Teaching of Mathematics Focused on Computer Supported Exploration

Outline of PhD Thesis

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Introduction

The formal feature and the deductive reasoning separate mathematics from other science. At first sight experiments have not had any role in mathematics. In spite the fact that formalism only helps in the processes of organizing, proving and understanding. The new knowledge becomes assessable only by examination and experiments in mathematics as well. Beside experiments computers have been kept behind. The improvement and spread of computers have allowed to do research quickly and efficiently moreover to analyse great amount of data immediately, which creates new environment for educational mathematic research. It is not computer *supported* rather computer *created* mathematics. The problem identification, definitions, thesis and proofs are created with the help of computer. Our aim is to present how experimental mathematics can be adapted to teaching mathematics.

The thesis is based on the following publications of the author:

- Eszter K. Horváth, Attila Máder, Andreja Tepavčević. *One-dimensional Czédli-type Islands*, The College Mathematical Journal, **42**(5), 374-378, 2011.
- Attila Máder. *Heads or Tails Gambling - What Can Be Learned about Probability?*, Teaching Mathematics and Computer Science, **6**(1), 15-41, 2008.
- Attila Máder, Róbert Vajda. *Elementary Approaches to the Teaching of the Combinatorial Problem of Rectangular Islands*, International Journal of Computers for Mathematical Learning, **15**(3), 267-281, 2010.
- Attila Máder, Géza Makay. *The maximum number of rectangular islands*, The Teaching of Mathematics, **14**(1), 31-44, 2011.
- Attila Máder. *The Use of Experimental Mathematics in the Classroom*, Interesting Mathematical Problems in Sciences and Everyday Life, 2011. `www.model.u-szeged.hu`

Additionally we have used the following publication as well:

- Máder Attila. *A „specmat” 40 éve a Ságváriban I.*, A matematika tanítása, **17**(4), 13-21, 2009.

In this outline we use the same index and label structure as in the thesis.

1. Present State of Mathematical Education in Hungary

The first wave of modernisation in modern Hungarian mathematics teaching started in 1963 [107]. The main idea of Tamas Varga's new mathematics teaching was to make children explore the knowledge that we expect them to learn. Mathematics learning became a thinking formulating process where students play active role. At that time mathematics teaching meant formal presentation of the assessed knowledge. Unfortunately, the situation is similar today which has motivated us to find new methodology using modern equipments.

The information explosion has basically changed our world. Possessing the information is not enough but organised knowledge is valued. Education in our new, digital world is in crisis. It cannot keep up with the rapidly changing expectations and is unable to find the way to digital aboriginals who has brought up in a different environment and as a result has become physiologically different. The necessity of knowledge was replaced by the importance of skill acquisition.

The concept of competence (Coolahan)

Competence and competencies should be regarded as the general capability based on knowledge, experience, values, dispositions which a person has developed through engagement with educational practices.

The main points of pedagogic reform of education system are the following [38]:

(1) the improvement of moral, aesthetic, reading comprehension, speaking, writing and infocommunicative skills should happen regardless of subjects;
(2) making experiments is key in scientific education;
(3) infocommunicative methodology - the new pedagogical technique.

The information explosion has great influence on education as well. This determined the change of objectives and curriculum, the modernization of tools and methodology [38]. The main expectations are motivated by globalisation towards education was to provide knowledge which can be modernised quickly and efficiently. As a result of the information explosion knowledge and mathematical knowledge (literacy) has been revalued. The fact that mathematics is used in different situations has an important role in the reinterpretation of mathematical literacy. The teacher is not the primary source of information anymore. The teacher's task is not simply to give information rather to help the students with finding directions in the huge amount of data and presenting how to use the acquired knowledge. The teacher's role has changed from source of information (frontal teaching) to students' helper in exploration. It has become important to give problems which make the students see that the world does not work without mathematics. So today we need more practical mathematics. The change, especially in case of Hungarian education, is greater and deeper than we expect. Since Hungarian mathematics education is rather theoretic and problem solving than practical.

2. Explorations in Mathematics

The methodology of experimental mathematics can be summarised in the following way [17].

