the content of section 2.2, in particular Proposition 2.2 and Theorem 2.3. Third, in section 2.3 we compute explicitly the gauge transformations $g(q)$ (from eq. (2.19)) and the resulting most general constant r-matrix, which is given by Theorem 2.6. It turns out that in the rational case the constant r-matrix is conjugate to the antisymmetric solution of the classical Yang–Baxter equation that belongs to the Frobenius subalgebra of gl_n consisting of the matrices with vanishing last row [45]. In the hyperbolic/trigonometric cases the sl_n-part of the most general $gl_n \wedge gl_n$-valued constant r-matrix (see Proposition 2.7) is equivalent to a multiple of the Cremmer–Gervais classical r-matrix [9, 46], and it can also be made equal to it by a choice of the gauge transformation. This identification of the constant Calogero–Moser r-matrices is presented in section 2.4. The details of some proofs are contained in appendix A, B, and C.

2.1 Momentum independent dynamical r-matrices

The standard (degenerate) Calogero–Moser–Sutherland models are defined by the Hamiltonian

$$h = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k<l} v(q_k - q_l), \tag{2.3}$$

where v is given as

$$v(x) = \begin{cases} x^{-2}, & \text{rational case} \\ a^2 \sinh^{-2}(ax), & \text{hyperbolic case} \\ a^2 \sin^{-2}(ax), & \text{trigonometric case.} \end{cases} \tag{2.4}$$

One has the canonical Poisson brackets $\{p_k, q_l\} = \delta_{k,l}$, the coordinates are restricted to a domain in $\mathbb{R}^n$ where $v(q_k - q_l) < \infty$, and $a > 0$ is a parameter.

Let us fix the following notation for elements of the Lie algebra gl_n:

$$H_k := e_{kk},\ E_\alpha := e_{kl},\ H_\alpha := (e_{kk} - e_{ll}),\ K_\alpha := (e_{kk} + e_{ll}) \text{ for } \alpha = \lambda_k - \lambda_l \in \Phi. \tag{2.5}$$

Here $\Phi = \{(\lambda_k - \lambda_l) | k \neq l\}$ is the set of roots of gl_n, λ_k operates on a diagonal matrix, $H = \mathrm{diag}(H_{1,1}, \dots, H_{n,n})$ as $\lambda_k(H) = H_{k,k}$, and e_{kl} is the $n \times n$ elementary matrix whose kl-entry is 1. Moreover, we denote the standard Cartan subalgebra of $sl_n \subset gl_n$ as $\mathcal{H}_n$, and put $p = \sum_{k=1}^n p_k H_k$, $q = \sum_{k=1}^n q_k H_k$, $\mathbf{1}_n = \sum_{k=1}^n H_k$.

From the list of known Lax representations we consider the original one [5, 6] for which L is the gl_n valued function

$$L(q,p) = p + \sqrt{-1} \sum_{\alpha \in \Phi} w(\alpha(q)) E_\alpha, \tag{2.6}$$

where the real function w is chosen according to

$$w(x) = \begin{cases} x^{-1} \\ a \sinh^{-1}(ax) \\ a \sin^{-1}(ax). \end{cases} \tag{2.7}$$

Then the function

$$F := -\frac{w'}{w} \tag{2.8}$$

enjoys the important identities

$$F' = -w^2, \tag{2.9}$$

$$F(x) + F(y) = \frac{w(x)w(y)}{w(x+y)}, \tag{2.10}$$

$$F(x-y)\left(F(x) - F(y)\right) + F(x)F(y) = \mathcal{B}, \tag{2.11}$$

where, respectively to the cases above,

$$\mathcal{B} = \begin{cases} 0 \\ a^2 \\ -a^2. \end{cases} \tag{2.12}$$

For any real function f (like v, w or F), we introduce the functions f_k and f_α of q as

$$f_k(q) := f(q_k), \qquad f_\alpha(q) = f(\alpha(q)), \tag{2.13}$$

and sometimes write f_{kl} for f_α if $\alpha = (\lambda_k - \lambda_l)$. As an $n \times n$ matrix $L_{k,l} = p_k\delta_{k,l} + \sqrt{-1}(1 - \delta_{k,l})w(q_k - q_l)$, but L can also be used in any other representation of gl_n. The r-matrix corresponding to this L was studied by Avan and Talon [26], who found that it is necessarily dynamical, and may be chosen so as to depend on the coordinates q_k only. We next describe a slight generalization of their result.

Theorem 2.1 *The most general $gl_n \otimes gl_n$-valued r-matrix that satisfies (2.2) with the Lax matrix in (2.6) and depends only on q is given by*

$$r(q) = -\sum_{\alpha\in\Phi} F_\alpha(q)E_\alpha \otimes E_{-\alpha} + \frac{1}{2}\sum_{\alpha\in\Phi} w_\alpha(q)(C_\alpha(q) - K_\alpha) \otimes E_\alpha + \mathbf{1}_n \otimes Q(q), \tag{2.14}$$

where the $C_\alpha(q)$ are $\mathcal{H}_n \subset sl_n$ valued functions subject to the conditions

$$C_{-\alpha}(q) = -C_\alpha(q), \qquad \beta(C_\alpha(q)) = \alpha(C_\beta(q)) \quad \forall\alpha,\beta \in \Phi \tag{2.15}$$

and $Q(q)$ is an arbitrary gl_n-valued function.

Remarks. The functions C_α can be given arbitrarily for the simple roots, and are then uniquely determined for the other roots by (2.15). The r-matrix found by Avan and Talon [26] is recovered from (2.14) with $C_\alpha \equiv 0$; and we refer to $r(q)$ in (2.14) as the *Avan-Talon r-matrix in its general form*. Given that this holds for the Avan-Talon r-matrix, the fact that $r(q)$ above satisfies (2.2) with any $Q(q)$ and $C_\alpha(q)$ subject to (2.15) is easy to verify. Theorem 2.1 can be proved by a careful calculation along the lines of [29]. For the details, see appendix A.

2.2 Is $r(q)$ gauge equivalent to a constant?

A gauge transformation (1.10) of a given Lax representation (2.1) has the form

$$L \mapsto L' = gLg^{-1}, \qquad M \mapsto M' = gMg^{-1} - \frac{dg}{dt}g^{-1}, \tag{2.16}$$

where g is an invertible matrix function on the phase space. If L satisfies (2.2), then L' will have similar Poisson brackets with a transformed r-matrix r'. The question now is whether one can remove the q-dependence of any of the r-matrices in (2.14) by a gauge transformation. It is natural to assume this gauge transformation to be p-independent, i.e. defined by some function $g : q \mapsto g(q) \in GL_n$. In this special case (1.16) reads as

$$\{L'_1, L'_2\} = [r'_{12}, L'_1] - [r'_{21}, L'_2], \tag{2.17}$$

where (1.17) takes the form

$$r'(q) = (g(q) \otimes g(q)) \left(r(q) + \sum_{k=1}^{n} A_k(q) \otimes H_k \right) (g(q) \otimes g(q))^{-1}, \tag{2.18}$$

$$A_k(q) := -g^{-1}(q)\partial_k g(q), \qquad \partial_k := \frac{\partial}{\partial q_k}. \tag{2.19}$$

The meaning of this formula is that if $r(q)$ is the most general p-independent r-matrix for which L (2.6) satisfies (2.2), then $r'(q)$ has the analogous property in relation to L'.

We wish to find $r(q)$ and $g(q)$ such that $\partial_k r' = 0$. On account of (2.18) this is equivalent to

$$\partial_k (r + \sum_{l=1}^{n} A_l \otimes H_l) + [r + \sum_{l=1}^{n} A_l \otimes H_l, A_k \otimes \mathbf{1}_n + \mathbf{1}_n \otimes A_k] = 0. \tag{2.20}$$

By using (2.19), whereby

$$\partial_k A_l - \partial_l A_k + [A_l, A_k] = 0, \tag{2.21}$$

it is useful to rewrite (2.20) as

$$\partial_k r + \sum_{l=1}^{n} \partial_l A_k \otimes H_l + [r, A_k \otimes \mathbf{1}_n + \mathbf{1}_n \otimes A_k] + \sum_{l=1}^{n} A_l \otimes [H_l, A_k] = 0. \tag{2.22}$$

Our strategy is to first find $A_k(q)$ and $r(q)$ from eqs. (2.21), (2.22), and then determine $g(q)$ and the resulting constant r-matrix. For this we now parametrize A_k as

$$A_k(q) = \sum_{l=1}^{n} A_k^l(q) H_l + \sum_{\alpha \in \Phi} A_k^\alpha(q) E_\alpha, \tag{2.23}$$

and expand the r-matrix from Theorem 2.1 in the form

$$r(q) = -\sum_{\alpha} F_\alpha(q) E_\alpha \otimes E_{-\alpha} + \sum_{i,\alpha} r_i^\alpha(q) H_i \otimes E_\alpha + \sum_i Q^i(q) \mathbf{1}_n \otimes H_i. \tag{2.24}$$

We here have

$$r_i^\alpha(q) = Q^\alpha(q) + \frac{1}{2} w_\alpha(q) \mathrm{tr}\left(H_i (C_\alpha(q) - K_\alpha) \right), \tag{2.25}$$

$$Q(q) = \sum_{i=1}^{n} Q^i(q) H_i + \sum_{\alpha \in \Phi} Q^\alpha(q) E_\alpha, \tag{2.26}$$

where $Q(q)$, $C_\alpha(q)$ and K_α appear in (2.14).

With reference to the conventions (2.5), we define the structure constants $c^{\alpha+\beta}_{\alpha,\beta}$ by writing $[E_\alpha, E_\beta] = c^{\alpha+\beta}_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, (\alpha+\beta)$ all belong to Φ, and $c^{\alpha+\beta}_{\alpha,\beta} := 0$ otherwise. Then (2.21) yields

$$\partial_l A^i_k - \partial_k A^i_l = \sum_{\alpha\in\Phi} \alpha_i A^\alpha_l A^{-\alpha}_k, \quad \forall i, k, l, \tag{2.27}$$

$$\partial_l A^\alpha_k - \partial_k A^\alpha_l = \sum_{i=1}^n \alpha_i (A^i_l A^\alpha_k - A^i_k A^\alpha_l) + \sum_{\gamma\in\Phi} c^\alpha_{\gamma,\alpha-\gamma} A^\gamma_l A^{\alpha-\gamma}_k, \quad \forall\alpha, \forall k, l. \tag{2.28}$$

The $H_i \otimes H_j$ and $H_i \otimes E_\alpha$ components of (2.22) require that

$$\partial_k Q^j + \partial_j A^i_k + \sum_{\alpha\in\Phi} \alpha_j r^\alpha_i A^{-\alpha}_k = 0, \qquad \forall i, j, k, \tag{2.29}$$

$$\partial_k r^\alpha_i - \alpha_i F_\alpha A^\alpha_k + \sum_{j=1}^n \alpha_j Q^j A^\alpha_k - \sum_{j=1}^n \alpha_j A^j_k r^\alpha_i + \sum_{\gamma\in\Phi} c^\alpha_{\gamma,\alpha-\gamma} r^\gamma_i A^{\alpha-\gamma}_k + \sum_{j=1}^n \alpha_j A^i_j A^\alpha_k = 0 \tag{2.30}$$

$\forall i, k, \alpha$. From the $E_\alpha \otimes H_i$ and $E_\alpha \otimes E_\beta$ components of (2.22) we find that

$$\partial_i A^\alpha_k + \alpha_i F_\alpha A^\alpha_k = 0, \quad \forall i, k, \alpha, \tag{2.31}$$

$$\delta_{\beta,-\alpha}\alpha_k w^2_\alpha - c^{\alpha+\beta}_{\alpha,\beta} \frac{w_\alpha w_\beta}{w_{\alpha+\beta}} A^{\alpha+\beta}_k + \sum_{j=1}^n \alpha_j r^\beta_j A^\alpha_k + \sum_{j=1}^n \beta_j A^\alpha_j A^\beta_k = 0 \tag{2.32}$$

$\forall k, \alpha, \beta$. Note that to derive (2.32) we have used the identities (2.9), (2.10) and the symmetry properties of the structure constants.

It is convenient to focus first on the last two equations, since they do not contain the Cartan components of A_k. Eq. (2.31) obviously implies that

$$A^\alpha_k(q) = w_\alpha(q) b^\alpha_k, \qquad b^\alpha_k : \text{some constants.} \tag{2.33}$$

The constants are then determined as follows.

Proposition 2.2 *Eq. (2.32) admits solution for the constants b^α_k only for those two families of $r(q)$ in (2.14) for which the C_α are chosen according to*

$$\textit{case I}: \quad C_\alpha = -H_\alpha \quad \forall\alpha\in\Phi, \qquad \textit{or} \qquad \textit{case II}: \quad C_\alpha = H_\alpha \quad \forall\alpha\in\Phi. \tag{2.34}$$

For $\alpha = \lambda_m - \lambda_l$, the b^α_k are respectively given by

$$b^{(\lambda_m-\lambda_l)}_k = \delta_{km} + \Omega \quad \textit{in case I,} \quad \textit{and} \quad b^{(\lambda_m-\lambda_l)}_k = \delta_{kl} + \Omega \quad \textit{in case II,} \tag{2.35}$$

where Ω is an arbitrary constant.

Proof. The statement is obtained by an elementary, but rather lengthy inspection of eq. (2.32). This is contained in appendix B. *Q.E.D.*

It is easy to explain why we got *two* series of solutions in the above. Namely, they arise due to the fact that L in (2.6) is a self-adjoint matrix. Indeed, $L^\dagger = L$ implies that if $r(q)$ solves (2.2) then $r^\dagger(q)$ also solves it, where $(u_1 \otimes u_2)^\dagger = u_1^\dagger \otimes u_2^\dagger$. Furthermore, if $r(q)$ is gauge transformed to a constant r' by $g(q)$, then $r^\dagger(q)$ is transformed to $(r')^\dagger$ by $(g^\dagger)^{-1}$. The two series of solutions described in Proposition 2.2 are exchanged by this symmetry. It is thus enough to consider only one of these series, and from now on we concentrate on case I.

As the main result of this section, we now give the most general 'gauge potential' A_k and $r(q)$ for which r' (2.18) will be constant.

Theorem 2.3 *The most general solution of eqs. (2.21), (2.22) for A_k and Q in case I of Proposition 2.2 can be described as follows. The root part of A_k is determined by Proposition 2.2, while its Cartan part has the form*[1]

$$A_k^l = F_{\lambda_l - \lambda_k} + \Omega \sum_{m\,(m \neq l)} F_{\lambda_l - \lambda_m} + \partial_k \theta \qquad (\forall k, l = 1, \dots, n), \tag{2.36}$$

where $\theta(q)$ is arbitrary smooth function. The function $Q(q) \in gl_n$ is given by

$$Q = -\sum_{k=1}^{n} A_k^k H_k - \Omega \sum_{\alpha \in \Phi} w_\alpha E_\alpha + g^{-1} Q' g, \tag{2.37}$$

where $g(q) \in GL_n$ denotes a solution of $\partial_k g = -g A_k$ and $Q' \in gl_n$ is an arbitrary constant.

Proof. The main steps of the proof can be outlined as follows. After choosing case I of Proposition 2.2, the right hand side of (2.27) can be calculated. The general solution of (2.27) for the unknowns A_k^l is then found to be

$$A_k^l = F_{\lambda_l - \lambda_k} + \Omega \sum_{m\,(m \neq l)} F_{\lambda_l - \lambda_m} + \partial_k \theta^l \qquad (\forall k, l = 1, \dots, n), \tag{2.38}$$

where the θ^l are arbitrary smooth functions of q. Next, it is verified that (2.38) solves (2.28) if and only if

$$\theta^1 = \theta^2 = \cdots = \theta^n := \theta. \tag{2.39}$$

[1] Note that $F_{\lambda_l - \lambda_l} = 0$ by the definition of $F_{\lambda_l - \lambda_k}$ in (2.13).

At this point we have the general solution for A_k and remaining task is to solve (2.29), (2.30) for Q. By using also (2.25) with $C_\alpha = -H_\alpha$, these are inhomogeneous linear differential equations for Q. It is an easy matter to check that (2.37) with $Q' = 0$ gives a particular solution, and that the difference δQ of two solutions must satisfy the equations

$$\partial_k(\delta Q) + [\delta Q, A_k] = 0 \qquad (\forall k = 1, \ldots, n). \tag{2.40}$$

The proof is completed by remarking that the last equation is equivalent to $\partial_k(g(\delta Q)g^{-1}) = 0$ with $\partial_k g = -gA_k$. *Q.E.D.*

We wish to make some observations on the above result. Firstly, note that if r' is the constant r-matrix obtained from (2.18) in the case

$$\theta = 0, \qquad Q' = 0, \tag{2.41}$$

then in the general case of Theorem 2.3 the same formula yields

$$r' + \mathbf{1}_n \otimes Q'. \tag{2.42}$$

This means that the free parameters θ and Q' in (2.36), (2.37) are irrelevant. Henceforth they will be set to zero. An additional convenience of this choice is that it guarantees the antisymmetry of r' (2.18). In fact, one can compute the symmetric part of $(r + \sum_k A_k \otimes H_k)$ and finds it to be zero if $Q' = 0$. Secondly, it is worth pointing out that

$$r' \in sl_n \wedge sl_n \qquad \Leftrightarrow \qquad \Omega = -\frac{1}{n}. \tag{2.43}$$

Indeed, the condition $r' \in sl_n \wedge sl_n$ is clearly equivalent to $(r + \sum_k A_k \otimes H_k) \in sl_n \wedge sl_n$, and this is easily calculated to hold if and only if $Q' = 0$ and $\Omega = -\frac{1}{n}$. Since for a given A_k the solution of (2.19) for $g(q) \in GL_n$ is unique up to a constant,

$$g(q) \to g_0 g(q), \qquad \forall g_0 \in GL_n, \tag{2.44}$$

we can also conclude that if the condition $r' \in sl_n \otimes sl_n$ is imposed, then r' is necessarily antisymmetric and is uniquely determined up to an automorphism of sl_n.

Finally, let us observe that our $r(q)$ and $A_k(q)$ for which r' will be a constant admit the interesting decompositions

$$r = \tilde{r} - \Omega \mathbf{1}_n \otimes \mathcal{A}, \qquad A_k = \tilde{A}_k + \Omega \mathcal{A}, \tag{2.45}$$

where

$$\mathcal{A} = \sum_{l,m\,(l \neq m)} \left(F_{\lambda_l - \lambda_m} H_l + w_{\lambda_l - \lambda_m} E_{\lambda_l - \lambda_m} \right). \tag{2.46}$$

Here $r(q)$, A_k are given by Theorem 2.3 together with (2.41). In the rest of the chapter we shall determine the corresponding constant r-matrices from (2.18). It will be convenient to consider first the $\Omega = 0$ special case, for which r, A_k, r' become $\tilde{r}$, $\tilde{A}_k$, $\tilde{r}'$, respectively.

2.3 Constant r-matrices from gauge transformation

If A_k is given so that (2.21) holds then the gauge transformation $g(q)$ can be determined from the differential equation in (2.19). By taking A_k and $r(q)$ from Theorem 2.3 with (2.41), this g will transform the dynamical r-matrix $r(q)$ into an antisymmetric constant (2.18). Here we shall determine $g(q)$ and r' explicitly. For an antisymmetric constant r' the (modified) classical Yang–Baxter equation is a sufficient condition for the Jacobi identity $\{\{L'_1, L'_2\}, L'_3\} + \text{cycl.} = 0$, which will be seen to hold for the r-matrices found below.

2.3.1 The case of $\Omega = 0$

Now we calculate the gauge transformation and the resulting constant r-matrix in the special case of Theorem 2.3 for which $\Omega = 0$ and (2.41) hold. In agreement with (2.45), the various quantities will carry a tilde in this case. We shall use the notation

$$I_k^n := \{1, \ldots, n\} \setminus \{k\}, \qquad \forall k = 1, \ldots, n, \tag{2.47}$$

and write the elements of gl_n as matrices. Then $\tilde{r}(q)$ and $\tilde{A}_k(q)$ take the following form:

$$\tilde{r} = -\sum_{1 \leq k \neq l \leq n} \left(F_{kl} e_{kl} \otimes e_{lk} + w_{kl} e_{kk} \otimes e_{kl} \right), \quad \tilde{A}_k = \sum_{l \in I_k^n} \left(w_{kl} e_{kl} + F_{lk} e_{ll} \right). \tag{2.48}$$

Let us start by defining the matrix function φ of q as follows: $\varphi_{nk} := 1$ for any $k = 1, \ldots, n$ and

$$\varphi_{jk} := \sum_{\substack{P \subset I_k^n \\ |P| = n-j}} \left(\prod_{l \in P} F_l \right) \qquad \forall k, \quad 1 \leq j \leq n-1, \tag{2.49}$$

where $|P|$ denotes the number of the elements of P. Moreover, let χ be the $n \times n$ matrix function of q given by

$$\chi_{jk} = \delta_{jk} \prod_{l \in I_k^n} \frac{1}{w_l}. \tag{2.50}$$

These formulas yield invertible matrices on the admissible domain of q, where $v(q)$ is finite. This is obvious for the diagonal matrix χ. By using the identity

$$\sum_{l=1}^{n}(-F_i)^{l-1}\varphi_{lj} = \prod_{\tau\in I_j^n}(F_\tau - F_i), \tag{2.51}$$

we can also find the inverse of φ explicitly

$$\left(\varphi^{-1}\right)_{jk} = (-F_j)^{k-1}\prod_{l\in I_j^n}\frac{1}{(F_l - F_j)}. \tag{2.52}$$

Proposition 2.4 *A gauge transformation $\tilde{g}(q) \in GL_n$ that satisfies*

$$\partial_k \tilde{g}(q) = -\tilde{g}(q)\tilde{A}_k(q) \tag{2.53}$$

with $\tilde{A}_k$ in (2.48) is given by $\tilde{g}(q) = \varphi(q)\chi(q)$, where φ and χ are defined by (2.49) and (2.50).

Proof. The componentwise form of (2.53) with $\tilde{A}_k$ in (2.48) reads

$$\partial_k \tilde{g}_{ik} = 0, \qquad \forall i,k \in \{1,\dots,n\}, \tag{2.54}$$

$$\partial_k \tilde{g}_{ij} = -\tilde{g}_{ij}F_{jk} - \tilde{g}_{ik}w_{kj}, \qquad \forall i,j,k \in \{1,\dots,n\}, \quad j \neq k. \tag{2.55}$$

We notice that the matrix

$$\tilde{g}_{ij}(q) = \prod_{l\in I_j^n}\frac{1}{w(q_l + c_i)}, \qquad i,j \in \{1,\dots,n\}, \tag{2.56}$$

where the $\{c_i\}_{i=1}^n$ are pairwise distinct constants, yields a solution of these equations. Indeed, (2.54) holds obviously, while (2.55) is checked with the aid of the identity (2.10). Using (2.10) again, we can rewrite the matrix $\tilde{g}(q)$ defined by (2.56) in the product form

$$\tilde{g}(q) = \mathbf{C}\varphi(q)\chi(q), \tag{2.57}$$

where $\mathbf{C}$ is the invertible constant matrix given by

$$\mathbf{C}_{ij} = \frac{1}{w(c_i)^{n-1}}\left(F(c_i)\right)^{j-1}. \tag{2.58}$$

Since equation (2.53) determines $\tilde{g}$ up to multiplication by a constant matrix form the left, the required statement follows. *Q.E.D.*

We can now calculate the gauge transformed r-matrix from (2.18). The result turns out to be an antisymmetric, constant solution of the (modified) classical Yang–Baxter equation,

$$[r'_{12}, r'_{13}] + [r'_{12}, r'_{23}] + [r'_{13}, r'_{23}] = -\mathcal{B}\hat{\mathcal{F}}, \tag{2.59}$$

where $\mathcal{B}$ appears in (2.12) and $\hat{\mathcal{F}} \in (gl_n)^{3\wedge}$ is given by

$$\hat{\mathcal{F}} := \sum_{i,j,k,l,r,s=1}^{n} \mathcal{F}^{rs}_{ij,kl} e_{ji} \otimes e_{lk} \otimes e_{rs} \quad \text{with} \quad [e_{ij}, e_{kl}] = \sum_{r,s=1}^{n} \mathcal{F}^{rs}_{ij,kl} e_{rs}. \tag{2.60}$$

Proposition 2.5 *The gauge transform of $\tilde{r}(q)$ in (2.48) by $\tilde{g}(q)$ in Proposition 2.4 is given by*

$$\tilde{r}' = \sum_{(a,b,c,d)\in S} \left(\mathcal{B} e_{ab} \wedge e_{cd} - e_{a+1,b} \wedge e_{c+1,d}\right), \tag{2.61}$$

$$S = \{\, (a,b,c,d) \in \mathbf{N}^4 \,|\, a+c+1 = b+d, \quad 1 \le b \le a < n, \quad b \le c < n, \quad 1 \le d \le n \,\}.$$

This formula defines an antisymmetric solution of (2.59).

Proof. The first statement is verified by a direct calculation, which is described in appendix C. The fact that $\tilde{r}'$ solves (2.59) can also be checked directly. Alternatively, it follows from the identification of $\tilde{r}'$ in terms of certain well-known solutions of (2.59), which is presented in section 2.4. *Q.E.D.*

It is clear from (2.59) that the two terms in (2.61) must separately satisfy the classical Yang–Baxter equation,

$$[b_{12}, b_{13}] + [b_{12}, b_{23}] + [b_{13}, b_{23}] = 0. \tag{2.62}$$

In fact, this holds since the first term

$$b_{gl_n} := \sum_{(a,b,c,d)\in S} e_{ab} \wedge e_{cd} \tag{2.63}$$

is nothing but the classical r-matrix associated with the Frobenius subalgebra of gl_n spanned by the matrices with vanishing last row, which is described as an example in [45]. More explicitly, it reads as

$$b_{gl_n} = \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} e_{jj} \wedge e_{n-k,n+1-k} + \sum_{1\le i<j\le n} \sum_{m=1}^{j-i-1} e_{n+1-i-m,n+1-j} \wedge e_{n+m-j,n+1-i}. \tag{2.64}$$

The second term is a transform of the first one according to

$$\sum_{(a,b,c,d)\in S} e_{a+1,b} \wedge e_{c+1,d} = -(\sigma \otimes \sigma) b_{gl_n}, \tag{2.65}$$

where $\sigma : gl_n \to gl_n$ is the inner automorphism

$$\sigma : e_{ij} \mapsto e_{n+1-i,n+1-j}. \tag{2.66}$$

Finally, we note for later purpose that

$$\tilde{r}' = \mathcal{B} b_{gl_n} + (\sigma \otimes \sigma) b_{gl_n} = \tilde{r}'_{sl_n} + X \wedge \mathbf{1}_n, \tag{2.67}$$

where $\tilde{r}'_{sl_n} \in sl_n \wedge sl_n$ and

$$X = -\frac{1}{n} \sum_{k=1}^{n-1} (n-k) e_{k+1,k} - \frac{\mathcal{B}}{n} \sum_{k=1}^{n-1} k e_{k,k+1}. \tag{2.68}$$

Of course, $\tilde{r}'_{sl_n}$ satisfies the same equation (2.59) as $\tilde{r}'$.

