Counting and generating in some classes of lattices

Ph.D. Thesis

Júlia Kulin

Supervisors:

Dr. Gábor Czédli and Dr. Kamilla Kátai-Urbán

Professor Emeritus — Assistant Professor

Doctoral School of Mathematics

Bolyai Institute, University of Szeged

2024

Contents

Chapter 1

Introduction

This dissertation is built around four of the author's papers. In these papers, we count specific objects of lattice theory, answering the following questions. What is the number of (the isomorphism classes of) slim rectangular lattices of a given length? How many elements does a minimum-sized generating set of a given quasiorder lattice have? How many congruences can a finite lattice have? The answer to the first question gave us an integer sequence not added previously to The On-Line Encyclopedia of Integer Sequences, OEIS for short. We were the first to publish it there; see A273988 at https://oeis.org/A273988. As in general, the motivation to enumerate mathematical objects (lattices, generating elements, lattice congruences in our case) is two-fold. First, research of this kind can, sometimes, contribute to a better insight into the objects we count. Second, the numbers we obtain can occur (now or possibly in the future) in the OEIS. If so, then the numerical coincidence can be accidental but it can also be a sign of a previously not known relation between distinct mathematical topics. For example, in a related paper, Czédli, Dékány, Ozsvárt, Szakács and Udvari [34, Proposition 3.4] (so not in one of the author's papers), the number of some lattices with a given parameter n turned out to be the n-th Catalan number. At the time of writing, the search for "Catalan number" in the OEIS returns 3356 results. Several of these research results are in connection with mathematical structures. It remains a task for the future whether the lattices enumerated in Czédli et al. [34] have some contentful connection with the just-mentioned mathematical structures. When the numbers of our sequence A273988 appear in the OEIS more than once, they might offer analogous tasks to deal with.

In the second chapter, we present our result about slim rectangular lattices from the paper [33]. Slim rectangular lattices are special planar semimodular lattices introduced by Grätzer and Knapp in [62]. After describing these lattices by permutations, we determine the number of these lattices of a given length n. Besides giving formulas, which are effective up to about $n = 1000$, we also prove that the number of these lattices of a given length n is asymptotically $(n-2)! \cdot e^2/2$, where e is Euler's famous number, $e \approx 2.71828$.

In Chapter 3, we aim at finding the sizes of minimum-sized generating sets of some well-known lattices, which consist of some relations. For every set A occurring in the description of Chapter 2, unless it is explicitly stated otherwise, we assume that A satisfies the condition that

$$\text{there is no inaccessible cardinal } \lambda \text{ such that } \lambda \leq |A|. \tag{1.0.1}$$

In connection with this (possibly strange) condition, one of Kuratowski's results is worth mentioning here. Namely, Kuratowski [73] has proved the following:

> If ZFC is consistent, which is generally believed, then ZFC augmented with the axiom that "there is no inaccessible cardinal at all" is also consistent. In other words: in ZFC, we cannot prove the existence of inaccessible cardinal numbers, simply because in the so-called *Kuratowski's model of ZFC*, there is no inaccessible cardinal. (1.0.2)

Being in the Bolyai Institute, we point out that (1.0.2) shows some similarity with János Bolyai's famous result (proved also by Nikolai Lobachevsky, independently); indeed, the "existence of inaccessible cardinals" in the (appropriately reformulated version of) (1.0.2) corresponds to "the failure of Euclid's fifth postulate" in Bolyai's result.

By Strietz [83] and [84], Zádori [87], and Czédli [7], the complete lattice $\mathrm{Equ}(A)$ of all equivalences on a set A is four-generated, provided that A satisfies (1.0.1).

Also, $\mathrm{Equ}(A)$ cannot be generated by less than four elements if $|A| \geq 4$; here we do not have to assume (1.0.1). A quasiorder (relation), also known as a preorder, is a reflexive and transitive relation. The quasiorders on a set A form the complete lattice $\mathrm{Quo}(A)$ with respect to set inclusion. Results of Chajda and Czédli [4], Takách [85], and Dolgos [48] show that both the lattice $\mathrm{Quo}(A)$ of all quasiorders on a set A satisfying (1.0.1) and, for $|A| \leq \aleph_0$, the lattice $\mathrm{Tran}(A)$ of all transitive relations on A have small generating sets.

In Chapter 3, based on our papers [72] and [37], we improve these results about the lattices of quasiorders and those of transitive relations by allowing larger sets A, but not larger than what (1.0.1) allows, and/or finding smaller generating sets. First, generalizing the 1996 result of Chajda and Czédli, and the 2015 result of Dolgos, we prove that (1.0.1) implies that the lattice of quasiorders on A is five-generated, as a complete lattice. Then, based on complicated earlier constructions, we derive some new results in a concise but not self-contained way. These results include showing that $\mathrm{Quo}(A)$ is four-generated if $|A| \neq 4$, furthermore it is $(1+1+2)$-generated in many (however not all) cases; of course, (1.0.1) is assumed; in fact, we do not know any idea how to attack the case when A fails to satisfy (1.0.1). Although (1.0.1) would mean no restriction at all if we worked in Kuratowski's model of ZFC, we admit that set theory usually assumes the opposite of (1.0.1).

In Chapter 4, for a fixed natural number n, we investigate the largest possible values of the numbers of congruences of n-element lattices; this section is taken from our paper [77]. Motivated by a result of Freese and continuing Czédli [21], we determine the third, fourth and fifth largest numbers of congruences of an n-element lattice. Furthermore, we determine the structures of those n-element lattices that witness these numbers.

Chapter 2

The number of slim rectangular lattices

This chapter is based on [33]. Compared to [33], which appeared in 2016, the introductory part of the chapter has changed a lot. The rest of the section is practically unchanged.

2.1 Outline and related results

The key definitions are given in Section 2.2. Some concepts in this historical mini-survey will not be defined with full details; their role is only to give a vague idea about motivations and earlier results.

Unless otherwise stated, all lattices occurring in this chapter are finite.

A lattice L is *semimodular* if for every $x, y, z \in L$ such that $x \preceq y$, we have that $x \vee z \preceq y \vee z$. Slim rectangular lattices and, in particular, slim patch lattices are of particular importance, because each planar semimodular lattice can be obtained from them easily; see Grätzer and Knapp [61], Czédli and Schmidt [46], and Grätzer [55]. (We say more about their importance later in the section.) The present chapter describes slim rectangular lattices by permutations. Using this description, we are going to enumerate slim rectangular lattices and slim patch lattices of a given length n. Also, we enumerate their planar diagrams in a reasonable sense. We give precise, however, involved formulas and asymptotic ones. By means of computer algebra, the precise formulas lead to concrete numbers for $n \le 1000$.

The rest of this section gives a brief historical overview of planar semimodular lattices, including slim rectangular and slim patch lattices. Section 2.2 recalls the main concepts and some tools we need from the theory of planar semimodular lattices. In Section 2.3, we describe slim rectangular lattices by certain permutations, and we prove several auxiliary statements that could be of separate interest. We count these lattices of a given height n and their diagrams in Section 2.4, and we give these numbers asymptotically for $n \to \infty$ in Section 2.5. Finally, Section 2.6 contains some concrete numerical values.

The concept of slim semimodular lattices and that of rectangular lattices appeared first in Grätzer and Knapp's pioneering papers [61] and [62]. These lattices are planar[1]. So far, the just-mentioned two papers have been followed by more than four dozen others devoted to planar semimodular lattices; see the

$$\begin{aligned}&\text{Appendix in Czédli's paper https://arxiv.org/abs/2107.10202v1}\\ &\qquad\text{for the 2021 list; for the up-to-date and longer list, see}\\ &\text{http://www.math.u-szeged.hu/\~{}czedli/m/listak/publ-psml.pdf}\end{aligned} \tag{2.1.1}$$

(Note that the paper occurring in the first line of (2.1.1) is the extended arXiv version of [28]; only this extended version contains the list in question.)

Next, we briefly discuss the role of planar semimodular lattices in lattice theory and related mathematical fields. By a classical (1942) result of N. Funayama and T. Nakayama, the congruence lattice $\mathrm{Con}(L)$ of a lattice L is necessarily distributive. For the finite case, the converse was first published by Grätzer and Schmidt [68]: for any finite distributive lattice K, there exists a finite lattice L such that $K \cong \mathrm{Con}(L)$. Following the terminology of Grätzer [57], we will reference this result as the *Basic Representation Theorem*. Note that in his monograph [57], Grätzer

[1] They are planar by their original definition given in Grätzer and Knapp [61]. Later, we will go by Czédli and Schmidt's setting, [43], where the concept of slimness is extended to all finite lattices including the non-semimodular ones, and planarity is a consequence of this general notion of slimness.

declares the Basic Representation Theorem in [68] as a rediscovery of an unpublished result of R. P. Dilworth. Later, some authors, mainly Grätzer and Schmidt, improved the Basic Representation Theorem by tailoring extra conditions to L and/or simultaneously representing two finite distributive lattices together with a function between them by means of lattice congruences; discussing such results would be very far from our targets. What is important in our aspect is that the Basic Representation Theorem remains true if we say "finite *planar semimodular* lattice L" instead of a "finite lattice L" in it; this was proved by Grätzer, Lakser, and Schmidt [65].

The first motivation for studying planar semimodular lattices is due to G. Grätzer and it is in connection with the Basic Representation Theorem: Can we put further restrictions on the finite lattice L (in addition to being planar and semimodular)? And if we put certain further restrictions on L, then what further properties will $\mathrm{Con}(L)$ have? So the first motivation is in connection with the *congruence lattices* of some special planar semimodular lattices. Indeed, 23 out of the 56 items on the list mentioned in (2.1.1) contain "congruence" in their titles. For brevity, we will refer to the first motivation as "*understanding the congruence lattices*" (of slim semimodular lattices).

The second motivation is that some special planar semimodular lattices, which we will call slim semimodular lattices, appeared to be the right tools in generalizing the classical Jordan–Hölder theorem for groups, see Czédli and Schmidt [43] and Grätzer and Nation [66]. That is, slim semimodular lattices have been *applied in group theory.*

The third motivation is that the purely lattice theoretic topic of slim semimodular lattices has lead to several papers in geometry; see the survey part of one of these papers, Czédli and Kurusa [38]; this paper is also on the list mentioned in (2.1.1).

The fourth motivation is somewhat weaker than the preceding three but it is still worth mentioning. At some cases, even though slim semimodular lattices are not applied at other fields of mathematics, they still have some connections with these fields: model theory and category theory; the titles of Czédli [27] and Czédli and Molkhasi [39] speak for themselves.

Based on the four motivations mentioned above, the class of planar semimodular lattices, that of slim planar semimodular lattices, and two other subclasses to be discussed later are natural classes of structures to study. Note at this point the just-mentioned classes contain only finite lattices, since any planar lattice is finite by definition.

To prove results about planar semimodular lattices and to apply these lattices outside lattice theory (see the second and third motivation), one should understand their structures. There are several approaches that offer insight into these lattices.

First of all, each planar semimodular lattice L has its *slimming*, which we obtain by removing the "inner doubly irreducible elements" of the cover-preserving M_3-sublattices of L. Here "inner" is understood in the geometric sense with respect to a fixed planar diagram of L. Thus, this concept and some other concepts that come later depend on a planar diagram of L rather than on L itself. In most of the cases, the choice of the diagram is irrelevant, at least up to left-right symmetry. Hence, we usually drop that "a fixed planar diagram of". According to Grätzer and Knapp [61], a planar semimodular lattice is *slim* if it is its own slimming. (Latter, we will define slimness in another but—in the presence of semimodularity—equivalent way.) We know from Grätzer and Knapp [61] that, to understand planar semimodular lattices, it suffices to describe the slim semimodular ones. This explains the importance of *slim planar semimodular lattices* among planar semimodular lattices.

Another important subclass of planar semimodular lattices is formed by *rectangular lattices*; see Grätzer and Knapp [62]. *Slim rectangular lattices* are also important. First, because we can obtain the slim semimodular lattices from slim rectangular ones; actually, we can do this in two ways. Namely, to obtain a slim semimodular lattice L, either we start with a "large" slim rectangular lattice L' and we can obtain L from L' in a particular way as a sublattice, see Czédli and Schmidt [44, Lemma 21], or we can glue "small slim rectangular lattices" (in fact, some rather special slim rectangular lattices called *slim patch lattices*) together, see Czédli and Schmidt [46]. As the construction and the concept on which it relies are quite involved in [46], it is worth noting that the patch lattices are exactly the Hall–Dilworth-gluing irreducible lattices in the class of the at least four-element planar semimodular lattices; see Czédli and Schmidt [46, Theorem 3.4].

In view of the paragraph above, the problem of understanding planar semimodular lattices has been reduced to the task of understanding slim semimodular and slim rectangular lattices. Partly, this task is reduced to understanding slim patch lattices, too. The tools for this task fall into two categories.

The tools in the first category yield a slim rectangular lattice L step-by-step, inductively. These steps either remove some special bundles of elements, see Czédli and Grätzer [35], or (and more importantly) the steps add special bundles of new elements, see Czédli and Schmidt [44, Lemma 22] and, for a more useful tool, Czédli [13, Theorem 3.7]. (In both cases, the bundles are called forks.) This step-by-step approach has often been useful in proving properties of slim rectangular or slim semimodular lattices.

The second category consists of two known tools that describe slim semimodular lattices by matrices or by permutations in an explicit way; see Czédli [11] and Czédli and Schmidt [45], respectively. The idea of using permutations for this purpose goes back to Abels [1].

This section uses permutations, the most advanced tool in the "explicit" category to describe slim semimodular lattices. We have not investigated whether the other explicit tool, the description by matrices, could be useful for our purpose, but note that matrices were used in another paper that enumerated some slim semimodular lattices; see Czédli, Ozsvárt, and Udvari [42]. (The title of this paper shows that, in addition to those two mentioned earlier, [42] is also a paper showing that slim semimodular lattices are applicable in group theory.) The advantage of permutations over matrices is that there is a considerable knowledge about the "Combinatorics of Permutations" (this is the title of the monograph of Bóna [3]).

The second and the third out of the four motivations, that is, the applicability in group theory and geometry, might look more interesting than the first motivation, understanding the congruence lattices (of slim semimodular lattices). To dispel this feeling and to strengthen the reputation of the first motivation, we mention that the tools outlined above were developed for the sake of understanding the congruence lattices of slim semimodular lattices. For example, Grätzer and Knapp [62] introduced slim rectangular lattices simply because:

- slim rectangular lattices are easier to describe than slim semimodular lattices;
- however, to describe the congruence lattice of slim semimodular lattices, it suffices to deal with the congruence lattices of slim rectangular lattices.

Indeed, it is implicit in Grätzer and Knapp [62] (and it is explicitly mentioned, say, in the Abstract of Czédli [26]), that the congruence lattices of *slim* semimodular lattices are, up to isomorphism, the same as the congruence lattices of *slim* rectangular lattices, provided that we disregard lattices with less than four elements.

The enumeration of slim semimodular lattices and their planar diarams started in Czédli, Ozsvárt and Udvari [42], and continued in Czédli, Dékány, Ozsvárt, Szakács and Udvari [34], and Czédli [16]. There are several earlier papers on counting other particular lattices; for example, see Erné, Heitzig and Reinhold [49] and [70], and Pawar and Waphare [78].

2.2 Preliminaries

Here, we overview some concepts and facts we need in the present chapter. For a more complex overview (but only up to 2014), the reader might be interested in Grätzer [54] and Czédli and Grätzer [36]. An element of a lattice is *join-irreducible* if it has exactly one lower cover. A finite lattice L is *slim*, if $\mathrm{Ji}\,L$, the set of the join-irreducible elements of L, is included in the union of two chains of L; see Czédli and Schmidt [43]. Note that, in the semimodular case, this concept was first introduced by Grätzer and Knapp [61] in a different way. We know from Czédli and Schmidt [43] that slim lattices are *planar*, that is, they possess planar diagrams. Remember that all lattices, and thus all diagrams, in this chapter are assumed to be finite. If D_1 and D_2 are planar diagrams and $\varphi\colon D_1 \to D_2$ is a bijective map such that φ is a lattice isomorphism and it preserves the left-right order of (upper) covers and that of lower covers of each element of D_1, then φ is called a *similarity map*. Two planar diagrams are *similar* if there exists a similarity map between them. We treat similar diagrams as equal ones. Therefore, when we count planar diagrams, we always do it up to similarity. Adjectives typically used for lattices, like semimodularity, will also be used for their planar diagrams; in this case the diagram is automatically a planar lattice diagram.

A minimal non-chain region of a planar lattice diagram D is called a *cell*. A four-element cell is a 4-*cell*. 4-cells are *covering squares*, that is, cover-preserving four-element Boolean sublattices. A diagram is a *4-cell diagram* if all of its cells are 4-cells. The following statement was proved in Grätzer and Knapp [61, Lemmas 4 and 5]; see also Czédli and Schmidt [44, Proposition 1] for the present form.

Lemma 2.2.1. *If D is a slim semimodular diagram, then it is a* 4*-cell diagram, and no two distinct* 4*-cells have the same bottom. Conversely, if D is a* 4*-cell lattice diagram in which no two distinct* 4*-cells have the same bottom, then D is a slim semimodular diagram.*

Following Grätzer and Knapp [62], a semimodular diagram D is *rectangular* if its *left boundary chain*, denoted by $\mathrm{C_l}(D)$, has exactly one doubly irreducible element, $\mathrm{lc}(D)$, its *right boundary chain*, $\mathrm{C_r}(D)$, has exactly one doubly irreducible element, $\mathrm{rc}(D)$, and these two elements, called the *corners* of D, are complementary, that is, $\mathrm{lc}(D) \wedge \mathrm{rc}(D) = 0$ and $\mathrm{lc}(D) \vee \mathrm{rc}(D) = 1$. It was noticed by Schmidt, see Czédli and Grätzer [36, Exercise 1.58], that a slim semimodular lattice L is rectangular iff $\mathrm{Ji}\, L$ is a union of two chains such that no element in the first chain is comparable with some element of the second chain. Associated with a slim rectangular diagram D, the following three numerical parameters will be of particular interest.

Notation 2.2.2. As usual, the *length* of D is denoted by $\mathrm{length}\, D$. The *left upper length* and the *right upper length* of D, denoted by $^{\mathrm{lu}}\mathrm{len}\, D$ and $^{\mathrm{ru}}\mathrm{len}\, D$, are the length of the interval $[\mathrm{lc}(D), 1]$ and that of $[\mathrm{rc}(D), 1]$, respectively; see Figure 2.1 for illustration.

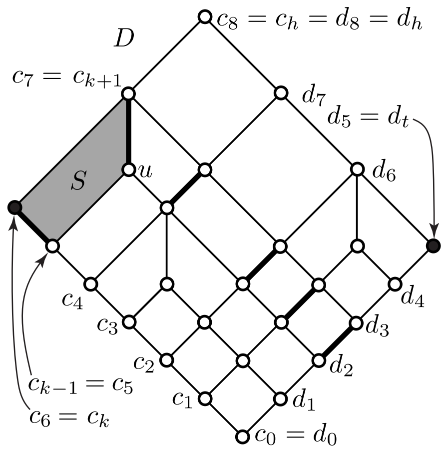


Figure 2.1: A rectangular diagram with length $D = 8$, ${}^{\mathrm{lu}}\mathrm{len}\, D = 2$, and ${}^{\mathrm{ru}}\mathrm{len}\, D = 3$.

A rectangular diagram D is a *patch diagram* if $\mathrm{lc}(D)$ and $\mathrm{rc}(D)$ are coatoms. Equivalently, if ${}^{\mathrm{lu}}\mathrm{len}\, D = {}^{\mathrm{ru}}\mathrm{len}\, D = 1$. A *patch lattice* is a lattice that has a patch diagram.

Two prime intervals of a slim semimodular diagram D are *consecutive* if they are opposite sides of a 4-cell. By Czédli and Schmidt [43, Lemma 2.3], covering squares and 4-cells in a slim semimodular diagram are the same, whence the previous sentence can be rephrased as follows: two prime intervals of a slim semimodular diagram D are consecutive if they are opposite sides of a covering square. Therefore, the consecutiveness of two prime intervals in slim semimodular lattice L does not depend on the planar diagram chosen. Maximal sequences of consecutive prime intervals form a *trajectory*, see Czédli and Schmidt [43]. In other words, a trajectory is a class of the equivalence relation generated by consecutiveness. In [43, Lemma 2.8], the following statement was derived from (the present) Lemma 2.2.1.

Lemma 2.2.3. *If T is a trajectory of a slim semimodular diagram D, then T contains exactly one prime interval of* $\mathrm{C_l}(D)$*, and the same holds for* $\mathrm{C_r}(D)$*. Going from left to right, T does not branch out. First T goes up* (*possibly in zero steps*)*, then it may turn to the lower right, and finally it goes down* (*possibly, in zero steps*)*. In particular, at most one turn is possible.*

Notation 2.2.4. We denote the set of (the similarity classes of) *slim rectangular* *diagrams* of length n and that of *slim semimodular diagrams* of length n by the acronyms $\mathrm{SRectD}(n)$ and $\mathrm{SSmodD}(n)$, respectively. Similarly, the set of the isomorphism classes of *slim rectangular lattices* of length n, that of *slim semimodular lattices* of length n, and that of *slim patch lattices* of length n are denoted by $\mathrm{SRectL}(n)$, $\mathrm{SSmodL}(n)$, and $\mathrm{SPatchL}(n)$.

For a given $n \in \{1, 2, \dots\} = \mathbb{N}$, these five sets above are finite, since we do not make a distinction between similar diagrams or between isomorphic lattices.

Jordan–Hölder permutations associated with semimodular lattices appeared first in Abels [1] and Stanley [82]. Here, following Czédli and Schmidt [45], we define them by means of trajectories. For a slim rectangular diagram D, let $n = \operatorname{length} D$, and let

$$\begin{aligned} \mathrm{C}_{\mathrm{l}}(D) &= \{0 = c_0 \prec c_1 \prec \dots \prec c_n = 1\}, \\ \mathrm{C}_{\mathrm{r}}(D) &= \{0 = d_0 \prec d_1 \prec \dots \prec d_n = 1\}. \end{aligned} \tag{2.2.1}$$

The set of all $\{1, \dots, n\} \to \{1, \dots, n\}$ permutation is denoted by S_n. The (*Jordan–Hölder*) *permutation* $\pi = \pi_D \in S_n$ is defined by the rule $\pi(i) = j$ iff $[c_{i-1}, c_i]$ and $[d_{j-1}, d_j]$ belong to the same trajectory. The following statement was proved in Czédli and Schmidt [45].

Lemma 2.2.5. *The map* $\mathrm{SSmodD}(n) \to S_n$*, defined by* $D \mapsto \pi_D$*, is a bijection.*

In what follows in this chapter, since this lemma above is obvious for $n = 1$ and since the length of a slim rectangular lattice is at least 2, we always assume that n denotes an integer greater than 1. Combining Lemma 2.2.5 with [45, Lemma 4.6] and the definition of π_D, we obtain that

Lemma 2.2.6. *Let* D_1 *and* D_2 *be slim rectangular diagrams. They determine the same lattice iff* $\pi_{D_1} \in \{\pi_{D_2}, \pi_{D_2}^{-1}\}$*.*

Planar lattice diagrams have several properties that are easy to believe but not so easy to prove. What we need from them is given by the following lemma, taken from Kelly and Rival [71, Lemmas 1.2 and 1.5, Propositions 1.6 and 1.7, and Theorem 2.5].

Lemma 2.2.7. *Let* D *be a planar lattice diagram, and let* $a, b \in D$*.*

(i) *If $a \le b$ and a and b are on different sides of a maximal chain C, then there exists an $x \in C$ such that $a \le x \le b$.*

(ii) *A closed interval of D is a planar subdiagram.*

(iii) *If $|D| \ge 3$, then D contains a doubly irreducible element distinct from* 0 *and* 1 *on its left boundary.*

(iv) *If $a \parallel b$, then either a is on the left of all maximal chains through b, or b is on the left of all maximal chains through a. The same holds with "right" instead of "left".*

Based on Lemma 2.2.7(iv), if $a \parallel b$ and a is on the left of some (equivalently, all) maximal chains through b, then we say that a is *on the left* of b; analogous terminology is used if "left" is replaced by "right".

2.3 Description by permutations

For convenience, we introduce the following concept; it is visualized by Figure 2.2, and our terminology will be explained by Proposition 2.3.3.

Definition 2.3.1. A permutation $\pi \in S_n$ is called *rectangular* if it satisfies the following three properties.

(i) For all i and j, if $\pi^{-1}(1) < i < j \le n$, then $\pi(i) < \pi(j)$.

(ii) For all i and j, if $\pi(1) < i < j \le n$, then $\pi^{-1}(i) < \pi^{-1}(j)$.

(iii) $\pi(n) < \pi(1)$.

Clearly, $\pi^{-1}(1) < i$ and $\pi(1) < i$ above can be replaced by $\pi^{-1}(1) \le i$ and $\pi(1) \le i$, respectively. In Figure 2.2, where $n = 16$, a permutation π is given as a bipartite graph; however, not all the 16 edges are drawn. The rectangularity of π means that neither the edges denoted by (i) nor those denoted by (ii) intersect, but the two thick solid edges do. (According to Remark 2.3.2 below, the two thick dotted edges also intersect.)