(a) Gaining insight and intuition.
(b) Discovering new patterns and relationships.
(c) Using graphical displays to suggest underlying mathematical principles.
(d) Testing and especially falsifying conjecture.
(e) Exploring a possible result to see if it is worth a formal proof.
(f) Suggesting approaches for formal proof.
(g) Replacing lengthy hand derivations with computer based derivations.
(h) Confirming analytically derived results.

Actually the above points build the process of mathematical creation split into steps that are available for computer experiments. The first three points ((a)-(c)) are related to the intuition the next two ((d)-(e)) to making conjectures the last three to prove.

We can make intuitive hypotheses from the results of experiments. We are able to use these suppositions in further continuous experiments to make other more punctual hypotheses. By further examination we can confirm, prove or confute our statements.

From Polya and Lakatos we know that experiments have unquestionable role in development of heuristics. According to Polya experiments are the basic source of mathematical knowledge. They drive the intuition in the good direction. However, this knowledge has not been combined with experimental mathematics yet. The examination and presentation of this relationship is the aim of this thesis. By the improvement and spread of computers we are able to perform large number of experiments, process great amount of data, manipulate these data, make real-time calculations and examine animations which have crucial role in the learning process. All these things now appear in context adequate to the 21st century's expectations. We think that our experiments help to integrate the traditionally separated exploration and proof.

By this creative and innovative way of computer usage, the research method adaptation to education was started by Borwein. The heuristics urged by Gyorgy Polya and experiments finally might achieve the deserved ratio in education. When we examine realistic and experimental problems with students computers help not only in representation but in definition of problems and in creation of concepts as well. Moreover through making and testing hypotheses they help in the whole mathematical creation. Experiment and theory are inseparable. Students can become explorers and this is the most important idea.

To be able to use the advantages offered by experimental mathematics teacher should consider the following practical views and forms. This new method makes possible or easier the

(a) teamwork;
(b) projectwork;
(c) differentation;
(d) active learning;
(e) game based learning;
(f) inquiry based learning.

Methodological advantages:

(1) generating exercises dynamically;
(2) examination;
(3) demonstrating applications;
(4) collective experiences;
(5) dynamization;
(6) increasing problem sensitivity;
(7) building need of proof;
(8) reducing mathematical tools;
(9) edutainment;
(10) let the children ask;
(11) meet society's expectations;
(12) helping inductive, deductive, critical thinking and problem-solving thinking;
(13) promoting problem-solving cycle;
(14) guessing;
(15) convergencing weaker.

Our method helps problem-solving and classroom exploration in the following points:

(i) making definitions;
(ii) applying definitions;
(iii) abstraction;
(iv) making a model;
(v) observation;
(vi) analogy;
(vii) recognizing patterns: proof without words;
(viii) recognizing patterns: induction;
(ix) making strategies;
(x) monotonic increase of the truth;
(xi) justification;
(xii) creating wrong proofs;
(xiii) examples and counterexamples;
(xiv) finding proving methods;
(xv) proving step by step;
(xvi) checking.

3. Exploration in Mathematics Lessons

In this chapter we examine the methodology of mathematical exploration where students participate actively and creatively. We get to knowledge acquisition where computers play creative role from the non-computerized explorations through computer helped explorations. Group experiments even the non computerized ones can easily be masked as games.

Exercise 3.9 Two players Anne and Bob (A and B from now on) play with a regular dice. A starts the game, they toss one after the other and the winner is the one who throws a six for the first time. What is the probability of A winning [104]?

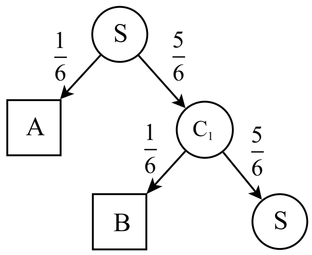


Figure 3.10 Tossing for the first 'six'

The figure grabs the gist of probabilistic problems and make the whole game traceable and enjoyable. During the game students by themselves find the odds, primarily trough experience. They realize the relationship among situations and probabilities. Moreover actually using their experiences achieved in several games they draw the above flow chart. Even by this simple example it can be easily shown that the most efficient way of mathematical creation is active participation in experiments.