2.3.2 The case of an arbitrary Ω

Now we tackle the general case by making use of the decompositions of $r(q)$ and A_k in (2.45).

It is natural to look for $g(q)$ as a product

$$g(q) = h(q)\tilde{g}(q), \tag{2.69}$$

where $\tilde{g}(q)$ is given in Proposition 2.4. Then the equation $\partial_k g = -(\tilde{A}_k + \Omega\mathcal{A})g$ reduces to

$$\partial_k h = -h\tilde{\mathcal{A}}\Omega \quad \text{with} \quad \tilde{\mathcal{A}} := \tilde{g}\mathcal{A}\tilde{g}^{-1}, \tag{2.70}$$

where $\mathcal{A}$ is given in (2.46). By using also the decomposition of $r(q)$ in (2.45) we obtain from (2.18) that

$$r' = (h(q) \otimes h(q)) \left(\tilde{r}' + \Omega\tilde{\mathcal{A}}(q) \wedge \mathbf{1}_n\right) (h(q) \otimes h(q))^{-1}, \tag{2.71}$$

where $\tilde{r}'$ is given by (2.61). The fact that r' and $\tilde{r}'$ are both constant permits us to prove the following result without further explicit calculation.

Theorem 2.6 *With the above notations and $\tilde{r}'$, X defined in (2.61), (2.67), we have*

$$h(q) = g_0 \exp\left(-Xn\Omega \sum_{i=1}^{n} q_i\right), \tag{2.72}$$

where $g_0 \in GL_n$ is an arbitrary constant, and

$$r' = (g_0 \otimes g_0) \left(\tilde{r}'_{sl_n} + (n\Omega + 1)X \wedge \mathbf{1}_n\right) (g_0 \otimes g_0)^{-1} \tag{2.73}$$

is the most general constant r-matrix resulting from gauge transformation.

Proof. By substituting (2.67), we can rewrite (2.71) as the sum $r' = r'_{sl_n} + r'_{\text{rest}}$ with

$$r'_{sl_n} = (h(q) \otimes h(q))\tilde{r}'_{sl_n}(h(q) \otimes h(q))^{-1} \tag{2.74}$$

and

$$r'_{\text{rest}} = \Big(h(q)(\Omega\tilde{\mathcal{A}}(q) + X)h^{-1}(q)\Big) \wedge \mathbf{1}_n. \tag{2.75}$$

Since r' is constant, these two terms must be constant separately. Recall now that $\tilde{\mathcal{A}}(q)$ is independent of Ω by its definition (2.70) and that for $\Omega = -\frac{1}{n}$ we must have $r' \in sl_n \wedge sl_n$ (2.43). This implies that $(X - \frac{1}{n}\tilde{\mathcal{A}}(q))$ must vanish, whereby

$$\tilde{\mathcal{A}} = nX. \tag{2.76}$$

Hence we obtain (2.72) from the differential equation in (2.70). But then the fact that r'_{sl_n} is constant shows that the relation

$$[X \otimes \mathbf{1}_n + \mathbf{1}_n \otimes X, \tilde{r}'_{sl_n}] = 0, \tag{2.77}$$

which is equivalent to

$$r'_{sl_n} = (g_0 \otimes g_0)\tilde{r}'_{sl_n}(g_0 \otimes g_0)^{-1}, \tag{2.78}$$

must be valid. By substituting these results back into (2.71) we arrive at (2.73). *Q.E.D.*

Incidentally, we have also verified by explicit calculation that (2.76) and (2.77) are indeed satisfied, which represents a reassuring check on the foregoing considerations in the work.

2.4 Identification of the constant r-matrices

The constant r-matrix (2.73) is a solution of (2.59). For the rational Calogero–Moser model, $\mathcal{B} = 0$, this is the classical Yang–Baxter equation. In this case the identification of the r-matrix in terms of a Frobenius subalgebra of gl_n has already been mentioned (2.67). In the hyperbolic/trigonometric cases (2.59) is the modified classical Yang–Baxter equation, whose antisymmetric solutions have been classified by Belavin and Drinfeld [45] for the complex simple Lie algebras. A well-known solution for the Lie algebra sl_n, with the normalization

$$[\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}] = -\hat{\mathcal{F}}, \tag{2.79}$$

is the so-called Cremmer–Gervais classical r-matrix, which we quote from [46] as

$$r_{CG} = \sum_{1\leq i<j\leq n} e_{ij} \wedge e_{ji} + 2 \sum_{1\leq i<j\leq n} \sum_{m=1}^{j-i-1} e_{i,j-m} \wedge e_{j,i+m} + \frac{1}{n} \sum_{1\leq i<j\leq n} (n + 2(i-j)) e_{ii} \wedge e_{jj}. \tag{2.80}$$

Note that $r_{CG} \in sl_n \wedge sl_n$ and $\hat{\mathcal{F}}$ (2.60) belongs to $(sl_n)^{3\wedge}$. Below we show that for $\mathcal{B} \neq 0$ the sl_n-part of the constant Calogero–Moser r-matrix (2.73) is equivalent to r_{CG}.

We shall need the following properties of r_{CG}. As in [46], first introduce $J_0, J_\pm \in sl_n$ by

$$J_0 = \frac{1}{2}\sum_{k=1}^{n}(n+1-2k)e_{kk}, \qquad J_+ = \sum_{k=1}^{n-1}(n-k)e_{k,k+1}, \qquad J_- = \sigma(J_+) = \sum_{k=1}^{n-1} k e_{k+1,k}. \tag{2.81}$$

They generate the principal sl_2 subalgebra of sl_n,

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0. \tag{2.82}$$

Then define the elements $b^\pm_{CG} := \mp\frac{1}{2}[J_\pm \otimes \mathbf{1}_n + \mathbf{1}_n \otimes J_\pm, r_{CG}] \in sl_n \wedge sl_n$. Explicitly,

$$b^+_{CG} = \sum_{k=1}^{n-1} d_k \wedge e_{k,k+1} + \sum_{1\leq i<j\leq n} \sum_{m=1}^{j-i-1} e_{i,j-m+1} \wedge e_{j,i+m}, \qquad d_k := \sum_{j=1}^{k} e_{jj} - \frac{k}{n}\mathbf{1}_n. \tag{2.83}$$

On account of $(\sigma\otimes\sigma)r_{CG} = -r_{CG}$, with σ defined in (2.66), $b^-_{CG} = (\sigma\otimes\sigma)b^+_{CG}$. It has been pointed out in [46] that the subspace of $sl_n \wedge sl_n$ spanned by r_{CG} and $b^\pm_{CG}$ is an irreducible representation of the principal sl_2 subalgebra. In fact, for the operators

$$\mathcal{J}_{0,\pm}(Y) := [J_{0,\pm} \otimes \mathbf{1}_n + \mathbf{1}_n \otimes J_{0,\pm}, Y] \qquad \forall Y \in gl_n \otimes gl_n, \tag{2.84}$$

one has the relations:

$$\mathcal{J}_0 \begin{pmatrix} b^+_{CG} \\ r_{CG} \\ b^-_{CG} \end{pmatrix} = \begin{pmatrix} b^+_{CG} \\ 0 \\ -b^-_{CG} \end{pmatrix}, \quad \mathcal{J}_+ \begin{pmatrix} b^+_{CG} \\ r_{CG} \\ b^-_{CG} \end{pmatrix} = \begin{pmatrix} 0 \\ -2b^+_{CG} \\ r_{CG} \end{pmatrix}, \quad \mathcal{J}_- \begin{pmatrix} b^+_{CG} \\ r_{CG} \\ b^-_{CG} \end{pmatrix} = \begin{pmatrix} -r_{CG} \\ 2b^-_{CG} \\ 0 \end{pmatrix}. \tag{2.85}$$

It follows from these relations that $b^\pm_{CG}$ satisfy the classical Yang–Baxter equation [46], and the identification of $b^\pm_{CG}$ in terms of Frobenius subalgebras of sl_n is also described in this reference.

Now we are prepared to establish the connection between r_{CG} and the r-matrix r' (2.73). The key observation is the following identity:

$$-(T \otimes T)\tilde{r}'_{sl_n} = b^+_{CG} + \mathcal{B}b^-_{CG}, \tag{2.86}$$

where $T : gl_n \to gl_n$ denotes matrix transposition. This can be checked directly from the formulas (2.67), (2.64), (2.83). It permits us to transform $\tilde{r}'_{sl_n}$ into a multiple of r_{CG} in a simple manner. To treat the hyperbolic/trigonometric cases together, we introduce the parameter

$$a' = \begin{cases} a, & \text{hyperbolic case,} \\ \sqrt{-1}a, & \text{trigonometric case,} \end{cases} \tag{2.87}$$

whose square $\mathcal{B} = (a')^2$ appears in (2.59). By using (2.85) it is not difficult to verify that

$$(u_-u_+ \otimes u_-u_+)\left(T \otimes T\tilde{r}'_{sl_n}\right)(u_-u_+ \otimes u_-u_+)^{-1} = a'r_{CG} \tag{2.88}$$

with

$$u_- := \exp\left(\frac{a'}{2}J_-\right), \qquad u_+ := \exp\left(-\frac{1}{a'}J_+\right). \tag{2.89}$$

According to (2.88) the sl_n-part of r' is equivalent to $a'r_{CG}$ under a Lie algebra automorphism.

In the end, notice from (2.68) and (2.81) that

$$X = -\frac{1}{n}(J_+^T + \mathcal{B}J_-^T). \tag{2.90}$$

This allows us to present the r-matrix associated with

$$L'(q,p) = g_0h(q)\tilde{g}(q)L(q,p)(g_0h(q)\tilde{g}(q))^{-1} \tag{2.91}$$

in a 'standard form'. Here $h(q)$ and $\tilde{g}(q)$ are the same as in Theorem 6, and our final result is formulated as follows.

Proposition 2.7 *Consider the hyperbolic/trigonometric Calogero–Moser models. If in Theorem 2.6 the constant g_0 is chosen to be*

$$g_0 = \exp\left(-\frac{a'}{2}J_-^T\right)\exp\left(\frac{1}{a'}J_+^T\right), \tag{2.92}$$

then the r-matrix (2.73) becomes

$$r' = a'(T \otimes T)(r_{CG} + 2(\Omega + \frac{1}{n})J_0 \wedge \mathbf{1}_n). \tag{2.93}$$

Proof. By means of the sl_2 algebra (2.82) and (2.90) it is easy to check that $g_0 X g_0^{-1} = \frac{2a'}{n} J_0^T$. The statement is obtained by combining this with (2.88). *Q.E.D.*

This proposition describes the precise relationship between the most general constant r-matrices of the hyperbolic/trigonometric Calogero–Moser models and the standard Cremmer–Gervais classical r-matrices.

The outcome of our direct analysis of the degenerate Calogero–Moser models is consistent with the previous results [10, 44, 42]. In addition to the advantage that our analysis is elementary, we also clarify the extent to which the constant r-matrix is unique in the degenerate cases. In principle, this uniqueness question cannot be answered by studying the limits of the elliptic case, even though in the final analysis it follows that all our constant r-matrices can be regarded as various degenerations (see also [47]) of Belavin's elliptic r-matrix.

Chapter 3

Canonical dynamical r-matrices

The present chapter contains a detailed study of a particular dynamical r-matrix, which is an important special case of the classical dynamical r-matrices introduced in [15]. Let us recall that Etingof–Varchenko type dynamical r-matrices [15] are associated with any subalgebra $\mathcal{H}$ of any Lie algebra $\mathcal{G}$ (see section 1.2). In most applications $\mathcal{G}$ is a simple Lie algebra and $\mathcal{H}$ is (a subalgebra of) a Cartan subalgebra. Another distinguished special case is is obtained by taking $\mathcal{H} := \mathcal{G}$. We consider this latter case, and allow $\mathcal{G}$ to be any self-dual Lie algebra for which $\mathcal{G}^*$ can be identified with $\mathcal{G}$ by means of an invariant scalar product $\langle\ ,\ \rangle$. We here study the dynamical r-matrix given by the formula

$$r : \omega \mapsto r(\omega) := \langle T_j, f(\mathrm{ad}\omega) T_k\rangle T^j \otimes T^k, \qquad \omega \in \check{\mathcal{G}}, \tag{3.1}$$

where $\check{\mathcal{G}} \subset \mathcal{G}$ is an open subset, $\{T_j\}$ and $\{T^k\}$ denote dual bases of $\mathcal{G}$, $\langle T_j, T^k\rangle = \delta_j^k$, and f is the complex analytic function defined by

$$f(z) := \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z}, \qquad z \in \mathbb{C} \setminus 2\pi i \mathbb{Z}^*. \tag{3.2}$$

It is known that this r-matrix is a solution of the mCDYBE (1.23) for $\mathcal{H} = \mathcal{G} \simeq \mathcal{G}^*$ with

$$\varphi = -\frac{1}{4} f^l_{jk} T^j \otimes T^k \otimes T_l, \qquad [T_j, T_k] = f^l_{jk} T_l. \tag{3.3}$$

If $\mathcal{G}$ is a simple Lie algebra, then the mCDYBE for r in (3.1) follows from a general result (Theorem 3.14) in [15]. Remarkably, this r-matrix came to light naturally in two different applications, namely in the context of equivariant-cohomology [12] and in the description of a Poisson structure on the chiral WZNW phase space compatible with classical $\mathcal{G}$-symmetry [13]. A further reason for which the r-matrix in (3.1) is important is that it can be reduced to certain self-dual subalgebras of $\mathcal{G}$, and thereby serves as a common 'source' for a large family of dynamical r-matrices [48]. We call it the canonical r-matrix of the self-dual Lie algebra $\mathcal{G}$.

The authors of [12] assumed $\mathcal{G}$ to be compact, while in [13] $\mathcal{G}$ was taken to be a simple Lie algebra. In these papers the mCDYBE for the canonical r-matrix was proved, independently, using the additional assumption that ω is near to zero, so that $f(\mathrm{ad}\omega)$ is given with the aid of the power series expansion of $f(z)$ around $z = 0$. Though this is not obvious, the proofs found in [12, 13] (see also [49, 50]) can in fact be adapted to cover the case of a general self-dual Lie algebra as well. In this case, a different proof of the mCDYBE appeared in [51]. This proof is indirect and uses the restriction of ω to a neighbourhood of the origin. The maximal domain of definition of $f(\mathrm{ad}\omega)$ contains all ω for which the eigenvalues of $\mathrm{ad}\omega$ lie in $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. Although the above-mentioned local proofs and the analyticity of $r(\omega)$ together imply the mCDYBE on this domain, it could be enlightening to have an alternative direct proof, too.

After this introduction, the chapter consists of 2 sections. The proof of the mCDYBE is described in section 3.1. It relies on some technical material collected in appendix D, E, and F. Appendix D is a recall of relevant basics of the functional calculus from [52]. Section 3.2 is devoted to a discussion of consequences of the proof, including the above-mentioned uniqueness result for the function f, and some comments.

3.1 Proof of the mCDYBE for the canonical r-matrix

Let $\mathcal{G}$ be a finite-dimensional complex Lie algebra equipped with an invariant, symmetric, nondegenerate bilinear form $\langle\ ,\ \rangle$. For the structure of such Lie algebras, see e.g. [53]. We call these Lie algebras self-dual, since we identify $\mathcal{G}$ with $\mathcal{G}^*$ by means of the 'scalar product' $\langle\ ,\ \rangle$. Defining the transpose A^T of an operator $A \in \mathrm{End}\,(\mathcal{G})$ by $\langle A^T X, Y\rangle = \langle X, AY\rangle$ $(\forall X, Y \in \mathcal{G})$, the invariance property of $\langle\ ,\ \rangle$ means that $(\mathrm{ad}\omega)^T = -\mathrm{ad}\omega$ $(\forall \omega \in \mathcal{G})$, where $(\mathrm{ad}\omega)(X) = [\omega, X]$.

Consider a map $r : \check{\mathcal{G}} \to \mathcal{G} \otimes \mathcal{G}$, where $\check{\mathcal{G}} \subset \mathcal{G}$ is a nonempty open subset. Then there exists a unique map $R : \check{\mathcal{G}} \to \mathrm{End}(\mathcal{G})$ for which

$$r(\omega) = \langle T_j, R(\omega) T_k\rangle T^j \otimes T^k, \qquad \forall \omega \in \check{\mathcal{G}}, \tag{3.4}$$

where $\{T_j\}$ and $\{T^k\}$ denote dual bases of $\mathcal{G}$. The directional derivatives of R are given by

$$(\nabla_S R)(\omega) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R(\omega + tS)\,, \qquad \forall S \in \mathcal{G},\, \omega \in \check{\mathcal{G}}, \tag{3.5}$$

and the ‘gradient’ of R is defined by

$$\langle X, (\nabla R)(\omega) Y \rangle := T^j \left\langle X, \left(\nabla_{T_j} R\right)(\omega) Y \right\rangle, \qquad \forall X, Y \in \mathcal{G},\, \omega \in \check{\mathcal{G}}. \tag{3.6}$$

If r is antisymmetric, i.e., $R^T(\omega) = -R(\omega)$, then the mCDYBE (1.23) for r with φ in (3.3) is in fact equivalent to the following equation for R:

$$\begin{aligned}&\frac{1}{4}[X, Y] + [R(\omega) X, R(\omega) Y] - R(\omega)([R(\omega) X, Y] + [X, R(\omega) Y]) \\ &\quad + \langle X, (\nabla R)(\omega) Y\rangle + (\nabla_Y R)(\omega) X - (\nabla_X R)(\omega) Y = 0, \quad \forall X, Y \in \mathcal{G},\, \omega \in \check{\mathcal{G}}.\end{aligned} \tag{3.7}$$

The $\mathcal{G}$-equivariance of the map $r : \check{\mathcal{G}} \to \mathcal{G} \otimes \mathcal{G}$ can be expressed as

$$(\nabla_{[S,\omega]} R)(\omega) = [\mathrm{ad}S, R(\omega)] \qquad \forall S \in \mathcal{G},\, \omega \in \check{\mathcal{G}}. \tag{3.8}$$

After these remarks, we are ready to study the canonical r-matrix. From now on we use

$$\check{\mathcal{G}} := \{\, \omega \in \mathcal{G} \mid \sigma(\mathrm{ad}\omega) \cap 2\pi i \mathbb{Z}^* = \emptyset \,\}, \tag{3.9}$$

which is a nonempty open subset in $\mathcal{G}$. Here and below $\sigma(\mathrm{ad}\omega)$ denotes the spectrum of $\mathrm{ad}\omega$ ($\forall \omega \in \mathcal{G}$), and sometimes we use the notation $\bar{\omega} := \mathrm{ad}\omega$ for brevity. With the aid of the familiar holomorphic functional calculus (see appendix D), we can define an operator valued dynamical r-matrix $R\colon \check{\mathcal{G}} \to \mathrm{End}(\mathcal{G})$ by

$$\omega \mapsto R(\omega) := f(\mathrm{ad}\omega) = \frac{1}{2\pi i} \int_C d\xi f(\xi)(\xi I - \mathrm{ad}\omega)^{-1}, \tag{3.10}$$

where f is given in (3.2). The curve C encircles each eigenvalue of $\mathrm{ad}\omega$ and I is the identity operator on $\mathcal{G}$. Now our main theorem can be formulated as follows.

Theorem 3.1 *The mapping (3.10) with f in (3.2) defines an antisymmetric r-matrix which satisfies the equivariance condition (3.8) and the mCDYBE given by (3.7).*

The antisymmetry of the r-matrix follows from (3.10) by using that f is an odd function, and the equivariance condition (3.8) is also an immediate consequence of (3.10) (cf. (D.3)). Before verifying (3.7), we gather some useful information and lemmas that make the calculations easier.

Let ω be an arbitrary fixed element of $\check{\mathcal{G}}$. For every $\lambda \in \mathbb{C}$, let $b_\lambda := \mathrm{ad}\omega - \lambda I = \bar{\omega} - \lambda I \in \mathrm{End}(\mathcal{G})$. Thanks to the derivation property of $\mathrm{ad}\omega$, the b_λ's enjoy the identities

$$b^n_{\alpha+\beta}[X, Y] = \sum_{j=0}^{n} \binom{n}{j} \left[b^j_\alpha X, b^{n-j}_\beta Y\right], \quad \forall X, Y \in \mathcal{G}, \forall \alpha, \beta \in \mathbb{C}. \tag{3.11}$$

By means of the $\mathcal{G} = \oplus_{\lambda\in\sigma(\bar{\omega})} N_\lambda$ Jordan decomposition, where $N_\lambda = \mathrm{Ker}\left(b_\lambda^{\nu(\lambda)}\right)$ (see appendix D), the r-matrix (3.10) can be written as

$$R(\omega) = f(\bar{\omega}) = \sum_{\lambda\in\sigma(\bar{\omega})} \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k E_\lambda. \tag{3.12}$$

We can regard this equation as the application of (D.4) to the operator $\mathrm{ad}\omega$. Here $E_\lambda \in \mathrm{End}(\mathcal{G})$ means the projection corresponding to the subspace N_λ. Note also that $[N_\lambda, N_\mu] \subset N_{\lambda+\mu}$ is implied by (3.11), with $N_\mu = \{0\}$ for any $\mu \notin \sigma(\bar{\omega})$.

The mCDYBE (3.7) is linear in X and Y. Therefore it is enough to prove this equation when $X \in N_\lambda$, $Y \in N_\mu$ are arbitrary elements of the subspaces associated with the eigenvalues $\lambda, \mu \in \sigma(\bar{\omega})$. So, from now on let λ, μ be arbitrary, fixed eigenvalues of $\bar{\omega}$ and $X \in N_\lambda$, $Y \in N_\mu$ arbitrary, but fixed vectors. Applying the r-matrix (3.12) on these vectors, we obtain

$$\begin{aligned} R(\omega) X &= f(\bar{\omega}) X = \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k X, \\ R(\omega) Y &= f(\bar{\omega}) Y = \sum_{l=0}^{\nu(\mu)-1} \frac{f^{(l)}(\mu)}{l!} b_\mu^l Y. \end{aligned} \tag{3.13}$$

In the subsequent four lemmas we calculate the various terms of the mCDYBE (3.7) in a form that will prove convenient for verifying this equation. In all expressions containing $\left[b_\lambda^k X, b_\mu^l Y\right]$ it is understood that the indices k, l vary as in (3.13).