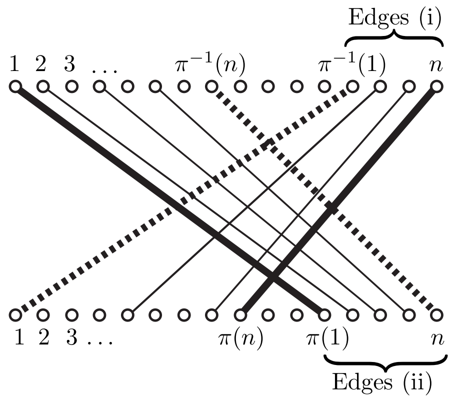


Figure 2.2: The rectangularity of a permutation

Remark 2.3.2. *If $\pi \in S_n$ is rectangular, then we have*

(iv) $\pi^{-1}(n) < \pi^{-1}(1)$.

So, π is rectangular iff π^{-1} is rectangular.

Proof of Remark 2.3.2. Assume that $\pi \in S_n$ satisfies (i)–(iii). Since π and π^{-1} are injective, (iii) implies that

$$1 < \pi(1), \qquad \pi(n) < n, \qquad 1 < \pi^{-1}(1), \qquad \pi^{-1}(n) < n. \tag{2.3.1}$$

Suppose, for a contradiction, that (iv) fails. Then $n \geq 2$, and we have that $\pi^{-1}(1) < \pi^{-1}(n)$. By the last inequality of (2.3.1), (i) applies for the pair $\langle i, j \rangle = \langle \pi^{-1}(n), n \rangle$, and we obtain that $n = \pi(\pi^{-1}(n)) < \pi(n)$, a contradiction. $\square$

Now, we are in the position to formulate the main result of this section.

Proposition 2.3.3. *A slim, semimodular, planar diagram D of length $n \geq 2$ is rectangular if and only if $\pi = \pi_D \in S_n$ is rectangular. Furthermore, if D is rectangular, then*

$$\pi_D(1) = \mathrm{length}\, D - {}^{\mathrm{ru}}\mathrm{len}\, D + 1, \quad \pi_D^{-1}(1) = \mathrm{length}\, D - {}^{\mathrm{lu}}\mathrm{len}\, D + 1.$$

This proposition trivially implies the following statement.

Corollary 2.3.4. *A slim, semimodular, planar diagram D of length n is a patch diagram if and only if $\pi_D(1) = n = \pi_D^{-1}(1)$. Therefore, the number of these diagrams is $(n-2)!$.*

Combining Proposition 2.3.3 and Corollary 2.3.4 with Lemmas 2.2.5 and 2.2.6, we obtain a new description of slim rectangular (or patch) diagrams and lattices by permutations. This description is effective, because Czédli and Schmidt [45, Proposition 2.7 and Theorem 3.3] tell us how to construct D from π_D; however, we do not need these long details here.

The rest of this section is devoted to the proof of Proposition 2.3.3. The following definition is taken from Grätzer and Quackenbush [67].

Definition 2.3.5. An element x of a lattice L is called a *narrows* if $L = \downarrow x \cup \uparrow x$. If, in addition, $x \notin \{0, 1\}$, then x is a *proper narrows.* The set of narrows of L is denoted by $\mathrm{Nar}(L)$. A lattice L is called (*glued sum*) *indecomposable* if $|L| \geq 3$ and $\mathrm{Nar}(L) = \{0, 1\}$.

We know from Czédli and Schmidt [45, after (1.2)] that the set $\mathrm{Nar}(D)$ of narrows of D is $\mathrm{C_l}(D) \cap \mathrm{C_r}(D)$. Note that, by definitions, a glued sum indecomposable diagram is of length at least 2.

Obviously, Lemma 2.2.1 implies the following statement.

Corollary 2.3.6. *If D is a (glued sum) indecomposable, slim, semimodular diagram, then for each $c \in \mathrm{C_l}(D) \setminus \{0, 1\}$, there exists a unique c' such that $\{c \wedge c', c, c', c \vee c'\}$ is a 4-cell.*

Lemma 2.3.7. *If D is an indecomposable, slim, semimodular diagram, $a \prec b$, and $a, b \in \mathrm{C_l}(D)$, then exactly one of the following two possibilities holds.*

(i) *a is meet-reducible and b is join-irreducible. (In this case, we say that $[a, b]$ is an* up-edge.)

(ii) *a is meet-irreducible and b is join-reducible. (In this case, we say that $[a, b]$ is an* down-edge.)

Proof. Since D is indecomposable, the trajectory starting at $[a, b]$ is not a singleton. In other words, $[a, b]$ is a left edge of a 4-cell S. This implies that a is meet-reducible or b is join-reducible. Hence, Czédli and Schmidt [44, Lemma 4], which says that each of these two cases excludes the other one, completes the proof. □

The name "down-edge" is motivated by the following lemma.

Lemma 2.3.8. *Let D be a slim semimodular diagram of length n, and assume that $1 \leq i < j \leq n$.*

(i) *If D is glued sum indecomposable and, with the notation given in* (2.2.1), *$[c_{i-1}, c_i]$ is a down-edge, then $\pi_D(i) < \pi_D(j)$ and $\pi_D(i) < i$.*

(ii) *If c_i is a narrows, then $\pi_D(i) < \pi_D(j)$.*

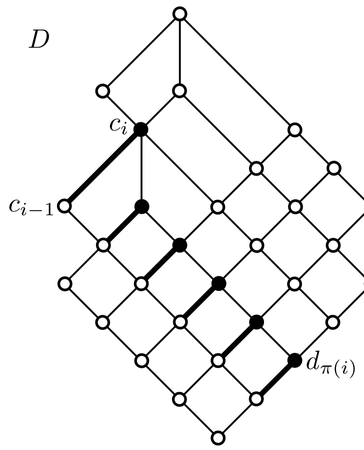


Figure 2.3: Illustrating the proof of Lemma 2.3.8

Proof. (i): Assume that D is indecomposable. Denote π_D by π. Let T_i be the trajectory that contains $[c_{i-1}, c_i]$; see Figure 2.3, where T_i consists of the thick edges. Note that T_i consists of at least two edges, because D is indecomposable. Since $[c_{i-1}, c_i]$ is a down-edge, T_i launches to the lower right, and keeps going to this direction without any turn by Lemma 2.2.3. Hence, the top elements of the edges of T_i, which are the black-filled elements in the figure, form a descending, nontrivial chain. This implies that $d_{\pi(i)} < c_i$, and we conclude that $\pi(i) < i$.

Suppose, for a contradiction, that $\pi(i) > \pi(j)$. This implies that $c_{j-1} \geq c_i > d_{\pi(i)} > d_{\pi(j)}$. Hence, $[c_{j-1}, c_j]$ and $[d_{\pi(j)-1}, d_{\pi(j)}]$ are two comparable prime intervals of the same trajectory. This is a contradiction, since a trajectory cannot have comparable prime intervals by Czédli [12, Lemma 3.3]. This proves (i).

(ii): Assume that c_i is a narrows. Clearly, for every 4-cell S, either we have that $S \cap (\downarrow c_i \setminus \{c_i\}) = \varnothing$, or $S \cap (\uparrow c_i \setminus \{c_i\}) = \varnothing$. Hence, no trajectory can cross c_i, and part (ii) follows immediately. $\square$

Next, we generalize some parts of Grätzer and Knapp [62, Lemmas 3 and 4]. By Lemma 2.2.7(iii), the element c in the following lemma exists.

Lemma 2.3.9. *Let D be a glued sum indecomposable, planar lattice diagram. If c is the least doubly irreducible element on the left boundary of D, then the ideal $\downarrow c$ is a chain.*

Proof. Let $\mathrm{C_l}(D) \cap \downarrow c = \{0 = c_0 \prec c_1 \prec \cdots \prec c_k = c\}$. It suffices to prove that

$$\{c_1, \ldots, c_k\} \subseteq \mathrm{Ji}\, D.$$

Suppose, for a contradiction, that there is an $i \in \{1, \ldots, k\}$ such that c_i is join-reducible. Let i be minimal with respect to this property. The ideal $\downarrow c_i$ is a planar subdiagram by Lemma 2.2.7(ii). Let $U = \mathrm{C_r}(\downarrow c_i)$. Take the largest $j \in \{0, \ldots, i-1\}$ such that $c_j \in U$; this j exists, since $c_0 = 0 \in U$. Note that $j \leq i-2$, since c_i is join-reducible. By Lemma 2.2.7(ii), $D' := [c_j, c_i]$ is a planar subdiagram. Clearly, $|D'| \geq 3$, $\mathrm{C_l}(D') = \{c_j, c_{j+1}, \ldots, c_i\}$, and $\mathrm{C_r}(D') = U \cap [c_j, c_i]$. By Lemma 2.2.7(iii), there is an $s \in \{j+1, \ldots, i-1\}$ such that c_s is doubly irreducible in D'. By the choice of k, the element c_s is not doubly irreducible in D. The minimality of i yields that c_s is meet-reducible in D. By Czédli and Schmidt [44, Lemma 4], mentioned already in the proof of Lemma 2.3.7, the join-reducibility of c_i implies that $s \neq i-1$. Hence, $s \leq i-2$. The element c_s has a cover $v \in D$, distinct from c_{s+1}. Since c_s is meet-irreducible in D', we have that $v \notin D'$. We have that $\mathrm{height}\, v = s+1 < i = \mathrm{height}\, c_i$, whence $c_i \not\leq v$. We also have that $v \not\leq c_i$, since $v \notin D' = [c_j, c_i]$. Thus, $c_i \parallel v$. We conclude from Lemma 2.2.7(iv) that c_i is on the left of v. That is, v is on the right of all maximal chains through c_i. In particular, if we extend $\mathrm{C_r}(D')$ to a maximal chain V of D, then v is strictly on the right of V. On the other hand, c_s, which belongs to $\mathrm{C_l}(D') \setminus \mathrm{C_r}(D')$, is strictly on the left of $\mathrm{C_r}(D')$, whence it is strictly on the left of V. Thus, c_s and v are strictly on different sizes of V while $c_s \prec v$. This contradicts Lemma 2.2.7(i). $\square$

Lemma 2.3.10. *Let D be a glued sum indecomposable, slim semimodular diagram of length n. If, with notation* (2.2.1)*, c_k is the least doubly irreducible element of D on the left boundary chain, then $\pi_D(k+1) = 1$.*

Proof. Clearly, $k \geq 1$. We prove the lemma by induction on k.

First, assume that $k = 1$. Since D is indecomposable, $0 \notin \mathrm{Mi}\, D$. By Czédli and Schmidt [44, Lemma 2],

$$\text{each element of a slim lattice has at most two covers.} \tag{2.3.2}$$

Hence, there are exactly two atoms, and $c_k = c_1$ is one of them. This clearly implies that $\pi_D(k+1) = \pi_D(2) = 1$.

Next, assume that $k > 1$, and the lemma holds for smaller values. Let $u = c'_k$ by Corollary 2.3.6. Since c_k has only one cover, and this cover belongs to $\mathrm{C_l}(D)$, we have that $c_k \vee u = c_{k+1}$. Similarly, $c_k \wedge u = c_{k-1}$. Hence,

$$S = \{c_{k-1}, c_k, u, c_{k+1}\} \quad \text{is a 4-cell.} \tag{2.3.3}$$

This 4-cell (or Lemma 2.3.7) shows that c_{k-1} is meet-reducible; see Figure 2.1 for an illustration. Let $D' = D \setminus \{c_k\}$; it consists of the empty-filled elements in the figure. Clearly, $c_{k-1} \in \mathrm{C_l}(D')$. By (2.3.2), $c_{k-1} \in \mathrm{Mi}\, D'$. We also have that $c_{k-1} \in \mathrm{Ji}\, D'$, because $c_{k-1} \in \mathrm{Ji}\, D$ by Lemma 2.3.9. Thus, c_{k-1} is a doubly irreducible element in D'.

Suppose, for a contradiction, that there exists an $i < k-1$ such that c_i is doubly irreducible in D'. Obviously, it is join-irreducible in D. By the choice of k, c_i is meet-reducible in D. However, its covers are of height $i+1$, which is less than $k = \mathrm{height}\, c_k$. Hence, these covers belong to D', contradicting the assumption that c_i is doubly irreducible in D'. This proves that c_{k-1} is the least doubly irreducible element of D' that belongs to $\mathrm{C_l}(D')$.

Let T' be the trajectory of D' such that T' contains $[c_{k-1}, u]$. Obviously, or by Czédli [12, Lemma 3.1], the trajectory of D that contains $[c_k, c_{k+1}]$ is $T := T' \cup \{[c_k, c_{k+1}]\}$. Note that the element of height k in $\mathrm{C_l}(D')$ is u. By the induction hypothesis, $\pi_{D'}(k) = 1$. This means that $[d_0, d_1] \in T'$. Thus, $[d_0, d_1] \in T$, proving that $\pi_D(k+1) = 1$. $\square$

Proof of Proposition 2.3.3. By definitions, $\mathrm{SRectD}(n) \subseteq \mathrm{SSmodD}(n)$. Therefore, by Lemma 2.2.5, it suffices to prove that, for $D \in \mathrm{SSmodD}(n)$, the diagram D is rectangular iff so is the permutation π_D.

To prove the "only if" part of Proposition 2.3.3, assume that $D \in \mathrm{SRectD}(n)$. Let $k \in \{1, \dots, n-1\}$ denote the height of $\mathrm{lc}(D)$, that is, $\mathrm{lc}(D) = c_k$. By the rectangularity of D, c_k is the only doubly irreducible element that belongs to the left boundary chain. Thus, Lemma 2.3.10 yields that

$$\pi(k+1) = 1, \text{ that is, } k+1 = \pi^{-1}(1). \tag{2.3.4}$$

Next, to verify condition 2.3.1(i), assume that $\pi^{-1}(1) < i < j \le n$. That is, we assume that $k+1 < i < j \le n$. Since $\mathrm{lc}(D) = c_k < c_i$ and c_k is the only doubly irreducible element on the left boundary chain, the element c_i is join-reducible by Grätzer and Knapp [62, Lemma 3]. Hence, $[c_{i-1}, c_i]$ is a down-edge by Lemma 2.3.7. Thus, Lemma 2.3.8(i) yields that $\pi(i) < \pi(j)$, proving that π satisfies 2.3.1(i).

Next, let t be the height of $\mathrm{rc}(D)$. Again by [62, Lemma 3], d_j is join-reducible for all $t < j \le n$. Hence, for these j, no trajectory can arrive at $[d_{j-1}, d_j]$ from the upper left. On the other hand, c_{n-1} is meet-irreducible and $1 = c_n$ is join-reducible by [62, Lemma 3]. Hence, $[c_{n-1}, c_n]$ is a down-edge, and the trajectory T_n containing this edge goes downwards by Lemma 2.2.3. Hence, T_n arrives at the right boundary chain from the upper left. Consequently, it cannot arrive at $[d_{j-1}, d_j]$ if $t < j$, and we conclude that $\pi(n) \le t$. If we interchange $\langle \text{left}, \pi, k\rangle$ and $\langle \text{right}, \pi^{-1}, t\rangle$ in the argument proving (2.3.4), we obtain that $\pi(1) = t+1$. Consequently, 2.3.1(iii) holds.

Similarly, interchanging $\langle \text{left}, \pi\rangle$ and $\langle \text{right}, \pi^{-1}\rangle$ in the proof of 2.3.1(i), we obtain that 2.3.1(ii) holds. Therefore, if D is rectangular, then so is π_D.

Next, to prove the "if" part of Proposition 2.3.3, assume that $D \in \mathrm{SSmodD}(n)$ but $D \notin \mathrm{SRectD}(n)$. We have to prove that $\pi = \pi_D$ is not rectangular.

First, we assume that D has a nontrivial narrows v. Since $v \in \mathrm{C_l}(D) \cap \mathrm{C_r}(D)$, it is of the form $v = c_s = d_s$ for some $s \in \{1, \dots, n-1\}$. Let T_1' denote the trajectory of the subdiagram $\downarrow v$ that begins with the prime interval $[c_0, c_1]$ of the left boundary chain. It reaches the right boundary of $\downarrow v$ at some $[d_{i-1}, d_i]$, where $i \leq s$. Clearly, T_1' is also a trajectory of D, and so $\pi(1) = i \leq s$. The dual argument shows that $\pi(n) \geq s$. (Note, however, that the concept of slim rectangular lattices is not selfdual.) Hence, 2.3.1(iii) fails and π is not rectangular.

Next, we can assume that D is glued sum indecomposable. Since $n \geq 2$, we conclude that 0 is meet-reducible and 1 is join-reducible. By Lemma 2.2.7(iii), each of $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$ has at least one doubly irreducible element. Since D is not rectangular, we obtain from Grätzer and Knapp [62, Lemma 6] that at least one of $\mathrm{C_l}(D)$ and $\mathrm{C_r}(D)$ has at least two doubly irreducible elements. Note that if we reflect D to a vertical axis, then π turns into π^{-1}. Thus, since the rectangularity of π is equivalent to that of π^{-1} by Remark 2.3.2, we can assume that, with notation (2.2.1), there are $1 \leq i < j < n$ such that c_i and c_j are the smallest and the largest doubly irreducible elements that belong to $\mathrm{C_l}(D)$, respectively. We have that

$$\pi^{-1}(1) = i + 1 \tag{2.3.5}$$

by Lemma 2.3.10. To prove that π is not rectangular, we intend to show that 2.3.1(i) fails.

First of all, we show that $i+1 < j$. Suppose, for a contradiction, that $j = i+1$. Then $[c_i, c_j]$ is a prime interval. Let T denote the trajectory that begins with $[c_i, c_j]$. Since c_i is meet-irreducible, T cannot make its first step to the upper right. Similarly, it cannot make the first step to the lower right since c_j is join-irreducibly. Thus, T makes no first step, and it consists only of $[c_i, c_j]$. By Lemma 2.2.3, $\{c_i, c_j\} \in \mathrm{C_r}(D)$. Hence, c_i and c_j are nontrivial narrows of D, contradicting our assumption. This proves that $i + 1 < j$.

Next, let c_j' be as in Lemma 2.3.6, that is, $c_j = \mathrm{lc}(S)$ and $c_j' = \mathrm{rc}(S)$ for a unique 4-cell S. Since c_j is doubly irreducible, the subdiagram $D' = D \setminus \{c_j\}$ is a slim semimodular lattice diagram by Lemma 2.2.1. Similarly to (2.3.3), we have that $\{c_{j-1} = c_j \wedge c'_j,\, c_j,\, c'_j,\, c_{j+1} = c_j \vee c'_j\}$ is a 4-cell. Let T_{j+1} and T_j denote the trajectories of D beginning with $[c_j, c_{j+1}]$ and with $[c_{j-1}, c_j]$, respectively. Also, let T'_{j+1} and T'_j be the trajectories of D' through $[c_{j-1}, c'_j]$ and $[c'_j, c_{j+1}]$, respectively. Clearly,

$$T_j = T'_j \cup \{[c_{j-1}, c_j]\} \quad \text{and} \quad T_{j+1} = T'_{j+1} \cup \{[c_j, c_{j+1}]\}. \tag{2.3.6}$$

By Lemma 2.3.7, the double irreducibility of c_j in D yields that $[c_{j-1}, c_j]$ is an up-edge and $[c_j, c_{j+1}]$ is a down-edge. Hence, by Lemma 2.2.3, T_{j+1} goes down, without any turn. This, together with (2.3.6), yields that T'_{j+1} is also a "down-going" trajectory of D'. Thus, either D' is indecomposable and $[c_{j-1}, c'_j]$ is a down-edge, or c'_j is a narrows of D'. In both cases, Lemma 2.3.8 implies that $\pi_{D'}(j) < \pi_{D'}(j+1)$. This inequality and (2.3.6) imply that

$$\pi_D(j+1) = \pi_{D'}(j) < \pi_{D'}(j+1) = \pi_D(j).$$

This, together with (2.3.5) and $i+1 < j$, shows that 2.3.1(i) fails. $\square$

2.4 Enumeration

For a rectangular permutation $\pi \in S_n$, we let

$$^{\mathrm{lu}}\mathrm{len}\,\pi = n + 1 - \pi^{-1}(1) \quad \text{and} \quad {}^{\mathrm{ru}}\mathrm{len}\,\pi = n + 1 - \pi(1).$$

By Proposition 2.3.3, $^{\mathrm{lu}}\mathrm{len}\,\pi_D = {}^{\mathrm{lu}}\mathrm{len}\,D$ and $^{\mathrm{ru}}\mathrm{len}\,\pi_D = {}^{\mathrm{ru}}\mathrm{len}\,D$ hold for all $D \in \mathrm{SRectD}(n)$. For $2 \le n \in \mathbb{N}$ and $a, b \in \mathbb{N}$, we let

$$\begin{aligned} \mathrm{RPerm}(n) &= \{\pi \in S_n : \pi \text{ is rectangular}\} \text{ and} \\ \mathrm{RPerm}(n; a, b) &= \{\pi \in \mathrm{RPerm}(n) : {}^{\mathrm{lu}}\mathrm{len}\,\pi = a \text{ and } {}^{\mathrm{ru}}\mathrm{len}\,\pi = b\}. \end{aligned}$$

It follows from Definition 2.3.1 that $\mathrm{RPerm}(n; a, b) \neq \varnothing$ iff $a + b \le n$.

Lemma 2.4.1. *For $a, b, n \in \mathbb{N}$ with $a + b \le n$,*

$$|\mathrm{RPerm}(n; a, b)| = \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,. \tag{2.4.1}$$

Proof. For $\pi \in \mathrm{RPerm}(n;a,b)$, we have $\pi^{-1}(1) = n-\big(n+1-\pi^{-1}(1)\big)+1 = n-a+1$ and, similarly, $\pi(1) = n-b+1$. Since $\pi(n) < \pi(1)$ and $\pi^{-1}(n) < \pi^{-1}(1)$ by 2.3.1(iii) and 2.3.2(iv), conditions 2.3.1(i) and 2.3.1(ii) can be rephrased as follows:

$$\pi(n-a+1) = 1 < \pi(n-a+2) < \cdots < \pi(n) < n-b+1, \text{ and} \tag{2.4.2}$$

$$\pi^{-1}(n-b+1) = 1 < \pi^{-1}(n-b+2) < \cdots < \pi^{-1}(n) < n-a+1. \tag{2.4.3}$$

Conversely, if $\pi \in S_n$ satisfies (2.4.2) and (2.4.3), then $\pi \in \mathrm{RPerm}(n;a,b)$. The first and the second binomial coefficients in (2.4.1) show how many ways conditions (2.4.3) and (2.4.2) can be fulfilled, respectively. These conditions take care of the images of $a+b$ elements in $\{1,\dots,n\}$. Hence, there are $(n-a-b)!$ possibilities for the rest of elements. $\square$

From Lemmas 2.2.5 and 2.4.1 and Proposition 2.3.3, we immediately obtain that

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\le n\\ a,b\in\mathbb{N}}} |\mathrm{RPerm}(n;a,b)|. \tag{2.4.4}$$

Consequently, the following statement holds.

Proposition 2.4.2. *For $2 \le n \in \mathbb{N}$, the number of slim rectangular diagrams of length n is*

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\le n\\ a,b\in\mathbb{N}}} \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,.$$

The following lemma belongs to the folklore; see the first sentence in the proof of Proposition 7.13 in Bóna [3, page 256], or see Czédli, Ozsvárt and Udvari [42, Lemma 6.1]. As usual, $(2t-1)!!$ denotes $1\cdot3\cdot5\cdot\dots\cdot(2t-1) = (2t)!/(2^t\cdot t!)$. Note that $(-1)!! = 1$ by definition. An *involution* is a permutation π such that $\pi^{-1} = \pi$. Let $\mathrm{Invl}(k) = \{\pi \in S_k : \pi = \pi^{-1}\}$ denote the set of involutions acting on the set $\{1,\dots,k\}$.

Lemma 2.4.3. *For $k \in \mathbb{N}$, the number of involutions in S_k is*

$$|\mathrm{Invl}(k)| = \sum_{j=0}^{\lfloor k/2\rfloor} \binom{k}{k-2j}\cdot(2j-1)!!\ .$$

Now, after that $|\mathrm{SRectD}(n)|$ has been determined by Proposition 2.4.2 and we also have Lemma 2.4.3, we formulate the following statement.

Proposition 2.4.4. *For $2 \le n \in \mathbb{N}$, the number of (the isomorphism classes of) slim rectangular lattices of length n is*

$$|\mathrm{SRectL}(n)| = \frac{1}{2} \cdot \left(|\mathrm{SRectD}(n)| + \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)| \right). \tag{2.4.5}$$

Proof. By Lemmas 2.2.5 and 2.2.6, two distinct slim rectangular diagrams, D_1 and D_2, determine the same rectangular lattice iff $\pi_{D_1} = (\pi_{D_2})^{-1}$. Hence, if we count every involution twice and any other permutation once, then we count each lattice in question twice, that is,

$$\begin{aligned} 2 \cdot |\mathrm{SRectL}(n)| &= |\mathrm{RPerm}(n) \setminus \mathrm{Invl}(n)| + 2 \cdot |\mathrm{RPerm}(n) \cap \mathrm{Invl}(n)| \\ &= |\mathrm{RPerm}(n)| + |\mathrm{RPerm}(n) \cap \mathrm{Invl}(n)| \\ &= |\mathrm{SRectD}(n)| + |\mathrm{RPerm}(n) \cap \mathrm{Invl}(n)|. \end{aligned} \tag{2.4.6}$$

Therefore, to obtain (2.4.5), it suffices to prove that

$$|\mathrm{RPerm}(n) \cap \mathrm{Invl}(n)| = \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)|. \tag{2.4.7}$$

The argument we need is similar to the one used in the proof of Lemma 2.4.1. If $\pi = \pi^{-1}$, then $a = b \le n/2$. Hence, an involution π is in $\mathrm{RPerm}(n)$ iff it satisfies (2.4.2) with $b = a$. There are $\binom{n-a-1}{a-1}$ ways to select the values $\pi(n-a+2) < \cdots < \pi(n)$ from $\{2, \dots, n-a\}$. Since π is an involution, each of these selections determines the action of π on the $2a$-element set

$$\begin{aligned} \{1 = \pi(n-a+1) &< \pi(n-a+2) < \cdots < \pi(n) \\ &< \pi(1) = \pi^{-1}(1) = n-a+1 < n-a+2 < \cdots < n\}. \end{aligned}$$

Clearly, π acts as an involution on the $n-2a$ remaining elements. Hence, there are $|\mathrm{Invl}(n-2a)|$ ways to continue the above-mentioned selection to an involution on the whole set $\{1, \dots, n\}$. Finally, $2a = a + b \le n$ gives that $a \le \lfloor n/2 \rfloor$, and we conclude (2.4.7). $\square$

The situation for slim patch lattices is much easier.