In exploration the role of teacher is not classical. The teacher is responsible for, among other things, preparation and active supervision of the experiment. Hereinafter we are going to show some examples to it. On the figure there is the graph of the $\cos x$ function, and one point animated along the graph. The point goes along the $\cos x$ function's reflection related to x axis, so according to our experiences based on sufficient number of values the hidden curve is the graph of $y = -\cos x$ function.

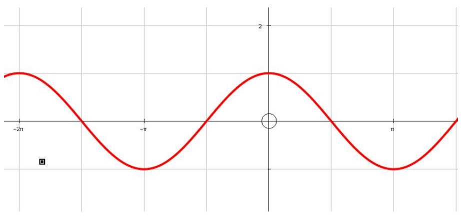


Figure 3.22 Where is the point going...?

Plotting the graph of $y = -\cos x$ function, reanimating the point our hypothesis gained in empirical way becomes irrefutable.

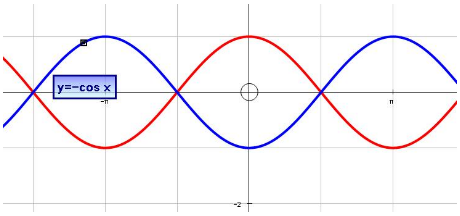


Figure 3.23 Along the $y = -\cos x$ curve

If we now display the real coordinates of the moving point, it becomes apparent that the graph determined by the point is $y = \cos(x - \pi)$ [112].

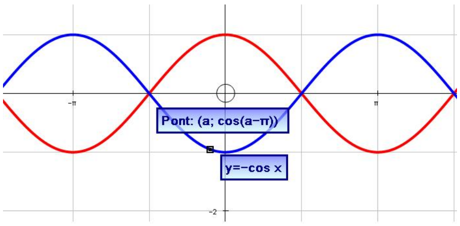


Figure 3.24 Finding: $-\cos x = \cos(x - \pi)$

By this creative way of computer using we can combine student's active experience acquisition with exploration and at the same time teacher can meet the expectations toward infocommunitive education.

4. Islands

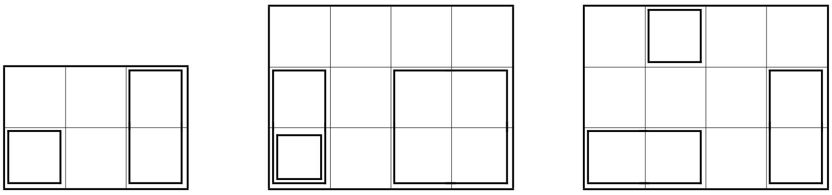
In this section we make an attempt to introduce the area known as islands through consciously supervised connected exercises which help student's self exploration. The historical roots go back to 2007. The concept introduced by Földes was generalised by Gábor Czédli [34]. However the definition used here is too formal. We based on intuitive, exclusively demonstrative concept refined by exploration have created a definition which makes possible to conduct arbitrary amount and deepness of further mathematical examinations proving that by exploration work not only on the level of hypotheses but on the level of concept creation is also possible [110].

Definition 4.3 An island rises up out of the sea, each of its points is higher than any of the surrounding points.

Definition 4.12 Let a rectangular $m \times n$ board be given. We associate an integer number to each cell of the board. We can think of this number as a height above sea level. A rectangular part of the board is called a rectangular island if and only if there is a possible water level such that the rectangle is an island in the usual sense.

Here the role of teacher is to encourage students to create more concrete, tangible concepts and characteristics. Obviously the aim is to set up a mathematical model. From the above example it is clear how it is possible to go through the process of concept creation independently with the help of exploration and guided experiments. In the following section we present how the previously created definitions can be used; the aim is to deepen the concept. To highlight the important characteristics from different prospective it is crucial to demonstrate false and counterexamples.

Exercise 4.35 Give height levels, such that exactly the marked rectangular regions constitute islands, if the maximum height h of an island is 2!



(a) (b) (c)

Figure 4.25 Give appropriate height levels!

Beside counterexamples, false proofs and incorrect solutions are very important tools of our methodology. They rouse sceptical and critical thinking. The problems given can be grouped in the following way:

(a) intuitive approach of the concept of island;
(b) creation and abstraction of the mathematical island concept;
(c) looking for islands related to a given height function;
(d) inverse problems for deeper understanding, grouped in the following way:
   - looking for height function to a given system of islands;
   - looking for height function to a given number of islands;
   - looking for bounded height function to a given system of islands;
   - looking for bounded height function to a given number of islands;

(e) system of islands, canonical representation;
(f) further definitions, examples, creation of counter examples, testing relevance of proofs.