Lemma 3.2 *If $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, then*

$$\frac{1}{4}[X, Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{1}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}, \tag{3.14}$$

$$[f(\bar{\omega}) X, f(\bar{\omega}) Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha) f(\beta) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}, \tag{3.15}$$

$$f(\bar{\omega})[f(\bar{\omega}) X, Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha+\beta) f(\alpha) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}, \tag{3.16}$$

$$f(\bar{\omega})[X, f(\bar{\omega}) Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha+\beta) f(\beta) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{3.17}$$

Proof. First, identity (F.3) from appendix F leads immediately to (3.14) as

$$\frac{1}{4}[X, Y] = \frac{1}{4}\left[b_\lambda^0 X, b_\mu^0 Y\right] = \sum_{k,l} \frac{\delta_{k,0}\delta_{l,0}}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} = \sum_{k,l} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{1}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{3.18}$$

Second, with the aid of (3.13) and (F.4), we easily obtain (3.15)

$$
\begin{aligned}
\left[f\left(\bar{\omega}\right)X, f\left(\bar{\omega}\right)Y\right] &= \sum_{k,l} f^{(k)}\left(\lambda\right)f^{(l)}\left(\mu\right)\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \\
&= \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l} f\left(\alpha\right)f\left(\beta\right)\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \qquad (3.19)
\end{aligned}
$$

Third, the calculation of

$$
f\left(\bar{\omega}\right)\left[f\left(\bar{\omega}\right)X, Y\right] = f\left(\bar{\omega}\right)\left[\sum_k \frac{f^{(k)}\left(\lambda\right)}{k!} b_\lambda^k X, Y\right] \qquad (3.20)
$$

goes as follows. Since $\left[\sum_k \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k X, Y\right] \in N_{\lambda+\mu}$, (3.13) yields

$$
\begin{aligned}
f\left(\bar{\omega}\right)\left[f\left(\bar{\omega}\right)X, Y\right] &= \sum_{k,l} \frac{f^{(k)}\left(\lambda\right)f^{(l)}\left(\lambda+\mu\right)}{k!l!} b_{\lambda+\mu}^l \left[b_\lambda^k X, Y\right] \\
&= \sum_{k,l} \frac{f^{(k)}\left(\lambda\right)f^{(l)}\left(\lambda+\mu\right)}{k!l!} \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} \left[b_\lambda^{k+l-j} X, b_\mu^j Y\right], \qquad (3.21)
\end{aligned}
$$

where we used (3.11). Introducing a new variable $s := k + l$ for the summation, we have

$$
\begin{aligned}
f\left(\bar{\omega}\right)\left[f\left(\bar{\omega}\right)X, Y\right] &= \sum_s \sum_{j=0}^{s} \sum_{l=j}^{s} \begin{pmatrix} l \\ j \end{pmatrix} \frac{f^{(s-l)}\left(\lambda\right)f^{(l)}\left(\lambda+\mu\right)}{(s-l)!l!} \left[b_\lambda^{s-j} X, b_\mu^j Y\right] \qquad (3.22) \\
&= \sum_s \sum_{j=0}^{s} \sum_{l=0}^{s-j} \begin{pmatrix} l+j \\ j \end{pmatrix} \frac{f^{(j+l)}\left(\lambda+\mu\right)f^{(s-j-l)}\left(\lambda\right)}{(l+j)!\left(s-j-l\right)!} \left[b_\lambda^{s-j} X, b_\mu^j Y\right] \\
&= \sum_s \sum_{j=0}^{s} \frac{1}{j!\left(s-j\right)!} \sum_{l=0}^{s-j} \begin{pmatrix} s-j \\ l \end{pmatrix} f^{(j+l)}\left(\lambda+\mu\right)f^{(s-j-l)}\left(\lambda\right)\left[b_\lambda^{s-j} X, b_\mu^j Y\right].
\end{aligned}
$$

Using the Leibniz rule and introducing new summation variables as $l := j$, $k := s - j$, we obtain

$$
\begin{aligned}
f\left(\bar{\omega}\right)\left[f\left(\bar{\omega}\right)X, Y\right] &= \sum_s \sum_{j=0}^{s} \frac{1}{j!\left(s-j\right)!} \left.\frac{\mathrm{d}^{s-j}}{\mathrm{d}\xi^{s-j}}\right|_{\xi=\lambda} f^{(j)}\left(\xi+\mu\right)f\left(\xi\right)\left[b_\lambda^{s-j} X, b_\mu^j Y\right] \\
&= \sum_{k,l} \left.\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\right|_{\xi=\lambda} f^{(l)}\left(\xi+\mu\right)f\left(\xi\right)\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \qquad (3.23)
\end{aligned}
$$

By (F.5), this gives (3.16). Finally, (3.17) is trivial consequence of (3.16). *Q.E.D.*

Lemma 3.3 *If $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, then*

$$\langle X, (\nabla R)(\omega) Y\rangle = -\sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{f(\alpha)+f(\beta)}{\alpha+\beta} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{3.24}$$

Proof. We obtain directly from the definitions (3.5), (3.6), (3.10) (see also (D.3)) that

$$\langle X, (\nabla R)(\omega) Y\rangle = \frac{1}{2\pi i} \int_C \mathrm{d}\xi f(\xi)\, T^j \langle X, \rho_\xi(\bar{\omega}) [T_j, \rho_\xi(\bar{\omega}) Y]\rangle, \tag{3.25}$$

where $\rho_\xi(\bar{\omega}) = (\xi I - \bar{\omega})^{-1}$. By using that $\rho_\xi(\bar{\omega})^T = -\rho_{-\xi}(\bar{\omega})$ and the invariance of $\langle\ ,\ \rangle$, this expression is easily converted into

$$\langle X, (\nabla R)(\omega) Y\rangle = \frac{1}{2\pi i} \int_C \mathrm{d}\xi f(\xi) \left[\rho_{-\xi}(\bar{\omega}) X, \rho_\xi(\bar{\omega}) Y\right]. \tag{3.26}$$

We can apply the functional calculus to the holomorphic function $\rho_\xi : (\mathbb{C} \setminus \{\xi\}) \to \mathbb{C}$ defined by $\rho_\xi : z \mapsto (\xi - z)^{-1}$. Thus we have

$$\rho_{-\xi}(\bar{\omega}) X = \sum_k \frac{\rho_{-\xi}^{(k)}(\lambda)}{k!} b_\lambda^k X, \qquad \rho_\xi(\bar{\omega}) Y = \sum_l \frac{\rho_\xi^{(l)}(\mu)}{l!} b_\mu^l Y, \tag{3.27}$$

similarly to (3.13). Since $\rho_{-\xi}^{(k)}(\lambda) = k!\,(-\xi-\lambda)^{-(k+1)} = (-1)^{k+1} \rho_\xi(-\lambda)$, this leads to

$$\langle X, (\nabla R)(\omega) Y\rangle = \sum_{k,l} \left(\frac{(-1)^{k+1}}{2\pi i} \int_C \mathrm{d}\xi f(\xi)\, \rho_\xi^{(k)}(-\lambda)\, \rho_\xi^{(l)}(\mu) \right) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{3.28}$$

Now our task is to determine these integrals. Obviously, two different cases can appear. When $-\lambda = \mu$, the integrands have poles only at the point μ. Alternatively, when $-\lambda \neq \mu$, the integrands have poles at the point $-\lambda$ and at the point μ.

The $\lambda + \mu = 0$ case. In this case $\rho_\xi^{(k)}(-\lambda)\, \rho_\xi^{(l)}(\mu) = k!l!\,(\xi - \mu)^{-(k+l+1)-1}$. Thanks to Cauchy's theorem, the integrals can be written as

$$\begin{aligned} \frac{(-1)^{k+1}}{2\pi i} \int_C \mathrm{d}\xi f(\xi)\, \rho_\xi^{(k)}(-\lambda)\, \rho_\xi^{(l)}(\mu) &= \frac{(-1)^{k+1} k!l!}{2\pi i} \int_C \mathrm{d}\xi \frac{f(\xi)}{(\xi-\mu)^{(k+l+1)+1}} = \\ = \frac{(-1)^{k+1} k!l!}{(k+l+1)!} f^{(k+l+1)}(\mu) &= -\lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{f(\alpha)+f(\beta)}{\alpha+\beta}, \end{aligned} \tag{3.29}$$

where we used the identity (F.8). Thus (3.24) is valid in this case.

The $\lambda+\mu\neq 0$ *case.* By C_α we denote a sufficiently small circle around the eigenvalue $\alpha\in\sigma(\bar{\omega})$, which encircles this point in the positive sense. Using Cauchy's theorem in (3.28), we can write

$$\begin{aligned}
&\frac{(-1)^{k+1}}{2\pi i}\int_C \mathrm{d}\xi f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) = \\
&\quad = (-1)^{k+1}\left\{\frac{1}{2\pi i}\int_{C_\mu}\mathrm{d}\xi f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) + \frac{1}{2\pi i}\int_{C_{-\lambda}}\mathrm{d}\xi f(\xi)\,\rho_\xi^{(l)}(\mu)\,\rho_\xi^{(k)}(-\lambda)\right\} \\
&\quad = (-1)^{k+1}\left\{\left.\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\right|_{\xi=\mu} f(\xi)\,(-1)^{k+1}\,\rho_{-\lambda}^{(k)}(\xi) + \left.\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\right|_{\xi=-\lambda} f(\xi)\,(-1)^{l+1}\,\rho_\mu^{(l)}(\xi)\right\} \\
&\quad = (-1)^{k}\left\{\sum_{a=0}^{l}(-1)^k\begin{pmatrix} l \\ a\end{pmatrix} f^{(a)}(\mu)\,\rho_{-\lambda}^{(k+l-a)}(\mu) + \sum_{b=0}^{k}(-1)^l\begin{pmatrix} k \\ b\end{pmatrix} f^{(b)}(-\lambda)\,\rho_{\mu}^{(k+l-b)}(-\lambda)\right\} \\
&\quad = -(-1)^{k+l}\sum_{a=0}^{l}\begin{pmatrix} l \\ a\end{pmatrix}(k+l-a)!\,(-1)^a\,\frac{f^{(a)}(\mu)}{(\lambda+\mu)^{k+l+1-a}} \\
&\qquad -(-1)^{k+l}\sum_{b=0}^{k}\begin{pmatrix} k \\ b\end{pmatrix}(k+l-b)!\,(-1)^b\,\frac{f^{(b)}(\lambda)}{(\lambda+\mu)^{k+l+1-b}}.
\end{aligned}\tag{3.30}$$

Comparing this equation with (F.7), we see that when $\lambda+\mu\neq 0$

$$\frac{(-1)^{k+1}}{2\pi i}\int_C \mathrm{d}\xi f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) = -\left.\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\right|_{(\alpha,\beta)=(\lambda,\mu)}\frac{f(\alpha)+f(\beta)}{\alpha+\beta}.\tag{3.31}$$

Thus the proof of the lemma is complete. *Q.E.D.*

Lemma 3.4 *If* $\lambda,\mu\in\sigma(\bar{\omega})$, $X\in N_\lambda$, $Y\in N_\mu$, *then*

$$(\nabla_X R)(\omega)\,Y = \sum_{k,l}\lim_{(\alpha,\beta)\to(\lambda,\mu)}\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\frac{f(\alpha+\beta)-f(\beta)}{\alpha}\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}.\tag{3.32}$$

Proof. As a consequence of (D.3), the left hand side of (3.32) can be written as

$$(\nabla_X R)(\omega)\,Y = \frac{1}{2\pi i}\int_C \mathrm{d}\xi f(\xi)\,\rho_\xi(\bar{\omega})\left[X,\rho_\xi(\bar{\omega})\,Y\right].\tag{3.33}$$

The application of the functional calculus (see also (3.17) and (F.6)) gives

$$\rho_\xi(\bar{\omega})\left[X,\rho_\xi(\bar{\omega})\,Y\right] = \sum_{k,l}\left.\frac{\mathrm{d}^l}{\mathrm{d}\eta^l}\right|_{\eta=\mu}\rho_\xi^{(k)}(\lambda+\eta)\,\rho_\xi(\eta)\,\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}.\tag{3.34}$$

Therefore,

$$\left(\nabla_X R\right)(\omega) Y = \sum_{k,l} \left\{ \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} \frac{1}{2\pi i} \int_C \mathrm{d}\xi f(\xi) \rho_\xi^{(k+l-j)} (\lambda+\mu) \rho_\xi^{(j)} (\mu) \right\} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k! l!}. \quad (3.35)$$

When $\lambda = 0$, the integrands have poles only at the point μ. If $\lambda \neq 0$, then the integrands have poles at the points $\lambda + \mu$ and μ.

The $\lambda = 0$ case. In this case $\rho_\xi^{(k+l-j)} (\lambda+\mu) \rho_\xi^{(j)} (\mu) = (k+l-j)! j! (\xi-\mu)^{-(k+l+1)-1}$. Thus

$$\begin{aligned}
&\sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} \frac{1}{2\pi i} \int_C \mathrm{d}\xi f(\xi) \rho_\xi^{(k+l-j)} (\lambda+\mu) \rho_\xi^{(j)} (\mu) = \\
&\quad = \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} (k+l-j)! j! \frac{f^{(k+l+1)}(\mu)}{(k+l+1)!} = \frac{k! l! f^{(k+l+1)}(\mu)}{(k+l+1)!} \sum_{j=0}^{l} \begin{pmatrix} (k+l)-j \\ (k+l)-l \end{pmatrix} \\
&\quad = \frac{k! l! f^{(k+l+1)}(\mu)}{(k+l+1)!} \begin{pmatrix} k+l+1 \\ l \end{pmatrix} = \frac{f^{(k+l+1)}(\mu)}{k+1} \\
&\quad = \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial \alpha^k \partial \beta^l} \frac{f(\alpha+\beta) - f(\beta)}{\alpha},
\end{aligned} \quad (3.36)$$

where we used the combinatorial identity (E.2) and (F.14). So in this case (3.32) holds.

The $\lambda \neq 0$ case. Denote by C_α a sufficiently small circle around $\alpha \in \sigma(\bar{\omega})$. Then, by Cauchy's theorem, the relevant integrals in (3.35) give

$$\begin{aligned}
&\frac{1}{2\pi i} \int_C \mathrm{d}\xi f(\xi) \rho_\xi^{(k+l-j)} (\lambda+\mu) \rho_\xi^{(j)} (\mu) = \\
&\quad = \frac{1}{2\pi i} \int_{C_\mu} \mathrm{d}\xi f(\xi) \rho_\xi^{(k+l-j)} (\lambda+\mu) \rho_\xi^{(j)} (\mu) + \frac{1}{2\pi i} \int_{C_{\lambda+\mu}} \mathrm{d}\xi f(\xi) \rho_\xi^{(j)} (\mu) \rho_\xi^{(k+l-j)} (\lambda+\mu) \\
&\quad = \left.\frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\right|_{\xi=\mu} f(\xi) \rho_\xi^{(k+l-j)} (\lambda+\mu) + \left.\frac{\mathrm{d}^{k+l-j}}{\mathrm{d}\xi^{k+l-j}}\right|_{\xi=\lambda+\mu} f(\xi) \rho_\xi^{(j)} (\mu) \\
&\quad = (-1)^{k+l-j+1} \sum_{a=0}^{j} \begin{pmatrix} j \\ a \end{pmatrix} (k+l-a)! \frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}} \\
&\qquad + \sum_{b=0}^{k+l-j} \begin{pmatrix} k+l-j \\ b \end{pmatrix} (j+b)! (-1)^b \frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}}.
\end{aligned} \quad (3.37)$$

Thus the coefficient of $\left[b_\lambda^k X, b_\mu^l Y\right]/k!l!$ in (3.35) is equal to the following expression:

$$\sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\left\{(-1)^{k+l-j+1}\sum_{a=0}^{j}\begin{pmatrix} j \\ a \end{pmatrix}(k+l-a)!\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}\right.$$
$$\left.+\sum_{b=0}^{k+l-j}\begin{pmatrix} k+l-j \\ b \end{pmatrix}(j+b)!\,(-1)^b\,\frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}}\right\}. \tag{3.38}$$

Firstly, do the summation of the first part of (3.38):

$$\begin{aligned}
\mathrm{Part1}(k,l) &:= \sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}(-1)^{k+l+1-j}\sum_{a=0}^{j}\begin{pmatrix} j \\ a \end{pmatrix}(k+l-a)!\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}} = \\
&= (-1)^{k+l+1}\sum_{a=0}^{l}\frac{(k+l-a)!l!}{a!\,(l-a)!}\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}(-1)^a\sum_{j=0}^{l-a}\begin{pmatrix} l-a \\ j \end{pmatrix}(-1)^j \\
&= (-1)^{k+l+1}\sum_{a=0}^{l}\frac{(k+l-a)!l!}{a!\,(l-a)!}\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}(-1)^a\,\delta_{l-a,0} = -(-1)^k\,k!\frac{f^{(l)}(\mu)}{\lambda^{k+1}}.
\end{aligned} \tag{3.39}$$

Secondly, do the summation of the second part of (3.38). Introducing a new variable $m := j+b$, we obtain

$$\begin{aligned}
\mathrm{Part2}(k,l) &:= \sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\sum_{b=0}^{k+l-j}\begin{pmatrix} k+l-j \\ b \end{pmatrix}(j+b)!\,(-1)^b\,\frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}} = \\
&= -\sum_{j=0}^{l}\sum_{b=0}^{k+l-j}(-1)^{m+1}\,\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} k+l-j \\ k \end{pmatrix} \\
&= -\sum_{m=0}^{l}(-1)^{m+1}\,\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} k+l-j \\ k \end{pmatrix} \\
&\quad -\sum_{m=l+1}^{k+l}(-1)^{m+1}\,\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}\sum_{j=0}^{l}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} k+l-j \\ k \end{pmatrix}
\end{aligned} \tag{3.40}$$

By means of the combinatorial identities (E.3), (E.10), we can simplify this formula. In fact, after a straightforward further computation, we get

$$\mathrm{Part2}(k,l) = -\sum_{m=0}^{k}(-1)^{m+1}\,\frac{k!}{(k-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}. \tag{3.41}$$

Now collecting equations (3.41), (3.39), (3.38), (3.35), in the $\lambda \neq 0$ case we can write

$$\begin{aligned}
\left(\nabla_X R\right)(\omega) Y &= \sum_{k,l} \{\mathrm{Part1}(k,l) + \mathrm{Part2}(k,l)\} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \\
&= \sum_{k,l} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l}\bigg|_{(\alpha,\beta)=(\lambda,\mu)} \frac{f(\alpha+\beta) - f(\beta)}{\alpha} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}, \qquad (3.42)
\end{aligned}$$

since equation (F.13) is valid. Hence Lemma 3.4 is proved. *Q.E.D.*

Lemma 3.5 *If $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, then*

$$\left(\nabla_Y R\right)(\omega) X = -\sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{f(\alpha+\beta) - f(\alpha)}{\beta} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \qquad (3.43)$$

Proof. This is a trivial consequence of the preceding lemma.

Now we are in the position to verify the mCDYBE (3.7) for the canonical r-matrix (3.10).

Proof of Theorem 3.1. Let $\lambda, \mu \in \sigma(\bar{\omega})$ and $X \in N_\lambda$, $Y \in N_\mu$. By applying the four lemmas, the left hand side of (3.7) can be written as

$$\begin{aligned}
&\frac{1}{4}[X,Y] + [R(\omega)X, R(\omega)Y] - R(\omega)\left([R(\omega)X, Y] + [X, R(\omega)Y]\right) \\
&\quad + \langle X, (\nabla R)(\omega) Y\rangle + (\nabla_Y R)(\omega) X - (\nabla_X R)(\omega) Y = \\
&= \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \left(\frac{1}{4} + f(\alpha) f(\beta) - f(\alpha+\beta)\left(f(\alpha) + f(\beta)\right)\right. \\
&\quad \left. - \frac{f(\alpha) + f(\beta)}{\alpha+\beta} - \frac{f(\alpha+\beta) - f(\alpha)}{\beta} - \frac{f(\alpha+\beta) - f(\beta)}{\alpha}\right) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \qquad (3.44)
\end{aligned}$$

This equals zero since the 'addition formula' (F.1) is valid for the function f in (3.2). *Q.E.D.*

3.2 Discussion

We have shown that the canonical r-matrix defined by (3.10) with f in (3.2) satisfies the mCDYBE (3.7). It is worth noticing that our proof implies a uniqueness result as well. Suppose that we wish to define an antisymmetric solution of the mCDYBE (3.7) by the functional calculus, i.e., by using *some* holomorphic complex function in formula (3.10) now considered as an *ansatz*. For this formula to be well defined, the domain of holomorphicity of the function f

must contain zero, since this is always an eigenvalue of $\mathrm{ad}\omega$. Moreover, for R to be antisymmetric, which is in turn necessary for the equivalence of (3.7) to (1.23) with φ in (3.3), f must be an odd function. Under these assumptions, the mCDYBE (3.7) for the ansatz (3.10) is in fact equivalent to the functional equation (F.1) for the unknown function f. Indeed, the whole calculation described in section 2.1 is valid for such an ansatz up to the equality in (3.44). The point then is that the functional equation (F.1) has a unique odd solution around the origin. The proof of this statement is quite easy. By taking the $y \to 0$ limit in (F.1) we obtain the differential equation for f which appears in (F.2). With the initial value $f(0) = 0$, which is implied by f being odd, this differential equation has a unique, holomorphic solution around the origin, namely the function $f(x) = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x}$.

So far we assumed the Lie algebra $\mathcal{G}$ to be complex, but the mCDYBE can be considered for a real self-dual Lie algebra, too. The real case arises naturally in applications [12, 13]. Let us now suppose that $\mathcal{G}$ is the complexification of a real self-dual Lie algebra, say $\mathcal{G}_r$. Then it is not difficult to see that $R(\omega)$ given by (3.10) maps $\mathcal{G}_r$ to $\mathcal{G}_r$ if $\omega \in \mathcal{G}_r$. This is obviously the case if ω is near to zero, where one can apply the power series expansion of f around zero to define $R(\omega)$. More generally, if $\omega \in \mathcal{G}_r$ then one may take the curve C in (3.10) to be invariant under complex conjugation as the eigenvalues of $\mathrm{ad}\omega$ occur in conjugate pairs. By using this and $f(\bar{z}) = \bar{f}(z)$, complex conjugation of (3.10) shows that $R(\omega)X \in \mathcal{G}_r$ if $\omega \in \mathcal{G}_r$ and $X \in \mathcal{G}_r$. Thus the canonical r-matrix is a solution of the mCDYBE (3.7) in the real case as well.

Our use of the functional calculus, which is applicable to Banach algebras in general [52], in the definition (3.10) might serve as a starting point for future work towards generalizations of this canonical r-matrix to certain infinite-dimensional self-dual Banach Lie algebras. However, this represents a nontrivial problem since the above-presented proof of Theorem 3.1 relies heavily on the finite-dimensionality of $\mathcal{G}$.

Chapter 4

Generalizations of Felder's elliptic dynamical r-matrices

The classical dynamical Yang–Baxter equation (1.23) introduced in its general form by Etingof and Varchenko [15] is a remarkable generalization of the classical Yang–Baxter equation (CYBE). Currently we are witnessing intense research on the theory and the applications of the CDYBE to integrable systems [16, 37, 54]. For a review, see [22].

The aim of this chapter is to study infinite-dimensional generalizations of a certain class of finite-dimensional classical dynamical r-matrices. Next we briefly recall these finite-dimensional r-matrices, which appear naturally in the chiral WZNW model (see e.g. [50] and references therein).

Let $\mathcal{A}$ be a finite-dimensional complex Lie algebra equipped with a nondegenerate, symmetric, invariant bilinear form $\langle\ ,\ \rangle$. Such a Lie algebra is called self-dual [53]. Consider a self-dual subalgebra $\mathcal{K} \subset \mathcal{A}$, on which $\langle\ ,\ \rangle$ remains nondegenerate. Introduce the complex analytic functions f and F by

$$f : z \mapsto \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z}, \qquad F : z \mapsto \frac{1}{2}\coth\frac{z}{2}. \tag{4.1}$$

Suppose that $\check{\mathcal{K}}$ is a nonempty open subset of $\mathcal{K}$ on which the operator valued function $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ is defined by

$$R(\kappa) := \begin{cases} f(\mathrm{ad}\,\kappa) & \text{on } \mathcal{K} \\ F(\mathrm{ad}\,\kappa) & \text{on } \mathcal{K}^\perp \end{cases} \qquad \forall \kappa \in \check{\mathcal{K}}. \tag{4.2}$$

The decomposition $\mathcal{A} = \mathcal{K} + \mathcal{K}^\perp$ is induced by $\langle\ ,\ \rangle$. $R(\kappa)$ is a well defined linear operator on $\mathcal{A}$ if and only if the spectrum of $\mathrm{ad}\,\kappa$, acting on $\mathcal{A}$, does not intersect $2\pi i\mathbb{Z}^*$, and $(\mathrm{ad}\,\kappa)\,|_{\mathcal{K}^\perp}$

is invertible. On $\check{\mathcal{K}} \subset \mathcal{K}$ subject to these conditions, the following (modified) version of the CDYBE holds:

$$[RX, RY] - R([X, RY] + [RX, Y]) + \langle X, (\nabla R)Y\rangle + (\nabla_{Y_{\mathcal{K}}} R)X - (\nabla_{X_{\mathcal{K}}} R)Y \\ = -\frac{1}{4}[X, Y], \qquad \forall X, Y \in \mathcal{A}. \tag{4.3}$$

Here the 'dynamical variable' κ is suppressed for brevity, $\forall X \in \mathcal{A}$ is decomposed as $X = X_{\mathcal{K}} + X_{\mathcal{K}^\perp}$, and

$$(\nabla_T R)(\kappa) := \frac{d}{dt} R(\kappa + tT)|_{t=0} \qquad \forall T \in \mathcal{K}, \quad \kappa \in \check{\mathcal{K}}, \tag{4.4}$$

$$\langle X, (\nabla R)(\kappa)Y\rangle := \sum_i K^i \langle X, (\nabla_{K_i} R)(\kappa)Y\rangle, \qquad \forall X, Y \in \mathcal{A}, \tag{4.5}$$

where K_i and K^i denote dual bases of $\mathcal{K}$, $\langle K_i, K^j\rangle = \delta_i^j$. $R(\kappa)$ is antisymmetric, $\langle R(\kappa)X, Y\rangle = -\langle X, R(\kappa)Y\rangle$, and is $\mathcal{K}$-equivariant in the sense that

$$(\nabla_{[T,\kappa]} R)(\kappa) = [\mathrm{ad}\, T, R(\kappa)], \qquad \forall T \in \mathcal{K}, \kappa \in \check{\mathcal{K}}. \tag{4.6}$$

These properties of R have been established in this general setting in [50, 48]. In various special cases — in particular the case $\mathcal{K} = \mathcal{A}$ — they were proved earlier in [15, 12, 13]; this was the main topic of Chapter 3. If one introduces $r^\pm : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ by

$$r^\pm(\kappa) := (R(\kappa)T_\alpha) \otimes T^\alpha \pm \frac{1}{2} T_\alpha \otimes T^\alpha, \tag{4.7}$$

where $\{T_\alpha\}$ and $\{T^\alpha\}$ are dual bases of $\mathcal{A}$, and uses the identification $\mathcal{K} \simeq \mathcal{K}^*$ induced by $\langle\ ,\ \rangle$, then the above properties of R become the CDYBE for $r^\pm$ with respect to the pair $\mathcal{K} \subset \mathcal{A}$ as defined in [15] (see also (1.23)).

It is natural to suspect that whenever (4.2) is a well defined formula, the resulting r-matrix always satisfies (4.3). For this it is certainly not necessary to assume that $\mathcal{A}$ is finite dimensional. For example, Etingof and Varchenko [15] verified the CDYBE in the situation for which $\mathcal{A}$ is an affine Lie algebra based on a simple Lie algebra and $\mathcal{K} \subset \mathcal{A}$ is a Cartan subalgebra. Moreover, by applying evaluation homomorphisms to these r-matrices they recovered Felder's celebrated spectral-parameter-dependent elliptic dynamical r-matrices [16]. Without presenting proofs, this construction was generalized in [48] to any affine Lie algebra, $\mathcal{A}(\mathcal{G}, \mu)$, defined by adding the derivation to the central extension of a twisted loop algebra, $\ell(\mathcal{G}, \mu)$, based on an appropriate automorphism, μ, of a self-dual Lie algebra, $\mathcal{G}$. Namely, such an affine Lie algebra automatically comes equipped with the integral gradation associated with the powers of the loop parameter, and it can be shown that (4.2) provides a solution of (4.3) if one takes $\mathcal{K}$ to be the grade zero subalgebra in this gradation. In this chapter, this solution will arise as a special case of a general theorem, which ensures the validity of (4.3) for (4.2) under the assumption that $\mathcal{K} = \mathcal{A}_0$ where $\mathcal{A} = \oplus_{n\in\mathcal{Z}}\mathcal{A}_n$ is graded by finite-dimensional subspaces and carries an invariant scalar product that is compatible with the grading in the sense that $\mathcal{A}_n \perp \mathcal{A}_m$ unless $(n+m) = 0$. Here $\mathcal{Z}$ is some abelian group, in our examples $\mathcal{Z} = \mathbb{Z}$. The precise statement, which is our first main result, is given by Theorem 4.1 in section 4.1. We shall use this result to obtain dynamical r-matrices on the twisted loop algebras $\ell(\mathcal{G},\mu)$ with the dynamical variable lying in the fixed point set $\mathcal{G}_0 \subset \mathcal{G}$ of the automorphism μ of $\mathcal{G}$. By means of evaluation homomorphisms, these r-matrices then yield spectral-parameter-dependent $\mathcal{G}\otimes\mathcal{G}$-valued dynamical r-matrices generalizing Felder's elliptic r-matrices. The latter are recovered if $\mathcal{G}$ is taken to be a simple Lie algebra and μ a Coxeter automorphism, consistently with the derivation found in [15]. The existence of the above-mentioned family of elliptic dynamical r-matrices was announced in [48]. Our second main result is their derivation presented in section 4.2. See in particular Proposition 4.2 and Proposition 4.3 in subsection 4.2.3. We shall also find a relationship between the underlying $\ell(\mathcal{G},\mu)\otimes\ell(\mathcal{G},\mu)$-valued r-matrices with dynamical variables in $\mathcal{G}_0$, and certain $\mathcal{G}\otimes\mathcal{G}$-valued dynamical r-matrices on $\mathcal{G}_0$ introduced in [51]. This is contained in appendix H.