Proposition 2.4.5. *For $2 \le n \in \mathbb{N}$, the number of (the isomorphism classes of) slim patch lattices of length n is* $|\mathrm{SPatchL}(n)| = \big((n-2)! + |\mathrm{Invl}(n-2)|\big)/2$.

Proof. A permutation π from Corollary 2.3.4 is an involution iff so is its restriction to $\{2, \dots, n-2\}$. Hence, using the idea of (2.4.6) with "patch" instead of "rectangular", we can obviously conclude our statement from Lemma 2.2.5 and Corollary 2.3.4 . ☐

2.5 Asymptotic results

For functions $f, g : \mathbb{N} \to \{x \in \mathbb{R} : x > 0\}$, we say that f *is asymptotically* g, denoted by $f(n) \sim g(n)$, if $\lim_{n\to\infty}\big(f(n)/g(n)\big) = 1$. In this section, a and b always denote positive integers. Hence, we will not indicate $a, b \in \mathbb{N}$ in range specifications. As usual, e denotes $\sum_{k=0}^{\infty}(k!)^{-1} \approx 2.7182818285$.

Proposition 2.5.1. *The number of slim rectangular diagrams of length n is asymptotically $(n-2)! \cdot e^2$, that is,* $|\mathrm{SRectD}(n)| \sim (n-2)! \cdot e^2$.

Proof. Based on (2.4.1), we can compute as follows.

$$
\begin{aligned}
|\mathrm{RPerm}(n; a, b)| &= \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)! \\
&= \frac{(n-a-1)\cdots(n-a-b+1)}{(b-1)!} \cdot \frac{(n-b-1)\cdots(n-a-b+1)}{(a-1)!} \\
&\times (n-a-b)! \\
&= \frac{(n-a-1)\cdots(n-a-b+1)}{(b-1)!} \cdot \frac{(n-2)!}{(a-1)!\,(n-2)\cdots(n-b)} \\
&= \frac{(n-2)!}{(a-1)!\,(b-1)!} \cdot \frac{n-a-1}{n-2} \cdot \frac{n-a-2}{n-3} \cdots \frac{n-a-b+1}{n-b}\,.
\end{aligned}
\tag{2.5.1}
$$

Denote by $q(n, a, b)$ the product of the last $b-1$ factors in (2.5.1), that is, the product of all but the first factor. In particular, $q(n, a, 1) = 1$. Hence,

$$|\mathrm{RPerm}(n; a, b)| = q(n, a, b) \cdot \frac{(n-2)!}{(a-1)!\,(b-1)!}. \tag{2.5.2}$$

Since $1 \leq a$, $q(n,a,b)$ is the product of factors not greater than 1. Hence, $q(n,a,b) \leq 1$ and $|\mathrm{RPerm}(n;a,b)| \leq (n-2)!\,((a-1)!\,(b-1)!)^{-1}$. Combining this estimate with (2.5.2) and using (2.4.4), we obtain that

$$\begin{aligned}|\mathrm{SRectD}(n)| &\overset{(2.4.4)}{=} \sum_{a+b\leq n} |\mathrm{RPerm}(n;a,b)| \leq \sum_{a+b\leq n} \frac{(n-2)!}{(a-1)!\,(b-1)!}\\ &\leq (n-2)! \cdot \sum_{a=1}^{\infty} \frac{1}{(a-1)!} \cdot \sum_{b=1}^{\infty} \frac{1}{(b-1)!} = (n-2)! \cdot e^2.\end{aligned} \tag{2.5.3}$$

Next, let ε be an arbitrary (small) positive real number. Since

$$\sum_{a=1}^{\lfloor n/2\rfloor} \frac{1}{(a-1)!} \cdot \sum_{b=1}^{\lfloor n/2\rfloor} \frac{1}{(b-1)!} \leq \sum_{a+b\leq n} \frac{1}{(a-1)!\,(b-1)!},$$

there exists an $r_1 \in \mathbb{N}$ such that

$$(1-\varepsilon)e^2 \leq \sum_{a+b\leq n} \frac{1}{(a-1)!\,(b-1)!} \quad \text{for all} \quad n \geq r_1. \tag{2.5.4}$$

Since each of the $b-1$ factors of $q(n,a,b)$ tends to 1 as $n \to \infty$ while a and b are fixed, and since there are finitely many pairs $(a,b) \in \{1,\dots,r_1\}^2$, there exists an $r_2 \in \mathbb{N}$ such that

$$1-\varepsilon \leq q(n,a,b) \quad \text{for all } a \leq r_1,\ b \leq r_1 \text{ and } n \geq r_2. \tag{2.5.5}$$

By the previous achievements as indicated below, if n is an arbitrary integer greater than $r = \max\{r_1, r_2\}$, then

$$\begin{aligned}|\mathrm{SRectD}(n)| &\overset{(2.4.4)}{=} \sum_{a+b\leq n} |\mathrm{RPerm}(n;a,b)|\\ &\overset{(2.5.2)}{=} (n-2)! \sum_{a+b\leq n} \frac{q(n,a,b)}{(a-1)!\,(b-1)!} \geq (n-2)! \sum_{a+b\leq r_1} \frac{q(n,a,b)}{(a-1)!\,(b-1)!}\\ &\overset{(2.5.5)}{\geq} (n-2)! \sum_{a+b\leq r_1} \frac{1-\varepsilon}{(a-1)!\,(b-1)!} \overset{(2.5.4)}{\geq} (n-2)! \cdot (1-\varepsilon)^2 e^2.\end{aligned}$$

This and (2.5.3) imply Proposition 2.5.1, since $(1-\varepsilon)^2 \to 1$ as $\varepsilon \to 0$. $\square$

Now, we are in the position to formulate and prove our main result.

Theorem 2.5.2. *The number of (the isomorphism classes of) slim rectangular lattices of length n is asymptotically $(n-2)! \cdot e^2/2$, that is,*

$$\lim_{n\to\infty} \frac{|\mathrm{SRectL}(n)|}{(n-2)! \cdot e^2/2} = 1.$$

Proof. If we divide (2.4.5) by $(n-2)! \cdot e^2/2$, then the theorem follows from Proposition 2.5.1, provided we can show that

$$\lim_{n\to\infty} \frac{f(n)}{(n-2)!} = 0, \text{ where } f(n) = \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)|. \tag{2.5.6}$$

Hence, it suffices to deal with (2.5.6). In order to prove it, recall from Chowla, Herstein and Moore [5, Theorem 8] that

$$|\mathrm{Invl}(k)| \sim \frac{1}{\sqrt[4]{4e}} \cdot (k/e)^{k/2} \cdot e^{\sqrt{k}}. \tag{2.5.7}$$

Since $\sqrt{k} \le k/2$ for $k \ge 4$, (2.5.7) implies that

$$|\mathrm{Invl}(k)| \le k^{k/2}, \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{2.5.8}$$

Stirling's formula, $k! \sim \sqrt{2\pi k} \cdot (k/e)^k$, implies that

$$(k/e)^k \le k! \le (k/e)^{k+1}\ , \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{2.5.9}$$

Denote $n-2$ by m, and assume that m is sufficiently large. Besides (2.5.8) and (2.5.9), the following obvious estimates are also needed below. Since the sum of the $\binom{m}{i}$ is 2^m, we have that $\binom{n-a-1}{a-1} \le 2^m$. Since $|\mathrm{Invl}(k)|$ is clearly an increasing function of k, we obtain that $|\mathrm{Invl}(n-2a)| \le |\mathrm{Invl}(m)|$. Clearly, $m \cdot 2^m \le 2^m \cdot 2^m = 4^m$ and $\lfloor n/2 \rfloor \le m$. Let us compute:

$$\begin{aligned}
\frac{f(n)}{(n-2)!} &= \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n-a-1}{a-1} \cdot \frac{|\mathrm{Invl}(n-2a)|}{(n-2)!} \le \sum_{a=1}^{m} 2^m \frac{|\mathrm{Invl}(m)|}{m!} \\
&= m \cdot 2^m \cdot \frac{|\mathrm{Invl}(m)|}{m!} \overset{(2.5.8,\, 2.5.9)}{\le} m \cdot 2^m \cdot \frac{m^{m/2}}{(m/e)^m} \qquad (2.5.10) \\
&\le 4^m \cdot \frac{(\sqrt{m})^m}{(m/e)^m} = \frac{1}{\left(\frac{\sqrt{m}}{4e}\right)^m} \to 0, \quad \text{as } n \to \infty. \qquad (2.5.11)
\end{aligned}$$

This completes the proof. □

Remember that SSmodD(n) and SSmodL(n) denote the set of slim semimodular diagrams of length n and that of slim semimodular lattices of length n, respectively. In Czédli, Ozsvárt and Udvari [42, Proposition 7.1], it was proved that $|\mathrm{SSmodL}(n)| \sim n!/2$. This result, $(n-1)/n \sim 1$, Lemma 2.2.5 and Theorem 2.5.2 immediately yield the following statement.

Corollary 2.5.3.

$$\frac{|\mathrm{SRectD}(n)|}{|\mathrm{SSmodD}(n)|} \sim (e/n)^2 \quad \textit{and} \quad \frac{|\mathrm{SRectL}(n)|}{|\mathrm{SSmodL}(n)|} \sim (e/n)^2.$$

Next, we give the asymptotic number of slim patch lattices.

Proposition 2.5.4. *The number* $|\mathrm{SPatchL}(n)|$ *of* (*the isomorphism classes of*) *slim patch lattices of length* n *is asymptotically* $(n-2)!/2$.

Proof. That $|\mathrm{Invl}(n-2)|/((n-2)!) = |\mathrm{Invl}(m)|/(m!) \to 0$ as $n \to \infty$ follows from (2.5.10) and (2.5.11). This and Proposition 2.4.5 imply the statement. □

2.6 Results by computer algebra

Based on Propositions 2.4.2 and 2.4.4, $|\mathrm{SSmodD}(n)|$ and $|\mathrm{SSmodL}(n)|$ can easily be determined for $n \leq 1000$. Appropriate programs (Maple 5) are available from the website of the first author of [33]. The numbers in the first two rows of Table 2.1 are also given in https://oeis.org/A273596 and https://oeis.org/A273988, respectively. For much more extensive lists, see https://oeis.org/A273596/b273596.txt and https://oeis.org/A273988/b273988.txt. Our computer algebraic calculations show that $|1-|\mathrm{SPatchL}(n)|/((n-2)!/2)|$ and $|1/2-|\mathrm{SRectL}(n)|/|\mathrm{SRectD}(n)||$ are smaller than 10^{-40} for $n \in \{64, \ldots, 100, 200, 600, 1000\}$. This fact and Table 2.2 indicate (but do not prove) that the convergence in Proposition 2.5.4 is much faster than that in Proposition 2.5.1 and Theorem 2.5.2.

n	2	3	4	5	6	7	8	9	10	11	12
$\lvert\mathrm{SRectD}(n)\rvert$	1	3	9	32	139	729	4 515	32 336	263 205	2 401 183	24 275 037
$\lvert\mathrm{SRectL}(n)\rvert$	1	2	6	19	78	387	2 327	16 384	132 336	1 203 145	12 146 959
$\lvert\mathrm{SPatchL}(n)\rvert$	1	1	2	5	17	73	398	2 636	20 542	182 750	1 819 148

Table 2.1: Computational results for $2 \leq n \leq 12$

n	200	600	1000
$\lvert\mathrm{SRectD}(n)\rvert$	$1.4568041 \cdot 10^{371}$	$2.5975960 \cdot 10^{1403}$	$2.9732576 \cdot 10^{2562}$
$\lvert\mathrm{SRectL}(n)\rvert$	$7.2840205 \cdot 10^{370}$	$1.2987980 \cdot 10^{1403}$	$1.4866288 \cdot 10^{2562}$
$\lvert\mathrm{SPatchL}(n)\rvert$	$9.9077622 \cdot 10^{369}$	$1.7606738 \cdot 10^{1402}$	$2.0139503 \cdot 10^{2561}$
$\dfrac{\lvert\mathrm{SRectL}(n)\rvert}{(n-2)! \cdot e^2/2}$	0.99496227	0.99832914	0.99899847

Table 2.2: Computational results for $n \in \{200, 600, 1000\}$

Chapter 3

Small generating sets of lattices of quasiorders and transitive relations

3.1 Basic concepts and historical overview

Quasiorders, also known as preorders, on a set A form a complete lattice $\mathrm{Quo}(A)$. So do the transitive relations on A; their complete lattice is denoted by $\mathrm{Tran}(A)$. Similarly, $\mathrm{Equ}(A)$ will stand for the lattice of all equivalences on A. The natural involution, which maps a relation ρ to its inverse, $\rho^* := \rho^{-1} = \{(x, y) : (y, x) \in \rho\}$, is an automorphism of each of the three lattices mentioned above. If, besides arbitrary joins and meets, the involution is an operation of the structure, then we speak of the *complete involution lattices* $\mathrm{Quo}(A)$ and $\mathrm{Tran}(A)$. However, it would not be worth considering the involution on $\mathrm{Equ}(A)$, because it is the identity map.

As usual, Δ_A stands for the diagonal relation $\{(x, x) \mid x \in A\}$ on the set A.

For $a \neq b \in A$, let

$$\begin{aligned}\langle a, b\rangle^e &:= \{(a,b),(b,a)\} \cup \Delta_A \in \mathrm{Equ}(A),\\ \langle a, b\rangle &:= \{(a,b)\} \cup \Delta_A \in \mathrm{Quo}(A), \text{ and}\\ \{\!\{a, b\}\!\}^{\mathrm{tr}} &:= \{(a,b)\} \in \mathrm{Tran}(A);\end{aligned}$$

they are the smallest equivalence, the smallest quasiorder, and the smallest transitive relation, respectively, containing the ordered pair (a, b). While $\{\!\{a, b\}\!\}^{\mathrm{tr}}$ is always an atom of $\mathrm{Tran}(A)$ and all atoms of $\mathrm{Tran}(A)$ are of this form, $\langle a, b\rangle^e$ is an atom of $\mathrm{Equ}(A)$ iff $\langle a, b\rangle$ is an atom of $\mathrm{Quo}(A)$ iff $a \neq b$, and all atoms of $\mathrm{Equ}(A)$ and $\mathrm{Quo}(A)$ are of this form. Typically, we use the notation $\langle a, b\rangle$ only for $a \neq b$.

Unless otherwise stated, *generation* is understood in the complete sense. That is, for a subset X of $\mathrm{Equ}(A)$, $\mathrm{Quo}(A)$, or $\mathrm{Tran}(A)$, we say that X generates the complete (involution) lattice in question if the only complete sublattice (closed with respect to involution) including X is the whole lattice itself. For $k \in \mathbb{N} := \{1, 2, 3, \dots\}$, we say that a complete lattice L is *k-generated* if it can be generated by a k-element subset X; k-generated complete involution lattices are understood similarly. Since the involution commutes with infinitary joins and meets, we obtain easily that

$$\text{if a complete involution lattice } L \text{ is } k\text{-generated and } |L| \geq 2k, \text{ then the complete lattice we obtain from } L \text{ by disregarding the involution is } 2k\text{-generated.} \tag{3.1.1}$$

Note that when dealing with *finite* sets A or finite lattices, then the adjectives "complete" and "infinitary" are superfluous; this trivial fact will not be repeated all the time later.

If a complete lattice is generated by a four-element subset $X = \{x_1, x_2, x_3, x_4\}$ such that $x_1 < x_2$ but both $\{x_1, x_3, x_4\}$ and $\{x_2, x_3, x_4\}$ are antichains, then we say that this lattice is $(1 + 1 + 2)$-*generated.*

Next, we introduce the concept of *accessible cardinals.* Shortly saying, a cardinal κ is accessible if there is no inaccessible cardinal λ such that $\lambda \leq \kappa$. (So the adjective "accessible" in this chapter is not the opposite of "inaccessible".) Instead of recalling the concept of inaccessible cardinals from, say, the monograph of Levy [74], we define accessible cardinals directly. A cardinal κ is accessible if it is finite, or it is infinite and for every $\lambda \le \kappa$,

- either $\lambda \le 2^\mu$ for some cardinal $\mu < \lambda$,
- or there is a set $I = I(\lambda)$ of cardinals such that $\lambda \le \sum_{\mu \in I} \mu$, $|I| < \lambda$, and $\mu < \lambda$ for all $\mu \in I$.

In this chapter, all sets will be assumed to be of accessible cardinalities. As we mentioned in (1.0.2), it impossible to prove in ZFC that there are other cardinals. For more about other (that is, inaccessible) cardinals, the reader can resort to standard textbooks on set theory, for example, to Levy [74, pages 138–141].

3.2 The history of motivating results

3.2.1 Related results from the twentieth century

In 1975 and 1977, Strietz [83] and [84] proved that for a finite set A with at least 3 elements, the lattice $\mathrm{Equ}(A)$ of all equivalences on a set A, the *equivalence lattice* of A for short, is four-generated. (If $|A| \le 2$, then $|\mathrm{Equ}(A)| < 4$, so this is not an interesting case.) Furthermore, these two papers prove also that, for $10 \le [A| \in \mathbb{N}$, $\mathrm{Equ}(A)$ is $(1+1+2)$-generated and, for $3 \le |A| \in \mathbb{N}$, $\mathrm{Equ}(A)$ is not three-generated.

In 1983, Zádori [87] gave a new proof of Strietz's above-mentioned results; in fact, he proved a stronger statement instead of the second result by showing that $\mathrm{Equ}(A)$ is $(1+1+2)$-generated for $7 \le [A| \in \mathbb{N}$. His proofs are visual, and they are simpler and more powerful than Strietz's ones. It is Zádori's idea that many of the subsequent proofs in this topic develop further.

To formulate the next result in the chronological order, define the cardinal number $\beth_n$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ by induction: $\beth_0 := \aleph_0$ and $\beth_{n+1} := 2^{\beth_n}$. In their 1996 paper, Chajda and Czédli [4] proved that if A is a set with at least two elements such that $|A| \le \beth_n$ for some $n \in \mathbb{N}_0$, then $\mathrm{Quo}(A)$ is three-generated as a complete involution lattice, and so it is six-generated as a complete lattice by

(3.1.1). Their result was soon generalized by Takách [85], appeared in 1996; this paper replaces the assumption "$|A| \leq \beth_n$ for some $n \in \mathbb{N}_0$" with the much less restrictive assumption that "$|A|$ is an accessible cardinal". The year 1996 brought some progress for the equivalence lattices, too: Czédli [7] proved that $\mathrm{Equ}(A)$ is four-generated, provided that $3 \leq |A|$ is an accessible cardinal. In the same year, Czédli [8] proved that if B is a countably infinite set, then $\mathrm{Equ}(B)$ has a four-generated sublattice S such that S contains all atoms of $\mathrm{Equ}(B)$; here "generated" has its usual meaning based on the binary join and the binary meet. Finally, in his 1999 paper, Czédli [9] proved that if A is a set with accessible cardinality and $7 \leq |A|$, then $\mathrm{Equ}(A)$ is $(1+1+2)$-generated.

3.2.2 Results from 2015–2016

The first results in the twenty-first century were proved by Dolgos [48], an M.Sc. student of that time supervised by Miklós Maróti. Subsequent results came so soon that, exceptionally, we give the precise dates of the relevant papers. These dates are taken from the "Submitted" or "Received" lines from the published papers and from the "declaration page" (last page) of Dolgos [48].

Dolgos [48], submitted on May 16, 2015, proved that for a set A with $2 \leq |A| \leq \aleph_0$, the quasiorder lattice $\mathrm{Quo}(A)$ is five-generated while the lattice $\mathrm{Tran}(A)$ of transitive relations of A is eight-generated.

We proved in [72], submitted on October 29, 2015, that if A is a set with at least two elements such that $|A|$ is an accessible cardinal, then $\mathrm{Quo}(A)$ is five-generated as a complete lattice; see Theorem 3.3.1 later. This result improves both Takách's "six-generated" from 1996 and Dolgos's "$\leq \aleph_0$".

Czédli [19], submitted on November 6, 2015, observed that in some cases, even the number 5 can be reduced. Namely, he proved that if

$$|A| \in \{n \in \mathbb{N} : n \geq 11\} \cup \{2, 3, 5, 7, 9, \aleph_0\}, \tag{3.2.1}$$

then $\mathrm{Quo}(A)$ has a four-generated sublattice (in the ordinary, non-complete sense) that contains all atoms of $\mathrm{Quo}(A)$. Therefore, (3.2.1) implies that $\mathrm{Quo}(A)$ is four-generated as a complete lattice. Moreover, it is implicit in Czédli [19] that whenever

$|A| \geq 3$ and S is a 3-generated sublattice of $\mathrm{Quo}(A)$ in the complete sense, then S cannot contain all atoms of $\mathrm{Quo}(A)$ and, in particular, S cannot be $\mathrm{Quo}(A)$.

Finally, in our joint paper [37], submitted on October 4, 2016, we managed to combine the merits of [72] and Czédli [19] by proving that $\mathrm{Quo}(A)$ is four-generated as a complete lattice for all non-singleton sets A with accessible cardinalities except for $|A| = 4$. [37] contained some other results on $\mathrm{Quo}(A)$, too; see Theorem 3.4.9 later. Furthermore, [37] improved Dolgos's result on $\mathrm{Tran}(A)$ by reducing the number of generators by 2 and allowing that A is of an accessible cardinality not just at most $\aleph_0$; see Lemma 3.4.10.

The most recent related results will be surveyed later, in Section 3.5.

3.2.3 The aim of (this) Chapter 3

We are going to present the results and the proofs published in [72] and [37]. This target needs some explanation, as we know from Subsection 3.2.2 that [37] supersedes [72] in several aspects.

According to its title, [37] gives a *concise approach*. In this case, conciseness means that [37] is far from being self-contained. Although the proofs given in [37] are short, sometimes very short, these proofs rely on nontrivial earlier constructions mentioned in Subsection 3.2.1. To give *self-contained* proofs of the theorems of [37], one should add several additional pages to each of these proofs; about dozen pages to a proof dealing with all accessible cardinals.

As opposed to [37], [72] is a self-contained and single-authored paper. A statement of [72], which is Lemma 3.3.2 here, was needed in [37]. These facts explain that, in addition to [37], [72] is also included in the dissertation.

3.3 Quasiorder lattices are five-generated

Apart from introductory features and the fact that now Lemma 3.3.2 is an explicit statement rather than an implicit one hidden in a proof, (this) Section 3.3 is almost the same as [72]. We are going to prove the following theorem.

Theorem 3.3.1. *Let A be a set with at least three elements.*

(i) *If $|A|$ is an accessible cardinal, then* $\mathrm{Quo}(A)$ *is five-generated as a complete lattice.*

(ii) *If $\aleph_0 \le |A| \le 2^{\aleph_0}$, then* $\mathrm{Quo}(A)$ *is five-generated as a complete lattice.*

Of course, part (ii) is a particular case of part (i). While the proof of (i) relies heavily on Czédli [7], which is a long paper, the proof of (ii) is self-contained. Even (ii) strengthens the corresponding result of Dolgos [48]. Developing the proof of (ii) to a self-contained proof of (i) would probably be possible, but this is not targeted.

First of all, we prove the following lemma; it will be needed also in (the next) Section 3.4. Following the traditions of lattice theory, $\subset$ stands for proper set inclusion, that is, $X \subset Y \iff (X \subseteq Y$ and $X \neq Y)$.

Lemma 3.3.2. *If $3 \le |A|$ and L is a complete sublattice of* $\mathrm{Quo}(A)$ *such that* $\mathrm{Equ}(A) \subset L$*, then* $L = \mathrm{Quo}(A)$.

Proof. For the sake of contradiction, suppose that $L \neq \mathrm{Quo}(A)$. We know that $\varrho = \bigvee\{\langle x, y\rangle : (x, y) \in \varrho\}$ for every $\varrho \in \mathrm{Quo}(A)$. Hence, if L contained all the atoms of $\mathrm{Quo}(A)$, that is, all $\langle x, y\rangle$ with $x \neq y \in A$, then L would equal $\mathrm{Quo}(A)$ and this would be the required contradiction to complete the proof.

Observe that, for any $p, q, x \in A$ such that $|\{p, q, x\}| = 3$,

$$\langle p, x\rangle = \langle p, x\rangle^e \wedge (\langle p, q\rangle \vee \langle q, x\rangle^e) \text{ and}$$
$$\langle x, q\rangle = \langle x, q\rangle^e \wedge (\langle x, p\rangle^e \vee \langle p, q\rangle).$$

Thus, since all equivalences belong to L, we obtain that, for pairwise distinct $p, q, x \in A$,

$$\langle p, q\rangle \in L \Longrightarrow \big(\langle p, x\rangle \in L \text{ and } \langle x, q\rangle \in L\big). \tag{3.3.1}$$

Next, we show the following rule, in which a and b denote distinct elements of A.

$$\begin{aligned}&\text{If } \langle a, b\rangle \in L \text{ and } c \in A \setminus \{a, b\}, \text{ then } \langle x, y\rangle \in L\\ &\text{for all } x, y \in \{a, b, c\} \text{ such that } x \neq y.\end{aligned} \tag{3.3.2}$$

Indeed, (3.3.1) applied to $\langle a, b\rangle \in L$ yields that $\langle a, c\rangle, \langle c, b\rangle \in L$. This allows us to apply (3.3.1) to $\langle a, c\rangle \in L$ and $\langle c, b\rangle \in L$ to obtain that $\langle b, c\rangle \in L$ and $\langle c, a\rangle \in L$, respectively. Finally, (3.3.1) applied to $\langle b, c\rangle$ implies that $\langle b, a\rangle \in L$, proving (3.3.2).