With the help of Mathematica application which was developed together with Robert Vajda ([110]) and presented in this chapter, the above objectives can be fulfilled in a playful and active experimental way.

After discussing the concept of island we show the structure of system of islands. We can prove our previous conjectures which was acquired in an experimental way

Conjecture 4.52 Two different islands can have a common cell or boundary only if one contains the other; i.e. two different islands either are far from each other (we can walk away between them) or the bigger one contains completely the cells of the smaller one in an island configuration.

We have proof that this property is true for system of islands as well. We have created the concept of maximal and minimal number of islands, introduced the graph of system of islands and we have examined the relationship between system of islands and the graphs in details trough problems. One of our most important results is the following theorem.

Theorem 4.64 Islands on a given grid with inclusion form a partially order set.

Problems related to the structure of system of islands can be grouped in the following way:

(a) mutual position of islands;
(b) special islands;
(c) creation of the mathematical model;
(d) examination of our model;
(e) relationship between special islands and our model;
(f) examination of system of islands through our graphs.

We have presented another Mathematica application ([110]) by which students themselves can investigate the model of system of islands on higher level of abstraction. Thus we can see that computerised experiments can be used for abstraction as well. With the help of Mathematica demonstration we can easily increase the number of problems and efficiently motivate digital aboriginals. We can see that great number of experiments have important role in preparation of proofs.

To determine the maximal number of islands on an $m \times n$ sized board we investigated special cases. After proving our conjectures related to $1 \times n$ sized board we continued with our investigation on larger boards.

Conjecture 4.77 The maximum number of rectangular islands on an $1 \times n$ grid is n.

To find the maximal number of island on a $2 \times n$ board we use two sided approximation method. We are going to show computerised investigation alternatives for determining the lower bound and the upper bound. In case of lower bound we have used the cutting method ([110]), by which we can write the recursion.

After examining the $3 \times n$ sized board we determine the lower bound of $m \times n$ sized boards. Here we need to solve a parametric second-order recursion which can be easily fulfilled if we change the previously used Mathematica command. If b_n notates the maximal number of islands on $m \times n$ sized board determined by the cutting technique, then $b_n = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$ [110]. When determining upper bound we have introduced the concept of completed graphs related to dummy islands and system of islands [11, 110]. As a result of examining the number of islands, graphs and vertices we have got a surprising upper board: $\left\lfloor \frac{mn+m+n-1}{2} \right\rfloor$. Thus we have the maximal number of islands on a $m \times n$ board. Our computer supported experiments have helped us with making and confirming hypothesis. Examples are presented where the findings of mass experiments can predict the main steps of the proof. We have demonstrated how to use computers in a creative way with some easily adjustable codes for calculations.

In previous discussions height was unbounded. The author has introduced the investigation of system of islands with bounded heights [72, 111]. The investigation has been started on special boards. The most important findings are the following [111]:

Conjecture 4.125 The maximum number of islands in a $1 \times n$ board using heights $1, 2, ..., h$ is

$$I_h(n) = n + 1 - \left[\frac{n}{2^{h-1}}\right]^+ .$$

where $[\cdot]^+ = \max\{1, [\cdot]\}$.

Theorem 4.133 The maximum number of islands in a $2 \times n$ board using heights $1, 2, ..., h$ only $(h \geq 3)$ is

$$\left[\frac{3n+1}{2}\right] + 1 - \left[\frac{n}{2^{h-2}}\right]^+ .$$

Theorem 4.138 The maximum number of islands in a $3 \times n$ board using heights $1, 2, ..., h$ only $(h \geq 3)$ is

$$2n + 2 - \left[\frac{n}{2^{h-2}}\right]^+ .$$

The first conjecture has been found with the help of a self-developed program [111]. Without our previous findings we would not have been able to write the program, by which we can not only use an existing technique but make possible a computerised investigation. This is a new way of exploration.

The proof of the conjecture is dual induction according to n and to h [72, 111].

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