4.1 r-matrices on graded, self-dual Lie algebras

In this section we apply formula (4.2) to infinite-dimensional Lie algebras that are decomposed into finite-dimensional subspaces in such a way that the r-matrix leaves these subspaces invariant. The definition of the r-matrix on these subspaces will be given in terms of the well known holomorphic functional calculus of linear operators [52]. The relevant basics of functional calculus are contained in appendix D.

We now consider a complex Lie algebra $\mathcal{A}$ equipped with a gradation based on some abelian group $\mathcal{Z}$. We use the additive notation to denote the group operation on $\mathcal{Z}$. The zero as a number and the unit element of $\mathcal{Z}$ are both denoted simply by 0, but this should not lead to any confusion. We assume that as a linear space

$$\mathcal{A} = \oplus_{n\in\mathcal{Z}}\mathcal{A}_n, \quad 0 \leq \dim(\mathcal{A}_n) < \infty, \quad \dim(\mathcal{A}_0) \neq 0, \tag{4.8}$$

and

$$[\mathcal{A}_m, \mathcal{A}_n] \subset \mathcal{A}_{m+n} \qquad \forall m,n \in \mathcal{Z}. \tag{4.9}$$

The elements of $\mathcal{A}$ are finite linear combinations of the elements of the homogeneous subspaces, and we permit the possibility that $\dim(\mathcal{A}_n) = 0$ for some $n \in \mathcal{Z}$. We further assume that $\mathcal{A}$ has a nondegenerate, symmetric, invariant bilinear form $\langle\ ,\ \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, which is compatible with the gradation in the sense that

$$\mathcal{A}_m \perp \mathcal{A}_n \quad \text{unless} \quad (m+n) = 0. \tag{4.10}$$

This means that if $(m+n) \neq 0$ then $\langle X, Y \rangle = 0$ for any $X \in \mathcal{A}_m$, $Y \in \mathcal{A}_n$, and the dual space of $\mathcal{A}_n$ can be identified with $\mathcal{A}_{-n}$ by means of the pairing given by $\langle\ ,\ \rangle$. In particular, $\mathcal{A}_0$ is a finite-dimensional self-dual subalgebra of $\mathcal{A}$. Since $[\mathcal{A}_0, \mathcal{A}_n] \subset \mathcal{A}_n$ and $\mathcal{A}_n$ is finite dimensional, $e^{\operatorname{ad}\kappa}$ is a well defined linear operator on $\mathcal{A}$ for any $\kappa \in \mathcal{A}_0$. The invariance of the bilinear form, $\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0$, $\forall X, Y, Z \in \mathcal{A}$, implies that $\langle e^{\operatorname{ad}\kappa} Y, e^{\operatorname{ad}\kappa} Z \rangle = \langle Y, Z \rangle$ for any $Y, Z \in \mathcal{A}$ and $\kappa \in \mathcal{A}_0$.

Now we wish to apply formula (4.2) to

$$\mathcal{K} := \mathcal{A}_0, \qquad \mathcal{K}^\perp = \oplus_{n \in \mathcal{Z} \setminus \{0\}} \mathcal{A}_n. \tag{4.11}$$

For any $\kappa \in \mathcal{K}$ and $n \in \mathcal{Z}$, introduce $(\operatorname{ad}\kappa)_n := \operatorname{ad}\kappa|_{\mathcal{A}_n}$ and let σ_κ^n denote the spectrum of this finite-dimensional linear operator ($\sigma_\kappa^n = \emptyset$ if $\dim(\mathcal{A}_n) = 0$). Our crucial assumption is that there exists a *nonempty, open* subset $\check{\mathcal{K}} \subset \mathcal{K}$ for which

$$\sigma_\kappa^n \cap 2\pi i \mathbb{Z} = \emptyset \quad \forall n \neq 0 \quad \text{and} \quad \sigma_\kappa^0 \cap 2\pi i \mathbb{Z}^* = \emptyset \quad \forall \kappa \in \check{\mathcal{K}}, \tag{4.12}$$

where $\mathbb{Z}$ and $\mathbb{Z}^*$ are the set of all integers, and nonzero integers, respectively. It is clear that if such a $\check{\mathcal{K}}$ exists, then there exists also a maximal one. If this assumption is satisfied, then we can define the map $R : \check{\mathcal{K}} \to \operatorname{End}(\mathcal{A})$ by requiring that the homogeneous subspaces $\mathcal{A}_n$ be invariant with respect to $R(\kappa)$ in such a way that $\forall \kappa \in \check{\mathcal{K}}$

$$R(\kappa)|_{\mathcal{A}_0} := f((\operatorname{ad}\kappa)_0), \quad R(\kappa)|_{\mathcal{A}_n} := F((\operatorname{ad}\kappa)_n) \quad \forall n \in \mathcal{Z} \setminus \{0\}. \tag{4.13}$$

For $n \in \mathcal{Z}$ for which $\dim(\mathcal{A}_n) \neq 0$, these finite-dimensional linear operators are given similarly to (D.1). The assumption (4.12) guarantees that the spectra σ_κ^n do not intersect the poles of the corresponding meromorphic functions f and F in (4.1), whereby $R(\kappa)$ is well defined for $\kappa \in \check{\mathcal{K}}$. If $\dim(\mathcal{A}_n) = 0$, then $R(\kappa)|_{\mathcal{A}_n}$ is of course understood to be the zero linear operator. Somewhat informally, we summarize (4.13) by saying that $R(\kappa)$ equals $f(\operatorname{ad}\kappa)$ on $\mathcal{K}$ and $F(\operatorname{ad}\kappa)$ on $\mathcal{K}^\perp$.

Theorem 4.1 *Let $\mathcal{A}$ be a graded, self-dual, complex Lie algebra satisfying the assumptions given by (4.8)–(4.10). Take $\mathcal{K} := \mathcal{A}_0$ and suppose the existence a nonempty, open domain $\check{\mathcal{K}} \subset$*

$\mathcal{K}$ for which (4.12) holds. Then the r-matrix $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ defined by (4.13) satisfies the CDYBE (4.3). Moreover, $R(\kappa)$ is an antisymmetric operator $\forall\kappa \in \check{\mathcal{K}}$, and the $\mathcal{K}$-equivariance condition (4.6) holds.

Proof. Since the CDYBE (4.3) is linear in $X, Y \in \mathcal{A}$, it is enough to verify it case by case for all possible choices of homogeneous elements X and Y. As a preparation, let us write the function F in (4.1) as

$$F(z) = \frac{1}{2}\frac{Q_+(z)}{Q_-(z)} \quad \text{with} \quad Q_\pm(z) = e^{\frac{z}{2}} \pm e^{-\frac{z}{2}}, \tag{4.14}$$

and define the linear operators $Q_\pm(\kappa)$ on $\mathcal{A}$ by

$$Q_\pm(\kappa) = e^K \pm e^{-K} \quad \text{with} \quad K := \frac{1}{2}\mathrm{ad}\,\kappa \quad \forall\kappa \in \check{\mathcal{K}}. \tag{4.15}$$

$Q_\pm(\kappa)$ are well defined operators on $\mathcal{A}$ since their restrictions to any $\mathcal{A}_n$ are obviously well defined. It follows from the definitions of the domain $\check{\mathcal{K}}$ and that of $R(\kappa)$ that $Q_-(\kappa)$ is an invertible operator on $\mathcal{A}_n$ for any $n \neq 0$ and that we have

$$R(\kappa)Q_-(\kappa) = Q_-(\kappa)R(\kappa) = \frac{1}{2}Q_+(\kappa) \quad \text{on} \quad \mathcal{A}_n \quad \forall n \neq 0. \tag{4.16}$$

We first consider the simplest case,

$$X \in \mathcal{A}_m, \quad Y \in \mathcal{A}_n, \quad m \neq 0, \quad n \neq 0, \quad (m+n) \neq 0, \tag{4.17}$$

for which the derivative terms drop out from (4.3). Without loss of generality, we can now write

$$X = Q_-(\kappa)\xi, \qquad Y = Q_-(\kappa)\eta \tag{4.18}$$

with some $\xi \in \mathcal{A}_m$, $\eta \in \mathcal{A}_n$. If we multiply (4.3) from the left by the invertible operator $4Q_-(\kappa)$ on $\mathcal{A}_{m+n}$, then by using (4.16) the required statement becomes

$$\begin{aligned} &Q_-(\kappa)[Q_-(\kappa)\xi, Q_-(\kappa)\eta] + Q_-(\kappa)[Q_+(\kappa)\xi, Q_+(\kappa)\eta] \\ &\qquad -Q_+(\kappa)\left([Q_-(\kappa)\xi, Q_+(\kappa)\eta] + [Q_+(\kappa)\xi, Q_-(\kappa)\eta]\right) = 0. \end{aligned} \tag{4.19}$$

We further spell out this equation by using that $e^{\pm K}$ are Lie algebra automorphism, and thereby (4.19) is verified in a straightforward manner.

Second, let us consider the case for which

$$X \in \mathcal{A}_0, \quad Y \in \mathcal{A}_n, \quad n \neq 0. \tag{4.20}$$

Then the derivative term $(\nabla_X R)(\kappa)(Y)$ appears in equation (4.3). To calculate this, we need the holomorphic complex function h given by

$$z \mapsto h(z) := \frac{e^z - 1}{z}. \tag{4.21}$$

We recall (e.g. [55], page 35) that for a curve $t \mapsto A(t)$ of finite-dimensional linear operators one has the identity

$$\frac{de^{\pm A(t)}}{dt} = \pm e^{\pm A(t)} h(\mp \mathrm{ad}_{A(t)})(\dot{A}(t)), \qquad \dot{A}(t) := \frac{dA(t)}{dt}. \tag{4.22}$$

The right hand side of the above equation is defined by means of the Taylor expansion of h around 0, and of course

$$(\mathrm{ad}_{A(t)})^j(\dot{A}(t)) = [A(t), (\mathrm{ad}_{A(t)})^{j-1}(\dot{A}(t))], \quad j \in \mathbb{N}, \quad (\mathrm{ad}_{A(t)})^0(\dot{A}(t)) = \dot{A}(t). \tag{4.23}$$

In our case we consider the curve of linear operators on $\mathcal{A}_n$ given by

$$t \mapsto \mathrm{ad}\,\kappa + t(\mathrm{ad}\,X). \tag{4.24}$$

Then (4.22) leads to the formula

$$\left(\nabla_X e^{\pm K}\right)(Y) = \pm\frac{1}{2} e^{\pm K}[h(\mp K)X, Y], \tag{4.25}$$

where $K = \frac{1}{2}\mathrm{ad}\,\kappa$. From this, by taking the derivative of the identity $2Q_- R = Q_+$ on $\mathcal{A}_n$ along the curve (4.24) at $t = 0$, we obtain

$$4Q_-(\kappa)(\nabla_X R)(\kappa)Y = e^K[h(-K)X, Y - 2R(\kappa)Y] - e^{-K}[h(K)X, Y + 2R(\kappa)Y]. \tag{4.26}$$

On the other hand, for (4.20) the CDYBE (4.3) is equivalent to

$$\begin{aligned} 4Q_-(\kappa)(\nabla_X R)(\kappa)Y = {} & Q_-(\kappa)[X, Y] + 4Q_-(\kappa)[R(\kappa)X, R(\kappa)Y] \\ & -2Q_+(\kappa)\left([X, R(\kappa)Y] + [R(\kappa)X, Y]\right). \end{aligned} \tag{4.27}$$

We fix $\kappa \in \check{\mathcal{K}}$ arbitrarily, and write $Y = Q_-(\kappa)\eta$ with some $\eta \in \mathcal{A}_n$. Then by a straightforward calculation, using that $e^{\pm K}$ are Lie algebra automorphisms and collecting terms, we obtain that the required equality of the right hand sides of the last two equations is equivalent to

$$\left[\left(e^K h(-K) + e^{-K} h(K) - e^K - e^{-K}\right) X, \eta\right] = 2\left[\left(e^{-K} R(\kappa) - e^K R(\kappa)\right) X, \eta\right]. \tag{4.28}$$

Here $R(\kappa)X = f(2K)X$ with (4.2), and the statement follows from the equality of the corresponding complex analytic functions, namely

$$e^z \frac{1 - e^{-z}}{z} + e^{-z}\frac{e^z - 1}{z} - e^z - e^{-z} = e^{-z}\left(\coth z - \frac{1}{z}\right) - e^z\left(\coth z - \frac{1}{z}\right), \tag{4.29}$$

which is checked in the obvious way.

The third case to deal with is that of

$$X \in \mathcal{A}_{-n}, \quad Y \in \mathcal{A}_n, \quad n \neq 0, \tag{4.30}$$

for which the derivative term $\langle X, (\nabla R)(\kappa)Y\rangle$ occurs in (4.3). At any fixed $\kappa \in \check{\mathcal{K}}$, we may write

$$X = Q_-(\kappa)\xi, \qquad Y = Q_-(\kappa)\eta \tag{4.31}$$

with some $\xi \in \mathcal{A}_{-n}$, $\eta \in \mathcal{A}_n$. We introduce the holomorphic function

$$z \mapsto g(z) := \frac{e^z - e^{-z}}{z}, \tag{4.32}$$

and define $g(K)$ by the Taylor series of $g(z)$ around $z = 0$. Then we can calculate that

$$\langle X, (\nabla R)(\kappa)Y\rangle = \frac{1}{2} g(K)[\eta, \xi]. \tag{4.33}$$

To obtain this, note that

$$\langle X, (\nabla R)(\kappa)Y\rangle = T^i \langle X, (\nabla_{T_i} R)(\kappa)Y\rangle \tag{4.34}$$

with dual bases T_i and T^i of $\mathcal{A}_0$, where $(\nabla_{T_i} R)(\kappa)Y$ is determined by (4.26). By using these and the invariance of the scalar product of $\mathcal{A}$, it is not difficult to rewrite (4.34) in the form (4.33). As for the non-derivative terms in (4.3), with X, Y in (4.31) we find

$$\begin{aligned} &[R(\kappa)X, R(\kappa)Y] - R(\kappa)\left([X, R(\kappa)Y] + [R(\kappa)X, Y]\right) + \frac{1}{4}[X, Y] = \\ &\frac{1}{2}\left(Q_+(\kappa) - 2R(\kappa)Q_-(\kappa)\right)[\xi, \eta]. \end{aligned} \tag{4.35}$$

It is easy to check that the sum of the right hand sides of (4.33) and (4.35) is zero, which finishes the verification of the CDYBE (4.3) in the case (4.30).

The remaining case is that of $X, Y \in \mathcal{A}_0$. Then the variable κ as well as all terms in (4.3) lie in the subalgebra $\mathcal{A}_0$, and it is known [15, 12, 13] that the formula $\kappa \mapsto f(\operatorname{ad}\kappa)$ (4.2) defines a solution of the CDYBE on any finite-dimensional self-dual Lie algebra (it was the main object of Chapter 3). This completes the verification of the CDYBE (4.3).

The antisymmetry of $R(\kappa)$ follows from (4.13) since $\operatorname{ad}\kappa$ is antisymmetric by the invariance of $\langle\ ,\ \rangle$ and both f and F are odd functions. Finally, the equivariance property (4.6) is also easily verified from (4.13) by using that for any finite-dimensional linear operator given by (D.1) one has

$$\frac{dH(A(t))}{dt}|_{t=0} = \frac{1}{2\pi i}\oint_\Gamma dz H(z)(zI_V - A)^{-1}\dot{A}(0)(zI_V - A)^{-1} \tag{4.36}$$

along any smooth curve $t \mapsto A(t)$ for which $A(0) = A$. *Q.E.D.*

We conclude this section by describing the tensorial interpretation of the CDYBE (4.3) for the r-matrices of Theorem 4.1. For this, consider dual bases $T_i[n]$ and $T^j[n]$ of $\mathcal{A}$ ($n \in \mathcal{Z}, i, j = 1, \ldots, \dim(\mathcal{A}_n)$), which satisfy $T_i[n] \in \mathcal{A}_n$ and $\langle T_i[m], T^j[n]\rangle = \delta_{m,-n}\delta_i^j$. Then introduce $r^\pm : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ by[1]

$$r^\pm(\kappa) := \sum_{n\in\mathcal{Z}} \sum_{i=1}^{\dim(\mathcal{A}_n)} \Big((R(\kappa)T_i[n]) \otimes T^i[-n] \pm \frac{1}{2} T_i[n] \otimes T^i[-n] \Big) . \tag{4.37}$$

In fact, as a consequence of the properties of R established in Theorem 4.1, $r^\pm$ satisfies the tensorial version of the CDYBE given by

$$\begin{aligned}&[r^s_{12}(\kappa), r^s_{13}(\kappa)] + [r^s_{12}(\kappa), r^s_{23}(\kappa)] + [r^s_{13}(\kappa), r^s_{23}(\kappa)] \\ &\quad + T_j[0]^1 \frac{\partial}{\partial\kappa_j} r^s_{23}(\kappa) - T_j[0]^2 \frac{\partial}{\partial\kappa_j} r^s_{13}(\kappa) + T_j[0]^3 \frac{\partial}{\partial\kappa_j} r^s_{12}(\kappa) = 0, \quad s = \pm,\end{aligned} \tag{4.38}$$

where $\kappa_j := \langle \kappa, T_j[0]\rangle$. Here the standard notations are used, $T_j[0]^1 := T_j[0] \otimes 1 \otimes 1$, $r^s_{12} := r^s \otimes 1$ etc. The expression on the left hand side of (4.38) belongs to a completion of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$; it has a unique expansion in the basis $T_{i_1}[n_1] \otimes T_{i_2}[n_2] \otimes T_{i_3}[n_3]$ of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$. Similarly to the CYBE, the CDYBE (4.38) is compatible with homomorphisms of $\mathcal{A}$. This means that if $\pi_i : \mathcal{A} \to \mathcal{G}^i$ ($i = 1, 2, 3$) are (possibly different) homomorphisms of $\mathcal{A}$ into (possibly different) Lie algebras $\mathcal{G}^i$, then we can obtain a $\mathcal{G}^1 \otimes \mathcal{G}^2 \otimes \mathcal{G}^3$-valued equation from (4.38) by the obvious application of the map $\pi_1 \otimes \pi_2 \otimes \pi_3$ to all objects on the left hand side of (4.38). More precisely, to take into account the unit element 1, here one uses the extensions of these Lie algebra homomorphisms to the corresponding universal enveloping algebras.

4.2 Applications to affine Lie algebras

Let $\mathcal{G}$ be a finite-dimensional complex, self-dual Lie algebra equipped with an invariant 'scalar product' denoted as $B : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$. Suppose that μ is a finite order automorphism of $\mathcal{G}$ that preserves the bilinear form B and has nonzero fixed points[2]. With this data, one may associate the twisted loop algebra $\ell(\mathcal{G}, \mu)$ and the affine Lie algebra $\mathcal{A}(\mathcal{G}, \mu)$ obtained by adding the natural derivation to the central extension of $\ell(\mathcal{G}, \mu)$. We below show that Theorem 4.1

[1]Here $\mathcal{A}\otimes\mathcal{A}$ is a completion of the algebraic tensor product containing the elements that are associated with the linear operators on $\mathcal{A}$.

[2]The last two properties are automatic if $\mathcal{G}$ is simple or $\mu = \mathrm{id}$, which are included as special cases.

is directly applicable to $\mathcal{A}(\mathcal{G},\mu)$. Then we explain that the resulting dynamical r-matrices on $\mathcal{A}(\mathcal{G},\mu)$ admit a reinterpretation as one-parameter families of r-matrices on $\ell(\mathcal{G},\mu)$. By applying evaluation homomorphisms to the corresponding $\ell(\mathcal{G},\mu)\otimes\ell(\mathcal{G},\mu)$-valued r-matrices, we finally derive spectral-parameter-dependent $\mathcal{G}\otimes\mathcal{G}$-valued dynamical r-matrices. These results were announced in [48] without presenting proofs, which are provided here.

4.2.1 Application of Theorem 4.1 to $\mathcal{A}(\mathcal{G},\mu)$

Any automorphism μ of order N, $\mu^N = \mathrm{id}$, gives rise to a decomposition of $\mathcal{G}$ as

$$\mathcal{G} = \oplus_{a\in\mathcal{E}_\mu}\mathcal{G}_a, \qquad \mathcal{E}_\mu \subset \{0,1,\ldots,(N-1)\,\}, \tag{4.39}$$

with the eigensubspaces

$$\mathcal{G}_a := \{\xi\in\mathcal{G}\,|\,\mu(\xi)=\exp(\frac{ia2\pi}{N})\xi\,\}\neq\{0\}. \tag{4.40}$$

Since we assumed that $B(\mu\xi,\mu\eta)=B(\xi,\eta)$ $(\forall\xi,\eta\in\mathcal{G})$, $\mathcal{G}_a$ is perpendicular to $\mathcal{G}_b$ with respect to the form B unless $a+b=N$ or $a=b=0$. This implies that if a nonzero a belongs to the index set $\mathcal{E}_\mu$ then so does $(N-a)$. We assume that $0\in\mathcal{E}_\mu$, and thus $\mathcal{G}_0\neq\{0\}$ is a self-dual subalgebra of $\mathcal{G}$.

The twisted (or untwisted if we choose $\mu=\mathrm{id}$) loop algebra $\ell(\mathcal{G},\mu)$ is the subalgebra of $\mathcal{G}\otimes\mathbb{C}[t,t^{-1}]$ generated by the elements of the form

$$\xi^{n_a} := \xi\otimes t^{n_a} \quad\text{with}\quad \xi\in\mathcal{G}_a,\quad n_a = a+mN,\;\; m\in\mathbb{Z}, \tag{4.41}$$

where t is a formal variable. The 'affine Lie algebra' $\mathcal{A}(\mathcal{G},\mu)$ is then introduced as

$$\mathcal{A}(\mathcal{G},\mu) := \ell(\mathcal{G},\mu)\oplus\mathbb{C}d\oplus\mathbb{C}\hat{c} \tag{4.42}$$

with the Lie bracket of its generators defined by

$$[\xi^{n_a},\eta^{p_b}] = [\xi,\eta]^{n_a+p_b} + n_a\delta_{n_a,-p_b}B(\xi,\eta)\hat{c}, \quad \forall\xi\in\mathcal{G}_a,\;\; \eta\in\mathcal{G}_b, \tag{4.43}$$

$$[d,\xi^{n_a}] = n_a\xi^{n_a}, \quad [\hat{c},d]=[\hat{c},\xi^{n_a}]=0. \tag{4.44}$$

$\mathcal{A}(\mathcal{G},\mu)$ is a self-dual Lie algebra as it carries the scalar product $\langle\;,\;\rangle$ given by

$$\langle\xi^{n_a},\eta^{p_b}\rangle = \delta_{n_a,-p_b}B(\xi,\eta), \quad \langle\hat{c},d\rangle=1, \quad \langle d,\xi^{n_a}\rangle=\langle\hat{c},\xi^{n_a}\rangle=0. \tag{4.45}$$

We obtain a $\mathbb{Z}$-gradation of $\mathcal{A}(\mathcal{G},\mu)$ by the decomposition

$$\mathcal{A}(\mathcal{G},\mu) = \oplus_{n\in(\mathcal{E}_\mu+N\mathbb{Z})}\mathcal{A}(\mathcal{G},\mu)_n = \oplus_{n\in\mathbb{Z}}\mathcal{A}(\mathcal{G},\mu)_n, \tag{4.46}$$

where $\mathcal{A}(\mathcal{G},\mu)_n$ is the eigensubspace of $\mathrm{ad}\, d$ with eigenvalue n if $n \in (\mathcal{E}_\mu + N\mathbb{Z})$, and $\mathcal{A}(\mathcal{G},\mu)_n = \{0\}$ if $n \notin (\mathcal{E}_\mu + N\mathbb{Z})$. We need to introduce these zero subspaces for notational consistency, since $(\mathcal{E}_\mu + N\mathbb{Z})$ is not necessarily a group in general. This is also consistent with the fact that (4.43) gives zero if $(n_a + p_b) \notin (\mathcal{E}_\mu + N\mathbb{Z})$. The gradation given by (4.46) clearly satisfies equations (4.8)–(4.10), where now $\mathcal{Z} := \mathbb{Z}$. We below regard $\mathcal{G}_0$ as a subspace of $\mathcal{A}(\mathcal{G},\mu)$ by identifying $\xi \in \mathcal{G}_0$ with $\xi \otimes t^0 \in \mathcal{A}(\mathcal{G},\mu)$, whereby we can write

$$\mathcal{A}(\mathcal{G},\mu)_0 = \mathcal{G}_0 \oplus \mathbb{C}d \oplus \mathbb{C}\hat{c}. \tag{4.47}$$

Since we wish to apply Theorem 4.1, we now set $\mathcal{A} := \mathcal{A}(\mathcal{G},\mu)$ and $\mathcal{K} := \mathcal{A}(\mathcal{G},\mu)_0$. We parametrize the general element $\kappa \in \mathcal{K}$ as

$$\kappa = \omega + kd + l\hat{c}, \qquad \omega \in \mathcal{G}_0, \quad k, l \in \mathbb{C}. \tag{4.48}$$

It follows from the above that formula (4.13) provides us with a dynamical r-matrix $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ if we can find a nonempty, open domain $\check{\mathcal{K}} \subset \mathcal{K}$ whose elements satisfy the conditions given in (4.12). The point is that we can indeed find such a domain, and actually the maximal domain has the form

$$\check{\mathcal{K}} = \{\, \kappa = \omega + kd + l\hat{c} \,|\, l \in \mathbb{C},\ k \in (\mathbb{C} \setminus \mathbb{R}i),\ \omega \in \mathcal{B}_k\}, \tag{4.49}$$

where $\mathcal{B}_k \subset \mathcal{G}_0$ is described as follows. Let λ_a denote an eigenvalue of the operator $\mathrm{ad}\,\omega|_{\mathcal{G}_a}$ associated with $\omega \in \mathcal{G}_0$. By definition, the subset $\mathcal{B}_k \subset \mathcal{G}_0$ consists of those $\omega \in \mathcal{G}_0$ whose eigenvalues satisfy the following conditions:

$$(\lambda_a + k(a + mN)) \notin 2\pi i\mathbb{Z} \quad \forall m \in \mathbb{Z}, \quad \forall a \in \mathcal{E}_\mu \setminus \{0\}, \tag{4.50}$$

$$\lambda_0 \notin 2\pi i\mathbb{Z}^* \quad \text{and} \quad (\lambda_0 + kmN) \notin 2\pi i\mathbb{Z} \quad \forall m \in \mathbb{Z}^*. \tag{4.51}$$

If we note that for ξ^{n_a} in (4.41) and $\kappa \in \mathcal{K}$ written as in (4.48) one has

$$(\mathrm{ad}\,\kappa)(\xi^{n_a}) = kn_a\xi^{n_a} + [\omega,\xi]^{n_a}, \tag{4.52}$$

then the conditions in (4.50) and (4.51) are recognized to be the translation of the condition in (4.12) to our case. The set $\check{\mathcal{K}}$ defined by these requirements obviously contains the elements of the form $\kappa = kd + l\hat{c}$ for any $k \in (\mathbb{C} \setminus i\mathbb{R})$, $l \in \mathbb{C}$, and therefore it is nonempty. It is not difficult to see that $\check{\mathcal{K}} \subset \mathcal{K}$ in (4.49) is an open subset, for which one needs k to have a nonzero real part, and $\mathcal{B}_k \subset \mathcal{G}_0$ is a nonempty open subset as well. For completeness, we present a proof of these statements in appendix G.