Pick a $\rho \in L \setminus \mathrm{Equ}(A)$. As ρ is not symmetric, there is a pair $(a, b) \in \rho$ such that $(b, a) \notin \rho$. Since $\langle a, b\rangle^e \in \mathrm{Equ}(A) \subseteq L$, we obtain that $\langle a, b\rangle = \rho \wedge \langle a, b\rangle^e \in L$. Let $\langle p, q\rangle$ be an arbitrary atom of $\mathrm{Quo}(A)$. There are two cases. First, if $\{p, q\} \cap \{a, b\} \neq \varnothing$, then (3.3.2) immediately implies that $\langle p, q\rangle \in L$. (Here we exploit that $|A| \geq 3$.) Second, assume that $\{p, q\} \cap \{a, b\} = \varnothing$. Letting $c := q$, (3.3.2) implies that $\langle a, q\rangle \in L$. From now on, b and q play the same role. Thus, we can assume that $q = b$, whereby $\langle p, q\rangle \in L$ follows by the first case. We have shown that L contains all atoms of $\mathrm{Quo}(A)$, as required. $\square$

Proof of part (i) *of Theorem 3.3.1.* We know from Czédli [7] that $\mathrm{Equ}(A)$ is generated by a four-element set X as a complete lattice. Pick a quasiorder $\rho \in \mathrm{Quo}(A) \setminus \mathrm{Equ}(A)$. By Lemma 3.3.2, $\mathrm{Quo}(A)$ is generated by its five-element subset $X \cup \{\rho\}$ as a complete lattice. $\square$

Next, we give a self-contained proof for part (ii).

Proof of part (ii) *of Theorem 3.3.1.* Let $A_0 = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\}$. The subsets $\{a_0, a_1, a_2, \dots\}$ and $\{b_0, b_1, b_2, \dots\}$ are called *rows*, the a-row and the b-row, respectively. For a technical reason, which will be clear soon, we denote a_{3i+10} and b_{3i+11} by e_i and e'_i, respectively; these elements will be black-filled in our figures. In Figure 3.1, e_i and e'_i are connected by a dotted edge whose role will be explained in due time. Furthermore, sometimes we even use the notation (e_{-1}, e'_{-1}) for (a_7, b_8) in our computations.

We are going to define five quasiorders on A_0, denoted by α_0^0, α_1^0, α_2^0, β^0, and β_*^0; in fact, the first three will be equivalences. (The upper subscripts 0 refer to the fact that they are defined on A_0; later we will also introduce α_0, α_1, α_2, β, and β_*, which will be defined on a larger set A.) Besides (or instead of) their formal definition below, the reader is advised to understand them from Figure 3.1. For

$i \in \{0,1,2\}$, we define α_i^0 by the corresponding partition

$$\begin{aligned} &\big\{\{a_{3k+i} : k \in \mathbb{Z}\}\big\} \cup \big\{\{a_{3k+i+1}, a_{3k+i+2}\} : k \in \mathbb{Z}\big\} \cup \\ &\quad \big\{\{b_{3k+i+1} : k \in \mathbb{Z}\}\big\} \cup \big\{\{b_{3k+i+1+1}, b_{3k+i+1+2}\} : k \in \mathbb{Z}\big\}. \end{aligned} \tag{3.3.3}$$

Also, let

$$\beta^0 = \langle a_0, a_2\rangle \vee \langle b_0, b_2\rangle \vee \langle a_4, b_5\rangle \vee \langle b_8, a_7\rangle$$

and, finally, let

$$\beta_*^0 = (\beta^0)^{-1}.$$

For $\delta \in \{\alpha_0^0, \alpha_1^0, \alpha_2^0, \beta^0\}$ and $x, y \in A_0$, we have $(x,y) \in \delta$ iff the vertices x and y can be connected by a δ-colored directed path in Figure 3.1; this is the meaning of the figure. (Almost all edges but (a_0, a_2), (b_0, b_2), (a_4, b_5) and (b_8, a_7) are directed in both ways.) Since β_*^0 is the inverse of β^0, the β_*^0-colored edges are not indicated. At present, the dotted edges belong neither to β^0, nor to β_*^0; however, some of these edges (directed upwards or downwards) will be added to β^0 or β_*^0 at a later stage of the construction.

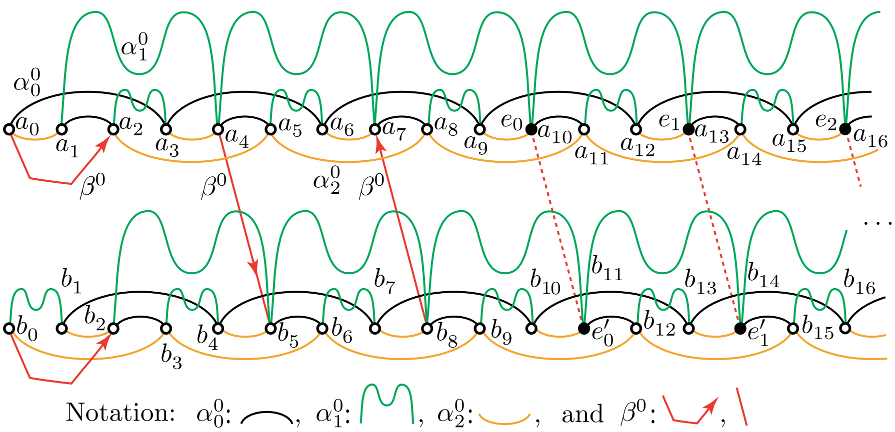


Figure 3.1: Quasiorders on A_0

Later we will need $\kappa \le 2^{\aleph_0}$ copies of A_0. Note that Dolgos [48] used only the upper row of a single copy of A_0. When we work in a single row, we often follow his arguments.

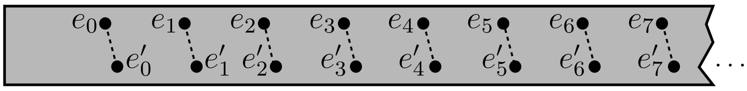


Figure 3.2: A_0 in a concise form

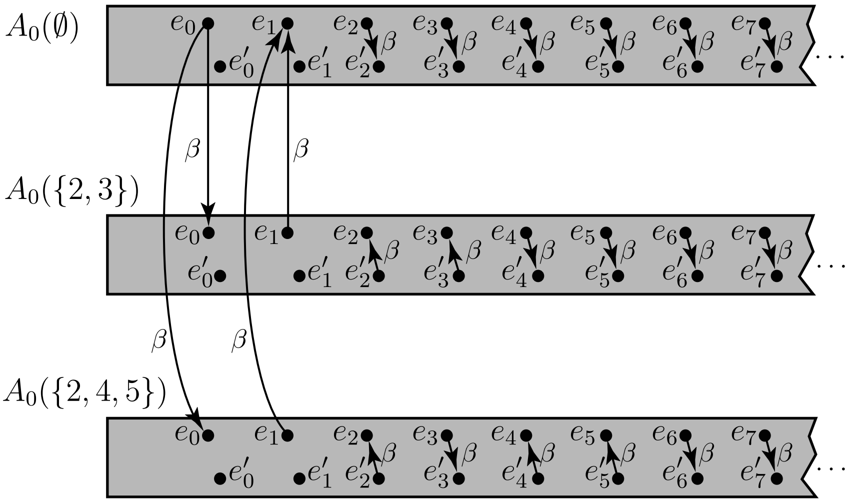


Figure 3.3: A part of $\beta \in \mathrm{Quo}(A)$ if $H = \big\{\varnothing, \{2,3\}, \{2,4,5\}\big\}$

Starting from the $\aleph_0$-sized graph A_0, we are going to define a more involved graph. (Note at this point that our graphs and their vertex sets are usually denoted in the same way.) Let κ be an arbitrary cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Let $I = \{2, 3, 4, \dots\}$, and take a subset H of $\mathcal{P}(I)$ such that $|H| = \kappa$. For simplicity, assume that $\varnothing \in H$. Next, for $U \in H$, we modify the graph A_0 to obtain a colored graph $A_0(U)$ with vertex set $\{a_0(U), b_0(U), a_1(U), b_1(U), a_2(U), b_2(U), \dots\}$ as follows. When it is not confusing, we drop the parameter U and simply write a_0, b_0, a_1, b_1, $\dots$. In particular, $e_i(U)$ and $e'_i(U)$ are denoted by e_i and e'_i in our figures. However, $A_0(U)$ is given in the figures and it refers to all these elements. Of course, we assume that $A_0(U) \cap A_0(V) = \varnothing$ whenever $U \neq V \in H$. Now, to obtain $A_0(U)$ from A_0, we replace the dotted edges with "real" edges (e'_i, e_i) for $i \in U$ and (e_i, e'_i) for $i \in I \setminus U$. For $U \in H$, the set $A_0(U)$ is called a box. In Figure 3.3, boxes are grey. For example, the lower grey box in our figure is $A_0(\{2, 4, 5\})$.

Now, we are in the position to define a new colored graph, A, as follows. Its vertex set is the union of the disjoint sets $A_0(U)$, that is, $A = \{A_0(U) : U \in H\}$.

Besides that all the previous edges are preserved, we add the β-colored directed edges $(e_0(\varnothing), e_0(U))$ and $(e_1(U), e_1(\varnothing))$ for all $U \in H$. In this way, we obtain our new graph, A; see Figure 3.3 for the particular case $H = \big\{\varnothing, \{2,3\}, \{2,4,5\}\big\}$. As before, for $\delta \in \{\alpha_0, \alpha_1, \alpha_2, \beta\}$, we let $(x, y) \in \delta$ iff the vertices x and y can be connected by a δ-colored directed path in the graph A. In this way, we have defined four quasiorders, α_0, α_1, α_2, and β on A; the fifth one is $\beta_* := \beta^{-1}$. Notice that if $\delta, \varepsilon \in \{\alpha_0, \alpha_1, \alpha_2, \beta, \beta_*\}$ and $\delta \neq \varepsilon$, then $\delta \wedge \varepsilon = \Delta_A$. Notice also that all the α_i are *row-preserving*; this means that whenever $(x, y) \in \alpha_i$ for some $i \in \{0, 1, 2\}$, then there is a unique $U \in H$ such that either $x, y \in \{a_0(U), a_1(U), \dots\}$, or $x, y \in \{b_0(U), b_1(U), \dots\}$. For an equivalence ϱ on A and $x \in A$, the ϱ-block $\{y \in A : (x, y) \in \varrho\}$ will be denoted by x/ϱ.

Now, let L denote the smallest complete sublattice of $\mathrm{Quo}(A)$ such that $\{\alpha_0, \alpha_1, \alpha_2, \beta, \beta_*\} \subseteq L$; our task is to show that $L = \mathrm{Quo}(A)$. As it was pointed out at the beginning of the previous proof, it suffices to show that L contains all atoms $\langle x, y\rangle$, where $x \neq y \in A$.

For $U \in H$ and distinct $x, y \in A(U)$, we introduce the notation

$$\langle x, y\rangle_H := \bigvee_{V \in H} \langle x(V), y(V)\rangle.$$

Let us emphasize that this notation is only permitted if x and y belong to the same copy of A_0, that is, to the same grey box in Figure 3.3.

We claim that

$$\langle a_3, a_2\rangle_H = (\alpha_0 \vee \beta) \wedge \alpha_1 \in L. \tag{3.3.4}$$

To show the "$\supseteq$" inclusion, assume that $x \neq y$ and $(x, y) \in (\alpha_0 \vee \beta) \wedge \alpha_1$. Then $(x, y) \in \alpha_1$ and there is a *shortest* path P from x to y in the graph whose edges are colored with α_0 and β. Since α_1 is row-preserving, x and y belong to the same row. Suppose, for a contradiction, that this row is $\{b_0(U), b_1(U), \dots\}$. If P goes entirely within this row, then it is clear by definitions, or by our figures, that either $(x, y) \in \alpha_0 \cup \beta$, or $(x, y) = (b_0(U), b_3(U))$. In both cases, $(x, y) \notin \alpha_1$, which is a contradiction. On the other hand, if P leaves this b-row, then it arrives at some $e_i(V)$ in the next step, where $V \in H$ and $i \in \{-1, 2, 3, 4, \dots\}$. But the only new vertex we can go from $e_i(V)$ via an $(\alpha_0 \cup \beta)$-colored path is the neighboring vertex to the right of $e_i(V)$. Then, in the next step of the path, we must turn back. This contradicts the minimality of P. Therefore, x and y belong to an a-row, $\{a_0(U), a_1(U), \dots\}$. Observe that our path P lies entirely in the same a-row. Really, if not, then P contains a β-colored edge $(e_i(U), e'_i(U))$, $(e_0(\varnothing), e_0(U))$ or $(e_1(U), e_1(\varnothing))$. However, all $e_i(U)$ and all $e'_i(U)$ belong to distinct two-element α_0-classes. All of these α_0-classes have the property that either at most one β-colored edge's endpoint belongs to the class or if two β-colored edge's endpoints are in the class, then these edges are directed in the same way. Hence, P can not leave this latter row, which is a contradiction. Thus, P lies in the a-row containing x and y. Since $\alpha_0 \wedge \alpha_1 = \Delta_A = \alpha_1 \wedge \beta$, both colors, α_0 and β, occur in our path P. Since P is the shortest path and the a-row of x and y contains only one β-colored edge, P contains exactly one β-colored edge, $(a_0(U), a_2(U))$. Therefore, $x \in a_0(U)/\alpha_0$ and $y \in a_2(U)/\alpha_0$. Using (3.3.3), we have that $x \in \{a_{3k}(U) : k \in \mathbb{Z}\}$ and $y \in \{a_1(U), a_2(U)\}$. Thus, taking $(x, y) \in \alpha_1$ into account, $(x, y) = (a_3(U), a_2(U)) \in \langle a_3, a_2 \rangle_H$. This proves the "$\supseteq$" inclusion in (3.3.4); the reverse inclusion is obvious. This proves (3.3.4).

Next, we assert that

$$\langle a_0, a_2 \rangle_H = (\langle a_3, a_2 \rangle_H \vee \alpha_0) \wedge \beta \in L. \tag{3.3.5}$$

To see this, let $(x, y) \in (\langle a_3, a_2 \rangle_H \vee \alpha_0) \wedge \beta$ such that $x \neq y$. Since both $\langle a_3, a_2 \rangle_H$ and α_0 are row-preserving, x and y belong to the same row. In the shortest path connecting x and y, both of the colors α_0 and $\langle a_3, a_2 \rangle_H$ occur, because the intersections of these colors with β is Δ_A. The presence of $\langle a_3, a_2 \rangle_H$ yields that we are in an a-row, say, in $A_0(U)$. Since the restriction of β to this a-row is $\langle a_0(U), a_2(U) \rangle$, we obtain that $(x, y) = (a_0(U), a_2(U)) \in \langle a_0, a_2 \rangle_H$. This proves the "$\supseteq$" inclusion in (3.3.5), while the converse inclusion is evident.

Next, we show that

$$\langle b_0, b_1 \rangle_H = (\alpha_2 \vee \beta) \wedge \alpha_1 \in L. \tag{3.3.6}$$

Assume that $(x, y) \in (\alpha_2 \vee \beta) \wedge \alpha_1$ and $x \neq y$. Again, since α_1 is row-preserving and $\alpha_2 \wedge \alpha_1 = \Delta_A = \beta \wedge \alpha_1$, x and y are in the same row and the shortest $(\alpha_2 \cup \beta)$-path

P connecting them contains both colors, α_2 and β. As in the argument verifying (3.3.4), exactly one edge of this path is β-colored and P does not leave the row of x and y. Suppose, for a contradiction, that we are in an a-row. It follows easily from definitions that either $(x, y) \in \alpha_2 \cup \beta$, or $x \in \{a_0, a_1\}$ and $y \in \{a_{3k+2} : k \in \mathbb{N}_0\}$, but this contradicts $(x, y) \in \alpha_1$. Hence, x and y are in a b-row. So the only β-colored edge in P is $(b_0(U), b_2(U))$. After its β-colored edge, P consists of at most one edge. This gives that $y \in \{b_1(U), b_2(U)\}$. There can be arbitrary many α_2-colored edges before the only β-colored one, but we have that $x \in \{b_{3k} : k \in \mathbb{N}_0\}$. Taking $(x, y) \in \alpha_1$ into account, we conclude that $(x, y) = (b_0(U), b_1(U)) \in \langle b_0, b_1\rangle_H$, as required. The converse inclusion is obvious, so we have proved (3.3.6).

Similarly to (3.3.5), we obtain the following containment easily:

$$\langle b_0, b_2\rangle_H = (\langle b_0, b_1\rangle_H \vee \alpha_2) \wedge \beta \in L. \tag{3.3.7}$$

Since the involutory automorphism $L \to L$, defined by $\varrho \mapsto \varrho^{-1}$, maps β to β_*, it follows that L is closed with respect to this automorphism, that is, for all $x, y \in A$, $U \in H$, and $u, v \in A_0(U)$,

$$\langle x, y\rangle \in L \iff \langle y, x\rangle \in L \quad \text{and} \quad \langle u, v\rangle_H \in L \iff \langle v, u\rangle_H \in L. \tag{3.3.8}$$

Combining (3.3.5) and (3.3.7) with (3.3.8), we obtain that $\langle a_2, a_0\rangle_H \in L$ and $\langle b_2, b_0\rangle_H \in L$. For a subset X of $\mathrm{Quo}(A)$, the smallest complete sublattice including X will be denoted by $[X]$. Our next task is to show that, for all $k \in \mathbb{N}_0$,

$$\langle a_k, a_{k+1}\rangle_H \in \big[\langle a_k, a_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{3.3.9}$$

Observe that, for every $U \in H$, there exists a unique $i \in \{0, 1, 2\}$ such that the pair $(a_{k+1}(U), a_{k+2}(U))$ is in α_i, and this i depends only on k but not on U. As it is clear from definitions, for all $s, t, j \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$,

$$(a_s, a_t) \in \alpha_i \iff (a_{s+j}, a_{t+j}) \in \alpha_{i+j}, \tag{3.3.10}$$

where the addition in the subscript of α is understood modulo 3. This allows us to assume that i above is 0, that is, $(a_{k+1}(U), a_{k+2}(U)) \in \alpha_0$ for all $U \in H$. This means that $k \equiv 3 \pmod 3$.

To prove (3.3.9), it suffices to show that

$$\langle a_k, a_{k+1}\rangle_H = (\alpha_0 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_2. \tag{3.3.11}$$

The "$\subseteq$ " inclusion is obvious. To verify the reverse inclusion, assume that $(x, y) \in (\alpha_0 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_2$. Since α_2 is row-preserving, there is a $U \in H$ such that x and y are in the same row of $A_0(U)$. Using that $\alpha_0 \wedge \alpha_2 = \Delta_A$, every $(\alpha_0 \cup \langle a_k, a_{k+2}\rangle_H)$-path P from x to y must contain an $\langle a_k, a_{k+2}\rangle_H$-colored edge. So, since α_0 is also row-preserving, both x and y are in the a-row of $A_0(U)$. Let P above be a shortest path, then it contains an $\langle a_k, a_{k+2}\rangle_H$-colored edge only once. Thinking of the segments of P after this edge, it follows that $y \in \{a_{k+1}(U), a_{k+2}(U)\}$, while the segment before this edge yields that $x \in \{a_i(U) : i \equiv 0 \pmod 3\}$. Now the definition of α_2 gives that $(x, y) = (a_k(U), a_{k+1}(U)) \in \langle a_k, a_{k+1}\rangle_H$, proving (3.3.9).

Since $\langle a_{k+1}, a_{k+2}\rangle_H = (\alpha_2 \vee \langle a_k, a_{k+2}\rangle_H) \wedge \alpha_0$ follows basically in the same way as (3.3.11), we obtain that

$$\langle a_{k+1}, a_{k+2}\rangle_H \in \big[\langle a_k, a_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{3.3.12}$$

Similarly, we obtain $\langle a_{k+2}, a_{k+3}\rangle_H = \big(\alpha_0 \vee \langle a_{k+2}, a_k\rangle_H\big) \wedge \alpha_1$, whence

$$\langle a_{k+2}, a_{k+3}\rangle_H \in \big[\langle a_{k+2}, a_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \tag{3.3.13}$$

Using the rule

$$(b_{s+1}, b_{t+1}) \in \alpha_i \iff (a_s, a_t) \in \alpha_i, \tag{3.3.14}$$

one concludes easily from (3.3.9), (3.3.12), and (3.3.13) that

$$\begin{aligned} \langle b_k, b_{k+1}\rangle_H &\in \big[\langle b_k, b_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big], \\ \langle b_{k+1}, b_{k+2}\rangle_H &\in \big[\langle b_k, b_{k+2}\rangle_H, \alpha_0, \alpha_1, \alpha_2\big], \text{ and} \\ \langle b_{k+2}, b_{k+3}\rangle_H &\in \big[\langle b_{k+2}, b_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big]. \end{aligned} \tag{3.3.15}$$

If we combine the generators occurring in (3.3.9), (3.3.12), and (3.3.13), then we obtain a larger subset of $\mathrm{Quo}(A)$ that is closed with respect to the involutory automorphism ϱ mentioned right after (3.3.7). Therefore, (3.3.9), (3.3.12), (3.3.13), and (3.3.15) yield that

$$\begin{aligned}
&\big\{\langle a_{k+1}, a_k\rangle_H, \langle a_{k+2}, a_{k+1}\rangle_H, \langle a_{k+3}, a_{k+2}\rangle_H, \langle a_k, a_{k+1}\rangle_H,\\
&\ \langle a_{k+1}, a_{k+2}\rangle_H, \langle a_{k+2}, a_{k+3}\rangle_H, \langle b_{k+1}, b_k\rangle_H, \langle b_{k+2}, b_{k+1}\rangle_H,\\
&\ \langle b_{k+3}, b_{k+2}\rangle_H, \langle b_k, b_{k+1}\rangle_H, \langle b_{k+1}, b_{k+2}\rangle_H, \langle b_{k+2}, b_{k+3}\rangle_H\big\}\\
&\ \subseteq \big[\langle a_k, a_{k+2}\rangle_H, \langle a_{k+2}, a_k\rangle_H, \langle b_k, b_{k+2}\rangle_H,\\
&\qquad\qquad\qquad \langle b_{k+2}, b_k\rangle_H, \alpha_0, \alpha_1, \alpha_2\big] =: \widehat{L}.
\end{aligned} \tag{3.3.16}$$

Here $\widehat{L}$ denotes the sublattice on the right of "$\subseteq$". We say that two sequences, $(x = p_0, p_1, \ldots, p_k = y)$ and $(x = q_0, q_1, \ldots, q_n = y)$, are *internally disjoint sequences from x to y* if $\{p_1, \ldots, p_{k-1}\} \cap \{q_1, \ldots, q_{n-1}\} = \varnothing$. The following lemma is straightforward.