4.2.2 One-parameter family of r-matrices on $\ell(\mathcal{G},\mu)$

We now reinterpret the dynamical r-matrices $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A}(\mathcal{G},\mu))$ constructed in subsection 4.2.1 as a family of r-matrices

$$R_k : \mathcal{B}_k \to \mathrm{End}(\ell(\mathcal{G},\mu)), \tag{4.53}$$

where the parameter k varies in $(\mathbb{C} \setminus i\mathbb{R})$ and the k-dependent domain $\mathcal{B}_k \subset \mathcal{G}_0$ appears in (4.49). For any $\omega \in \mathcal{B}_k$, the operator $R_k(\omega)$ is given by

$$R_k(\omega)\eta := f(\mathrm{ad}\,\omega)\eta, \qquad R_k(\omega)\xi^{n_a} := F(kn_a + \mathrm{ad}\,\omega)\xi^{n_a} \tag{4.54}$$

$\forall \eta \in \mathcal{G}_0 = \ell(\mathcal{G},\mu)_0$ and $\forall \xi^{n_a} \in \ell(\mathcal{G},\mu)_{n_a}$ with $n_a \neq 0$. In other words, by regarding $\ell(\mathcal{G},\mu)$ as a subspace of $\mathcal{A}(\mathcal{G},\mu)$, we have $R_k(\omega)X = R(\kappa)X$ for $X \in \ell(\mathcal{G},\mu)$ and $\kappa \in \check{\mathcal{K}}$.

It is an easy consequence of Theorem 4.1 that R_k satisfies the operator version of the CDYBE for any fixed k:

$$\begin{aligned}&[R_kX, R_kY] - R_k([X, R_kY] + [R_kX, Y]) + \langle X, (\nabla R_k)Y\rangle \\ &\quad +(\nabla_{Y_0}R_k)X - (\nabla_{X_0}R_k)Y = -\frac{1}{4}[X,Y], \qquad \forall X, Y \in \ell(\mathcal{G},\mu).\end{aligned} \tag{4.55}$$

Here the Lie brackets are evaluated in $\ell(\mathcal{G},\mu)$, X_0 is the grade 0 part of X, and the scalar product on $\ell(\mathcal{G},\mu)$ is given by the restriction of (4.45). This equation is verified by a simplified version of the calculation done in the proof of Theorem 4.1, the simplification being that $\hat{c}$ has now been set to zero. It is also clear that $R_k : \mathcal{B}_k \to \mathrm{End}(\ell(\mathcal{G},\mu))$ is a $\mathcal{G}_0$-equivariant map in the natural sense.

For later purpose, we here introduce the shifted r-matrices

$$R_k^{\pm} := R_k \pm \frac{1}{2}I, \tag{4.56}$$

where I is the identity operator on $\ell(\mathcal{G},\mu)$. By using the scalar product, we associate with these operator valued maps the corresponding $\ell(\mathcal{G},\mu)\otimes\ell(\mathcal{G},\mu)$-valued maps. These are denoted respectively as

$$r^{k,\pm} : \mathcal{B}_k \to \ell(\mathcal{G},\mu)\otimes\ell(\mathcal{G},\mu). \tag{4.57}$$

By translating the CDYBE into tensorial terms, (4.55) becomes

$$\begin{aligned}&[r_{12}^{k,s}(\omega), r_{13}^{k,s}(\omega)] + [r_{12}^{k,s}(\omega), r_{23}^{k,s}(\omega)] + [r_{13}^{k,s}(\omega), r_{23}^{k,s}(\omega)] \\ &\quad +T_j^1\frac{\partial}{\partial\omega_j}r_{23}^{k,s}(\omega) - T_j^2\frac{\partial}{\partial\omega_j}r_{13}^{k,s}(\omega) + T_j^3\frac{\partial}{\partial\omega_j}r_{12}^{k,s}(\omega) = 0, \quad s = \pm,\end{aligned} \tag{4.58}$$

where $\omega_j := B(\omega, T_j)$ with a basis T_j of $\mathcal{G}_0$.

4.2.3 Spectral-parameter-dependent r-matrices

The loop algebra $\ell(\mathcal{G},\mu)$ admits an 'evaluation homomorphism' $\pi_v : \ell(\mathcal{G},\mu) \to \mathcal{G}$ for any fixed $v \in \mathbb{C}^*$,

$$\pi_v : \xi \otimes t^n \mapsto v^n \xi \qquad \forall (\xi \otimes t^n) \in \ell(\mathcal{G},\mu). \tag{4.59}$$

It is well known that spectral-parameter-dependent $\mathcal{G} \otimes \mathcal{G}$-valued r-matrices may be obtained by applying these homomorphisms to $\ell(\mathcal{G},\mu) \otimes \ell(\mathcal{G},\mu)$-valued r-matrices. In the context of dynamical r-matrices, Etingof and Varchenko [15] used this method to derive Felder's elliptic dynamical r-matrices from the basic trigonometric dynamical r-matrices of the (untwisted) affine Kac-Moody Lie algebras. We here apply the same procedure to the general family of dynamical r-matrices introduced in eq. (4.57). As for the presentation below, we find it convenient to first provide a self-contained definition of the spectral-parameter-dependent r-matrices and show afterwards how they are obtained from the evaluation homomorphisms.

We start by collecting some meromorphic functions and identities that will be useful. Consider the standard theta function[3]

$$\theta_1(z|\tau) := -\sum_{j\in\mathbb{Z}} \exp\left(\pi i (j+\frac{1}{2})^2\tau + 2\pi i (j+\frac{1}{2})(z+\frac{1}{2})\right), \quad \Im(\tau) > 0, \tag{4.60}$$

which is holomorphic on $\mathbb{C}$ and has simple zeros at the points of the lattice

$$\Omega := \mathbb{Z} + \tau\mathbb{Z}. \tag{4.61}$$

Recall that θ_1 is odd in z and satisfies

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z+\tau|\tau) = -q^{-1}e^{-2\pi i z}\theta_1(z|\tau), \quad q := e^{\pi i \tau}. \tag{4.62}$$

Define now the function

$$\chi(w,z|\tau) := \frac{1}{2\pi i}\frac{\theta_1(\frac{w}{2\pi i}+z|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(\frac{w}{2\pi i}|\tau)}. \tag{4.63}$$

This function is holomorphic in w and in z at the points

$$(w,z) \in (\mathbb{C} \setminus 2\pi i\Omega) \times (\mathbb{C} \setminus \Omega). \tag{4.64}$$

The following important identity holds:

$$\chi(w,z) = \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{2\pi i z n}\left[1 + \coth(\frac{w}{2} + \pi i \tau n)\right] \tag{4.65}$$

[3]We have $\theta_1(z|\tau) = \vartheta_1(\pi z|\tau)$ with ϑ_1 in [56].

on the domain

$$D := \{(w,z)\,|\, w \in (\mathbb{C}\setminus 2\pi i\Omega),\ \ -\Im(\tau) < \Im(z) < 0\,\}. \tag{4.66}$$

All terms in the sum are holomorphic on D, the convergence is absolute at any fixed $(w,z) \in D$, and is uniform on compact subsets of D. The verification of (4.65) is a routine matter, example 13 on page 489 of [56] contains a closely related statement.

We also need the functions

$$\chi_a(w,z|\tau) := e^{\frac{2\pi i a z}{N}} \left(\chi(w + 2\pi i \frac{a}{N}\tau, z|\tau) - \frac{\delta_{a,0}}{w} \right), \tag{4.67}$$

where $a \in \{0, 1, \ldots, (N-1)\}$ with some positive integer N. The function $\chi_a(w,z|\tau)$ is holomorphic in w and in z if (w,z) belongs to the domain

$$(\mathbb{C}\setminus 2\pi i\Omega_a) \times (\mathbb{C}\setminus\Omega) \quad \text{where} \quad \Omega_a := \left(\Omega - \frac{a}{N}\tau\right) \setminus \{0\}. \tag{4.68}$$

By using the notation

$$f_a(w) := \frac{1}{2}\left[1 + \coth\frac{w}{2}\right] - \frac{\delta_{a,0}}{w}, \tag{4.69}$$

we have the identity

$$\chi_a(w,z|\tau) = e^{\frac{2\pi i a z}{N}} \left(f_a(w + 2\pi i\frac{a}{N}\tau) + \frac{1}{2}\sum_{n\in\mathbb{Z}^*} e^{2\pi i z n}\left[1 + \coth(\frac{w}{2} + \pi i \frac{a}{N}\tau + \pi i \tau n)\right]\right) \tag{4.70}$$

on the domain

$$D_a := \{(w,z)\,|\, w \in (\mathbb{C}\setminus 2\pi i\Omega_a),\ \ -\Im(\tau) < \Im(z) < 0\,\} \tag{4.71}$$

for any $a \in \{0, 1, \ldots, (N-1)\}$. All terms in the sum are holomorphic on D_a, the convergence is absolute at any $(w,z) \in D_a$, and is uniform on compact subsets of D_a.

Let now μ be an automorphism of $\mathcal{G}$ of order N as considered previously and fix τ with $\Im(\tau) > 0$. For any $\omega \in \mathcal{G}_0$ and $a \in \mathcal{E}_\mu$, let $\sigma((\mathrm{ad}\,\omega)_a)$ be the spectrum of the linear operator $(\mathrm{ad}\,\omega)_a := \mathrm{ad}\,\omega|_{\mathcal{G}_a}$. Define $\mathcal{B}^\tau \subset \mathcal{G}_0$ by

$$\mathcal{B}^\tau := \{\,\omega \in \mathcal{G}_0 \,|\, \sigma((\mathrm{ad}\,\omega)_a) \cap 2\pi i \Omega_a = \emptyset \quad \forall a \in \mathcal{E}_\mu\,\}. \tag{4.72}$$

It is easy to verify that

$$\mathcal{B}^\tau = \mathcal{B}_k \quad \text{if} \quad \tau = \frac{kN}{2\pi i}, \tag{4.73}$$

where $\mathcal{B}_k \subset \mathcal{G}_0$ appears in (4.49). In particular, $\mathcal{B}^\tau$ is an open subset of $\mathcal{G}_0$ that contains the origin. By using the above notations, we now define the function $\mathcal{R}_\tau$ as

$$\mathcal{R}_\tau : \mathcal{B}^\tau \times (\mathbb{C}\setminus\Omega) \to \mathrm{End}(\mathcal{G}), \qquad \mathcal{R}_\tau(\omega,z)|_{\mathcal{G}_a} := \chi_a((\mathrm{ad}\,\omega)_a, z|\tau). \tag{4.74}$$

It follows from the properties of the holomorphic functional calculus on Banach algebras [52] that $\mathcal{R}_\tau$ is well defined and is holomorphic in its variables. Next we introduce also the holomorphic function

$$r^\tau : \mathcal{B}^\tau \times (\mathbb{C} \setminus \Omega) \to \mathcal{G} \otimes \mathcal{G}, \qquad r^\tau(\omega, z) := B(T_\alpha, \mathcal{R}_\tau(\omega, z) T_\beta) T^\alpha \otimes T^\beta, \tag{4.75}$$

where T_α, T^β are dual bases of $\mathcal{G}$. We now state one of our main results.

Proposition 4.2 *The function r^τ introduced above satisfies the spectral-parameter-dependent version of the CDYBE:*

$$\begin{aligned}[][r^\tau_{12}(\omega, z_{12}), r^\tau_{13}(\omega, z_{13})] + [r^\tau_{12}(\omega, z_{12}), r^\tau_{23}(\omega, z_{23})] + [r^\tau_{13}(\omega, z_{13}), r^\tau_{23}(\omega, z_{23})] \\ + T^1_j \frac{\partial}{\partial \omega_j} r^\tau_{23}(\omega, z_{23}) - T^2_j \frac{\partial}{\partial \omega_j} r^\tau_{13}(\omega, z_{13}) + T^3_j \frac{\partial}{\partial \omega_j} r^\tau_{12}(\omega, z_{12}) = 0,\end{aligned} \tag{4.76}$$

where $z_{\alpha\beta} = (z_\alpha - z_\beta) \in (\mathbb{C} \setminus \Omega)$, $\omega \in \mathcal{B}^\tau$, and $\omega_j := B(\omega, T_j)$ with a basis T_j of $\mathcal{G}_0$. Furthermore, r^τ has the properties

$$\mathrm{Res}_{z=0} r^\tau(\omega, z) = \frac{1}{2\pi i} T^\alpha \otimes T_\alpha, \qquad\qquad (r^\tau(\omega, z))^T + r^\tau(\omega, -z) = 0, \tag{4.77}$$

where $(r^\tau(\omega, z))^T := B(T_\alpha, \mathcal{R}_\tau(\omega, z) T_\beta) T^\beta \otimes T^\alpha$ with dual bases T_α, T^β of $\mathcal{G}$, and

$$\frac{d}{dx} r^\tau(e^{\mathrm{ad}\, Tx}(\omega), z)|_{x=0} = [T \otimes 1 + 1 \otimes T, r^\tau(\omega, z)] \qquad \forall T \in \mathcal{G}_0. \tag{4.78}$$

The statements in (4.77) follow immediately from the definition (4.74), (4.75) and the properties of the meromorphic functions χ_a in (4.67). For the first equality in (4.77), one can check that

$$\mathrm{Res}_{z=0} \chi_a(w, z|\tau) = \frac{1}{2\pi i}, \qquad 0 \le a < N. \tag{4.79}$$

For the second statement, one uses the invariance of the scalar product B on $\mathcal{G}$ and

$$\chi_0(-w, z|\tau) = -\chi_0(w, -z|\tau), \quad \chi_a(-w, z|\tau) = -\chi_{N-a}(w, -z|\tau), \quad 0 < a < N. \tag{4.80}$$

The $\mathcal{G}_0$-equivariance property (4.78) is obvious from the definition of r^τ. As for the CDYBE (4.76), it is consequence of the following result.

Proposition 4.3 *The dynamical r-matrix r^τ given by (4.74), (4.75) results by evaluation homomorphism from the dynamical r-matrix $r^{k,+}$ in (4.57). More precisely, if we set*

$$\tau = \frac{kN}{2\pi i} \quad \textit{and} \quad \frac{v_1}{v_2} = \exp(\frac{2\pi i z}{N}) \quad \textit{with} \quad -\Im(\tau) < \Im(z) < 0, \tag{4.81}$$

then the evaluation homomorphism (4.59) yields the relation

$$(\pi_{v_1} \otimes \pi_{v_2})(r^{k,+}(\omega)) = r^\tau(\omega, z) \qquad \forall \omega \in \mathcal{B}_k = \mathcal{B}^\tau. \tag{4.82}$$

Proof. The left hand side of (4.82) gives only a formal infinite sum in general. Below we first calculate this sum, and then notice that it converges to the function on the right hand side of (4.82) if the variables satisfy (4.81).

Let $T_{a,j}$ and T_a^j ($j = 1, \ldots, \dim(\mathcal{G}_a)$) denote bases of $\mathcal{G}_a$ ($a \in \mathcal{E}_\mu$) subject to the relations

$$\langle T_{0,j}, T_0^l \rangle = \delta_j^l, \qquad \langle T_{a,j}, T_{N-a}^l \rangle = \delta_j^l, \quad \forall a \in \mathcal{E}_\mu \setminus \{0\}. \tag{4.83}$$

Introduce corresponding bases of $\ell(\mathcal{G}, \mu)$:

$$T_{a,j}[n_a] := T_{a,j} \otimes t^{n_a}, \qquad T_a^j[n_a] := T_a^j \otimes t^{n_a}, \quad \forall a \in \mathcal{E}_\mu,\ n_a \in (a + N\mathbb{Z}). \tag{4.84}$$

By definition, we then have

$$\begin{aligned} r^{k,+}(\omega) = & \sum_{j,l=1}^{\dim(\mathcal{G}_0)} \sum_{n_0 \in N\mathbb{Z}} \langle T_{0,j}[-n_0], R_k^+(\omega) T_{0,l}[n_0] \rangle\, T_0^j[n_0] \otimes T_0^l[-n_0] \\ & + \sum_{a \in \mathcal{E}_\mu \setminus \{0\}} \sum_{j,l=1}^{\dim(\mathcal{G}_a)} \sum_{n_a \in (a+N\mathbb{Z})} \langle T_{N-a,j}[-n_a], R_k^+(\omega) T_{a,l}[n_a] \rangle\, T_a^j[n_a] \otimes T_{N-a}^l[-n_a]. \end{aligned} \tag{4.85}$$

By substituting the definition of $R_k^+(\omega)$, (4.56) with (4.54), we obtain that

$$\langle T_{N-a,j}[-n_a], R_k^+(\omega) T_{a,l}[n_a] \rangle = B(T_{N-a,j}, (F(kn_a + \operatorname{ad}\omega) + \frac{1}{2}) T_{a,l}) \tag{4.86}$$

for $a \in \mathcal{E}_\mu \setminus \{0\}$, and

$$\langle T_{0,j}[-n_0], R_k^+(\omega) T_{0,l}[n_0] \rangle = B(T_{0,j}, (F(kn_0 + \operatorname{ad}\omega) + \frac{1}{2}) T_{0,l}), \quad n_0 \neq 0, \tag{4.87}$$

$$\langle T_{0,j}[0], R_k^+(\omega) T_{0,l}[0] \rangle = B(T_{0,j}, (f(\operatorname{ad}\omega) + \frac{1}{2}) T_{0,l}), \tag{4.88}$$

where the functions f and F are given in (4.1). This implies that the left hand side of (4.82) can be written in the following form:

$$\begin{aligned} (\pi_{v_1} \otimes \pi_{v_2})(r^{k,+}(\omega)) = & \sum_{j,l=1}^{\dim(\mathcal{G}_0)} B(T_{0,j}, \psi_0((\operatorname{ad}\omega)_0, z|k) T_{0,l})\, T_0^j \otimes T_0^l \\ & + \sum_{a \in \mathcal{E}_\mu \setminus \{0\}} \sum_{j,l=1}^{\dim(\mathcal{G}_a)} B(T_{N-a,j}, \psi_a((\operatorname{ad}\omega)_a, z|k) T_{a,l})\, T_a^j \otimes T_{N-a}^l \end{aligned} \tag{4.89}$$

with

$$\psi_a((\operatorname{ad}\omega)_a, z|k) = \frac{\exp(\frac{2\pi i a z}{N})}{2} \sum_{m \in \mathbb{Z}} e^{2\pi i z m} \left[1 + \coth \frac{kNm + ka + (\operatorname{ad}\omega)_a}{2} \right], \quad a \neq 0, \tag{4.90}$$

and

$$\psi_0((\mathrm{ad}\,\omega)_0, z|k) = \left[\frac{1}{2} + f((\mathrm{ad}\,\omega)_0)\right] + \frac{1}{2}\sum_{m\in\mathbb{Z}^*} e^{2\pi izm}\left[1 + \coth\frac{kNm + (\mathrm{ad}\,\omega)_0}{2}\right]. \tag{4.91}$$

To obtain the $a \neq 0$ terms in (4.89) from (4.85), we used (4.86) and the parametrization $\frac{v_1}{v_2} = \exp(\frac{2\pi iz}{N})$, whereby

$$\begin{aligned}
&\sum_{n_a\in(a+N\mathbb{Z})} \langle T_{N-a,j}[-n_a], R_k^+(\omega)T_{a,l}[n_a]\rangle\,(\pi_{v_1}\otimes\pi_{v_2})\left(T_a^j[n_a]\otimes T_{N-a}^l[-n_a]\right)\\
&= \frac{1}{2}e^{\frac{2\pi iaz}{N}}\sum_{m\in\mathbb{Z}} e^{2\pi izm}B(T_{N-a,j}, [1+2F(ka+kmN+\mathrm{ad}\,\omega)]T_{a,l})T_a^j\otimes T_{N-a}^l\\
&= B(T_{N-a,j}, \frac{1}{2}e^{\frac{2\pi iaz}{N}}\sum_{m\in\mathbb{Z}} e^{2\pi izm}[1+2F(ka+kmN+\mathrm{ad}\,\omega)]T_{a,l})T_a^j\otimes T_{N-a}^l.
\end{aligned}\tag{4.92}$$

This leads to (4.89) with (4.90) by inserting the definition of F (4.1) and noting that $(\mathrm{ad}\,\omega)T_{a,l} = (\mathrm{ad}\,\omega)_a T_{a,l}$. The $a = 0$ term is dealt with in a similar manner.

Now we come to the main point. We notice that if on the right hand sides of (4.90) and (4.91) $(\mathrm{ad}\,\omega)_a$ is replaced by a complex variable w and one uses also $\tau = \frac{kN}{2\pi i}$, then these series become precisely identical with the corresponding series in (4.70), which are convergent on the domain D_a (4.71) for any $a \in \mathcal{E}_\mu$. Since these are absolute convergent series and the convergence is uniform on compact subsets of D_a, it follows that the corresponding operator series in (4.90), (4.91) converge, too. Therefore, if

$$\tau = \frac{kN}{2\pi i}, \qquad \omega \in \mathcal{B}^\tau, \qquad -\Im(\tau) < \Im(z) < 0, \tag{4.93}$$

then $\psi_a((\mathrm{ad}\,\omega)_a, z|k) \in \mathrm{End}(\mathcal{G}_a)$ is well defined by the corresponding series in (4.90), (4.91), and on this domain we obtain

$$\psi_a((\mathrm{ad}\,\omega)_a, z|k) = \chi_a((\mathrm{ad}\,\omega)_a, z|\tau), \qquad \forall a \in \mathcal{E}_\mu. \tag{4.94}$$

If we now compare (4.89) with the definition of r^τ given by (4.74), (4.75), then (4.94) allows us to conclude that $(\pi_{v_1}\otimes\pi_{v_2})(r^{k,+}(\omega)) = r^\tau(\omega, z)$ holds indeed on the domain given by (4.81). *Q.E.D.*

It is clear from the proof that (4.81) is necessary for (4.82); the series appearing in (4.90) and (4.91) do not converge if z lies outside the strip in (4.81). Thus, by applying $\pi_{v_1}\otimes\pi_{v_2}\otimes\pi_{v_3}$ to the CDYBE (4.58), Proposition 4.3 directly implies Proposition 4.2 if z_{12}, z_{13}, z_{23} all lie in this strip. However, by the holomorphicity of the function r^τ, (4.76) is then necessarily valid for any ω, z for which r^τ is defined by eqs. (4.74), (4.75).

Of course, it is possible to calculate $(\pi_{v_1} \otimes \pi_{v_2})(r^{k,-}(\omega))$ as well on an appropriate domain of v_1, v_2. This is left as an exercise.

4.2.4 Recovering Felder's r-matrices

In this subsection $\mathcal{G}$ is a complex simple Lie algebra, and we start by fixing a Cartan subalgebra and a corresponding set Φ^+ of positive roots. We also choose root vectors E_α ($\alpha \in \Phi$) and dual bases of the Cartan subalgebra, H_i and H^j, normalized so that

$$B(H_i, H^j) = \delta_i^j, \qquad B(E_\alpha, E_{-\alpha}) = 1. \tag{4.95}$$

If $\alpha_i \in \Phi^+$ are the simple roots, then there is a unique element, J, of the Cartan subalgebra for which

$$\alpha_i(J) = 1 \qquad \forall i = 1, \dots, \mathrm{rank}(\mathcal{G}). \tag{4.96}$$

Let N be the largest eigenvalue of $(\mathrm{ad}\, J)$ plus 1, i.e., the Coxeter number of $\mathcal{G}$. We wish to show that the application of our preceding construction to the automorphism

$$\mu := \exp(\frac{2\pi i}{N} \mathrm{ad}\, J) \tag{4.97}$$

provides an r-matrix that is equivalent to Felder's solution of the CDYBE [16]. The fixed point set $\mathcal{G}_0$ of this μ is the chosen Cartan subalgebra of $\mathcal{G}$, and Felder's r-matrix is in fact equivalent to

$$S^\tau(\omega, z) := \frac{1}{2\pi i} \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} H_i \otimes H^i + \sum_{\alpha \in \Phi} \chi(\alpha(\omega), z|\tau) E_\alpha \otimes E_{-\alpha}. \tag{4.98}$$

To be precise, Felder's original r-matrix, $\mathcal{F}^\tau$, is given by $\mathcal{F}^\tau(\omega, z) := 2\pi i S^\tau(2\pi i\omega, z)$, which is a substitution that leaves the CDYBE invariant. Referring to the corresponding terms in (4.98), below we also write $S^\tau := S^\tau_{\mathrm{Cartan}} + S^\tau_{\mathrm{root}}$.

It is well known that μ (4.97) acts as a Coxeter element on a Cartan subalgebra which is 'in opposition' to the Cartan subalgebra $\mathcal{G}_0$ and that $\mathcal{A}(\mathcal{G}, \mu)$ with its natural gradation is isomorphic to the untwisted affine Lie algebra of $\mathcal{G}$ equipped with its principal gradation [57]. In [15] the homogeneous realization of the untwisted affine Lie algebra was used to recover Felder's r-matrix with the aid of evaluation homomorphisms. The principal realization provided by $\mathcal{A}(\mathcal{G}, \mu)$ must of course give an equivalent result. It is enlightening to see how this works, and it also provides a useful check on our foregoing calculations.

By using the above notations, now we can spell out r^τ from (4.74), (4.75) explicitly as $r^\tau = r^\tau_{\text{Cartan}} + r^\tau_{\text{root}}$ with

$$r^\tau_{\text{Cartan}}(\omega, z) = B(H_i, \chi_0(\operatorname{ad}\omega, z|\tau)H_j)H^i \otimes H^j = \chi_0(0, z|\tau)H_i \otimes H^i. \tag{4.99}$$

The second equality holds because $\chi_0(\operatorname{ad}\omega, z|\tau)H_j = \chi_0(0, z|\tau)H_j$, which in turn follows from $(\operatorname{ad}\omega)H_j = 0$. It is easy to compute that

$$\chi_0(0, z|\tau) = \lim_{w\to 0} \chi_0(w, z|\tau) = \frac{1}{2\pi i}\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}. \tag{4.100}$$

Thus the Cartan parts of S^τ and r^τ are equal, and are ω-independent.

As for the root part, by using that $(\operatorname{ad}\omega)E_\alpha = \alpha(\omega)E_\alpha$, the definitions give

$$\begin{aligned} r^\tau_{\text{root}}(\omega, z) = &\sum_{\alpha\in\Phi^+} e^{\frac{2\pi i\alpha(J)z}{N}} \chi(\alpha(\omega) + 2\pi i\frac{\alpha(J)}{N}\tau, z|\tau)E_\alpha \otimes E_{-\alpha} \\ &+ \sum_{\alpha\in\Phi^+} e^{\frac{2\pi i(N-\alpha(J))z}{N}} \chi(-\alpha(\omega) + 2\pi i\frac{N-\alpha(J)}{N}\tau, z|\tau)E_{-\alpha} \otimes E_\alpha. \end{aligned} \tag{4.101}$$

Then we use the identity

$$\chi(w + 2\pi i\tau, z|\tau) = e^{-2\pi i z}\chi(w, z|\tau), \tag{4.102}$$

which permits us to rewrite r^τ_{root} as

$$r^\tau_{\text{root}}(\omega, z) = \sum_{\alpha\in\Phi} e^{\frac{2\pi i\alpha(J)z}{N}} \chi(\alpha(\omega) + 2\pi i\frac{\alpha(J)}{N}\tau, z|\tau)E_\alpha \otimes E_{-\alpha}. \tag{4.103}$$

By comparing the above expressions of r^τ and S^τ, we conclude that

$$r^\tau(\omega, z) = \left(e^{\frac{2\pi i}{N}z_1 \operatorname{ad} J} \otimes e^{\frac{2\pi i}{N}z_2 \operatorname{ad} J}\right) S^\tau(\omega + 2\pi i\frac{\tau}{N}J, z|\tau) \quad \text{with} \quad z = z_1 - z_2. \tag{4.104}$$

If the dynamical variable ω belongs to a Cartan subalgebra, $\mathcal{G}_0$, then the constant shifts of ω and the similarity transformations by $e^{z_1 \operatorname{ad} H} \otimes e^{z_2 \operatorname{ad} H}$ for any $H \in \mathcal{G}_0$, $z_1 - z_2 = z$ map the solutions of the CDYBE to other solutions. In fact, these transformations are special cases of the gauge transformations considered in section 4.2 of [15].

In summary, we have shown that the solution of the CDYBE provided by Proposition 4.2 in the principal case of μ in (4.97) is gauge equivalent to Felder's dynamical r-matrix in the sense of (4.104).

Recently generalizations of Felder's r-matrices have been found [58] for which the dynamical variables belong to a subalgebra of a Cartan of a simple Lie algebra $\mathcal{G}$. The subalgebra in question is the fixed point set of an outer automorphism of $\mathcal{G}$ of finite order, and the r-matrices given by proposition 4.2 in [58] contain the same elliptic functions that appear in (4.74). These r-matrices are very likely to be gauge equivalent to those special cases of the r-matrices constructed in subsection 4.2.3 for which $\mathcal{G}$ is simple and $\mathcal{G}_0$ is a contained in a Cartan subalgebra.

4.3 On a possible application to spin Calogero–Moser models

An interesting problem is to find applications of the generalizations of Felder's r-matrices provided by Proposition 4.2 in integrable systems. In this respect, it appears promising to seek for generalized Calogero–Moser type systems, since certain spin Calogero–Moser systems are known to be closely related to Felder's r-matrices [37, 54].

Let $(\mathcal{G}, \langle\ ,\ \rangle)$ be a self-dual Lie algebra and $\mathcal{H} \subset \mathcal{G}$ a self-dual, Abelian subalgebra. Keeping the notations of Proposition 4.2, let us consider a solution $r(q,z) \in \mathcal{G} \otimes \mathcal{G}$ of the spectral parameter dependent version of the CDYBE (4.76):

$$\begin{aligned}[r_{12}(q,z_{12}), r_{13}(q,z_{13})] + [r_{12}(q,z_{12}), r_{23}(q,z_{23})] + [r_{13}(q,z_{13}), r_{23}(q,z_{23})]\\ +T_j^1 \frac{\partial}{\partial q_j} r_{23}(q,z_{23}) - T_j^2 \frac{\partial}{\partial q_j} r_{13}(q,z_{13}) + T_j^3 \frac{\partial}{\partial q_j} r_{12}(q,z_{12}) = 0,\end{aligned} \tag{4.105}$$

where $z_{\alpha\beta} = (z_\alpha - z_\beta)$ and $q_j := \langle q, T_j \rangle$ with a basis T_j of $\mathcal{H}$. Assume in addition that

$$r_{21}(q,-z) = -r_{12}(q,z) \quad \text{and} \quad [r_{12}(q,z), H_1 + H_2] = 0 \quad \forall H \in \mathcal{H}. \tag{4.106}$$

Let $\mathcal{R}(q,z)$ denote the End$(\mathcal{G})$ valued function associated with $r(q,z)$ in the natural way, i.e., $\mathcal{R}(q,z)X = T_a r^{ab}(q,z)\langle T_b, X\rangle$ for $r(q,z) = r^{ab}(q,z) T_a \otimes T_b$, where $\{T_a\}$ and $\{T^a\}$ are dual bases of $\mathcal{G}$. The 'dynamical variable' q varies in an appropriate open subset of $\mathcal{H}^*$, denoted by $\check{\mathcal{H}}^*$, and we denote the spectral parameter by z.

Consider the phase space

$$\mathcal{M} := T^*\check{\mathcal{H}}^* \times \mathcal{G}^* \simeq \check{\mathcal{H}}^* \times \mathcal{H} \times \mathcal{G} \simeq \{(q,p,\xi)\} \tag{4.107}$$

equipped with the direct product of the natural Poisson brackets on $T^*\check{\mathcal{H}}^*$ and on $\mathcal{G}^*$. In coordinates,

$$\{q^i, p_j\} = \delta^i_j, \qquad \{\xi_a, \xi_b\} = f^c_{ab}\xi_c, \tag{4.108}$$

where $f^c_{ab} = \langle T^c, [T_a, T_b]\rangle$ are the structure constants of $\mathcal{G}$.

Define the functions $\mathcal{L}(z) : \mathcal{M} \to \mathcal{G}$ by

$$\mathcal{L}(z) : (q, p, \xi) \mapsto p - \mathcal{R}(q, z)\xi. \tag{4.109}$$

Using the decomposition $\mathcal{G} = \mathcal{H} + \mathcal{H}^\perp$ induced by $\langle , \rangle$, $\forall \xi \in \mathcal{G}$ is decomposed as $\xi = \xi_\mathcal{H} + \xi_{\mathcal{H}^\perp}$. We can now state

Proposition 4.4 *The $\mathcal{G}$-valued functions $\mathcal{L}(z)$ on $\mathcal{M}$ verify the Poisson bracket relation*

$$\{\mathcal{L}_1(z), \mathcal{L}_2(w)\} = [r_{12}(z-w), \mathcal{L}_1(z) + \mathcal{L}_2(w)] - \nabla_{\xi_\mathcal{H}} r_{12}(z-w). \tag{4.110}$$

Proof. An easy calculation gives that

$$\begin{aligned} \{\mathcal{L}_1(z), \mathcal{L}_2(w)\} &- \big([r_{12}(z-w), \mathcal{L}_1(z) + \mathcal{L}_2(w)] - \nabla_{\xi_\mathcal{H}} r_{12}(z-w)\big) \\ &= \langle \xi, \mathcal{E}(r, T^a, T^b, z, w)\rangle T_a \otimes T_b, \end{aligned} \tag{4.111}$$

where

$$\begin{aligned} \mathcal{E}(r, X, Y, z, w) := {} & [\mathcal{R}^T(z)X, \mathcal{R}^T(w)Y] + \mathcal{R}^T(z)[X, \mathcal{R}(z-w)Y] \\ & - \mathcal{R}^T(w)[\mathcal{R}^T(z-w)X, Y] + \langle X, (\nabla \mathcal{R}(z-w))Y\rangle \\ & - (\nabla_{Y_\mathcal{H}} \mathcal{R}^T(z))X + (\nabla_{X_\mathcal{H}} \mathcal{R}^T(w))Y \qquad \forall X, Y \in \mathcal{G}. \end{aligned} \tag{4.112}$$

Due to the first relation in (4.106),

$$\mathcal{E}(r, X, Y, z, w) = 0 \qquad \forall X, Y \in \mathcal{G} \tag{4.113}$$

can be checked to be equivalent to the CDYBE (4.105), whereby the proposition is proved. *Q.E.D.*

The main message of this proposition is that the Poisson brackets of $\mathcal{L}(z)$ are almost St Petersburg type (1.18), up to a derivative term. The natural question is how to construct integrable systems, based on the relation (4.110). In the special case when $\mathcal{G}$ is simple Lie algebra and $\mathcal{H}$ is a Cartan subalgebra, Li and Xu have given a detailed analysis [54]. (The statement of Proposition 4.4 in this case is contained in [54], but their proof is much more complicated.) The essential point of the construction of 'integrable spin Calogero–Moser type systems' can be summarized as follows: *Let us impose such constraints on the phase space $\mathcal{M}$*

(4.107) whereby the second term on the right hand side of (4.110) vanishes. The simplest (and in many cases the only) way to do this is to impose the constraint

$$\xi_{\mathcal{H}} = 0. \tag{4.114}$$

These constraints are first class in Dirac's terminology. Perhaps first restricting to an open submanifold $\check{\mathcal{M}}$ of $\mathcal{M}$, one has to determine the associated reduced phase space. In fact, this reduced phase space has the structure

$$\check{\mathcal{M}}_{red} = T^*\check{\mathcal{H}}^* \times \check{\mathcal{G}}^*_{red}, \tag{4.115}$$

where $\check{\mathcal{G}}^*_{red}$ is the reduced phase space coming from $\check{\mathcal{G}}^* \subset \mathcal{G}^*$ associated with $\check{\mathcal{M}}$. By restricting to appropriate symplectic leaves in $\check{\mathcal{G}}^*_{red}$, what one gets may be called a (generalized) integrable spin Calogero–Moser system.

In the near future we wish to list a set of (new) integrable spin Caloger–Moser systems in correspondence with our solutions of the CDYBE (see Proposition 4.2.) There are lots of further problems. For example to prove the integrability of these systems and to integrate them.

Summary

In conclusion, let us summarize the content of the present work, chapter by chapter.

Chapter 1: Overview of the theory of integrable systems. In this chapter we introduced those notions that are necessary to treat the non-dynamical r-matrix structure of the degenerate Calogero–Moser models and to understand the concept of dynamical r-matrices. We briefly reviewed the definition of Liouville integrability, Lax pairs and r-matrices. The definition of the classical Yang–Baxter equation and its dynamical generalization was also presented.

Chapter 2: Degenerate Calogero–Moser models. In this chapter we have determined the most general constant r-matrices that may be obtained by coordinate dependent gauge transformations of the standard Lax representation (2.6) of the degenerate Calogero–Moser models associated with gl_n. Up to automorphisms of gl_n (i.e. up to conjugation by constants $g_0 \in GL_n$ and transpose) and addition of an irrelevant term $\mathbf{1}_n \otimes Q'$ with any constant $Q' \in gl_n$, the most general such r-matrix turned out to have the form

$$r' = \sum_{(a,b,c,d)\in S} (\mathcal{B} e_{ab} \wedge e_{cd} - e_{a+1,b} \wedge e_{c+1,d}) + n\Omega X \wedge \mathbf{1}_n,$$

where

$$X = -\frac{1}{n}\sum_{k=1}^{n-1}(n-k)e_{k+1,k} - \frac{\mathcal{B}}{n}\sum_{k=1}^{n-1} k e_{k,k+1},$$

$\mathcal{B}$ is given according to (2.12) in correspondence with the rational, hyperbolic and trigonometric potential functions (2.4), Ω is an arbitrary constant scalar, and

$$S = \{\, (a,b,c,d) \in \mathbf{N}^4 \,|\, a+c+1 = b+d, \quad 1 \le b \le a < n, \quad b \le c < n, \quad 1 \le d \le n \,\}.$$

We have seen that r' solves the classical (modified) Yang–Baxter equation (2.59), and have identified it in terms of well-known solutions of this equation. In particular, we have shown that in the hyperbolic and trigonometric cases the above r' with $\Omega = -\frac{1}{n}$ is equivalent to a multiple of the Cremmer–Gervais classical r-matrix under an automorphism of gl_n. We obtained these results by an explicit determination of the gauge transformations $g(q) \in GL_n$ for which the Poisson brackets of $L'(q,p) = g(q)L(q,p)g^{-1}(q)$, where L is the standard Lax matrix (2.6), can be written in the form (2.2) with a constant r-matrix. The gauge transformation $g(q)$ for which the Poisson brackets of L' are encoded by r' in (4.116) was found as the product

$$g(q) = \exp\left(-Xn\Omega\sum_{i=1}^{n} q_i\right)\varphi(q)\chi(q),$$

where the matrices $\varphi(q)$ and $\chi(q)$ are defined by (2.49) and (2.50), with the notations fixed by equations (2.7), (2.8), (2.13) in section 2.1. The outcome of our direct analysis of the degenerate Calogero–Moser models is consistent with the results obtained in [44, 10, 42] by different means.

Chapter 3: Canonical dynamical r-matrices. In this chapter we have presented a direct proof of the mCDYBE for the canonical r-matrix. As opposed to the local power series expansion around 0, we here use the well known [52] holomorphic functional calculus of linear operators to define the canonical r-matrix as $R(\omega) = f(\mathrm{ad}\omega)$, and thus our proof is valid globally on the maximal domain of the 'dynamical variable' ω. An advantage of our proof is that it also yields a uniqueness result for the holomorphic function $f(z)$ that enters the definition of the r-matrix in (3.1). Namely, by taking formula (3.1) as an ansatz the mCDYBE translates into a functional equation (eq. (F.1)) for the holomorphic function f that has (3.2) as its unique solution under certain further natural conditions.

Capter 4: Generalizations of Felder's elliptic dynamical r-matrices. In this chapter we have further developed the construction of dynamical r-matrices building mainly on the seminal paper [15] and the work [48]. Here our first main result is Theorem 4.1, whereby a dynamical r-matrix is associated with any graded self-dual Lie algebra subject to the rather mild conditions in (4.8)–(4.10) and the strong spectral condition described in (4.12). Our second main result is the application of this construction to the general class of affine Lie algebras $\mathcal{A}(\mathcal{G},\mu)$ corresponding to the automorphisms of the finite-dimensional self-dual Lie algebras that preserve the scalar product and are of finite order. The resulting dynamical r-matrices are generalizations of the basic trigonometric dynamical r-matrices of [15], which are recovered if μ is a Coxeter automorphism of a simple Lie algebra. Motived by the derivation of Felder's elliptic dynamical r-matrices [16] found in [15], we have also determined the spectral-parameter-dependent $\mathcal{G}\otimes\mathcal{G}$-valued dynamical r-matrices that correspond to the $\mathcal{A}(\mathcal{G},\mu)\otimes\mathcal{A}(\mathcal{G},\mu)$-valued r-matrices directly obtained from Theorem 4.1. The result is given explicitly by Proposition 4.2 and Proposition 4.3 is subsection 4.2.3.

It is worth noting that the conditions of Theorem 4.1 are satisfied also if $\mathcal{A}$ is an arbitrary

Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix, equipped with the principal gradation [57]. In this case one recovers the r-matrices given by equation (3.4) in [15]. It would be interesting to find applications of Theorem 4.1 outside the aforementioned classes of Lie algebras. It would be also interesting to find applications of the r-matrices given by Proposition 4.2 in the context of spin Calogero–Moser models; some ideas in this direction are collected in section 4.3.

Összefoglalás

Végezetül, foglaljuk össze a dolgozat tartalmát, fejezetenkénti lebontásban.

1. Fejezet: Az integrálható rendszerek elméletének áttekintése. Ebben a fejezetben bevezettük azokat az objektumokat, amelyek szükségesek a degenerált Calogero–Moser modellek r-mátrix struktúrájának kezeléséhez, illetve a dinamikai r-mátrix fogalmának megértéséhez. Röviden áttekintettük a Liouville integrálhatóság fogalmát, illetve a Lax pár és r-mátrix definícióját. Ismertettük a klasszikus Yang–Baxter egyenletet, és annak egy lehetséges (dinamikai) általánosítását is.

2. Fejezet: Degenerált Calogero–Moser modellek. Ebben a fejezetben meghatároztuk a gl_n Lie algebrán alapuló degenerált Calogero–Moser modellek szokványos Lax reprezentációjából (2.6) koordináta függő mérték transzformációval nyerhető legáltalánosabb konstans r-mátrixot. Kiderítettük, hogy egy irreleváns $\mathbf{1}_n \otimes Q'$ tag ($Q' \in gl_n$), illetve a gl_n Lie algebra egy tetszőleges automorfizmusának erejéig a legáltalánosabb ilyen tulajdonságú r-mátrix alakja

$$r' = \sum_{(a,b,c,d)\in S} (\mathcal{B} e_{ab} \wedge e_{cd} - e_{a+1,b} \wedge e_{c+1,d}) + n\Omega X \wedge \mathbf{1}_n,$$

ahol

$$X = -\frac{1}{n}\sum_{k=1}^{n-1}(n-k)e_{k+1,k} - \frac{\mathcal{B}}{n}\sum_{k=1}^{n-1} k e_{k,k+1}.$$

A formulákban szereplő $\mathcal{B}$ paraméter értéke modellfüggő (2.12), Ω tetszőleges konstans skalár, az összegzésben szereplő S halmaz definíciója pedig

$$S = \{\, (a,b,c,d) \in \mathbf{N}^4 \,|\, a+c+1 = b+d, \quad 1 \le b \le a < n, \quad b \le c < n, \quad 1 \le d \le n \,\}.$$

Beláttuk, hogy r' megoldja a klasszikus (modifikált) Yang–Baxter egyenletet (2.59). Ezen egyenlet jól ismert megoldásainak birtokában a megoldásunk beazonosítása is megtörtént. Ennek eredményeként arra jutottunk, hogy hiperbolikus, illetve trigonometrikus esetben a nyert

r' megoldásunk $\Omega = \frac{1}{n}$ választás esetén, a gl_n Lie algebra egy automorfizmusának erejéig ekvivalens a Cremmer–Gervais-féle klasszikus r-mátrix többszörösével. Ezen eredményeket úgy kaptuk, hogy meghatároztuk azon $g(q) \in GL_n$ mérték transzformációkat, melynek eredményeként a transzformált $L'(q,p) = g(q)L(q,p)g^{-1}(q)$ Lax-mátrix Poisson zárójele a (1.16) alakot ölti, de már egy konstans r'-mátrix segítségével. Amennyiben a $g(q)$ mérték transzformációnak a

$$g(q) = \exp\left(-Xn\Omega\sum_{i=1}^{n} q_i\right)\varphi(q)\chi(q)$$

alakot választjuk, a transzformált L' Lax-mátrix Poisson zárójelét konstans r-mátrix kódolja, pontosan a mi r' megoldásunk. Érdemes megjegyezni, hogy a direkt analízisünk eredményeként nyert megoldások összhangban vannak más módszerekkel nyert [44, 10, 42] megfontolásokkal is.

3. Fejezet: Kanonikus dinamikai r-mátrixok. Ezen fejezet célja az, hogy egy új, direkt bizonyítását adjuk annak, hogy a kanonikus r-mátrix kielégíti a klasszikus dinamikai modifikált Yang–Baxter egyenletet. A lokális érvényű hatványsor módszerrel szemben, a lineáris operátorok jól ismert holomorf függvény kalkulusát [52] használjuk a kanonikus r-mátrix $R(\omega) = f(\mathrm{ad}\omega)$ definíciójában. Ennek eredményeként a bizonyításunk globálisan érvényes az ω *dinamikai változó* egy maximális tartományán. Bizonyításunk erősségének tekinthető az a tény, hogy az r-mátrix (3.1) definíciójában szereplő $f(z)$ függvényre egyértelműségi eredmény is nyerhető. Nevezetesen, ha az r-mátrix alakjára vonatkozólag az (3.1) feltevéssel élünk, akkor a klasszikus dinamikai modifikált Yang–Baxter egyenlet a holomorf f függvényre nézve egy függvényegyenletbe (F.1) megy át, ami további természetes feltevések mellett az egyértelmű (3.2) megoldással rendelkezik.

4. Fejezet: A Felder-féle elliptikus dinamikai r-mátrixok általánosításai. Ezen fejezet célja az, hogy Etingof és Varchenko nagyhatású dolgozata [15] és egy korábbi munkánk [48] alapján továbbfejlesszük a dinamikai r-mátrixok konstrukciójára vonatkozó ismereteket. Ezzel kapcsolatos első eredményünket a 4.1 Tétel tartalmazza. Beláttuk, hogy a meglehetősen enyhe (4.8)–(4.10) feltételek, és az erős (4.12) *spektrális* feltevés teljesülése esetén, tetszőleges önduális, gradált Lie algebrához dinamikai r-mátrix társítható. A következő érdekes eredményünk az, hogy ezen konstrukció az $\mathcal{A}(\mathcal{G},\mu)$ affin Lie algebrák általános osztályára is alkalmazható. (Az $\mathcal{A}(\mathcal{G},\mu)$ affin Lie algebrák alapjául olyan véges dimenziós önduális $\mathcal{G}$ Lie algebrák szolgálnak, melyeken adott egy véges rendű μ automorfizmus, ami megőrzi a skaláris szorzatot.) Az eredményül kapott dinamikai r-mátrixok a Felder-féle alapvető trigonometrikus dinamikai r-mátrixok általánosításaként tekinthetők. Valóban, konstrukciónk speciális eseteként visszanyerhető a Felder-féle megoldás, nevezetesen, ha $\mathcal{G}$ egyszerű Lie algebra, μ pedig Coxeter automorfizmus. Ugyancsak fontos eredmény, hogy a 4.1 Tétel következményeként nyerhető $\mathcal{A}(\mathcal{G},\mu)\otimes\mathcal{A}(\mathcal{G},\mu)$ értékű dinamikai r-mátrixokból, $\mathcal{G}\otimes\mathcal{G}$ értékű, spektrálparamétertől függő, elliptikus dinamikai r-mátrixokat kaptunk. Ennek alapjául az a megfigyelés szolgált, ahogy a [15]-es munkában levezették a Felder-féle elliptikus dinamikai r-mátrixot [16]. A pontos állítások a 4.2 és 4.3 Propozíciókban kerültek megfogalmazásra.

Érdemes megjegyezni, hogy a 4.1 Tétel feltételei akkor is fennállnak, ha az $\mathcal{A}$ Lie algebraként egy principális gradálással ellátott, tetszőleges Kac–Moody algebrát [57] választunk. Ilyen választás esetén az [15]-es dolgozat (3.4)-es egyenletében szereplő r-mátrixok is visszanyerhetők. Továbbra is nyitott kérdés Lie algebrák olyan újabb osztályainak feltárása, melyekre a 4.1 Tétel alkalmazható. Hasonlóan érdekes kérdés lenne alkalmazást találni a 4.2 Propozícióban leírt r-mátrixoknak a spin Calogero–Moser modellek keretei között. Ilyen irányú lehetőségeket a 4.3 szakasz tartalmaz.

Acknowledgements

I would like to express my deepest gratitude to my supervisor, Prof. László Fehér, for his kind encouragement and continuous help during my work at the Department of Theoretical Physics at University of Szeged. I also wish to thank my parents, grandparents and sister for everything.

Appendices

A Proof of Theorem 2.1

The proof given below relies on the general analysis of the momentum independent Calogero–Moser r-matrices presented by Braden and Suzuki in [29]. We first specialize the relevant results of [29] to our case and then further elaborate them to obtain the statement of Theorem 2.1.