Lemma 3.3.3. *If $(x = p_0, p_1, \ldots, p_k = y)$ and $(x = q_0, q_1, \ldots, q_n = y)$ are internally disjoint sequences from x to y, then*

$$(\langle p_0, p_1\rangle \vee \cdots \vee \langle p_{k-1}, p_k\rangle) \wedge (\langle q_0, q_1\rangle \vee \cdots \vee \langle q_{n-1}, q_n\rangle) = \langle x, y\rangle.$$

We claim that

$$\big\{\langle a_{k+1}, a_{k+3}\rangle_H, \langle b_{k+1}, b_{k+3}\rangle_H\big\} \subseteq \widehat{L}.$$

To see this, consider any $U \in H$ and the equivalence α_i with $(a_k(U), a_{k+3}(U)) \in \alpha_i$. As usual, (3.3.10) allows us to assume that $i = 0$, and $\langle a_k, a_{k+3}\rangle_H = (\langle a_k, a_{k+2}\rangle_H \vee \langle a_{k+2}, a_{k+3}\rangle_H) \wedge \alpha_0$ follows easily. So, according to (3.3.13),

$$\langle a_k, a_{k+3}\rangle_H \in \widehat{L}. \tag{3.3.17}$$

For every $U \in H$, Lemma 3.3.3 yields that

$$\begin{aligned}
\langle a_{k+1}(U), &a_{k+3}(U)\rangle\\
&= (\langle a_{k+1}(U), a_k(U)\rangle \vee \langle a_k(U), a_{k+3}(U)\rangle)\\
&\quad \wedge (\langle a_{k+1}(U), a_{k+2}(U)\rangle \vee \langle a_{k+2}(U), a_{k+3}(U)\rangle).
\end{aligned} \tag{3.3.18}$$

Since all the atoms occurring in (3.3.18) are row-preserving, we conclude that

$$\langle a_{k+1}, a_{k+3}\rangle_H =$$
$$(\langle a_{k+1}, a_k\rangle_H \vee \langle a_k, a_{k+3}\rangle_H)\wedge(\langle a_{k+1}, a_{k+2}\rangle_H \vee \langle a_{k+2}, a_{k+3}\rangle_H).$$

Hence, using (3.3.16) and (3.3.17) and (3.3.14), which says that the a-rows and b-rows play similar roles, we obtain that

$$\big\{\langle a_{k+1}, a_{k+3}\rangle_H, \langle b_{k+1}, b_{k+3}\rangle_H\big\} \subseteq \widehat{L}. \tag{3.3.19}$$

Combining $\widehat{L} \subseteq L$, (3.3.5), (3.3.7), (3.3.8), (3.3.16), and (3.3.19), we obtain that, for all $i, j \in \mathbb{N}_0$,

$$|i-j| \in \{1,2\} \Longrightarrow \big\{\langle a_i, a_j\rangle_H, \langle b_i, b_j\rangle_H\big\} \subseteq L. \tag{3.3.20}$$

Next, let $|i-j| > 2$. In the computation below, (3.3.8) allows us to assume, without loss of generality, that $i < j$. If $j - i$ is even, then

$$(a_i, a_{i+2}, a_{i+4}, \ldots, a_{j-2}, a_j) \text{ and}$$
$$(a_i, a_{i+1}, a_{i+3}, a_{i+5}, \ldots, a_{j-5}, a_{j-3}, a_{j-1}, a_j)$$

are internally disjoint sequences from a_i to a_j in A_0. So, Lemma 3.3.3 and (3.3.20) give that

$$\langle a_i, a_j\rangle_H \quad \text{and} \quad \langle b_i, b_j\rangle_H \quad \text{belong to} \quad L$$

in this case. The same holds for $j - i$ being odd, because then

$$(a_i, a_{i+1}, a_{i+3}, \ldots, a_{j-2}, a_j) \text{ and } (a_i, a_{i+2}, a_{i+4}, a_{j-3}, a_{j-1}, a_j)$$

are internally disjoint. Therefore,

$$\text{if } x, y \in A \text{ are in the same row, then } \langle x, y\rangle_H \in L. \tag{3.3.21}$$

As a first step to go beyond the limits of a single row, we claim that

$$\begin{aligned}\langle a_5, b_6\rangle_H &= \big(\langle a_5, a_4\rangle_H \vee \beta \vee \langle b_5, b_6\rangle_H\big) \wedge \big(\langle a_5, a_7\rangle_H \vee \beta_* \vee \langle b_8, b_6\rangle_H\big),\\ \langle a_6, b_7\rangle_H &= \big(\langle a_6, a_4\rangle_H \vee \beta \vee \langle b_5, b_7\rangle_H\big) \wedge \big(\langle a_6, a_7\rangle_H \vee \beta_* \vee \langle b_8, b_7\rangle_H\big).\end{aligned} \tag{3.3.22}$$

We only deal with the first equality, because the second one is analogous. We say that a β- or β_*-colored edge is *far on the right* if both of its endpoints belong to the set:

$$\{e_i(U), e'_i(U) \mid i \in \mathbb{N}_0,\ U \in H\}.$$

Observe that

$$\begin{aligned}\langle a_5, a_4\rangle_H \vee \beta \vee \langle b_5, b_6\rangle_H = \bigcup_{U\in H} \{&(a_0(U), a_2(U)), (b_0(U), b_2(U)),\\ &(a_4(U), b_5(U)), (b_8(U), a_7(U)), (a_5(U), a_4(U)),\\ &(b_5(U), b_6(U)), (a_5(U), b_5(U)), (a_4(U), b_6(U)),\\ &(a_5(U), b_6(U))\} \cup \{\text{some edges far on the right}\}.\end{aligned} \tag{3.3.23}$$

Similarly,

$$\begin{aligned}\langle a_5, a_7\rangle_H \vee \beta_* \vee \langle b_8, b_6\rangle_H = \bigcup_{U\in H} \{&(a_2(U), a_0(U)), (b_2(U), b_0(U)),\\ &(b_5(U), a_4(U)), (a_7(U), b_8(U)), (a_5(U), a_7(U)),\\ &(b_8(U), b_6(U)), (a_5(U), b_8(U)), (a_7(U), b_6(U)),\\ &(a_5(U), b_6(U))\} \cup \{\text{some edges far on the right}\}.\end{aligned} \tag{3.3.24}$$

By our construction, no edge far on the right occurs both in (3.3.23) and (3.3.24). Thus, we obtain (3.3.22).

Now, we are in the position to fully extend the validity of (3.3.21) as follows:

$$\text{if } x, y \in A \text{ are in the same } A_0(U), \text{ then } \langle x, y\rangle_H \in L. \tag{3.3.25}$$

To see this, let $U \in H$ and $x, y \in A_0(U)$ such that $x \neq y$. Apart from x–y symmetry, (3.3.21) allows us to assume that $x = a_i(U)$ and $y = b_j(U)$. Since we obtain

$$\langle x, y\rangle_H = \big(\langle x, a_5\rangle_H \vee \langle a_5, b_6\rangle_H \vee \langle b_6, y\rangle_H\big) \wedge \big(\langle x, a_6\rangle_H \vee \langle a_6, b_7\rangle_H \vee \langle b_7, y\rangle_H\big)$$

from Lemma 3.3.3, (3.3.25) follows.

Next, we turn our attention to atoms. As a first step, we will show that, for every $U \in H$,

$$\langle a_1(U), b_1(U)\rangle \in L.$$

To see this, we claim that

$$\begin{aligned}\langle a_1(U), b_1(U)\rangle = \langle a_1, b_1\rangle_H \wedge & \\ & \bigwedge_{i\in U} \left(\langle a_1, e'_i\rangle_H \vee \beta \vee \langle e_i, b_1\rangle_H\right) \\ & \bigwedge_{i\in I\setminus U} \left(\langle a_1, e'_i\rangle_H \vee \beta_* \vee \langle e_i, b_1\rangle_H\right).\end{aligned} \tag{3.3.26}$$

The "$\subseteq$" inclusion is evident. To see the reverse inclusion, let $V \in H$, $V \neq U$. This means there is a $j \in I$ such that $j \in V \setminus U$ or $j \in U \setminus V$. Because of symmetry, we can assume that $j \in U$ and $j \notin V$. This means that $\langle a_1, e'_j\rangle_H \vee \beta \vee \langle e_j, b_1\rangle_H$ is a part of the right side of (3.3.26). It is clear that $(a_1(U), b_1(U)) \in \langle a_1, e'_j\rangle_H \vee \beta \vee \langle e_j, b_1\rangle_H$. However, $(a_1(V), b_1(V)) \notin \langle a_1, e'_j\rangle_H \vee \beta \vee \langle e_j, b_1\rangle_H$, because $\langle a_1, e'_j\rangle_H$ and $\langle e_j, b_1\rangle_H$ are box-preserving, $a_1(V)$ and $b_1(V)$ are the only elements of their β-blocks and, since $j \notin V$, $(e'_j(V), e_j(V)) \notin \beta$. Hence, (3.3.26) holds.

Next, we claim that if $U \in H$ and $\{w, x, y, z\} \subseteq A_0$ such that $|\{w, x, y, z\}| = 4$, then

$$\langle w(U), z(U)\rangle \in L \Longrightarrow \langle x(U), y(U)\rangle \in L. \tag{3.3.27}$$

Since each quasiorder occurring in the right-hand side of

$$\langle x(U), y(U)\rangle = \langle x, y\rangle_H \wedge \big(\langle x, w\rangle_H \vee \langle w(U), z(U)\rangle \vee \langle z, y\rangle_H\big) \tag{3.3.28}$$

is box-preserving, (3.3.28) holds and implies (3.3.27). Starting from (3.3.26) and applying (3.3.28) once or twice, we obtain that

$$\text{if } U \in H \text{ and } x, y \in A_0(U) \text{ with } x \neq y, \text{ then } \langle x, y\rangle \in L.$$

Next, we leave a single box similarly as we left a single row around (3.3.22)–(3.3.25). This justifies to give less details. First we obtain that, for $U \neq V \in H$,

$$\begin{aligned}&\langle a_5(U), a_5(V)\rangle = \\ &\quad \big(\langle a_5(U), e_0(U)\rangle \vee \beta_* \vee \langle e_0(\varnothing), e_1(\varnothing)\rangle \vee \beta_* \vee \langle e_1(V), a_5(V)\rangle\big) \wedge \\ &\quad \big(\langle a_5(U), e_1(U)\rangle \vee \beta \vee \langle e_1(\varnothing), e_0(\varnothing)\rangle \vee \beta \vee \langle e_0(V), a_5(V)\rangle\big)\end{aligned}$$

is in L. Note that the second occurrence of β_* and that of β could be omitted; they only serve a better understanding. Similarly, $\langle a_6(U), a_6(V)\rangle \in L$. Hence, Lemma 3.3.3 yields easily that for all $x \neq y \in A$, $\langle x, y\rangle \in L$. This proves part (ii) of Theorem 3.3.1. $\square$

3.4 A concise approach to small generating sets of lattices of quasiorders and transitive relations

Definition 3.4.1. By a *Zádori configuration* of rank $n \in \mathbb{N}$, we mean an edge-colored graph $F_n = \{a_0, a_1, \dots, a_n, b_0, \dots, b_{n-1}\}$ with α-colored *horizontal edges* (a_{i-1}, a_i) and (b_{j-1}, b_j) for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n-1\}$, β-colored *vertical edges* (a_i, b_i) for $i \in \{0, \dots, n-1\}$, and γ-colored *slanted edges* (*of slope* 45°) (a_{i-1}, b_i) for $i \in \{1, \dots, n\}$; these edges are *solid* edges in our figures. For example, F_6 is given in Figure 3.4 but we have to disregard the dotted edges. We do not make a notational distinction between the graph and its vertex set, F_n. The colors α, β, and γ are also members of $\mathrm{Equ}(F_n)$; we let $(a, b) \in \alpha$ if there is an α-colored path from a to b in the graph, and we define the equivalences $\beta, \gamma \in \mathrm{Equ}(F_n)$ analogously.

The following lemma is due to Zádori [87]. Note that this lemma is implicit in [87], and it was used, implicitly, in Czédli [7], [8], [9], and [19]. The lattice operations join and meet are also denoted by $+$ and $\cdot$ (or concatenation), respectively.

Lemma 3.4.2 (Zádori [87])**.** *If $n \in \mathbb{N}$ and A is the base set of the Zádori configuration F_n, then* $\mathrm{Equ}(A)$ *is generated by* $\{\alpha, \beta, \gamma, \langle a_0, b_0\rangle^e, \langle a_n, b_{n-1}\rangle^e\}$.

We already used part 3.4.1 of the following straightforward lemma in the previous section, with different notations, see Lemma 3.3.3. This lemma was also used, explicitly or implicitly, in several earlier papers; see Chajda and Czédli [4, second display on page 423], Czédli [7, last display on page 55], [8, circle principle on page 12], [9, first display on page 451], and [19, Lemma 2.1], Takách [85, page 90], and Zádori [87, second display on page 583].

Lemma 3.4.3. *For an arbitrary set A and $j, k \in \mathbb{N}$, if $\{u, v\}$, $\{x_1, \dots, x_{j-1}\}$, and $\{y_1, \dots, y_{k-1}\}$ are pairwise disjoint subsets of A, $u = x_0 = y_0$, and $v = x_j = y_k$,* *then*

$$\langle u,v\rangle = \Big(\sum_{i=1}^{j}\langle x_{i-1},x_i\rangle\Big)\cdot\Big(\sum_{i=1}^{k}\langle y_{i-1},y_i\rangle\Big), \tag{3.4.1}$$

$$\textit{and } \langle u,v\rangle^e = \Big(\sum_{i=1}^{j}\langle x_{i-1},x_i\rangle^e\Big)\cdot\Big(\sum_{i=1}^{k}\langle y_{i-1},y_i\rangle^e\Big). \tag{3.4.2}$$

Lemma 3.4.4. *Assume that* $|A|\geq 3$ *and that* $\alpha_1,\dots,\alpha_k\in\mathrm{Quo}(A)$ *are* anti-symmetric (*in other words, they are orderings*) *and* $\{\alpha_1,\ \dots,\ \alpha_k\}$ *generates the complete involution lattice* Quo(A). *Then* $\{\alpha_1\setminus\Delta_A,\dots,\alpha_k\setminus\Delta_A\}$ *is a generating set of the complete involution lattice* Tran(A). *The same holds if we consider* Quo(A) *and* Tran(A) *as complete lattices* (*without involution*).

Proof. Let Rel(A) stand for the complete involution lattice of all binary relations over A. The meet in this lattice is the usual intersection, the involution is the map $\rho\mapsto\rho^*:=\rho^{-1}$, but the join is defined in the following way: for $\rho_i\in\mathrm{Rel}(A)$ and $(x,y)\in A^2$, we have $(x,y)\in\bigvee\{\rho_i : i\in I\}$ iff there is an $n\in\mathbb{N}$, there exists a finite sequence $x=z_0,z_1,\dots,z_n=y$ of elements of A, and there are $i_1,\dots,i_n\in I$ such that $(z_{j-1},z_j)\in\rho_{i_j}$ for all $j\in\{1,\dots,n\}$. Note that Tran(A) and Quo(A) are complete involution sublattices of Rel(A). For a relation ρ, denote $\rho\setminus\Delta_A$ by ρ^-. Instead of $\langle\beta_1,\dots,\beta_k\rangle\in\mathrm{Rel}(A)^k$ and $\langle\beta_1^-,\dots,\beta_k^-\rangle$, we write $\vec\beta$ and $\vec\beta^-$, respectively. We need k-ary $|A|$-complete involution lattice terms, which are defined in the usual way by transfinite induction, see, for example, [6]; these terms are built from at most $|A|$-ary joins and meets and the involution operation *. For such a term t, $t^-(\vec\beta)$ and $t^-(\vec\beta^-)$ will stand for $(t(\vec\beta))^-$ and $(t(\vec\beta^-))^-$. Then, for every k-ary $|A|$-complete involution lattice term t, we have that

$$\text{for every } \vec\beta\in\mathrm{Rel}(A)^k,\quad t^-(\vec\beta)=t^-(\vec\beta^-). \tag{3.4.3}$$

If the rank of t is 0, then t is a variable and (3.4.3) holds obviously. If (3.4.3) holds for a term t, then it also holds for t^*, because * is a lattice automorphism.

Next, assume that $t = \bigwedge\{t_i : i \in I\}$ and (3.4.3) holds for all the t_i. Then

$$\begin{aligned} t^-(\vec{\beta}) &= t(\vec{\beta}) \setminus \Delta_A = \big(\bigcap\{t_i(\vec{\beta}) : i \in I\}\big) \setminus \Delta_A = \bigcap\{t_i(\vec{\beta}) \setminus \Delta_A : i \in I\} \\ &= \bigcap\{t_i^-(\vec{\beta}) : i \in I\} = \bigcap\{t_i^-(\vec{\beta}^-) : i \in I\} \\ &= \bigcap\{t_i(\vec{\beta}^-) \setminus \Delta_A : i \in I\} = \big(\bigcap\{t_i(\vec{\beta}^-) : i \in I\}\big) \setminus \Delta_A \\ &= t(\vec{\beta}^-) \setminus \Delta_A = t^-(\vec{\beta}^-), \end{aligned}$$

whereby (3.4.3) holds for t.

Next, assume that $t = \bigvee\{t_i : i \in I\}$. In order to show the validity of (3.4.3) for t, assume first that $(x, y) \in t^-(\vec{\beta})$. Then $x \neq y$ and $(x, y) \in t(\vec{\beta})$. So there is a *shortest* finite sequence $x = z_0, z_1, \dots, z_n = y$ of elements of A and there are $i_1, \dots, i_n \in I$ such that $(z_{j-1}, z_j) \in t_{i_j}(\vec{\beta})$ for all $j \in \{1, \dots, n\}$. Since $x \neq y$ and we use a shortest sequence, $n \in \mathbb{N}$ is at least 1 and $z_{j-1} \neq z_j$ for $j \in \{1, \dots, n\}$. Thus, $(z_{j-1}, z_j) \in t_{i_j}^-(\vec{\beta})$, whereby the induction hypothesis gives that $(z_{j-1}, z_j) \in t_{i_j}^-(\vec{\beta}^-) \subseteq t_{i_j}(\vec{\beta}^-)$. Therefore, $(x, y) \in t_{i_1}(\vec{\beta}^-) \vee \dots \vee t_{i_n}(\vec{\beta}^-) \subseteq \bigvee\{t_i(\vec{\beta}^-) : i \in I\} = t(\vec{\beta}^-)$. But $x \neq y$, whence $(x, y) \in t^-(\vec{\beta}^-)$. This proves that $t^-(\vec{\beta}) \subseteq t^-(\vec{\beta}^-)$. Conversely, since the lattice operations and the involution are monotone, $t(\vec{\beta}^-) \subseteq t(\vec{\beta})$. Subtracting Δ_A, we obtain that $t^-(\vec{\beta}^-) \subseteq t^-(\vec{\beta})$. This proves (3.4.3).

Armed with (3.4.3), let $a \neq b \in A$. Since $\{\alpha_1, \dots, \alpha_k\}$ generates the complete involution lattice $\mathrm{Quo}(A)$, there is a k-ary $|A|$-complete involution lattice term t such that $\langle a, b\rangle = t(\vec{\alpha})$. Subtracting Δ_A from both sides, we obtain that $\{\!\{a, b\}\!\}^{\mathrm{tr}} = \langle a, b\rangle \setminus \Delta_A = t(\vec{\alpha}) \setminus \Delta_A = t^-(\vec{\alpha})$. Thus, by (3.4.3), $\{\!\{a, b\}\!\}^{\mathrm{tr}} = t^-(\vec{\alpha}^-)$. This means that for all $a \neq b \in A$, the complete involution sublattice L generated by $\vec{\alpha}^-$ in $\mathrm{Rel}(A)$ contains $\{\!\{a, b\}\!\}^{\mathrm{tr}}$. But L is also what $\vec{\alpha}^-$ generates in $\mathrm{Tran}(A)$. Thus, what we need to prove is that $L = \mathrm{Tran}(A)$. For $a \neq b$, $\{\!\{a, b\}\!\}^{\mathrm{tr}} \in L$. Based on this containment, for each $c \in A$, we can pick $x, y \in A$ such that $|\{x, y, c\}| = 3$; then

$$\{\!\{c, c\}\!\}^{\mathrm{tr}} = (\{\!\{c, x\}\!\}^{\mathrm{tr}} \vee \{\!\{x, c\}\!\}^{\mathrm{tr}}) \wedge (\{\!\{c, y\}\!\}^{\mathrm{tr}} \vee \{\!\{y, c\}\!\}^{\mathrm{tr}}) \in L. \tag{3.4.4}$$

Finally, for an arbitrary $\rho \in \mathrm{Tran}(A)$, we obtain from $\rho = \bigvee\{\{\!\{a, b\}\!\}^{\mathrm{tr}} : (a, b) \in \rho\}$ that $\rho \in L$. Consequently, $L = \mathrm{Tran}(A)$ is generated by $\vec{\alpha}^-$ as required. □

The main result of this section, Theorem 3.4.9, relies on the following three lemmas.

Lemma 3.4.5. *For a set A such that* $13 \leq |A| < \aleph_0$ *and* $|A|$ *is odd,* $\mathrm{Quo}(A)$ *is* $(1+1+2)$*-generated.*

Proof. Take F_n for $6 \leq n \in \mathbb{N}$ from Lemma 3.4.2, see Figure 3.4.

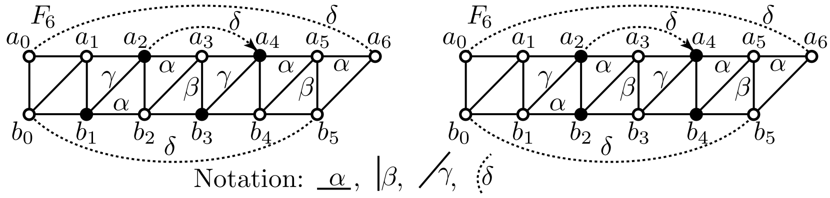


Figure 3.4: F_6 with dotted δ-edges, twice

Define

$$\delta = \langle a_0, a_n\rangle^e + \langle b_0, b_{n-1}\rangle^e + \langle a_2, a_4\rangle \in \mathrm{Quo}(A); \tag{3.4.5}$$

the join above is denoted by plus and it is taken in $\mathrm{Quo}(A)$. Note that (3.4.5) makes sense since, say, $\langle a_0, a_n\rangle^e \in \mathrm{Equ}(A) \subseteq \mathrm{Quo}(A)$. In the figure, δ is visualized by the dotted lines. Let $L := [\alpha, \dots, \delta] \leq \mathrm{Quo}(A)$. The $(\delta + \delta^{-1} + \gamma)$-block of a_2 is $\{b_1, a_2, b_3, a_4\}$, see the black-filled elements on the left, whereby it follows easily that $\langle a_0, b_0\rangle^e = \beta(\gamma + \delta)$. Similarly, the $(\delta + \delta^{-1} + \beta)$-block of a_2 consists of the black-filled elements on the right, and we conclude that $\langle a_n, b_{n-1}\rangle^e = \gamma(\beta + \delta)$. By Lemma 3.4.2, $\mathrm{Equ}(A) \subseteq L$. Actually, $\mathrm{Equ}(A) \subset L$, since $\delta \in L \setminus \mathrm{Equ}(A)$. Thus, the statement follows from Lemma 3.3.2. □

Let us agree that every infinite cardinal is even.

Lemma 3.4.6. *For* $56 \leq |A| \leq \aleph_0$*, if* $|A|$ *is even, then the complete lattice* $\mathrm{Quo}(A)$ *is* $(1+1+2)$*-generated.*

Proof. For $13 < t \in \mathbb{N}$, define the graph $F_{13} \oplus F_t$ in the same way (but with a new notation) as in Czédli [8]; see Figure 3.5 for $t = 16$.

Note that, for example, (b_9^0, a_{11}^1) is a γ-colored edge, no matter how large t is. Let $A := F_{13} \oplus F_t$. The dotted lines stand for δ again; note that because of $(a_2^0, a_4^0) \in \delta$ but $(a_4^0, a_2^0) \notin \delta$, $\delta \notin \mathrm{Equ}(A)$. Let $L := [\alpha, \dots, \delta] \leq \mathrm{Quo}(A)$. Clearly, $|A| = 2 \cdot 13 + 1 + 2t + 1$ ranges in $\{56, 58, 60, \dots\} \subset \mathbb{N}$. For $\aleph_0$, we let

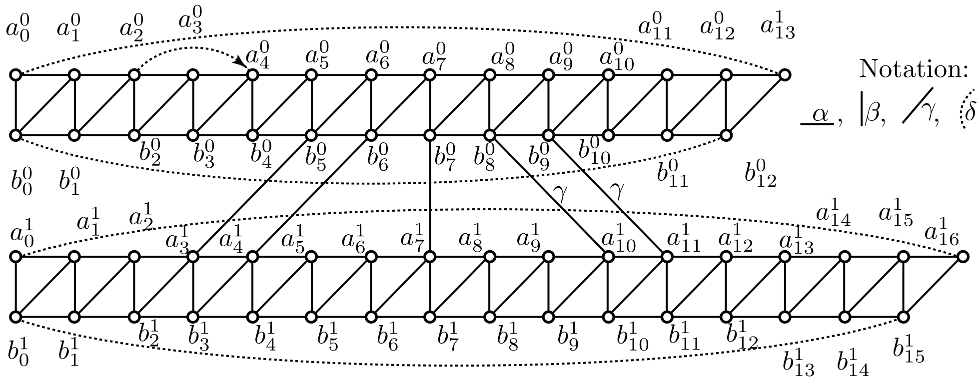


Figure 3.5: $F_{13} \oplus F_{16}$

$A := F_{13} \oplus F_{14} \oplus F_{15} \oplus \dots$ as in [8]. Since the δ-edge (a_2^0, a_4^0) does not disturb anything in the proof given in [8], $\mathrm{Equ}(A) \subseteq L$. This inclusion, $\delta \in L \setminus \mathrm{Equ}(A)$, and Lemma 3.3.2 yield the lemma. $\square$

Next, we formulate the “large accessible” counterpart of Lemma 3.4.6.

Lemma 3.4.7. *If* $\aleph_0 \leq |A|$ *is accessible, then* $\mathrm{Quo}(A)$ *is* $(1+1+2)$*-generated.*

Proof. Instead of F_{29} in Czédli [9, Figure 1], start with F_{34}. Instead of the five switches of F_{29}, designate six switches in F_{34}, but use only five of them exactly in the same way as in [9]. Follow the construction of [9] with F_{34} instead of F_{29} and, of course, not using the sixth switch. This change does not disturb the argument, and we obtain a $(1+1+2)$-generating set of the complete lattice $\mathrm{Equ}(A)$; the only difference is that very many unused switches remain by the end of the construction.

Now, we pick one of the unused switches and turn it to, say, the upper half of [9, Figure 4] but in a slightly modified form: instead of the non-oriented dotted arc (for δ), now we use an oriented arc. Since this arc changes neither $\beta(\gamma+\delta)$, nor $\gamma(\beta+\delta)$, $\delta \notin \mathrm{Equ}(A)$, we still have that $\mathrm{Equ}(A) \subseteq [\alpha, \dots, \delta]$. This fact, $\delta \notin \mathrm{Equ}(A)$ and Lemma 3.3.2 complete the proof. $\square$

The following lemma adds 6, 8, and 10 to the scope of the main result of Czédli [19]; unfortunately, the case $|A| = 4$ remains unsettled. Furthermore, it simplifies the approach of [19] for finite sets A with $|A|$ being even.

Lemma 3.4.8. *For* $6 \leq |A| \in \mathbb{N}$ *even, the (complete) lattice* $\mathrm{Quo}(A)$ *is four-generated.*

Proof. For $n \in \{6, 8, 10, 12, \dots\}$, in accordance with our previous constructs and notation, take the *one-point extension* $A := F_n \boxplus \{x\}$ of F_n; see Figure 3.6 for $n \in \{6, 8, 10\}$.

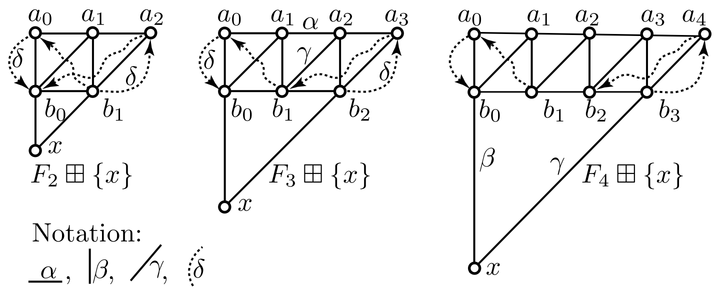


Figure 3.6: $F_n \boxplus \{x\}$ for $n \in \{2, 3, 4\}$

Let $L := [\alpha, \dots, \delta]$. Also, let $A' := A \setminus \{x\}$, and let $\mathrm{Quo}'(A) := \{\mu \in \mathrm{Quo}(A) :$ the $(\mu + \mu^{-1})$-block of x is $\{x\}\}$. For $\varepsilon \in \mathrm{Quo}(A)$, let $\varepsilon' := \varepsilon(\alpha + \delta) \in \mathrm{Quo}'(A)$. By Czédli [19] and $\mathrm{Quo}'(A) \cong \mathrm{Quo}(A')$, $\mathrm{Quo}'(A) \subseteq L$. Clearly, we have that $\langle a_0, x \rangle^e = \beta(\langle a_0, a_n \rangle^e + \gamma)$ and $\langle a_n, x \rangle^e = \gamma(\langle a_0, a_n \rangle^e + \beta)$ belong to L. Hence, Lemma 3.4.3 gives that $\mathrm{Equ}(A) \subseteq L$. Thus, Lemma 3.3.2 applies. $\square$

Now, the conclusion of this section is summarized in the following theorem.

Theorem 3.4.9. *Let A be a non-singleton set. Then the following statements hold.*

- *If $|A| \neq 4$ and $|A|$ is an accessible cardinal, then the complete lattice* $\mathrm{Quo}(A)$ *is four-generated.*

- *If $|A| \geq 13$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the complete lattice* $\mathrm{Quo}(A)$ *is $(1 + 1 + 2)$-generated.*

- *If $13 \leq |A| \leq \aleph_0$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the lattice* $\mathrm{Quo}(A)$ *(not a complete one now) contains a $(1+1+2)$-generated sublattice that includes all atoms of* $\mathrm{Quo}(A)$.

Lemma 3.4.10. *If* $3 \le |A|$ *and* $|A|$ *is an accessible cardinal, then the complete lattice* $\mathrm{Tran}(A)$ *is six-generated.*

Proof. By Czédli [7], there are $\alpha_1, \dots, \alpha_4 \in \mathrm{Equ}(A)$ such that $\{\alpha_1, \dots, \alpha_4\}$ generates $\mathrm{Equ}(A)$ as a complete lattice. Let ρ be a strict linear order on A; for example, it can be a well-ordering. In order to see that the complete sublattice $L := [\alpha_1, \dots, \alpha_4, \rho, \rho^{-1}]$ is actually $\mathrm{Tran}(A)$; it suffices to show that L contains all the atoms of $\mathrm{Tran}(A)$. Take an atom; it is of the form $\{\!(a,b)\!\}^{\mathrm{tr}}$. First, assume that $a \neq b$. Then either ρ, or ρ^{-1} contains the pair (a,b). Hence, $\{\!(a,b)\!\}^{\mathrm{tr}}$ is either $\langle a,b\rangle^{\mathrm{e}} \wedge \rho$, or $\langle a,b\rangle^{\mathrm{e}} \wedge \rho^{-1}$. In both cases, since $\langle a,b\rangle^{e} \in \mathrm{Equ}(A) = [\alpha_1, \dots, \alpha_4] \subseteq L$, we obtain that $\{\!(a,b)\!\}^{\mathrm{tr}} \in L$. Second, assume that $a = b$; that is, we need to deal with $\{\!(a,a)\!\}^{\mathrm{tr}}$. The assumption $3 \le |A|$ allows us to pick $x, y \in A$ such that $|\{a,x,y\}| = 3$. Using (3.4.4) with a in place of c, we obtain that $\{\!(a,a)\!\}^{\mathrm{tr}} \in L$, as required. $\square$

Lemma 3.4.11. *If* $3 \le |A|$ *and* $|A|$ *is an accessible cardinal, then the complete involution lattice* $\mathrm{Tran}(A)$ *is three-generated.*

Proof. Observe that the three generators constructed in Takách [85] are orderings. Thus, Lemma 3.4.4 applies. $\square$

Note that this proof is more complicated than the proof of Lemma 3.4.10, because this proof uses Lemma 3.4.4. Note also that (3.1.1) and Lemma 3.4.11 imply Lemma 3.4.10. Now, based on Lemmas 3.4.10 and 3.4.11, we are in the position to conclude this section and chapter with the following theorem.

Theorem 3.4.12. *If* A *is a set such that* $3 \le |A|$ *and* $|A|$ *is an accessible cardinal, then* $\mathrm{Tran}(A)$ *is six-generated as a complete lattice, and it is three-generated as a complete involution lattice.*

3.5 A mini-survey of recent related results

In Zádori [87], the problem of whether $\mathrm{Equ}(A)$ is $(1+1+2)$-generated for $|A| \in \{5, 6\}$ remained open. Czédli and Oluoch [41] solved this problem. Ahmed and

Czédli [2] proved that if A is a finite set such that $|A| \in \{3, 6, 11\}$ or $|A| \geq 13$, then $\mathrm{Quo}(A)$ is $(1+1+2)$-generated. By allowing 24 new values of $|A|$, this statement generalized the middle part of Theorem 3.4.9. This paper uses Lemma 3.3.2, which is [2, Lemma 2.4] in it. Czédli [25] and Czédli and Oluoch [41] prove that many direct products (in particular, direct powers) of finite equivalence lattices are four-generated. For example, Czédli [25] implies that if $|A| = 100$ and $k \leq 10^{34}$, then $\mathrm{Equ}(A)^k$ is $(1+1+2)$-generated. Czédli [25] and [31] point out that a large lattice with small generating set could be applied in cryptography. Finite Boolean lattices and, more generally, finite direct powers of small distributive lattices are large and we know from Czédli [30], [31], and [32] that these lattices can be generated by few elements. So are "large" principal filters F of $\mathrm{Quo}(A)$ by Czédli [29], in which "large" is appropriately defined.

Chapter 4

On the largest numbers of congruences of finite lattices

4.1 Introduction

This chapter is the same as our joint paper [77]. The problem of the existence of lattices with certain values for the cardinalities of their sets of congruences, filters, and ideals was raised in Mureşan [75, 76]. In Czédli and Mureşan [40], it was proved that the set of all the congruences of an infinite lattice can be of any size between 2 and the cardinality of the lattice, or it can have the same cardinality as the lattice's subsets. Thus, under the Generalized Continuum Hypothesis, the set of all the congruences of an infinite lattice can be of any size between 2 and the cardinality of the lattice's subsets. This does not hold for finite lattices, due to the limited number of configurations.

It has been proved in Freese [50] and Czédli [21] that a finite lattice can have at most as many congruences as the chain with the same cardinality, and in Czédli [21] that the second largest possible number of congruences is that of a glued sum of two (not necessarily nonsingleton) chains with the four-element Boolean algebra, and, moreover, that these are the only possible structures of finite lattices witnessing those numbers of congruences; in Czédli [23], the same problem has been investigated for semilattices, and the title of Czédli [24] speaks for itself.

In our main result of this chapter, Theorem 4.3.7, we determine the third, fourth and fifth largest possible numbers of congruences of a finite lattice, along with the structures of the finite lattices with these numbers of congruences, which also show the structures of their congruence lattices. The study of the representation of lattices in the form of congruence lattices of lattices goes back to Dilworth and was milestoned by Grätzer and Schmidt [68], Wehrung [86], Růžička [80], Grätzer and Knapp [62], and Ploščica [79], and surveyed in Grätzer [57] and Schmidt [81]. A lot of the results have been proved on the representation problem of two or more lattices and certain maps among them by (complete) congruences; for example, see Grätzer and Schmidt [69], Grätzer and Lakser [63], Czédli [10, 18]. Even the posets and monotone maps among them have been characterized by principal congruences of lattices; for example, see Grätzer [56, 58, 59, 60], Grätzer and Lakser [64], and Czédli [14, 15, 17, 20, 22]. Finally, the above-mentioned trends, focusing on the sizes of congruence lattices, on the structures formed by congruences, and on maps among these structures, have recently met in Czédli and Mureşan [40], enriching the first two trends and even related to the third one.

Regarding the determination of all possible numbers of congruences of an n-element lattice, we do not know whether, for an appropriately large finite number n of elements and an appropriately large natural number k_n, we can find n-element lattices with any number of congruences between 2 and the k_nth largest possible number of congruences of an n-element lattice. But in the older version of the paper this chapter is based on, available at arXiv:1801.05282v2, we have obtained some results on the smallest numbers of congruences of n-element lattices and we have laid down some ideas for bridging the gap between these and an appropriately chosen k_nth largest possible number of congruences.

4.2 Definitions, notations and immediate properties

As usual, $\dot\cup$ will be the disjoint union of sets. For any set M, we denote the bounded lattice of all partitions on M by $\mathrm{Part}(M)$. Just like in the previous chapter, $\mathrm{Equ}(M)$ stands for the lattice of all equivalences on M, and $\Delta_M = \{(x,x) : x \in M\}$; also, $\nabla_M = M^2$ as usual. We denote the canonical lattice isomorphism $\mathrm{Part}(M) \to \mathrm{Equ}(M)$ by eq; for any finite partition $\{M_1,\dots,M_k\}$, $\mathrm{eq}(\{M_1,\dots,M_k\})$ will simply be denoted by $\mathrm{eq}(M_1,\dots,M_k)$.

All lattices will be nonempty and they will be designated by their underlying sets. Let L and M be lattices, L is said to be *trivial* iff $|L| = 1$, L' stands for the dual of L, and if L and M are isomorphic, then we denote it by $L \cong M$.

The congruences, filters and ideals of L also form lattices, denoted by $\mathrm{Con}(L)$, $\mathrm{Filt}(L)$ and $\mathrm{Id}(L)$, respectively. Of course, $\mathrm{Con}(L) = \mathrm{Con}(L')$. Following [21], we use the notation $\mathrm{con}(a,b)$ for the principal congruence of L generated by the ordered pair (a,b). If L is a bounded lattice, then $\mathrm{Con}_{01}(L)$ stands for the set of the congruences of L where the classes of 0 and 1 are singletons: $\mathrm{Con}_{01}(L) = \{\theta \in \mathrm{Con}(L) : 0/\theta = \{0\}, 1/\theta = \{1\}\}$. As an immediate consequence of $\mathrm{Con}(L)$ being a complete sublattice of $\mathrm{Equ}(L)$ (see [53, Corollary 2, page 51]), $\mathrm{Con}_{01}(L)$ is a complete sublattice of $\mathrm{Con}(L)$ (see also [51, Lemma 2]).

For $a,b \in L$ arbitrary, $[a)_L$ and $(a]_L$ will be the principal filter and principal ideal of L generated by a, respectively, and we denote the interval $[a)_L \cap (b]_L$ by $[a,b]_L$. If L is the lattice of the natural numbers with the natural order, then the index L will be eliminated from the previous notations. Recall that $[a,b]_L$ is called a *prime interval* iff $a \prec b$, that is $a < b$ and $[a,b]_L = \{a,b\}$. We will call $[a,b]_L$ a *contractible edge* (in brief, *c-edge*) iff it is a prime interval such that a is meet-irreducible and b is join-irreducible in L, so that b is the only successor of a and a is the only predecessor of b in L. If L has a smallest element, then $\mathrm{At}(L)$ will denote the set of the atoms of L.

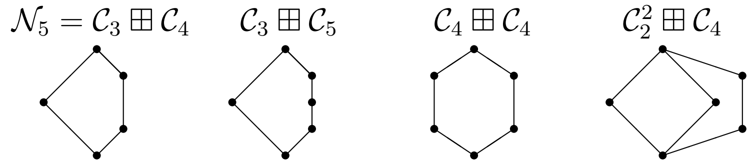
We denote the glued sum and the horizontal sum by $\dot{+}$ and $\boxplus$, respectively, whose constructions we briefly recall here; see [76, 51] for their rigorous definitions, but note that, in these papers, the operation $\dot{+}$ described below is denoted by $\oplus$ and called *ordinal sum*; see also the examples in the following diagrams.

If L has a largest element 1^L and M has a smallest element 0^M, then the *glued sum* of L and M is the lattice $L \dot{+} M$ whose underlying set is the quotient set of the equivalence of $L \dot{\cup} M$ which collapses only 1^L and 0^M, and $L \dot{+} M$ is obtained from L and M by identifying 1^L with 0^M and stacking M on top of L. Also, for any $\alpha \in \mathrm{Con}(L)$ and any $\beta \in \mathrm{Con}(M)$, the equivalence generated by $\alpha \cup \beta$ is denoted by $\alpha \dotplus \beta$, namely $\alpha \dotplus \beta = \mathrm{eq}((L/\alpha \setminus \{1^L/\alpha\}) \cup (M/\beta \setminus \{0^M/\beta\}) \cup \{1^L/\alpha \cup 0^M/\beta\})$, whose classes are the union of $1^L/\alpha$ and $0^M/\beta = 1^L/\beta$, along with all the other classes of α and all the other classes of β. Clearly, $\mathrm{Con}(L \dotplus M) = \{\alpha \dotplus \beta : \alpha \in \mathrm{Con}(L), \beta \in \mathrm{Con}(M)\} \cong \mathrm{Con}(L) \times \mathrm{Con}(M)$, and the glued sum of bounded lattices, also of congruences of those lattices, is associative.

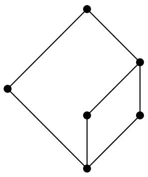
If L and M are nontrivial bounded lattices, then the *horizontal sum* of L and M is the nontrivial bounded lattice $L \boxplus M$ whose underlying set is the quotient set of the equivalence of $L \dot{\cup} M$ which collapses only 0^L with 0^M and 1^L with 1^M, and $L \boxplus M$ is obtained from L and M by identifying their bottom elements 0^L and 0^M, identifying their top elements 1^L and 1^M, and letting every element of $L \setminus \{0^L, 1^L\}$ be incomparable to every element of $M \setminus \{0^M, 1^M\}$ in $L \boxplus M$. Also, for any $\alpha \in \mathrm{Equ}(L)$ and any $\beta \in \mathrm{Equ}(M)$, $\alpha \boxplus \beta$ stands for the equivalence on $L \boxplus M$ generated by $\alpha \cup \beta$, so that, if $\alpha \neq \nabla_L$ and $\beta \neq \nabla_M$, then $\alpha \boxplus \beta = \mathrm{eq}((L/\alpha \setminus \{0/\alpha, 1/\alpha\}) \cup (M/\beta \setminus \{0/\beta, 1/\beta\}) \cup \{0/\alpha \cup 0/\beta, 1/\alpha \cup 1/\beta\})$, where $0 = 0^L = 0^M$ and $1 = 1^L = 1^M$ in $L \boxplus M$, whose classes are the union of $0/\alpha$ and $0/\beta$ and the union of $1/\alpha$ and $1/\beta$, along with all the other classes of α and all the other classes of β. Clearly, the horizontal sum of nontrivial bounded lattices, also of proper equivalences on those lattices, is both associative and commutative.

For any $n \in \mathbb{N}$, we denote the n-element chain by $\mathcal{C}_n$. Clearly, if L is a nontrivial bounded lattice, then $L \boxplus \mathcal{C}_2 = L$. Note that $\mathcal{C}_3 \boxplus \mathcal{C}_3 \boxplus \mathcal{C}_3$ is the five-element modular nondistributive lattice $\mathcal{M}_3$, while $\mathcal{C}_3 \boxplus \mathcal{C}_4$ is the five-element nonmodular lattice $\mathcal{N}_5$.

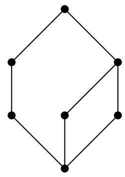
Now let us make some quick calculations in order to prove that the following lattices occurring in the proof of Theorem4.3.7 have the congruence lattices shown below them:



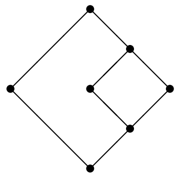
$\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)$



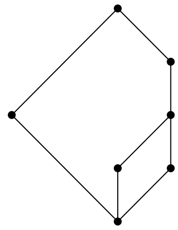
$\mathcal{C}_4 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)$



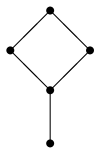
$\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_2)$



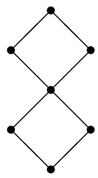
$\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_3)$



$\mathrm{Con}(\mathcal{N}_5) \cong \mathcal{C}_2 \dotplus \mathcal{C}_2^2$



$\mathrm{Con}(\mathcal{C}_3 \boxplus \mathcal{C}_5) \cong \mathrm{Con}(\mathcal{C}_4 \boxplus \mathcal{C}_4) \cong$
$\mathrm{Con}(\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_2)) \cong \mathcal{C}_2^2 \dotplus \mathcal{C}_2^2$



$\mathrm{Con}(\mathcal{C}_2^2 \boxplus \mathcal{C}_4) \cong$
$\mathrm{Con}(\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)) \cong \mathcal{C}_3$

$\mathrm{Con}(\mathcal{C}_4 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)) \cong$
$\mathrm{Con}(\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_3)) \cong \mathcal{C}_4$

For this, assume that L and M are bounded lattices and the length of each of them is at least three. Since L and M are sublattices of $L \boxplus M$, for every $\theta \in \mathrm{Con}(L \boxplus M)$, we have $\theta \cap L^2 \in \mathrm{Con}(L)$, $\theta \cap M^2 \in \mathrm{Con}(M)$, and, clearly, $\theta = (\theta \cap L^2) \boxplus (\theta \cap M^2)$. However, if $\alpha \in \mathrm{Con}(L)$ and $\beta \in \mathrm{Con}(M)$, then the equivalence $\alpha \boxplus \beta$ on $L \boxplus M$ is not always a congruence of this lattice; it is routine to prove (see also [76, 51]) that, whenever α and β are proper congruences of L and M, respectively, then the proper equivalence $\alpha \boxplus \beta$ is a congruence of $L \boxplus M$ iff either the α- or the β-classes of 0 and 1 are singletons or $\alpha \boxplus \beta$ is a two-class congruence obtained from two-class congruences of the form $\alpha = \mathrm{eq}(\{0\}, L \setminus \{0\})$ and $\beta = \mathrm{eq}(\{1\}, M \setminus \{1\})$, which, by the convexity of any congruence class, means that 0 is meet-irreducible in L and 1 is join-irreducible in M. Just note, for instance, that, if $\alpha \boxplus \beta \in \mathrm{Con}(L \boxplus M)$ and $0/\alpha$ is not a singleton, then $0/\beta$ is a singleton and $1/\beta = M \setminus \{0\}$. Therefore:

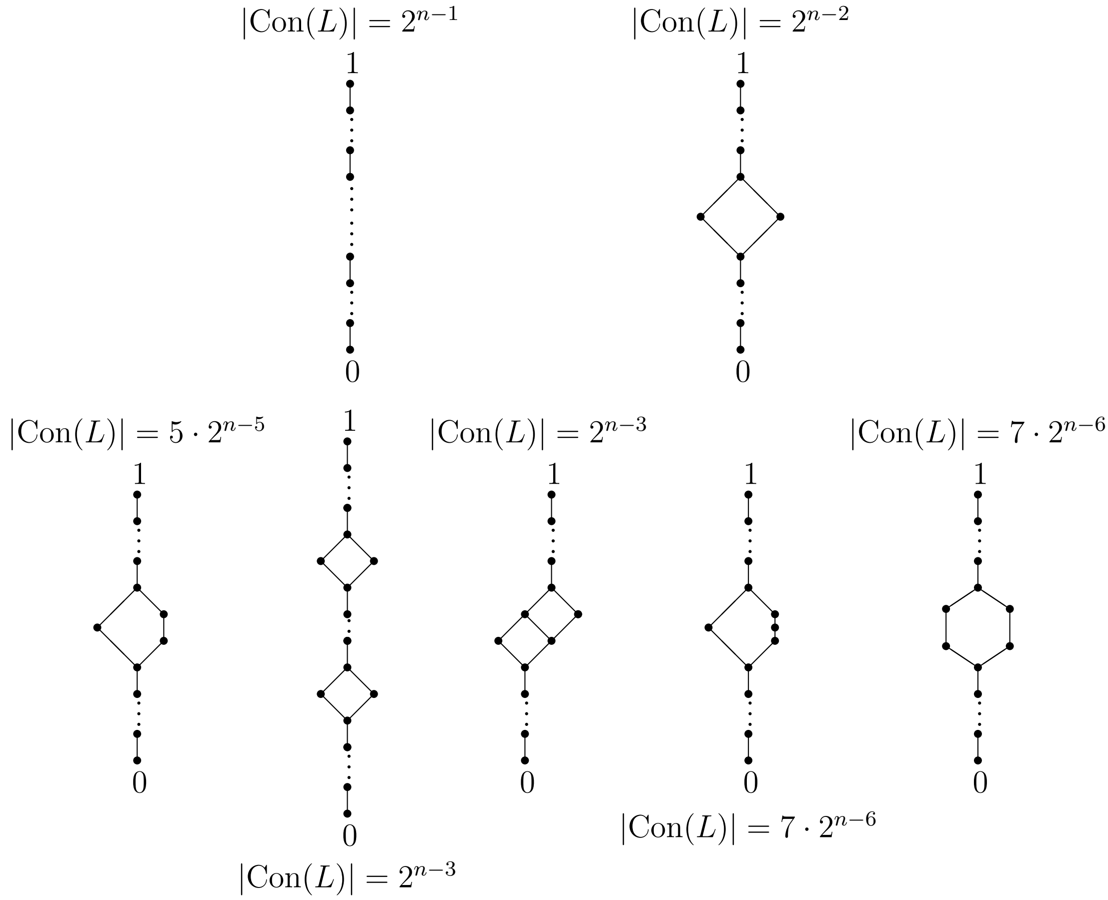
- $\mathrm{Con}_{01}(L \boxplus M) = \{\alpha \boxplus \beta \mid \alpha \in \mathrm{Con}_{01}(L), \beta \in \mathrm{Con}_{01}(M)\} \cong \mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)$,
- $\mathrm{Con}(L \boxplus M) = \mathrm{Con}_{01}(L \boxplus M) \cup \mathrm{Con}_{2cls}(L \boxplus M) \cup \{\nabla_{L \boxplus M}\}$, where $\mathrm{Con}_{2cls}(L \boxplus M)$ is the set of the two-class congruences of $L \boxplus M$,
- $\mathrm{Con}_{2cls}(L \boxplus M) \subseteq \{\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}), \mathrm{eq}(L \setminus \{1\}, M \setminus \{0\})\}$ and, out of these two two-class equivalences: $\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\}) \in \mathrm{Con}(L \boxplus M)$ iff 0 is meet-irreducible in L and 1 is join-irreducible in M, while, similarly, $\mathrm{eq}(L \setminus \{1\}, M \setminus \{0\}) \in \mathrm{Con}(L \boxplus M)$ iff 1 is join-irreducible in L and 0 is meet-irreducible in M.

Hence, noting that each of the equivalences $\mathrm{eq}(L \setminus \{0\}, M \setminus \{1\})$ and $\mathrm{eq}(L \setminus \{1\}, M \setminus \{0\})$ includes all members of $\mathrm{Con}_{01}(L \boxplus M)$, we get that $\mathrm{Con}(L \boxplus M)$ is isomorphic to the glued sum $(\mathrm{Con}_{01}(L) \times \mathrm{Con}_{01}(M)) \dotplus T$, where T is $\mathcal{C}_2$, $\mathcal{C}_3$ or $\mathcal{C}_2^2$, depending on whether the number of two-class congruences of $L \boxplus M$ is zero, one or two, respectively (see also [76]).

Noting that the four-element Boolean algebra is 0-regular and thus, for any bounded lattice K, $\mathrm{Con}_{01}(\mathcal{C}_2^2 \dotplus K \dotplus \mathcal{C}_2) = \{\Delta_{\mathcal{C}_2^2} \dotplus \theta \dotplus \Delta_{\mathcal{C}_2} \ : \ \theta \in \mathrm{Con}(K)\} \cong \mathrm{Con}_{01}(\mathcal{C}_2 \dotplus K \dotplus \mathcal{C}_2) = \{\Delta_{\mathcal{C}_2} \dotplus \theta \dotplus \Delta_{\mathcal{C}_2} \ : \ \theta \in \mathrm{Con}(K)\} \cong \mathrm{Con}(K)$, we obtain the congruence lattices displayed above.

4.3 The theorems

Let $n \in \mathbb{N}$ and L be an arbitrary lattice with $|L| = n$. By [21], the largest and the second largest possible numbers of congruences of L, along with the structures of the n-element lattices L with these numbers of congruences, are represented in the first row of the figure below. In this section, we will show that the third, fourth and fifth largest possible numbers of congruences of L, along with the structures of the n-element lattices L with these numbers of congruences, are as in the second row of the figure below:



Our observations on the smallest possible numbers of congruences of an n-element lattice suggest that, in order to fill the gap between these and the largest possible numbers of congruences of finite lattices, it might be useful to represent the numbers of congruences in base 2; this is why, in our main result below, we also indicate the numbers of congruences in base 2, apart from the fact that it helps to clarify the ordering of these numbers.