Consider the Lax matrix in (6) with a function w in (7). Our task is to find the most general momentum independent r-matrix, $r(q)$, which satisfies equation (2), i.e.,

$$\{L_1, L_2\}(q,p) = [r_{12}(q), L_1(q,p)] - [r_{21}(q), L_2(q,p)]. \tag{A.1}$$

Obviously, $r(q) = r_{12}(q) \in gl_n \otimes gl_n$ can be expanded in the form

$$r(q) = \sum_{i,j=1}^{n} r^{i,j}(q) H_i \otimes H_j + \sum_{\alpha\in\Phi}\sum_{i=1}^{n}\big(r^{i,\alpha}(q) H_i \otimes E_\alpha + r^{\alpha,i}(q) E_\alpha \otimes H_i\big) + \sum_{\alpha,\beta\in\Phi} r^{\alpha,\beta}(q) E_\alpha \otimes E_\beta. \tag{A.2}$$

Since the functions w in (7) are odd (and thus $w_{-\alpha}(q) = -w_\alpha(q)$), we can use the results of the third and fourth chapters of [12], where it has been shown that under our conditions the following equations hold:

$$r^{\alpha,i}(q) = 0 \qquad (\forall i \in \{1,\dots,n\},\, \forall \alpha \in \Phi), \tag{A.3}$$

$$r^{\alpha,\beta}(q) = \frac{w'_\alpha(q)}{w_\alpha(q)}\delta_{\alpha,-\beta} = -F_\alpha(q)\delta_{\alpha,-\beta} \qquad (\forall \alpha,\beta \in \Phi). \tag{A.4}$$

Moreover, according to [29], the remaining requirements on $r(q)$ reduce to the equations

$$\sum_{i=1}^{n} \alpha_i r^{i,j}(q) = 0 \qquad (\forall \alpha \in \Phi,\, \forall j \in \{1,\dots,n\}), \tag{A.5}$$

and

$$\sum_{i=1}^{n}(\alpha_i r^{i,\beta} w_\alpha - \beta_i r^{i,\alpha} w_\beta) = c^{\beta}_{-\alpha,\alpha+\beta}(r^{-\alpha,\alpha} w_{\alpha+\beta} + r^{-\beta,\beta} w_{\alpha+\beta}) \qquad (\forall \alpha, \beta \in \Phi). \tag{A.6}$$

We here use the basis of gl_n introduced in (5), $\alpha_i := \alpha(H_i)$, the structure constants $c^{\alpha+\beta}_{\alpha,\beta}$ satisfy $[E_\alpha, E_\beta] = c^{\alpha+\beta}_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, (\alpha+\beta)$ all belong to Φ, and $c^{\alpha+\beta}_{\alpha,\beta} := 0$ otherwise.

Now consider equation (A.5) for $\alpha := (\lambda_k - \lambda_l) \in \Phi$. From this we see that $r^{k,j}(q) - r^{l,j}(q) = 0$ $(k \neq l, \forall j)$, which means that the general solution of (A.5) is

$$r^{i,j}(q) = M^j(q) \qquad (\forall i, j \in \{1, \ldots, n\}), \tag{A.7}$$

where the M^j are arbitrary smooth functions of q. Let us next solve (A.6) for $r^{i,\alpha}(q)$. By substituting (A.4) into (A.6) and using the identity (10) and the symmetry properties of the structure constants we obtain

$$\sum_{i=1}^{n} \left(\alpha_i \hat{r}^{i,\beta}(q) - \beta_i \hat{r}^{i,\alpha}(q)\right) = c^{\alpha+\beta}_{\alpha,\beta} \qquad (\forall \alpha, \beta \in \Phi), \tag{A.8}$$

where we define $\hat{r}^{i,\gamma} := \frac{r^{i,\gamma}}{w_\gamma}$ for any $\gamma \in \Phi$. By introducing the notations

$$\hat{r}^\alpha_S := \sum_{i=1}^{n}(\hat{r}^{i,\alpha} + \hat{r}^{i,-\alpha}) H_i, \qquad \hat{r}^\alpha_A := \sum_{i=1}^{n}(\hat{r}^{i,\alpha} - \hat{r}^{i,-\alpha}) H_i, \tag{A.9}$$

we have

$$\hat{r}^\alpha := \sum_{i=1}^{n} \hat{r}^{i,\alpha} H_i = \frac{1}{2}(\hat{r}^\alpha_S + \hat{r}^\alpha_A), \quad \hat{r}^\alpha_S(q) = \hat{r}^{-\alpha}_S(q), \quad \hat{r}^{-\alpha}_A(q) = -\hat{r}^\alpha_A(q). \tag{A.10}$$

We now consider equation (A.8) for the pairs of roots (α, β) and $(\alpha, -\beta)$. By adding these two equations we get

$$\alpha(\hat{r}^\beta_S(q)) = c^{\alpha+\beta}_{\alpha,\beta} + c^{\alpha-\beta}_{\alpha,-\beta} \qquad (\forall \alpha, \beta \in \Phi). \tag{A.11}$$

It follows from the definition of K_α in (5) that $\alpha(K_\beta) = -(c^{\alpha+\beta}_{\alpha,\beta} + c^{\alpha-\beta}_{\alpha,-\beta})$ for any $\alpha, \beta \in \Phi$. Therefore the general solution of (A.11) is given by

$$\hat{r}^\alpha_S(q) = -K_\alpha + \tau^\alpha_S(q) \mathbf{1}_n \qquad (\forall \alpha \in \Phi), \tag{A.12}$$

where $\tau^\alpha_S(q) = \tau^{-\alpha}_S(q)$ are arbitrary smooth functions. On the other hand, by substituting (A.11) and the decomposition in (A.10) into (A.8) we obtain the relation

$$\alpha(\hat{r}^\beta_A(q)) = \beta(\hat{r}^\alpha_A(q)) \qquad (\forall \alpha, \beta \in \Phi). \tag{A.13}$$

Obviously, there exists the decomposition

$$\hat{r}^{\alpha}_{A}(q) = C_{\alpha}(q) + \tau^{\alpha}_{A}(q)\mathbf{1}_n, \tag{A.14}$$

where $C_\alpha(q) \in \mathcal{H}_n \subset sl_n$ and $\tau^\alpha_A(q)$ are smooth functions. The antisymmetry of $\hat{r}^\alpha_A(q)$ in α and (A.13) can be rewritten as

$$C_{-\alpha}(q) = -C_\alpha(q), \qquad \alpha(C_\beta(q)) = \beta(C_\alpha(q)), \qquad \tau^{-\alpha}_A(q) = -\tau^\alpha_A(q). \tag{A.15}$$

By the above, we have parametrized the most general $r(q)$ in terms of the functions M^j, τ^α_A, τ^α_S and C_α. If we now introduce the notation

$$Q(q) := \sum_{i=1}^{n} M^i(q)H_i + \frac{1}{2}\sum_{\alpha\in\Phi} \left(\tau^\alpha_S(q) + \tau^\alpha_A(q)\right) w_\alpha(q)E_\alpha, \tag{A.16}$$

then $r(q)$ in (A.2) takes precisely the form stated by Theorem 2.1, which completes the proof.

B Proof of Proposition 2.2

In this appendix we prove Proposition 2.2 by analyzing equation (2.32),

$$\alpha_k w_\alpha^2 \delta_{\beta,-\alpha} - c^{\alpha+\beta}_{\alpha,\beta}\frac{w_\alpha w_\beta}{w_{\alpha+\beta}} A^{\alpha+\beta}_k + (\alpha\cdot r^\beta)A^\alpha_k + (\beta\cdot A^\alpha)A^\beta_k = 0 \qquad (\forall k = 1,\dots,n), \tag{B.1}$$

whereby we determine the constants b^α_k that appear in $A^\alpha_k = w_\alpha b^\alpha_k$ (2.33). We here use the notation $\alpha\cdot r^\beta = \sum_{i=1}^n \alpha_i r^\beta_i$, $\beta\cdot A^\alpha = \sum_{i=1}^n \beta_i A^\alpha_i$ and similarly for all quantities with Cartan indices. For later reference, note from (2.25) that

$$\beta\cdot r^\alpha = \frac{1}{2} w_\alpha \beta\cdot(C_\alpha - K_\alpha), \qquad \forall \alpha,\beta\in\Phi, \tag{B.2}$$

where K_α is defined in (2.5) and $C_\alpha = \sum_{i=1}^n C^i_\alpha H_i$ enjoys the properties in (2.15).

If we fix $\alpha\in\Phi$, then (B.1) for the pairs of roots (α,β) given by (α,α), $(-\alpha,-\alpha)$, $(\alpha,-\alpha)$ and $(-\alpha,\alpha)$ leads respectively to the following relations:

$$(\alpha\cdot r^\alpha + \alpha\cdot A^\alpha)A^\alpha_k = 0, \tag{B.3}$$

$$(\alpha\cdot r^{-\alpha} + \alpha\cdot A^{-\alpha})A^{-\alpha}_k = 0, \tag{B.4}$$

$$\alpha_k w_\alpha^2 + (\alpha\cdot r^{-\alpha})A^\alpha_k - (\alpha\cdot A^\alpha)A^{-\alpha}_k = 0, \tag{B.5}$$

$$\alpha_k w_\alpha^2 + (\alpha \cdot r^\alpha) A_k^{-\alpha} - (\alpha \cdot A^{-\alpha}) A_k^\alpha = 0. \tag{B.6}$$

Since $\alpha \cdot r^\alpha = \alpha \cdot r^{-\alpha}$ by (B.2), these relations imply that

$$\alpha \cdot A^\alpha = \alpha \cdot A^{-\alpha} = -\alpha \cdot r^\alpha. \tag{B.7}$$

On account of (B.7) and (B.2), (B.5) can be written as

$$\alpha_k w_\alpha^2 = (\alpha \cdot A^\alpha)(A_k^\alpha + A_k^{-\alpha}). \tag{B.8}$$

This expression shows that

$$b_k^\alpha - b_k^{-\alpha} = \varepsilon^\alpha \alpha_k \tag{B.9}$$

with some constants ε^α. We then find from the above that

$$\alpha \cdot b^\alpha = \varepsilon^\alpha \tag{B.10}$$

and the ε^α must satisfy

$$\varepsilon^\alpha = \varepsilon^{-\alpha}, \qquad (\varepsilon^\alpha)^2 = 1. \tag{B.11}$$

Now it is convenient to introduce $\Pi_k^\alpha := (b_k^\alpha + b_k^{-\alpha})$, which results in

$$b_k^\alpha = \frac{1}{2}\varepsilon^\alpha \alpha_k + \frac{1}{2}\Pi_k^\alpha, \qquad \forall \alpha \in \Phi. \tag{B.12}$$

Let us put $\Pi_k^{ij} := \Pi_k^{(\lambda_i - \lambda_j)}$. Then the relations $\Pi_k^\alpha = \Pi_k^{-\alpha}$ and $\alpha \cdot \Pi^\alpha = 0$ (by (B.7)) give

$$\Pi_k^{ij} = \Pi_k^{ji}, \qquad \Pi_i^{ij} = \Pi_j^{ij}, \qquad \forall k,\, i \neq j. \tag{B.13}$$

Consider now such roots $\alpha = (\lambda_i - \lambda_j)$ and $\beta = \pm(\lambda_l - \lambda_m) \in \Phi$ that $\{i, j\} \cap \{l, m\} = \emptyset$. In this case (B.1) yields

$$(\alpha \cdot \hat{r}^\beta) b_k^\alpha + (\beta \cdot b^\alpha) b_k^\beta = 0, \tag{B.14}$$

$$(\alpha \cdot \hat{r}^{-\beta}) b_k^\alpha - (\beta \cdot b^\alpha) b_k^{-\beta} = 0, \tag{B.15}$$

where we use the notation $\hat{r}^\gamma := \frac{r^\gamma}{w_\gamma}$ for any $\gamma \in \Phi$. Adding these two equations, and using (B.7) and (B.12), we can easily get that now

$$\beta \cdot b^\alpha = 0, \qquad \beta \cdot \Pi^\alpha = 0. \tag{B.16}$$

The general form of Π_k^{ij} which obeys (B.13) and (B.16) is in fact the following:

$$\Pi_k^{ij} = \eta^\alpha (\delta_{ki} + \delta_{kj}) + 2\Omega^\alpha, \tag{B.17}$$

where η^α, Ω^α are constants. Notice that for $\alpha = (\lambda_i - \lambda_j)$ that element $K_\alpha = \sum_{k=1}^n K_\alpha^k H_k$ defined in (2.5) has precisely the components $K_\alpha^k = \delta_{ki} + \delta_{kj}$.

Now, let $\alpha, \beta, \alpha + \beta \in \Phi$ be roots. In this case $\alpha - \beta = \alpha + (-\beta) \notin \Phi$. Hence (B.1) for the (α, β) and the $(\alpha, -\beta)$ pairs reads as

$$c_{\alpha,\beta}^{\alpha+\beta} b_k^{\alpha+\beta} = (\alpha \cdot \hat{r}^\beta) b_k^\alpha + (\beta \cdot b^\alpha) b_k^\beta, \tag{B.18}$$

$$0 = (\alpha \cdot \hat{r}^{-\beta}) b_k^\alpha - (\beta \cdot b^\alpha) b_k^{-\beta}. \tag{B.19}$$

By adding these two equations making use of (B.2) and (B.9), we obtain

$$c_{\alpha,\beta}^{\alpha+\beta} b_k^{\alpha+\beta} = -(\alpha \cdot K_\beta) b_k^\alpha + \varepsilon^\beta (\beta \cdot b^\alpha) \beta_k. \tag{B.20}$$

If $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_j - \lambda_l)$ are chosen, then $\alpha \cdot K_\beta = -1$ and $c_{\alpha,\beta}^{\alpha+\beta} = 1$. Let us then substitute (B.12) with (B.17) into (B.20) and consider the resulting equation for $k \notin \{i, j, l\}$ and for $k \in \{i, j, l\}$. In this way we obtain the requirements[4]

$$\Omega^{\alpha+\beta} = \Omega^\alpha, \tag{B.21}$$

$$\varepsilon^{\alpha+\beta} + \eta^{\alpha+\beta} = \varepsilon^\alpha + \eta^\alpha, \tag{B.22}$$

$$\varepsilon^\alpha - \eta^\alpha = 2\varepsilon^\beta (\beta \cdot b^\alpha), \tag{B.23}$$

$$\eta^{\alpha+\beta} - \varepsilon^{\alpha+\beta} = -2\varepsilon^\beta (\beta \cdot b^\alpha). \tag{B.24}$$

These tell us that

$$\Omega^{\alpha+\beta} = \Omega^\alpha, \quad \varepsilon^{\alpha+\beta} = \varepsilon^\alpha, \quad \eta^{\alpha+\beta} = \eta^\alpha. \tag{B.25}$$

In conclusion, there exist some constants ε, η, Ω that

$$\varepsilon^\alpha = \varepsilon, \quad \eta^\alpha = \eta, \quad \Omega^\alpha = \Omega, \qquad \forall \alpha \in \Phi. \tag{B.26}$$

In addition, we can compute from (B.12) that in the above case $2\beta \cdot b^\alpha = (\eta - \varepsilon)$, and by substituting this back into (B.23) we obtain

$$(\varepsilon + 1)(\eta - \varepsilon) = 0. \tag{B.27}$$

[4] We here implicitly assume that $n \geq 4$, but the final solution is valid for any $n \geq 2$.

At the same time we know from (B.11) that ε must be equal to 1 or -1.

The first solution of (B.27) is $\varepsilon = 1 = \eta$. In this case we can determine b_k^α from (B.12) in terms of the arbitrary constant Ω as

$$b_k^{\lambda_i - \lambda_j} = \delta_{ki} + \Omega. \tag{B.28}$$

We can then also calculate $\beta \cdot r^\alpha$ from the above equations, and thereby find from (B.2) that $C_\alpha = -H_\alpha$ must hold. This is precisely the result stated in case I of Proposition 2.2. We have obtained it as a consequence of considering a subset of all cases of (B.1), but it can checked to satisfy this equation in all remaining cases (for $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_l - \lambda_i)$ etc.) as well.

The other solution of (B.27) is $\varepsilon = -1$, but then we still have to determine η. For this we consider $\alpha = (\lambda_i - \lambda_j)$, $\beta = (\lambda_j - \lambda_l)$ and calculate that

$$b_k^\alpha = \frac{1}{2}\left((\eta - 1)\,\delta_{ki} + (\eta + 1)\,\delta_{kj}\right) + \Omega, \tag{B.29}$$

$$b_k^\beta = \frac{1}{2}\left((\eta - 1)\,\delta_{kj} + (\eta + 1)\,\delta_{kl}\right) + \Omega. \tag{B.30}$$

We then look at (B.1) for the (α, β) and (β, α) pairs of roots and add these two equations, which gives

$$0 = (\alpha \cdot \hat{r}^\beta + \alpha \cdot b^\beta) b_k^\alpha + (\beta \cdot \hat{r}^\alpha + \beta \cdot b^\alpha) b_k^\beta. \tag{B.31}$$

Since b_k^α and b_k^β are linearly independent n-component vectors for any η, we obtain

$$\alpha \cdot \hat{r}^\beta + \alpha \cdot b^\beta = 0, \qquad \beta \cdot \hat{r}^\alpha + \beta \cdot b^\alpha = 0. \tag{B.32}$$

By subtracting these equations and taking into account that by (B.2) now

$$\alpha \cdot \hat{r}^\beta - \beta \cdot \hat{r}^\alpha = \frac{1}{2}\left(\beta \cdot K_\alpha - \alpha \cdot K_\beta\right) = 1, \tag{B.33}$$

we find that $\eta = 1$. So we have completely determined b_k^α again, and it is easy to confirm that the final formula agrees with case II of Proposition 2.2. Thus the proof is complete.

C Proof of Proposition 2.5

In this appendix we verify the statement of Proposition 2.5.

By combining eq. (2.18) and Proposition 2.4, the constant r-matrix that we wish to calculate can be written in the form

$$\tilde{r}' = (\varphi(q) \otimes \varphi(q))\, \rho(q)\, (\varphi(q) \otimes \varphi(q))^{-1} \tag{C.1}$$

with

$$\rho(q) = (\chi(q) \otimes \chi(q)) \left(\tilde{r}(q) + \sum_k A_k(q) \otimes H_k \right) (\chi(q) \otimes \chi(q))^{-1}. \tag{C.2}$$

The formulas in (2.48), (2.50) together with (2.10) and (2.11) result in

$$\begin{aligned}\rho = -\mathcal{B} \sum_{k \neq l} \frac{1}{F_k - F_l} (e_{kl} - e_{ll}) \otimes (e_{lk} - e_{kk}) + \sum_{k \neq l} \frac{F_k F_l}{F_k - F_l} (e_{kl} - e_{ll}) \otimes (e_{lk} - e_{kk}) \\ + \sum_{k \neq l} F_k e_{kk} \otimes e_{kl} - \sum_{k \neq l} F_l e_{lk} \otimes e_{ll}.\end{aligned} \tag{C.3}$$

Therefore, to prove Proposition 2.5 it is enough to verify that

$$(\varphi(q) \otimes \varphi(q)) \, \rho(q) = \tilde{r}' \, (\varphi(q) \otimes \varphi(q)) \tag{C.4}$$

holds for ρ in (C.3) and $\tilde{r}'$ in (2.61). We obtain in a straightforward manner that

$$(\varphi(q) \otimes \varphi(q)) \, \rho(q) = \sum_{a,b,c,d=1}^{n} \left(\mathcal{B} B_{abcd} + \tilde{B}_{abcd} \right) e_{ab} \otimes e_{cd}, \tag{C.5}$$

where

$$B_{abcd} = \frac{(\varphi_{ad} - \varphi_{ab})(\varphi_{cd} - \varphi_{cb})}{F_d - F_b}, \quad \text{if} \quad b \neq d, \tag{C.6}$$

$$\tilde{B}_{abcd} = \frac{F_d F_b}{F_d - F_b} (\varphi_{ad} - \varphi_{ab})(\varphi_{cd} - \varphi_{cb}) + F_d \varphi_{ad} \varphi_{cd} - F_b \varphi_{ab} \varphi_{cb}, \quad \text{if} \quad b \neq d, \tag{C.7}$$

and $B_{abcd} = \tilde{B}_{abcd} = 0$ if $b = d$. From (2.49) and (2.61), the right hand side of (C.4) is found to be

$$\tilde{r}' \, (\varphi(q) \otimes \varphi(q)) = \sum_{a,b,c,d=1}^{n} \left(\mathcal{B} D_{abcd} + \tilde{D}_{abcd} \right) e_{ab} \otimes e_{cd} \tag{C.8}$$

with

$$D_{abcd} = \sum_{(a,x,c,y) \in S} \varphi_{xb} \varphi_{yd} - \sum_{(c,y,a,x) \in S} \varphi_{xb} \varphi_{yd}, \tag{C.9}$$

$$\tilde{D}_{abcd} = \sum_{(a-1,x,c-1,y) \in S} \varphi_{xb} \varphi_{yd} - \sum_{(c-1,y,a-1,x) \in S} \varphi_{xb} \varphi_{yd}, \tag{C.10}$$

where the set S is defined in Proposition 2.5 and by an empty sum we mean zero.

We now observe that $\tilde{D}_{abcd} = 0 = \tilde{B}_{abcd}$ if $a = 1$ or $c = 1$, and

$$\tilde{D}_{a,b,c,d} = D_{a-1,b,c-1,d}, \qquad \tilde{B}_{a,b,c,d} = B_{a-1,b,c-1,d}, \qquad \text{if} \quad 2 \leq a, c \leq n. \tag{C.11}$$

These properties are obvious for $\tilde{D}$, while for $\tilde{B}$ they follow from the formula (2.49). In particular, the second equality in (C.11) is checked by inserting into (C.6) the identity

$$\varphi_{a-1,d} - \varphi_{a-1,b} = F_b \varphi_{ab} - F_d \varphi_{ad}, \qquad 2 \leq a \leq n, \tag{C.12}$$

which is consequence of (2.49). We conclude that it is sufficient to show that $B_{abcd} = D_{abcd}$.

Let us examine the expressions of B_{abcd} and D_{abcd}. First, we notice that for all indices

$$B_{abcd} = B_{cbad}, \qquad D_{abcd} = D_{cbad}, \tag{C.13}$$

and

$$B_{abcd} = 0 = D_{abcd} \quad \text{if} \quad a = n \quad \text{or} \quad c = n \quad \text{or} \quad b = d. \tag{C.14}$$

Hence it is enough to show that $B_{abcd} = D_{abcd}$ for such indices that $a \leq c < n$ and $b \neq d$. We now introduce the notation

$$F_P := \prod_{t \in P} F_t \qquad \forall P \subset \{1, \ldots, n\}, \tag{C.15}$$

and also put $F_P := 1$ if $P = \emptyset$, for which $|P| = 0$. We then rewrite B_{abcd} as

$$B_{abcd} = (F_d - F_b)\Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-1-a}} F_P \Big)\Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-1-c}} F_P \Big), \tag{C.16}$$

where I_k^n is defined in (2.47). This is derived from (C.6) by using that as a result of (2.49)

$$\varphi_{al} - \varphi_{ak} = (F_k - F_l) \sum_{\substack{P \subset I_k^n \cap I_l^n \\ |P| = n-1-a}} F_P. \tag{C.17}$$

Next, by inserting (2.49) into (C.9) and using that $a \leq c$, we get the expression

$$\begin{aligned} D_{abcd} = (F_d - F_b) \sum_{\substack{x+y=a+c+1 \\ 1 \leq x \leq a < y \leq n}} \Bigg(& \Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-1-x}} F_P \Big)\Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-y}} F_P \Big) \\ & - \Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-1-y}} F_P \Big)\Big(\sum_{\substack{P \subset I_b^n \cap I_d^n \\ |P| = n-x}} F_P \Big) \Bigg). \end{aligned} \tag{C.18}$$

The $x = a$, $y = (c+1)$ term in the first line of the right hand side of (C.18) clearly equals the right hand side of (C.16). The proof is completed by a close inspection of the ranges of the summation indices, which shows that all the remaining terms cancel pairwise between the two lines of (C.18) for any $a \leq c \leq (n-1)$.

D Functional calculus of linear operators

For convenient reference in the main text, in this appendix we collect some result from the theory of bounded operators based on chapter VII of the book [52].

Let $X \neq \{0\}$ be a complex Banach space. The space of bounded linear operators on X is denoted by $B(X)$, which is a Banach algebra in the usual way. Let $T \in B(X)$ be a bounded linear operator. The resolvent set of T is given by $\mathcal{R}(T) = \{\, \lambda \in \mathbb{C} \mid \lambda I - T \,\text{invertible operator}\,\}$, where I is the unit operator. The spectrum $\sigma(T)$ of T is the complement of $\mathcal{R}(T)$. The formula $\mathcal{R}(T) \ni \xi \mapsto \rho_\xi(T) = (\xi I - T)^{-1}$ defines the resolvent function of T. Denote by $\mathcal{F}(T)$ the set of all complex functions H that are holomorphic on some neighbourhood of $\sigma(T)$. Then one can define the functions $H(T)$ of the operator T as follows.

Definition D.1 *Let $H \in \mathcal{F}(T)$ and consider a closed, rectifiable curve C that lies in the domain of analyticity of H and encircles the spectrum $\sigma(T)$ in the positive sense customary in the theory of complex variables. Then the operator $H(T)$ is defined by the equation*

$$H(T) = \frac{1}{2\pi i} \int_C H(\xi)\, \rho_\xi(T)\, \mathrm{d}\xi. \tag{D.1}$$

It can be shown that $H(T)$ depends only on the function H, and not on the curve C. Some important properties of this functional calculus are gathered in the following theorem.

Theorem D.2 *If $f, g \in \mathcal{F}(T)$ and $\alpha, \beta \in \mathbb{C}$ then*

- *$\alpha f + \beta g \in \mathcal{F}(T)$ and $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$,*
- *$fg \in \mathcal{F}(T)$ and $(fg)(T) = f(T)\, g(T)$,*
- *if f has the power series expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$ valid in a neighbourhood of $\sigma(T)$, then $f(T) = \sum_{k=0}^{\infty} c_k T^k$.*

One can define the directional derivatives, $(\nabla_S H)(T) \in B(X)$, of $H(T)$ by

$$(\nabla_S H)(T) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H(T + tS), \qquad S \in B(X). \tag{D.2}$$

The integral formula (D.1) implies the equation

$$(\nabla_S H)(T) = \frac{1}{2\pi i} \int_C H(\xi)\, \rho_\xi(T)\, S \rho_\xi(T)\, \mathrm{d}\xi. \tag{D.3}$$

Now suppose that X is a finite dimensional Banach space. In this case the spectrum $\sigma(T)$ of the operator T has finitely many elements, which are just the eigenvalues of T. The *index* $\nu(\lambda)$ of an eigenvalue λ is the smallest positive integer ν such that $(\lambda I - T)^\nu x = 0$ for every vector x for which $(\lambda I - T)^{\nu+1} x = 0$. Introducing the invariant subspaces $N_\lambda := \mathrm{Ker}(T - \lambda I)^{\nu(\lambda)}$ $(\lambda \in \sigma(T))$, one has the usual $X = \oplus_{\lambda \in \sigma(T)} N_\lambda$ Jordan decomposition of X.

Theorem D.3 *If $\dim(X) < \infty$ and $H \in \mathcal{F}(T)$, then*

$$H(T) = \sum_{\lambda \in \sigma(T)} \sum_{k=0}^{\nu(\lambda)-1} \frac{1}{k!} H^{(k)}(\lambda)\, (T - \lambda I)^k E_\lambda, \tag{D.4}$$

where $E_\lambda \in B(X)$ is the projection operator of the subspace N_λ.

E Some combinatorial identities

We here gather some elementary combinatorial identities needed in section 3.1.

Identity E.1 *If $k, l \in \mathbb{N} := \{0, 1, 2, \ldots\}$, then*

$$\sum_{n=0}^{k} (-1)^n \frac{1}{n+l+1} \binom{k}{n} = \frac{k!\,l!}{(k+l+1)!}. \tag{E.1}$$

Proof. By induction, with respect to k.

Identity E.2 *If $k, n \in \mathbb{N}$ and $0 \le k \le n$, then*

$$\sum_{a=0}^{k} \binom{n-a}{n-k} = \binom{n+1}{k}. \tag{E.2}$$

Proof. By induction with respect to n.

Identity E.3 *Let* $k, l, m \in \mathbb{N}$ *and* $0 \leq m \leq l$, *then*

$$\sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} k+l-j \\ k \end{pmatrix} = \begin{cases} 0 & \textit{if } k < m, \\ \begin{pmatrix} k+l-m \\ l \end{pmatrix} & \textit{if } k \geq m. \end{cases} \tag{E.3}$$

Proof. Consider the smooth function

$$\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \ni (a, b) \mapsto b^{k+l-m} (a+b)^m . \tag{E.4}$$

Using the binomial theorem, we can write

$$b^{k+l-m} (a+b)^m = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} a^j b^{k+l-j}. \tag{E.5}$$

Let us differentiate this equation k-times with respect to b. Then the left hand side gives

$$\begin{aligned} \frac{\partial^k}{\partial b^k} \left(b^{k+l-m} (a+b)^m \right) &= \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(\frac{\partial^{k-i}}{\partial b^{k-i}} b^{k+l-m} \right) \frac{\partial^i}{\partial b^i} (a+b)^m \\ &= \sum_{i=0}^{\min(m,k)} \begin{pmatrix} k \\ i \end{pmatrix} \frac{(k+l-m)!\, m!}{(l+i-m)!\,(m-i)!} b^{l-m+i} (a+b)^{m-i} . \end{aligned} \tag{E.6}$$

By evaluating this equation at $a = -1$, $b = 1$, we obtain

$$\left. \frac{\partial^k}{\partial b^k} \left(b^{k+l-m} (a+b)^m \right) \right|_{a=-1,\, b=1} = \begin{cases} 0 & \text{if } k < m, \\ k! \begin{pmatrix} k+l-m \\ l \end{pmatrix} & \text{if } k \geq m. \end{cases} \tag{E.7}$$

At the same time, the right hand side of (E.5) gives

$$\frac{\partial^k}{\partial b^k} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} a^j b^{k+l-j} = k! \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} k+l-j \\ k \end{pmatrix} a^j b^{l-j}. \tag{E.8}$$

It follows that

$$\left. \frac{\partial^k}{\partial b^k} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} a^j b^{k+l-j} \right|_{a=-1,\, b=1} = k! \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} k+l-j \\ k \end{pmatrix} . \tag{E.9}$$

Comparing (E.7) and (E.9) we see that our statement is valid. *Q.E.D.*

Identity E.4 *Let $k, l, m \in \mathbb{N}$ and $l < m \leq k + l$, then*

$$\sum_{j=0}^{l} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} k+l-j \\ k \end{pmatrix} = \begin{cases} 0 & \text{if } k < m, \\ \begin{pmatrix} k+l-m \\ l \end{pmatrix} & \text{if } k \geq m. \end{cases} \tag{E.10}$$

Proof. Similar to the preceding identity.

F Addition formula and further identities

Let us consider the function $f(x) = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x}$. This function is holomorphic on the whole complex plane except the points $2\pi i\mathbb{Z}^*$, where it has first order poles. Using the familiar $\coth x \coth y - \coth(x+y)(\coth x + \coth y) + 1 = 0$ identity, the following 'addition formula' can be obtained:

Identity F.1 *If $x \neq 0$, $y \neq 0$, $x + y \neq 0$, then*

$$\begin{aligned} &\frac{1}{4} + f(x) f(y) - f(x+y)(f(x) + f(y)) \\ &\quad - \frac{f(x+y) - f(y)}{x} - \frac{f(x+y) - f(x)}{y} - \frac{f(x) + f(y)}{x+y} = 0. \end{aligned} \tag{F.1}$$

On its domain of holomorphicity, the function f satisfies also the relations

$$f^{(k)}(-x) = (-1)^{k+1} f^{(k)}(x), \qquad f'(x) + 2\frac{f(x)}{x} + f^2(x) = \frac{1}{4}. \tag{F.2}$$

The first relation in (F.2) uses only the fact that f is an odd function, while the second relation follows, for example, by taking the $y \to 0$ limit in (F.1).

For convenience, we now collect some further identities that give the results for the differentiation of expressions of the type appearing in (F.1). All these identities are obvious, and are actually valid for any odd holomorphic function f. They are used in section 3.1 to derive the equality in (3.44) for the r-matrix of the form in (3.10).

Identity F.2 *If $k, l \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then*

$$\begin{aligned}
\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{1}{4} &= \frac{1}{4} \delta_{k,0} \delta_{l,0}, && \text{(F.3)} \\
\frac{\partial^{k+l}}{\partial x^k \partial y^l} f(x) f(y) &= f^{(k)}(x) f^{(l)}(y), && \text{(F.4)} \\
\frac{\partial^{k+l}}{\partial x^k \partial y^l} f(x+y) f(x) &= \left. \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} \right|_{\xi = x} f^{(l)}(\xi + y) f(\xi), && \text{(F.5)} \\
\frac{\partial^{k+l}}{\partial x^k \partial y^l} f(x+y) f(y) &= \left. \frac{\mathrm{d}^l}{\mathrm{d}\xi^l} \right|_{\xi = y} f^{(k)}(\xi + x) f(\xi). && \text{(F.6)}
\end{aligned}$$

Identity F.3 *If $x + y \neq 0$, then*

$$\begin{aligned}
\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} &= (-1)^{k+l} \sum_{a=0}^{l} \begin{pmatrix} l \\ a \end{pmatrix} (k + l - a)! \, (-1)^a \frac{f^{(a)}(y)}{(x+y)^{k+l+1-a}} \\
&\quad + (-1)^{k+l} \sum_{b=0}^{k} \begin{pmatrix} k \\ b \end{pmatrix} (k + l - b)! \, (-1)^b \frac{f^{(b)}(x)}{(x+y)^{k+l+1-b}}. \qquad \text{(F.7)}
\end{aligned}$$

We also have

$$\lim_{x \to -y} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^k \frac{k! l!}{(k + l + 1)!} f^{(k+l+1)}(y). \qquad \text{(F.8)}$$

Proof. Equation (F.7) is a direct consequence of the Leibniz rule. To verify (F.8), let us introduce $u := x + y$, $y = u - x$. By using power series expansion around $u = 0$, we have

$$\begin{aligned}
\frac{f(x) + f(y)}{x + y} &= \frac{f(x) + f(u - x)}{u} = \frac{f(x) - f(x - u)}{u} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(x)}{(n+1)!} u^n = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(x)}{(n+1)!} (x + y)^n. \qquad \text{(F.9)}
\end{aligned}$$

Differentiating this equation l-times with respect to y, we get that

$$\frac{\partial^l}{\partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^l \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, (n + l + 1)} f^{(n+l+1)}(x) (x + y)^n. \qquad \text{(F.10)}$$

Then differentiating k-times with respect to x, we obtain

$$\begin{aligned}
\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} &= (-1)^l \sum_{n=0}^{k} \left(\frac{(-1)^n}{n + l + 1} \sum_{j=0}^{n} \begin{pmatrix} k \\ j \end{pmatrix} \frac{f^{(n+l+1+k-j)}(x)}{(n - j)!} (x + y)^{n-j} \right) \\
&\quad + (-1)^l \sum_{n=k+1}^{\infty} \left(\frac{(-1)^n}{n + l + 1} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{f^{(n+l+1+k-j)}(x)}{(n - j)!} (x + y)^{n-j} \right). \qquad \text{(F.11)}
\end{aligned}$$

Now, let us take the limit $x \to -y$. Using the combinatorial identity (E.1), we can see that

$$\begin{aligned} \lim_{x\to -y} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x)+f(y)}{x+y} &= (-1)^k f^{(k+l+1)}(y) \sum_{n=0}^{k} \frac{(-1)^n}{n+l+1} \begin{pmatrix} k \\ n \end{pmatrix} \\ &= (-1)^k \frac{k!l!}{(k+l+1)!} f^{(k+l+1)}(y), \end{aligned} \tag{F.12}$$

whereby the proof is complete. *Q.E.D.*

Identity F.4 *If $x \neq 0$, then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y)-f(y)}{x} = -\sum_{m=0}^{k} \frac{k!}{(k-m)!} (-1)^{m+1} \frac{f^{(k+l-m)}(x+y)}{x^{m+1}} - (-1)^k k! \frac{f^{(l)}(y)}{x^{k+1}}. \tag{F.13}$$

In the limit case, we have

$$\lim_{x\to 0} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y)-f(y)}{x} = \frac{f^{(k+l+1)}(y)}{k+1}. \tag{F.14}$$

Proof. The verification of (F.13) is trivial. As for (F.14), the power series expansion of f around $x=0$ implies that

$$\frac{f(x+y)-f(y)}{x} = \frac{1}{1!} f'(y) + \cdots + \frac{1}{(k+1)!} f^{(k+1)}(y)\, x^k + \mathcal{O}\left(x^{k+1}\right). \tag{F.15}$$

By taking the derivatives of this equation, we obtain that

$$\frac{\partial^k}{\partial x^k} \frac{f(x+y)-f(y)}{x} = \frac{f^{(k+1)}(y)}{k+1} + \mathcal{O}(x), \tag{F.16}$$

and

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y)-f(y)}{x} = \frac{f^{(k+l+1)}(y)}{k+1} + \mathcal{O}(x), \tag{F.17}$$

which implies (F.14). *Q.E.D.*

Identity F.5 *If $y \neq 0$, then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y)-f(x)}{y} = -\sum_{m=0}^{l} \frac{l!}{(l-m)!} (-1)^{m+1} \frac{f^{(k+l-m)}(x+y)}{y^{m+1}} - (-1)^l l! \frac{f^{(k)}(x)}{y^{l+1}}. \tag{F.18}$$

In the limit case,

$$\lim_{y\to 0} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y)-f(x)}{y} = \frac{f^{(k+l+1)}(x)}{l+1}. \tag{F.19}$$

Proof. This is an obvious consequence of the preceding identity.

G The maximal open domain $\check{\mathcal{K}} \subset \mathcal{A}(\mathcal{G}, \mu)_0$

In this appendix we show that if $\mathcal{A} = \mathcal{A}(\mathcal{G}, \mu)$, then the *maximal, nonempty, open* domain on which the r-matrix of Theorem 4.1 can be defined is given by $\check{\mathcal{K}}$ in (4.49), where $\omega \in \mathcal{B}_k$ is subject to the conditions in (4.50) and (4.51).

In general, the elements of the domain $\check{\mathcal{K}} \subset \mathcal{A}_0$ must satisfy the spectral conditions (4.12). If $\mathcal{A} = \mathcal{A}(\mathcal{G}, \mu)$ and $\kappa \in \mathcal{K}$ is parametrized as in (4.48), then these conditions are explicitly given by (4.50) and (4.51), where λ_a is an arbitrary eigenvalue of $\mathrm{ad}\,\omega|\mathcal{G}_a$. Since $\lambda_0 = 0$ is always one of the eigenvalues, the second condition in (4.51) implies that $k \neq 2\pi i \frac{n}{m}$ for any $n \in \mathbb{Z}$, $m \in \mathbb{Z}^*$. As $\check{\mathcal{K}}$ must be an *open* subset of $\mathcal{K}$, it follows that $k \in (\mathbb{C} \setminus i\mathbb{R})$ for any admissible $\kappa = \omega + kd + l\hat{c}$. Note that $\check{\mathcal{K}} \neq \emptyset$, since e.g. the elements of the form $\kappa = kd + l\hat{c}$ in (4.49) satisfy the conditions (4.50), (4.51). Hence we only have to show that (4.49) subject to these conditions is an *open* subset of $\mathcal{K}$.

If λ_a is an arbitrary eigenvalue of $\mathrm{ad}\,\omega$ on $\mathcal{G}_a$ and $k \in (\mathbb{C} \setminus i\mathbb{R})$, then let us consider the real line in $\mathbb{C}$ defined by

$$L_{\lambda_a,k}(t) = \lambda_a + kt, \qquad \forall t \in \mathbb{R}. \tag{G.1}$$

This line intersects the imaginary axis for $t = t_{\lambda_a,k}$ at the point $P_{\lambda_a,k} = L_{\lambda_a,k}(t_{\lambda_a,k})$,

$$t_{\lambda_a,k} = -\frac{\Re(\lambda_a)}{\Re(k)}, \qquad P_{\lambda_a,k} = \lambda_a - k\frac{\Re(\lambda_a)}{\Re(k)}. \tag{G.2}$$

Now the condition in (4.50) can be reformulated as follows:

$$P_{\lambda_a,k} \notin 2\pi i\mathbb{Z} \qquad \text{or} \qquad t_{\lambda_a,k} \notin (a + N\mathbb{Z}), \qquad \forall a \in \mathcal{E}_\mu \setminus \{0\}. \tag{G.3}$$

This can be further reformulated as the requirement

$$\left|e^{P_{\lambda_a,k}} - 1\right|^2 + \left|e^{\frac{2\pi i}{N}(t_{\lambda_a,k} - a)} - 1\right|^2 \neq 0. \tag{G.4}$$

It is also useful to rephrase the second condition in (4.51) as

$$P_{\lambda_0,k} \notin 2\pi i\mathbb{Z} \qquad \text{or} \qquad t_{\lambda_0,k} \notin N\mathbb{Z}^*. \tag{G.5}$$

Let $\mathcal{T} : \mathbb{C} \to \mathbb{C}$ be an arbitrary continuous function, which is zero precisely on $N\mathbb{Z}^*$. (For example, we may use $\mathcal{T}(z) = z^{-1}\sin(N^{-1}\pi z)$.) Then (G.5) is equivalent to

$$\left|e^{P_{\lambda_0,k}} - 1\right|^2 + |\mathcal{T}(t_{\lambda_0,k})|^2 \neq 0. \tag{G.6}$$

Since the left hand sides of (G.4) and (G.6) are given by continuous functions of k and the λ_a, it follows that these inequalities are stable with respect to small variations of k and the λ_a. The same is true for the first condition $\lambda_0 \notin 2\pi i\mathbb{Z}^*$ in (4.51). The statement that $\check{\mathcal{K}} \subset \mathcal{K}$ and $\mathcal{B}_k \subset \mathcal{G}_0$ subject to (4.49), (4.50), (4.51) are *open* subsets follows from this observation by taking into account that the position of the eigenvalues of $\operatorname{ad}\omega$ varies continuously with $\omega \in \mathcal{G}_0$. This means that by choosing ω near enough to say ω^*, any eigenvalue of $\operatorname{ad}\omega$ can be taken to be arbitrarily close to some eigenvalue of $\operatorname{ad}\omega^*$.

H A remark on some finite-dimensional r-matrices

We here describe some finite-dimensional dynamical r-matrices, which were first considered in the appendix of [51], and point out a relationship between these and the infinite-dimensional r-matrices described in subsection 4.2.2.

Let μ be an automorphism of a self-dual Lie algebra of the same type as in section 4.2 and recall the decomposition in (4.39), (4.40). For any $a \in \mathcal{E}_\mu$ and integer q specified below, introduce the meromorphic function $f_{a,q}$ by

$$f_{0,q}(w) := \frac{1}{2}\coth\frac{w}{2} - \frac{1}{w}, \quad f_{a,q}(w) := \frac{1}{2}\coth\frac{1}{2}(w + \frac{2\pi i}{N}qa) \qquad \text{if} \quad a \neq 0. \tag{H.1}$$

In order to guarantee that these functions are holomorphic in a neighbourhood of $w = 0$, we require the integer q to satisfy the conditions

$$1 \leq q \leq (N-1), \qquad qa \notin N\mathbb{Z}^* \quad \forall a \in \mathcal{E}_\mu \setminus \{0\}. \tag{H.2}$$

Then there exists a nonempty open domain $\check{\mathcal{G}}_0 \subset \mathcal{G}_0$, containing the origin, on which the map $\rho_q : \check{\mathcal{G}}_0 \to \operatorname{End}(\mathcal{G})$ can be defined by

$$\rho_q(\omega)\xi := f_{a,q}(\operatorname{ad}\omega)\xi \qquad \forall \xi \in \mathcal{G}_a, \quad \omega \in \check{\mathcal{G}}_0. \tag{H.3}$$

It can be shown that ρ_q satisfies the CDYBE (4.3), where $\mathcal{A}$ is replaced by $\mathcal{G}$ and $\mathcal{K}$ is taken to be $\mathcal{G}_0$. If $\mu = \mathrm{id}$, then ρ_q becomes the well known canonical (or Alekseev-Meinrenken) dynamical r-matrix [15, 12, 13]. In the case $q = 1$, which always satisfies (H.2), ρ_q has been introduced in [51], where it was proved that it solves the CDYBE. The proof given in [51] is very elegant and is very indirect. A direct proof in the case $\mu = \mathrm{id}$ is written down in Chapter 3. For general μ and q, a proof of the CDYBE for ρ_q can be extracted from the following observation. If we let $k := \frac{2\pi i}{N}q$, then we have

$$\rho_q(\omega)\eta = R_k(\omega)\eta \quad \text{and} \quad (\rho_q(\omega)\xi)^{n_a} = R_k(\omega)\xi^{n_a} \tag{H.4}$$

for any $\eta \in \mathcal{G}_0$ and $\xi \in \mathcal{G}_a$, $a \neq 0$, $n_a \in (a + N\mathbb{Z})$, where R_k refers to the formula (4.54). It should be stressed that this is a relationship purely at the level of formulas, since in the definition of the infinite-dimensional r-matrices in section 4.2 the imaginary values of k were excluded for domain reasons. Nevertheless, it follows from this coincidence of formulas that essentially the same algebraic computation that proves the CDYBE (4.55) can be repeated to verify the CDYBE for ρ_q. We have also verified the CDYBE for ρ_q by a direct calculation that proceeds analogously to the proof of our Theorem 4.1.

In certain cases ρ_q is equivalent to an r-matrix of the form in (4.2) by a shift of the dynamical variable. Namely, this happens if the automorphism μ can be written as

$$\mu = \exp(\frac{2\pi i}{N} \operatorname{ad} M), \qquad M \in \mathcal{G}, \tag{H.5}$$

where $\operatorname{ad} M$ is diagonalizable and the fixed point set $\mathcal{G}_0$ of μ satisfies

$$\mathcal{G}_0 = \operatorname{Ker}(\operatorname{ad} M). \tag{H.6}$$

In particular, by (H.5), μ is an inner automorphism of $\mathcal{G}$. If these assumptions hold, then we can define a new r-matrix $\tilde{\rho}_q$ by

$$\tilde{\rho}_q(\omega) := \rho_q(\omega - \frac{2\pi i}{N} q M), \tag{H.7}$$

and this r-matrix can be identified with R in (4.2) by taking $\mathcal{A} := \mathcal{G}$ and $\mathcal{K} := \mathcal{G}_0$. The $\mathcal{G}_0$-equivariance property of the dynamical r-matrices is respected by the shift of the variable in (H.7) on account of (H.6).

Bibliography

[1] O. Babelon and C.-M. Viallet, Phys. Lett. B **237** (1990) 411.

[2] O. Babelon and C.-M. Viallet, *Integrable Models, Yang–Baxter Equation, and Quantum Groups*, Trieste preprint SISSA 54 EP (1989).

[3] L. Fehér and **B.G. Pusztai**, Czech. J. Phys. **50** (2000) 59.

[4] L. Fehér and **B.G. Pusztai**, J. Phys. A: Math. Gen. **33** (2000) 7739.

[5] F. Calogero, Lett. Nuovo Cim. **13** (1975) 411.

[6] J. Moser, Adv. Math. **16** (1975) 197.

[7] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Vol. 1., Birkhäuser Verlag, 1990.

[8] J.F. van Diejen, L. Vinet (eds.), *Calogero–Moser–Sutherland Models*, CRM Series in Mathematical Physics (Springer-Verlag, New York 2000).

[9] E. Cremmer and J.-L. Gervais, Commun. Math. Phys. **134** (1990) 619.

[10] K. Hasegawa, Commun. Math. Phys. **187** (1997) 289.

[11] A.A. Belavin, Nucl. Phys. B **180** (1981) 189.

[12] A. Alekseev and E. Meinrenken, Invent. Math. **139** (2000) 135.

[13] J. Balog, L. Fehér and L. Palla, Phys. Lett. B **463** (1999) 83; Nucl. Phys. B **568** (2000) 503.

[14] **B.G. Pusztai** and L. Fehér, J. Phys. A: Math. Gen. **34** (2001) 10949.

[15] P. Etingof and A. Varchenko, Commun. Math. Phys. **192** (1998) 77.

[16] G. Felder, pp. 1247-1255 in: Proc. Int. Congr. Math. Zürich, 1994, hep-th/9407154; G. Felder and C. Wieczerkowski, Commun. Math. Phys. **176** (1996) 133, hep-th/9411004.

[17] L. Fehér and **B.G. Pusztai**, Nucl. Phys. B **621** (2002) 622.

[18] L. Fehér and **B.G. Pusztai**, Czech. J. Phys. **51** (2001) 1318.

[19] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Phys. Rev. Lett. **19** (1967) 1921.

[20] P. Lax, Commun. Pure Appl. Math. **21** (1968) 467.

[21] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics (Springer-Verlag, New York 1978).

[22] P. Etingof and O. Schiffmann, Lectures on the dynamical Yang–Baxter equations, preprint `math.QA/9908064`.

[23] M.A. Olshanetsky and A.M. Perelomov, Invent. Math. **37** (1976) 93.

[24] E. D'Hoker and D.H. Phong, Nucl. Phys. B **530** (1998) 537.

[25] A.J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. **100** (1998) 1107.

[26] J. Avan and M. Talon, Phys. Lett. B **303** (1993) 33.

[27] I. Krichever, Funct. Anal. Appl. **14** (1980) 282.

[28] E.K. Sklyanin, Alg. and Anal. **6** (1994) 227.

[29] H.W. Braden and T. Suzuki, Lett. Math. Phys. **30** (1994) 147.

[30] J. Avan, O. Babelon and M. Talon, Alg. and Anal. **6** (1995) 255.

[31] G.E. Arutyunov and P.B. Medvedev, Phys. Lett. A **223** (1996) 66.

[32] M. Forger and A. Winterhalder, Nucl. Phys. B **621** (2002) 529.

[33] J. Avan and G. Rollet, Phys. Lett. A **212** (1996) 50.

[34] F.W. Nijhoff, V.B. Kuznetsov, E.K. Sklyanin and O. Ragnisco, J. Phys. A **29** (1996) L333.

[35] Y.B. Suris, Phys. Lett. A **225** (1997) 253.

[36] S.N.M. Ruijsenaars and H. Schneider, Ann. Phys. (N.Y.) **170** (1986) 370.

[37] J. Avan, O. Babelon and E. Billey, Commun. Math. Phys. **178** (1996) 281.

[38] G. Felder and A. Varchenko, J. Statist. Phys. **89** (1997) 963.

[39] G.E. Arutyunov and S.A. Frolov, Commun. Math. Phys. **191** (1998) 15.

[40] G.E. Arutyunov, L.O. Chekhov and S.A. Frolov, Commun. Math. Phys. **192** (1998) 405.

[41] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.

[42] B.Y. Hou and W.L. Yang, J. Phys. A **32** (1999) 1475; J. Math. Phys. **41** (2000) 357.

[43] S.N.M. Ruijsenaars, Commun. Math. Phys. **110** (1987) 191.

[44] A. Antonov, K. Hasegawa and A. Zabrodin, Nucl. Phys. B **503** (1997) 747.

[45] A.A. Belavin and V.G. Drinfeld, Funct. Anal. Appl. **16** (1982) 159.

[46] M. Gerstenhaber and A. Giaquinto, Lett. Math. Phys. **40** (1997) 337.

[47] R. Endelman and T.J. Hodges, Quantization of certain skew-symmetric solutions of the classical Yang–Baxter equation, preprint `math.QA/0003066`.

[48] L. Fehér, A. Gábor and **B.G. Pusztai**, J. Phys. A: Math. Gen. **34** (2001) 7235.

[49] A. Alekseev and E. Meinrenken, pp. 9-17 in: L. D. Faddeev's Seminar on Mathematical Physics, M. Semenov-Tyan-Shansky (editor), Amer. Math. Soc. Trans. (2) Vol. 201, 2000.

[50] L. Fehér, Phys. Atom. Nucl. **65** (2002) 1023.

[51] P. Etingof and O. Schiffmann, Math. Res. Lett. **8** (2001) 157.

[52] N. Dunford and J.T. Schwartz, Linear operators, I. General theory, Interscience Publ. Inc., New York-London, 1958.

[53] J.M. Figueroa-O'Farrill and S. Stanciu, J. Math. Phys. **37** (1996) 4121.

[54] L.C. Li and P. Xu, Commun. Math. Phys. **231** (2002) 257.

[55] D.H. Sattinger and O.L. Weaver, Lie groups and algebras with applications to physics, geometry, and mechanics, Springer-Verlag, New York-Berlin, 1986.

[56] E.T. Whittaker and G.N. Wattson, A course of modern analysis, fourth edition, Cambridge University Press, Cambridge, 1927.

[57] V.G. Kac, Infinite dimensional Lie algebras, third edition, Cambridge University Press, Cambridge, 1990.

[58] P. Etingof and O. Schiffmann, Math. Res. Lett. **6** (1999) 593, math.QA/9908115.

[59] P. Etingof and I. Frenkel, Commun. Math. Phys. **165** (1994) 429, hep-th/9303047.