Lemma 4.3.1 ([21, 52, 57]). *If L is nontrivial, then:*

(i) $\varnothing \neq \mathrm{At}(\mathrm{Con}(L)) \subseteq \{\mathrm{con}(a,b) \ : \ a,b \in L, a \prec b\}$;

(ii) *for any* $\theta \in \mathrm{At}(\mathrm{Con}(L))$, $|\mathrm{Con}(L/\theta)| \geq |\mathrm{Con}(L)|/2$;

(iii) *for any* $a,b \in L$ *such that* $a \prec b$: $[a,b]_L$ *is a c-edge iff* $\{a,b\}$ *is the only nonsingleton block of* $\mathrm{con}(a,b)$ *iff* $|L/\mathrm{con}(a,b)| = |L|-1$;

(iv) *for any* $a, b \in L$ *such that* $a \prec b$ *and* $|L/\mathrm{con}(a,b)| = |L| - 2$, *we have the following situation or its dual (meaning the dual of the following for the case when* b *is join-reducible):* a *is meet-reducible,* $a \prec c$ *for some* $c \in L \setminus \{b\}$ *such that* $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ *and the only nonsingleton blocks of* $\mathrm{con}(a,b)$ *are* $\{a,b\}$ *and* $\{c, b \vee c\}$.

Remark 4.3.2. Let $a, b \in L$ with $a \neq b$. Also, let $\theta \in \mathrm{Con}(L)$. If $a \prec b$ and $a/\theta \neq b/\theta$, then, clearly, $a/\theta \prec b/\theta$. If $a/\theta \prec b/\theta$, then there exists no $u \in [a,b]_L \setminus (a/\theta \cup b/\theta)$, because otherwise we would have $a/\theta < u/\theta < b/\theta$. Let us also note that $a/\theta \leq b/\theta$ iff $a \vee b \in b/\theta$ iff $a \wedge b \in a/\theta$ iff $a \leq x$ for some $x \in b/\theta$ iff $w \leq b$ for some $w \in a/\theta$.

By Lemma 4.3.1(iii), if $[a,b]_L$ is a c-edge, then $\mathrm{con}(a,b)$ collapses a single pair of elements, thus, clearly, $\mathrm{con}(a,b) \in \mathrm{At}(\mathrm{Con}(L))$. Since $a/\mathrm{con}(a,b) = b/\mathrm{con}(a,b)$, we have $|L/\mathrm{con}(a,b)| \leq |L| - 1$, hence the second equivalence in Lemma 4.3.1(iii) is clear.

By Lemma 4.3.1(iii), if $|L/\mathrm{con}(a,b)| < |L| - 1$, as in Lemma 4.3.1(iv), then $[a,b]_L$ is not a c-edge, hence a is meet-reducible, so a has a successor different from b, or b is join-reducible, so b has a predecessor different from a. With the notations in Lemma 4.3.1(iv), if $|L| - |L/\mathrm{con}(a,b)| = 2$ and, for instance, a is meet-reducible, then, simply, the fact that $(a,b), (c, b \vee c) = (a \vee c, b \vee c) \in \mathrm{con}(a,b)$ implies that $L/\mathrm{con}(a,b) = \{\{a,b\},\{c, b \vee c\}\} \cup \{\{x\} : x \in L \setminus \{a,b,c,b \vee c\}\}$, with $a/\mathrm{con}(a,b) \neq x/\mathrm{con}(a,b) \neq c/\mathrm{con}(a,b)$ for all $x \in L \setminus \{a,b,c,b \vee c\}$ and $a/\mathrm{con}(a,b) \neq c/\mathrm{con}(a,b)$, which, along with the fact that $a \prec c$, as above, proves that $a/\mathrm{con}(a,b) \prec c/\mathrm{con}(a,b) = (b \vee c)/\mathrm{con}(a,b)$.

Lemma 4.3.3. *For any* $a, b \in L$ *such that* $a \prec b$ *and* $|L| - |L/\mathrm{con}(a,b)| = 3$, *we have the following situations or their duals (when* b *is join-reducible, as in Lemma 4.3.1(iv)):* a *is meet-reducible, so that* $a \prec c$ *for some* $c \in L \setminus \{b\}$, *and one of the following is fulfilled:*

(i) $b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ *and the only nonsingleton block of* $\mathrm{con}(a,b)$ *is* $\{a, b, c, b \vee c\}$;

(ii) *$b \prec b \vee c$, $c \prec b \vee c$ and, for some $d \in L \setminus \{a, b, c, b \vee c\}$, $d \prec a$, $[d, b \vee c]_L = \{d, a, b, c, b \vee c\} \cong \mathcal{C}_2 \dotplus \mathcal{C}_2^2$ and the only nonsingleton blocks of* $\mathrm{con}(a, b)$ *are $\{d, a, b\}$ and $\{c, b \vee c\}$;*

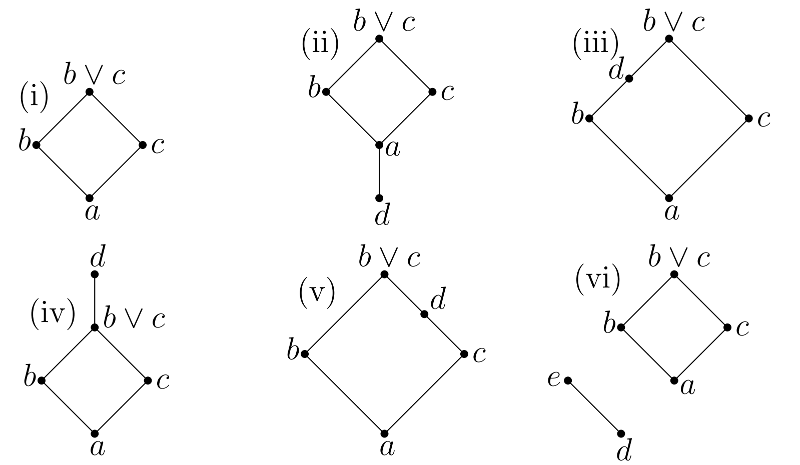
(iii) *$c \prec b \vee c$ and, for some $d \in L \setminus \{a, b, c, b \vee c\}$, $b \prec d \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and the only nonsingleton blocks of* $\mathrm{con}(a, b)$ *are $\{a, b, d\}$ and $\{c, b \vee c\}$;*

(iv) *$b \prec b \vee c$, $c \prec b \vee c$ and, for some $d \in L \setminus \{a, b, c, b \vee c\}$, $b \vee c \prec d$, $[a, d]_L = \{a, b, c, b \vee c, d\} \cong \mathcal{C}_2^2 \dotplus \mathcal{C}_2$ and the only nonsingleton blocks of* $\mathrm{con}(a, b)$ *are $\{a, b\}$ and $\{c, b \vee c, d\}$;*

(v) *$b \prec b \vee c$ and, for some $d \in L \setminus \{a, b, c, b \vee c\}$, $c \prec d \prec b \vee c$, the only nonsingleton blocks of* $\mathrm{con}(a, b)$ *are $\{a, b\}$ and $\{c, d, b \vee c\}$, and $[a, b \vee c]_L = \{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$;*

(vi) *$b \prec b \vee c$, $c \prec b \vee c$, $[a, b \vee c]_L = \{a, b, c, b \vee c\} \cong \mathcal{C}_2^2$ and, for some $d, e \in L \setminus \{a, b, c, b \vee c\}$ such that $d \prec e$, the only nonsingleton blocks of* $\mathrm{con}(a, b)$ *are $\{a, b\}$, $\{c, b \vee c\}$ and $\{d, e\}$.*

Proof. Let $\theta = \mathrm{con}(a, b)$. We have $a \prec b$ and $|L/\theta| = |L| - 3 = n - 3$, hence $[a, b]_L$ is not a c-edge, according to Lemma 4.3.1(iii), thus a is meet-reducible or b is join-reducible. We analyze the case when a is meet-reducible, that is $a \prec c$ for some $c \in L \setminus \{b\}$; the case when b is join-reducible is dual to this one. We depict the different situations that can appear in the following diagrams:



If $a/\theta = b/\theta = c/\theta$, thus $(b \vee c)/\theta = a/\theta$, then, since a/θ is a convex sublattice of L and $|L| - |L/\theta| = 3$, we have $a/\theta = \{a, b, c, b \vee c\} = [a, b \vee c]_L \cong \mathcal{C}_2^2$, that is $b \prec b \vee c$ and $c \prec b \vee c$, and $L/\theta = \{\{a, b, c, b \vee c\}\} \cup \{\{x\} \ : \ x \in L \setminus \{a, b, c, b \vee c\}\}$; this is case (i) in the statement of the present lemma.

If $a/\theta \neq c/\theta$, then, since $a \prec c$, it follows that $a/\theta \prec c/\theta = (b \vee c)/\theta$ by Remark 4.3.2. Since $|L| - |L/\theta| = 3 > 2$, we get that there exists $d \in L \setminus \{a, b, c, b \vee c\}$ such that $\{d\} \subsetneq d/\theta$. The fact that $|L| - |L/\theta| = 3$ shows that there are three possible situations:

- $d \in a/\theta$, in which case $a/\theta = \{a, b, d\}$, $c/\theta = \{c, b \vee c\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d\}$;
- $d \in c/\theta$, in which case $a/\theta = \{a, b\}$, $c/\theta = \{c, b \vee c, d\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d\}$;
- $d \notin a/\theta \cup c/\theta$, in which case $a/\theta = \{a, b\}$, $c/\theta = \{c, b \vee c\}$, $d/\theta = \{d, e\}$ for some $e \in L \setminus \{a, b, c, b \vee c, d\}$ and $x/\theta = \{x\}$ for all $x \in L \setminus \{a, b, c, b \vee c, d, e\}$.

If $d \in a/\theta$, then a/θ is a three-element lattice, thus $a/\theta = \{a, b, d\} \cong \mathcal{C}_3$, that is $d < a < b$ or $a < b < d$ since $a \prec b$. The convexity of a/θ ensures us that, if $d < a < b$, then $a/\theta = [d, b]_L$, so $d \prec a$, hence $\{d, a, b, c, b \vee c\} \cong \mathcal{C}_2 \dotplus \mathcal{C}_2^2$; this is case (ii) in the statement of the present lemma. If $a < b < d$, then $c \ngeq b < d \leq d \vee c \in (a \vee c)/\theta = c/\theta = \{c, b \vee c\}$, thus $b < d \leq d \vee c = b \vee c \neq d$, that is $b < d < b \vee c$. Therefore $\{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$, and, since $d/\theta \prec (b \vee c)/\theta$ and any $x \in L$ with $d < x < b \vee c$ would be such that $x \notin d/\theta \cup (b \vee c)/\theta$, Remark 4.3.2 shows that $d \prec b \vee c$; this is case (iii).

If $d \in c/\theta$, then $c/\theta = \{c, b \vee c, d\} \cong \mathcal{C}_3$, that is $d < c < b \vee c$ or $c < d < b \vee c$ or $c < b \vee c < d$. If $c < b \vee c < d$, then $\{a, b, c, b \vee c, d\} \cong \mathcal{C}_2^2 \dotplus \mathcal{C}_2$ and $\{c, b \vee c, d\} = c/\theta = [c, d]_L$, thus $b \vee c \prec d$; this is case (iv). If $c < d < b \vee c$, then $\{a, b, c, d, b \vee c\} \cong \mathcal{N}_5$ and $\{c, d, b \vee c\} = c/\theta = [c, b \vee c]_L$, that is $c \prec d \prec b \vee c$; this is case (v). Finally, if $d < c < b \vee c$, then $\{a, b\} = a/\theta = (a \wedge c)/\theta = (a \wedge d)/\theta$, hence $b > a \geq a \wedge d \in \{a, b\}$, thus $a \wedge d = a \neq d$, so we obtain $a < d < c$, which contradicts the fact that $a \prec c$.

The remaining possibility is that $d/\theta = e/\theta$ for some $e \in L \setminus \{a, b, c, b \vee c, d\}$, that is $c/\theta = \{c, b \vee c\} \cong \mathcal{C}_2$ and $d/\theta = \{d, e\} \cong \mathcal{C}_2$, thus $c \prec b \vee c$ and either $d \prec e$ or $e \prec d$; this is case (vi). $\square$

Remark 4.3.4. As pointed out by the anonymous referee of our paper [77], cases (ii) and (iv) in the previous lemma cannot occur, and they can be excluded by using [52, Lemma 229] to prove that. If $\mathrm{con}(a, b)$ collapses the elements from the nonsingleton blocks indicated in those cases, then it collapses more elements, that is $|L/\ \mathrm{con}(a, b)| \leq |L| - 4$, which contradicts the hypothesis of Lemma 4.3.3.

For the purpose of keeping self-containedness, while avoiding the lengthy projectivity arguments of [52, Lemma 229], we have kept these cases in the previous lemma, since they will be easily eliminated in the proof of Theorem 4.3.7 below.

For the Hasse diagrams of the lattices in the following theorems, see the figure at the beginning of this section.

Remark 4.3.5. Since the lattices with at most four elements are $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_2^2$, we notice that: if $|\mathrm{Con}(L)| < 2^{n-1}$, then $n \geq 4$, while, if $|\mathrm{Con}(L)| < 2^{n-2}$, then $n \geq 5$.

Theorem 4.3.6. (i) [50, 21] $|\mathrm{Con}(L)| \leq 2^{n-1}$ *and:* $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $L \cong \mathcal{C}_n$.

(ii) [21] *If* $|\mathrm{Con}(L)| < 2^{n-1}$, *then* $|\mathrm{Con}(L)| \leq 2^{n-2}$ *and:* $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-2}$ *for some* $k \in [1, n-3]$.

Following the line of the proof from [21] of Theorem 4.3.6, now we prove:

Theorem 4.3.7. *Let L be a finite lattice with n elements.*

(i) *If* $|\mathrm{Con}(L)| < 2^{n-2}$, *then* $n \geq 5$, $|\mathrm{Con}(L)| \leq 5 \cdot 2^{n-5} = 2^{n-3} + 2^{n-5}$, *and:* $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ *iff* $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$ *for some* $k \in [1, n-4]$.

(ii) *If* $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, *then* $|\mathrm{Con}(L)| \leq 2^{n-3}$, *and:* $|\mathrm{Con}(L)| = 2^{n-3}$ *iff either* $n \geq 6$ *and* $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$ *for some* $k \in [1, n-5]$, *or* $n \geq 7$ *and* $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-4}$ *for some* $k, m \in \mathbb{N}$ *such that* $k + m \leq n - 5$.

(iii) *If* $|\mathrm{Con}(L)| < 2^{n-3}$*, then* $|\mathrm{Con}(L)| \leq 7 \cdot 2^{n-6} = 2^{n-4} + 2^{n-5} + 2^{n-6}$*, and:* $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and, for some* $k \in [1, n-5]$*,* $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ *or* $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$*.*

Proof. Assume that $|\mathrm{Con}(L)| < 2^{n-2} < 2^{n-1}$, that is $n \geq 5$ by Remark 4.3.5. We will prove the statements of the theorem by induction on $n \in \mathbb{N}$, $n \geq 5$, identifying the lattices up to isomorphism.

The five-element lattices are: $\mathcal{M}_3$, $\mathcal{N}_5$, $\mathcal{C}_2 \dotplus \mathcal{C}_2^2$, $\mathcal{C}_2^2 \dotplus \mathcal{C}_2$ and $\mathcal{C}_5$, whose numbers of congruences are: 2, 5, 8, 8 and $2^4 = 16$, respectively. The five-element lattices with strictly less than $2^{5-2} = 8$ congruences are $\mathcal{M}_3$ and $\mathcal{N}_5$, out of which $\mathcal{N}_5 \cong \mathcal{C}_1 \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{5-1-3}$ is of the form in (i) and has $5 = 5 \cdot 2^{5-5}$ congruences, while $\mathcal{M}_3$ has $2 < 4 = 2^{5-3}$ congruences. From this fact and Remark 4.3.5, it follows that, if $|\mathrm{Con}(L)| = 2^{n-3}$, then $n \geq 6$.

The six-element lattices are: $\mathcal{M}_4 = \mathcal{M}_3 \boxplus \mathcal{C}_3$, $\mathcal{C}_4 \boxplus \mathcal{C}_2^2$, $(\mathcal{C}_2^2 \dotplus \mathcal{C}_2) \boxplus \mathcal{C}_3$, $(\mathcal{C}_2 \dotplus \mathcal{C}_2^2) \boxplus \mathcal{C}_3$, $\mathcal{M}_3 \dotplus \mathcal{C}_2$, $\mathcal{C}_2 \dotplus \mathcal{M}_3$, $\mathcal{C}_3 \boxplus \mathcal{C}_5$, $\mathcal{C}_4 \boxplus \mathcal{C}_4$, $\mathcal{C}_2 \times \mathcal{C}_3$, $\mathcal{N}_5 \dotplus \mathcal{C}_2$, $\mathcal{C}_2 \dotplus \mathcal{N}_5$, $\mathcal{C}_2^2 \dotplus \mathcal{C}_3$, $\mathcal{C}_3 \dotplus \mathcal{C}_2^2$, $\mathcal{C}_2 \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_2$ and $\mathcal{C}_6$, whose numbers of congruences are: 2, 3, 3, 3, 4, 4, 7, 7, 8, 10, 10, 16, 16, $16 = 2^{6-2}$ and $32 = 2^{6-1}$, respectively. So, the third largest number of congruences of a six-element lattice is $10 = 5 \cdot 2^{6-5}$, the fourth largest is $8 = 2^{6-3}$ and the fifth largest is $7 = 7 \cdot 2^{6-6}$. As above, we notice that $\mathcal{N}_5 \dotplus \mathcal{C}_2$ and $\mathcal{C}_2 \dotplus \mathcal{N}_5$ are of the form in (i), $\mathcal{C}_2 \times \mathcal{C}_3$ is of the first form in (ii) and $\mathcal{C}_3 \boxplus \mathcal{C}_5$ and $\mathcal{C}_4 \boxplus \mathcal{C}_4$ are of the forms in (iii).

It is easy to construct, as above, the 7-element lattices, and see that the ones with strictly less than $2^{7-2} = 32$ congruences are: the ones having $20 = 5 \cdot 2^{7-5}$ congruences, namely $\mathcal{N}_5 \dotplus \mathcal{C}_3$, $\mathcal{C}_3 \dotplus \mathcal{N}_5$ and $\mathcal{C}_2 \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_2$, all of the form in (i); the ones having $16 = 2^{7-3}$ congruences, namely $(\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_2$ and $\mathcal{C}_2 \dotplus (\mathcal{C}_2 \times \mathcal{C}_3)$, which are of the first form in (ii), as well as $\mathcal{C}_2^2 \dotplus \mathcal{C}_2^2$, which is of the second form in (ii); the ones having $14 = 7 \cdot 2^{7-6}$ congruences, namely $(\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_2$, $\mathcal{C}_2 \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5)$, $(\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_2$ and $\mathcal{C}_2 \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4)$, all of the forms in (iii); and the ones having strictly less than 14 congruences.

Now assume that $n \geq 8$ and the statements of the theorem hold for all lattices of cardinality at most $n-1$. Note that, in the rest of this proof, whenever $|\mathrm{Con}(L)| =$

$5 \cdot 2^{n-5}$, L is of the form in (i), whenever $|\mathrm{Con}(L)| = 2^{n-3}$, L is of one of the forms in (ii) and, whenever $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$, L is of one of the forms in (iii).

Let $\theta \in \mathrm{At}(\mathrm{Con}(L))$. By Lemma 4.3.1(i), at least one such θ exists, and $\theta = \mathrm{con}(a, b)$ for some $a, b \in L$ with $a \prec b$. Then $a/\theta = b/\theta$, that is $|L/\theta| \leq n - 1$, hence $|\mathrm{Con}(L/\theta)| \leq 2^{n-2}$ by Theorem 4.3.6(i). By Lemma 4.3.1(ii), $|\mathrm{Con}(L/\theta)| \geq |\mathrm{Con}(L)|/2$.

(i) By the hypothesis of the theorem, $|\mathrm{Con}(L)| < 2^{n-2}$. Assume by absurdum that $|\mathrm{Con}(L)| > 5 \cdot 2^{n-5}$, so that $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4} = 2^{(n-3)-1}$, thus $|L/\theta| > n - 3$ by Theorem 4.3.6(i), hence $|L/\theta| \in \{n - 1, n - 2\}$.

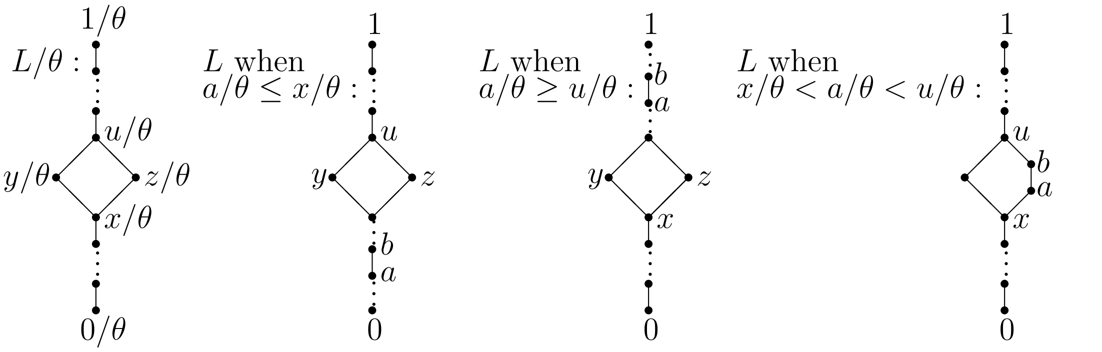
Case (i).1: Assume that $|L/\theta| = n - 1$, that is, according to Lemma 4.3.1(iii), $L/\theta = \{\{a, b\}\} \cup \{\{x\} \ : \ x \in L \setminus \{a, b\}\}$ and $[a, b]_L$ is a c-edge, thus b is the unique successor of a and a is the unique predecessor of b. Since $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} = 5 \cdot 2^{(n-1)-5}$, Theorem 4.3.6 and the induction hypothesis ensure us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}\}$.

Subcase (i).1.1: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-2} = 2^{(n-1)-1}$, that is $\{\{a, b\}\} \cup \{\{x\} \ : \ x \in L \setminus \{a, b\}\} = L/\theta \cong \mathcal{C}_{n-1}$ by Theorem 4.3.6(i), and thus, for any $x, y \in L \setminus \{a, b\}$, either $x/\theta \leq a/\theta$ or $a/\theta = b/\theta \leq x/\theta$, and either $x/\theta \leq y/\theta$ or $y/\theta \leq x/\theta$, that is, by the form of the classes of θ and Remark 4.3.2, either $x \leq a$ or $b \leq x$, and either $x \leq y$ or $y \leq x$, therefore $L \cong \mathcal{C}_n$. But then $|\mathrm{Con}(L)| = 2^{n-1}$, which contradicts the hypothesis of the present theorem that $|\mathrm{Con}(L)| < 2^{n-2}$.

Subcase (i).1.2: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-3} = 2^{(n-1)-2}$, that is, according to Theorem 4.3.6(ii), $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-3} \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-3}$ for some $k \in [1, n - 4]$. If we denote the elements of L/θ as in the first diagram below, with $x, y, z, u \in L$, and we also consider the facts that $|L| - |L/\theta| = 1$, b is the unique successor of a and a is the unique predecessor of b, $a/\theta = b/\theta = \{a, b\}$ and $v/\theta = \{v\}$ for all $v \in L \setminus \{a, b\}$, then we notice that L is in one of the following situations, represented in the three diagrams of L after that of L/θ:

- if $a/\theta = b/\theta \leq x/\theta$, then $b \leq x$ and $L \cong \mathcal{C}_2 \dotplus L/\theta \cong \mathcal{C}_{k+1} \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-3}$, while, if $a/\theta = b/\theta \geq u/\theta$, then $a \geq u$ and $L \cong L/\theta \dotplus \mathcal{C}_2 \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-2}$, but in these situations $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis of the theorem that $|\mathrm{Con}(L)| < 2^{n-2}$;

- if $x/\theta < a/\theta = b/\theta < u/\theta$, then $x < a < b < u$, hence $\{a,b\} \cap \{y,z\} \neq \varnothing$, that is $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-3} \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$, thus $|\mathrm{Con}(L)| = 2^{k-1} \cdot 5 \cdot 2^{n-k-4} = 5 \cdot 2^{n-5}$, which contradicts the assumption that $|\mathrm{Con}(L)| > 5 \cdot 2^{n-5}$.



Case (i).2: Now assume that $|L/\theta| = n-2$, which means that we are in the situation from Lemma 4.3.1(iv), and assume, for instance, that a is meet-reducible, that is $a \prec c$ for some $c \in L \setminus \{b\}$, and we have $b \prec b \vee c$ and $c \prec b \vee c$, that is $a/\theta = \{a,b\} \prec \{c, b \vee c\} = c/\theta$ by Remark 4.3.2, $[a, b \vee c]_L = \{a,b,c,b\vee c\} \cong \mathcal{C}_2^2$, and $x/\theta = \{x\}$ for all $x \in L \setminus \{a,b,c,b\vee c\}$; the dual case is analogous to this one. Since $|\mathrm{Con}(L/\theta)| > 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4} = 2^{(n-2)-2}$, Theorem 4.3.6 ensures us that $|\mathrm{Con}(L/\theta)| = 2^{(n-2)-1} = 2^{n-3}$ and $\{\{a,b\},\{c, b\vee c\}\} \cup \{\{x\} \ : \ x \in L \setminus \{a,b,c,b \vee c\}\} = L/\theta \cong \mathcal{C}_{n-2}$. So L/θ is a chain, thus, for all $x, y \in L \setminus \{a,b,c,b\vee c\}$, we have either $x/\theta \leq a/\theta \prec c/\theta$ or $a/\theta \prec c/\theta = (b \vee c)/\theta \leq x/\theta$, and either $x/\theta \leq y/\theta$ or $y/\theta \leq x/\theta$, hence, by the form of the classes of θ and Remark 4.3.2, we have either $x \leq a$ or $b \vee c \leq x$, and either $x \leq y$ or $y \leq x$, that is $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-2}$ for some $k \in [1, n-3]$, with $\{a,b,c,b\vee c\}$ being the sublattice of L isomorphic to $\mathcal{C}_2^2$; but then $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis of the theorem that $|\mathrm{Con}(L)| < 2^{n-2}$.

Therefore, indeed, $|\mathrm{Con}(L)| \leq 5 \cdot 2^{n-5}$. Now assume that $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$, that is $|\mathrm{Con}(L/\theta)| \geq 5 \cdot 2^{n-6} > 4 \cdot 2^{n-6} = 2^{n-4}$, thus, as above, $|L/\theta| \in \{n-1, n-2\}$. By Case (i).1, the equality $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ shows that, if $|L/\theta| = n-1$, then, for some $k \in [1, n-4]$, $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-3}$ and $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$. By Case (i).2, we cannot have $|L/\theta| = n-2$.

(ii) Assume that $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, and assume by absurdum that $|\mathrm{Con}(L)| > 2^{n-3}$, that is $|\mathrm{Con}(L/\theta)| > 2^{n-4} = 2^{(n-3)-1}$, hence $|L/\theta| > n-3$ by Theorem

4.3.6(i), thus $|L/\theta| \in \{n-1, n-2\}$. By Cases (i).1 and (i).2 above, in both of these situations we obtain that $|\mathrm{Con}(L)| \geq 5 \cdot 2^{n-5}$, contradicting the current assumption that $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$. Therefore $|\mathrm{Con}(L)| \leq 2^{n-3}$.

Now assume that $|\mathrm{Con}(L)| = 2^{n-3}$, that is $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-3)-1}$, hence $|L/\theta| \geq n-3$ by Theorem 4.3.6(i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

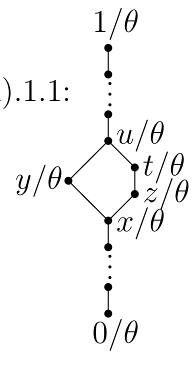
Case (ii).1: Assume that $|L/\theta| = n-1$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-1)-3}$, Theorem 4.3.6 and the induction hypothesis ensure us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5 \cdot 2^{n-6}, 2^{n-4}\}$. By Case (i).1, we cannot have $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}\}$.

Subcase (ii).1.1: Assume that $|\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-6}$, which, by the induction hypothesis, means that $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-4}$ for some $k \in [1, n-5]$, thus L is in one of the following situations, that we separate as above, where the elements of L/θ are denoted as in the diagram below, with $x, y, z, t, u \in L$:

- if $a/\theta = b/\theta \leq x/\theta$, then $a \prec b \leq x$ and $L \cong \mathcal{C}_2 \dotplus L/\theta \cong \mathcal{C}_{k+1} \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-4}$, while, if $a/\theta = b/\theta \geq u/\theta$, then $u \leq a \prec b$ and $L \cong L/\theta \dotplus \mathcal{C}_2 \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$, hence $|\mathrm{Con}(L)| = 2 \cdot |\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-5}$;

- if $x/\theta < a/\theta = b/\theta < u/\theta$, then $x < a \prec b < u$ and: either $\{a, b\} \cap \{z, t\} \neq \varnothing$, in which case a, b, z, t are pairwise comparable, because otherwise a would be meet-reducible or b would be join-reducible, thus $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$, or $y \in \{a, b\}$, thus $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$, hence $|\mathrm{Con}(L)| = 2^{k-1} \cdot (2^2 + 3) \cdot 2^{n-k-5} = 7 \cdot 2^{n-6}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 2^{n-3}$.

L/θ in Subcase (ii).1.1:



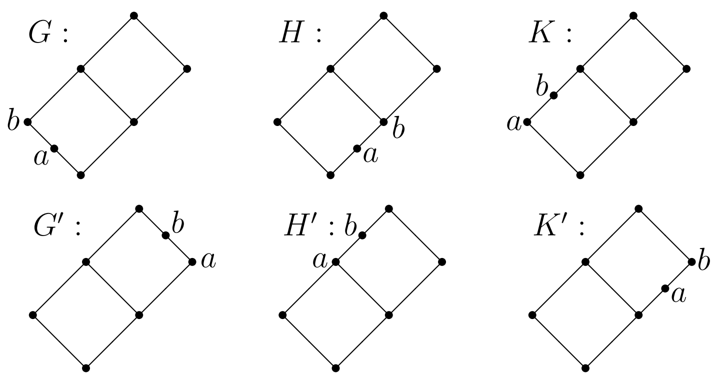
The following subcases can be treated exactly as above. For brevity, we only indicate the shapes of the lattices in the remaining part of the proof.

Subcase (ii).1.2: Assume that $|\mathrm{Con}(L/\theta)| = 2^{n-4} = 2^{(n-1)-3}$, which, by the induction hypothesis, means that either $L/\theta \cong \mathcal{C}_r \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-r-5}$ for some $r \in [1, n-6]$, or $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-5}$ for some $k, m \in \mathbb{N}$ such that $k + m \leq n - 6$, so that L is in one of the following situations:

- $L \cong \mathcal{C}_{r+1} \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-r-4}$ or $L \cong \mathcal{C}_{k+1} \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-5}$ or $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{m+1} \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-5}$ or $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-4}$, thus $|\mathrm{Con}(L)| = 2 \cdot |\mathrm{Con}(L/\theta)| = 2^{n-3}$;

- $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-5}$ or $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-m-5}$, in which case $|\mathrm{Con}(L)| = 5 \cdot 2^2 \cdot 2^{k-1+m-1+n-k-m-6} = 5 \cdot 2^{n-6} < 7 \cdot 2^{n-6} < 8 \cdot 2^{n-6} = 2^{n-3}$, contradicting the current assumption that $|\mathrm{Con}(L)| = 2^{n-3}$;

- $L \cong \mathcal{C}_r \dotplus G \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus G' \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus H \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus H' \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus K \dotplus \mathcal{C}_{n-r-5}$ or $L \cong \mathcal{C}_r \dotplus K' \dotplus \mathcal{C}_{n-r-5}$, where G, H and K are the following glueings of a pentagon with a four-element Boolean algebra and G', H' and K' are the duals of G, H and K, respectively, hence $|\mathrm{Con}(L)| = 9 \cdot 2^{r-1+n-r-6} = 9 \cdot 2^{n-7} < 14 \cdot 2^{n-7} = 7 \cdot 2^{n-6} < 2^{n-3}$, contradicting the current assumption that $|\mathrm{Con}(L)| = 2^{n-3}$, since $|\mathrm{Con}(G)| = |\mathrm{Con}(H)| = |\mathrm{Con}(K)| = 9$, which is simple to verify, and thus $|\mathrm{Con}(G')| = |\mathrm{Con}(H')| = |\mathrm{Con}(K')| = 9$ as well; in the diagrams below, we are indicating the positions of a and b in these copies of G, H, K, G', H' and K' from L, which, along with the shapes of these lattices, are easy to derive from the fact that, by the hypothesis of Case (ii).1, $\mathrm{con}(a, b)$ only collapses a and b:

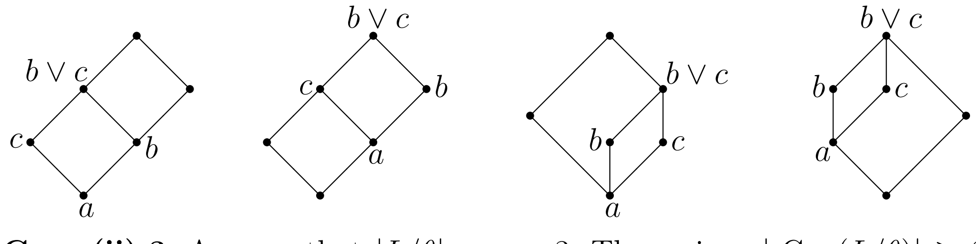


Case (ii).2: Assume that $|L/\theta| = n-2$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-2)-2}$, Theorem 4.3.6 ensures us that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}\}$. By Case (i).2, we cannot have $|\mathrm{Con}(L/\theta)| = 2^{n-3}$, thus $|\mathrm{Con}(L/\theta)| = 2^{n-4}$, hence $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-4}$ for some $k \in [1, n-5]$, according to Theorem 4.3.6(ii). We are in the situation from Lemma 4.3.1(iv), hence a is meet-reducible or b is join-reducible. We will assume that a is meet-reducible, that is $a \prec c$ for some $c \in L \setminus \{b\}$, and we will apply Lemma 4.3.1(iv) and Remark 4.3.2; the case when b is join-reducible shall follow by duality. Since $\{a,b\} = a/\theta \prec c/\theta = \{c, b\vee c\}$ and, for all $x \in L \setminus (a/\theta \cup c/\theta) = L \setminus \{a,b,c,b\vee c\}$, $x/\theta = \{x\}$ and $x \notin [a, b\vee c]_L$, L has one of the following forms:

- $L \cong \mathcal{C}_s \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_t \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-s-t-4}$ for some $s, t \in \mathbb{N}$ such that $s+t \leq n-5$; in this case, one of the two copies of $\mathcal{C}_2^2$ from L is $\{a,b,c,b\vee c\}$, $k \in \{s, s+t+2\}$, and, indeed, $|\mathrm{Con}(L)| = 2^{n-3}$;

- $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$, in which case, indeed, $|\mathrm{Con}(L)| = 2^{n-3}$, and a, b, c, $b\vee c$ belong to the copy of $\mathcal{C}_2 \times \mathcal{C}_3$ from L, in which they are situated as in one of the following first two diagrams, since $\theta = \mathrm{con}(a,b)$ only collapses a, b and $c, b\vee c$;

- $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)) \dotplus \mathcal{C}_{n-k-4}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)) \dotplus \mathcal{C}_{n-k-4}$, in which a, b, c and $b\vee c$ would be positioned in the copy of $\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_2)$, respectively $\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$, as in the third and fourth diagrams below, but then $|\mathrm{Con}(L)| = 3 \cdot 2^{k-1+n-k-5} = 3 \cdot 2^{n-6} < 7 \cdot 2^{n-6} < 8 \cdot 2^{n-6} = 2^{n-3}$, which contradicts the current hypothesis that $|\mathrm{Con}(L)| = 2^{n-3}$.



Case (ii).3: Assume that $|L/\theta| = n-3$. Then, since $|\mathrm{Con}(L/\theta)| \geq 2^{n-4} = 2^{(n-3)-1}$, by Theorem 4.3.6(i), it follows that $|\mathrm{Con}(L/\theta)| = 2^{n-4}$, so that $L/\theta \cong$

$\mathcal{C}_{n-3}$. We are in the case from Lemma 4.3.3; assume that a is meet-reducible, that is $a \prec c$ for some $c \in L \setminus \{b\}$; the case when b is join-reducible follows by duality.

In the situation from Lemma 4.3.3(i), since L/θ is a chain, it follows that, for any $x, y \in L \setminus \{a, b, c, b \vee c\}$, $\{x\} = x/\theta < a/\theta = \{a, b, c, b \vee c\}$ or $a/\theta = (b \vee c)/\theta < x/\theta$, and $x/\theta \leq y/\theta = \{y\}$ or $y/\theta \leq x/\theta$, so $x \leq y$ or $y \leq x$, and $x < z$ for every $z \in \{a, b, c, b \vee c\}$ or $z < x$ for every $z \in \{a, b, c, b \vee c\}$. Therefore $L \cong \mathcal{C}_k \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_{n-k-2}$ for some $k \in [1, n-3]$, thus $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis of the present theorem that $|\mathrm{Con}(L)| < 2^{n-2}$.

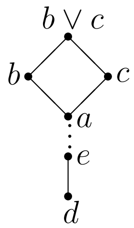
In the same way, in the situations (ii) and (iv) from Lemma 4.3.3, we obtain that $L \cong \mathcal{C}_k \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_{n-k-2}$ for some $k \in [1, n-3]$, thus $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts both the hypothesis of the theorem that $|\mathrm{Con}(L)| < 2^{n-2}$ and the fact that $\theta = \mathrm{con}(a, b)$. Similarly, in the situations (iii) and (v) from Lemma 4.3.3, we get that $L \cong \mathcal{C}_k \dot{+} \mathcal{N}_5 \dot{+} \mathcal{C}_{n-k-3}$ for some $k \in [1, n-4]$, hence $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 2^{n-3}$.

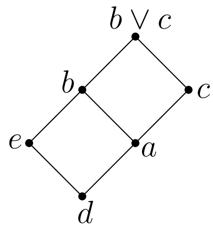
Now assume we are in the situation from Lemma 4.3.3(vi), with d and e as in the statement of the lemma. Since L/θ is a chain, without loss of generality, we may assume that $d/\theta < a/\theta \prec c/\theta$, because the other case is dual to this one. So, in L/θ, we will have $\{d, e\} < \{a, b\} \prec \{c, b \vee c\}$ and, for all $x \in L \setminus \{a, b, c, b \vee c, d, e\}$: either $x/\theta < a/\theta \prec c/\theta$ or $a/\theta \prec c/\theta = (b \vee c)/\theta < x/\theta$, and either $x/\theta \leq d/\theta$ or $d/\theta = e/\theta < x/\theta$, therefore, since $x/\theta = \{x\}$, Remark 4.3.2 ensures us that we have either $x < a$ or $b \vee c < x$, and either $x < d$ or $e < x$.

If we had $e < a$, then $L \cong \mathcal{C}_k \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_{n-k-2}$ for some $k \in [3, n-3]$, because $d, e, a, b, c, b \vee c$ would be positioned in L as in the first diagram below, thus $|\mathrm{Con}(L)| = 2^{n-2}$, which contradicts the hypothesis of the theorem that $|\mathrm{Con}(L)| < 2^{n-2}$, as well as the fact that $\theta = \mathrm{con}(a, b)$. We have $\{d, e\} = d/\theta < a/\theta = \{a, b\}$. Since a/θ and $d/\theta = e/\theta$ are convex, we cannot have $e > a$. Hence e and a are incomparable, $d < a$ and $e < b$. So $d \leq a \wedge e \leq e$, thus $a \wedge e \in d/\theta = \{d, e\}$ by the convexity of d/θ, hence $a \wedge e = d$ since $e \not< a$ by the above. Analogously, $a \vee e = b$. Hence $\{d, e, a, b, c, b \vee c\} \cong \mathcal{C}_2 \times \mathcal{C}_3$.

Recall that $d \prec e$, $a \prec b \prec b \vee c$, $a \prec c \prec b \vee c$ and $[a, b \vee c]_L = \{a, b, c, b \vee c\}$. Assume by absurdum that $[d, b]_L \neq \{d, e, a, b\}$, that is $x \in [d, b]_L$ for some $x \in$

$L \setminus \{d, e, a, b, c, b \vee c\} = L \setminus (d/\theta \cup a/\theta \cup c/\theta)$. If x is comparable to neither e, nor a, then $\{d, e, x, a, b\} \cong \mathcal{M}_3$, thus $(a, d) \in \mathrm{con}(a, b) = \theta$, which contradicts the fact that $a/\theta \neq d/\theta$. If x is comparable to a, then $d < x < a$, while, if x is comparable to e, then $e < x < b$, since $d < x < b$, $d \prec e$ and $a \prec b$; in each of these cases, $\{d, e, x, a, b\} \cong \mathcal{N}_5$, so $x \in a/\mathrm{con}(a, b) = a/\theta$ in the first of these two cases, and $x \in d/\mathrm{con}(a, b) = d/\theta$ in the second, and each of these situations contradicts the fact that $x \notin d/\theta \cup a/\theta \cup c/\theta$. Therefore $[d, b]_L = \{d, e, a, b\}$ and thus $[d, b \vee c]_L = \{d, e, a, b, c, b \vee c\} \cong \mathcal{C}_2 \times \mathcal{C}_3$, so d, e, a, b, c, $b \vee c$ are positioned in L as in the second diagram below, and, since L/θ is a chain, for all $x \in L \setminus \{d, e, a, b, c, b \vee c\}$, we have $\{x\} = x/\theta < d/\theta \prec a/\theta \prec c/\theta$ or $d/\theta \prec a/\theta \prec c/\theta < x/\theta$, thus $x < d$ or $b \vee c < x$ by Remark 4.3.2. Hence $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$ for some $k \in [1, n-5]$, which, indeed, has $|\mathrm{Con}(L)| = 2^{n-3}$.





(iii) Assume that $|\mathrm{Con}(L)| < 2^{n-3}$ and assume by absurdum that $|\mathrm{Con}(L)| > 7 \cdot 2^{n-6}$, so $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7} > 4 \cdot 2^{n-7} = 2^{n-5} = 2^{(n-4)-1}$ by Lemma 4.3.1(ii), hence $|L/\theta| > n-4$ by Theorem 4.3.6(i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

Case (iii).1: Assume that $|L/\theta| = n - 1$. Since $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} = 7 \cdot 2^{(n-1)-6}$, by Theorem 4.3.6 and the induction hypothesis, we get that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5 \cdot 2^{n-6}, 2^{n-4}\}$. By Case (i).1, $|\mathrm{Con}(L/\theta)| \notin \{2^{n-2}, 2^{n-3}\}$. By Subcase (ii).1.1, since $|\mathrm{Con}(L)| > 7 \cdot 2^{n-6}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 5 \cdot 2^{n-6}$. Finally, by Subcase (ii).1.2, since $2^{n-3} > |\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$.

Case (iii).2: Assume that $|L/\theta| = n - 2$. Since $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7}$, by Theorem 4.3.6 and the induction hypothesis, it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}\}$. By Case (i).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-3}$. By Case (ii).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$.

Case (iii).3: Assume that $|L/\theta| = n - 3$. Since $|\mathrm{Con}(L/\theta)| > 7 \cdot 2^{n-7} > 4 \cdot 2^{n-7} = 2^{n-5} = 2^{(n-3)-2}$, by Theorem 4.3.6, it follows that $|\mathrm{Con}(L/\theta)| = 2^{n-4} =$

$2^{(n-3)-1}$ and hence $L/\theta \cong \mathcal{C}_{n-3}$. By Case (ii).3, it follows that we cannot have $2^{n-3} > |\mathrm{Con}(L)| > 7 \cdot 2^{n-6}$.

Therefore $|\mathrm{Con}(L)| \leq 7 \cdot 2^{n-6}$.

Now assume that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$, so $|\mathrm{Con}(L/\theta)| \geq 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7} > 4 \cdot 2^{n-7} = 2^{n-5} = 2^{(n-4)-1}$ by Lemma 4.3.1(ii), hence $|L/\theta| > n-4$ by Theorem 4.3.6(i), thus $|L/\theta| \in \{n-1, n-2, n-3\}$.

Case 1: Assume that $|L/\theta| = n-1$. Since $|\mathrm{Con}(L/\theta)| \geq 7 \cdot 2^{n-7} = 7 \cdot 2^{(n-1)-6}$, by Theorem 4.3.6 and the induction hypothesis, it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-2}, 2^{n-3}, 5 \cdot 2^{n-6}, 2^{n-4}, 7 \cdot 2^{n-7}\}$. By Case (i).1, $|\mathrm{Con}(L/\theta)| \notin \{2^{n-2}, 2^{n-3}\}$.

Subcase 1.1: Assume that $|\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-6}$. Since we also have $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$, by Subcase (i).1.1, it follows that, for some $k \in [1, n-3]$, $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-4} \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$ and either $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$.

Subcase 1.2: Assume that $|\mathrm{Con}(L/\theta)| = 7 \cdot 2^{n-7}$, so, by the induction hypothesis, for some $k \in [1, n-6]$, $L/\theta \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-5}$ or $L/\theta \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-5}$, thus, since $[a, b]_L$ is a c-edge in this Case 1, we have one of the following situations:

- $L \cong \mathcal{C}_{k+1} \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-5}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ or $L \cong \mathcal{C}_{k+1} \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-5}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$;

- $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_6) \dotplus \mathcal{C}_{n-k-5}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-5}$, but in these cases $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-7} \times (\mathcal{C}_2^3 \dotplus \mathcal{C}_2^2)$, thus $|\mathrm{Con}(L)| = 2^{n-7} \cdot (2^3 + 3) = 11 \cdot 2^{n-7} < 14 \cdot 2^{n-7} = 7 \cdot 2^{n-6}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$.

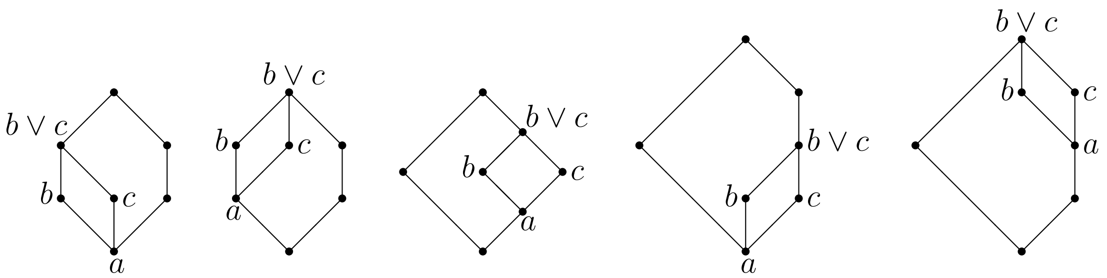
Case 2: Assume that $|L/\theta| = n-2$. Then, by Lemma 4.3.1(iv), we can assume that a is meet-reducible, that is $a \prec c$ for some $c \in L \setminus \{b\}$, since the other case is dual to this one. Since $|\mathrm{Con}(L/\theta)| \geq 7 \cdot 2^{n-7} > 5 \cdot 2^{n-7} = 5 \cdot 2^{(n-2)-5}$, by Theorem 4.3.6 and the induction hypothesis, it follows that $|\mathrm{Con}(L/\theta)| \in \{2^{n-3}, 2^{n-4}, 5 \cdot 2^{n-7}\}$. By Case (i).2, $|\mathrm{Con}(L/\theta)| \neq 2^{n-3}$. By Case (ii).2, since $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$, it follows that $|\mathrm{Con}(L/\theta)| \neq 2^{n-4}$. Hence $|\mathrm{Con}(L/\theta)| = 5 \cdot 2^{n-7}$, thus, by the induction hypothesis, for some $k \in [1, n-6]$, $L/\theta \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-5}$, hence L is in one of the following situations, as shown by Lemma 4.3.1(iv):

- either $k \geq 4$ and, for some $r, s \in \mathbb{N}$ such that $r + s = k$, $L \cong \mathcal{C}_r \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_s \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-5}$, or $n \geq k+9$ and, for some $r, s \in \mathbb{N}$ such that $r + s = n - k - 5$, $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_r \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_s$, but in these cases $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$, so $|\mathrm{Con}(L)| = 5 \cdot 2^2 \cdot 2^{n-8} = 5 \cdot 2^{n-6} < 7 \cdot 2^{n-6}$, which contradicts the current assumption that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$;

- $L \cong \mathcal{C}_k \dotplus ((\mathcal{C}_2^2 \dotplus \mathcal{C}_2) \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-5}$ or $L \cong \mathcal{C}_k \dotplus ((\mathcal{C}_2 \dotplus \mathcal{C}_2^2) \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $(\mathcal{C}_2^2 \dotplus \mathcal{C}_2) \boxplus \mathcal{C}_4$, respectively $(\mathcal{C}_2 \dotplus \mathcal{C}_2^2) \boxplus \mathcal{C}_4$ from L as depicted in the first two diagrams below, but in these cases $\mathrm{Con}(L) \cong \mathcal{C}_4 \times \mathcal{C}_2^{n-7}$, so $|\mathrm{Con}(L)| = 2^{n-5} = 2 \cdot 2^{n-6} < 7 \cdot 2^{n-6}$, which gives us another contradiction to the current assumption that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$;

- $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_2)) \dotplus \mathcal{C}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $\mathcal{C}_3 \boxplus (\mathcal{C}_2 \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_2)$ from L as depicted in the third diagram below, but in this case $\mathrm{Con}(L) \cong (\mathcal{C}_2^2 \dotplus \mathcal{C}_2^2) \times \mathcal{C}_2^{n-7}$, so $|\mathrm{Con}(L)| = 7 \cdot 2^{n-7} < 7 \cdot 2^{n-6}$, and, again, we obtain a contradiction to the assumption that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$;

- $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_3)) \dotplus \mathcal{C}_{n-k-5}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus (\mathcal{C}_3 \dotplus \mathcal{C}_2^2)) \dotplus \mathcal{C}_{n-k-5}$, with the positions of a, b, c and $b \vee c$ in the copy of $\mathcal{C}_3 \boxplus (\mathcal{C}_2^2 \dotplus \mathcal{C}_3)$, respectively $\mathcal{C}_3 \boxplus (\mathcal{C}_3 \dotplus \mathcal{C}_2^2)$ from L as depicted in the last two diagrams below, but in these cases $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-7} \times \mathcal{C}_4$, thus $|\mathrm{Con}(L)| = 4 \cdot 2^{n-7} = 2^{n-5} < 14 \cdot 2^{n-7} = 7 \cdot 2^{n-6}$, which gives us another contradiction to $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$.



Case 3: Assume that $|L/\theta| = n - 3$. Since $|\mathrm{Con}(L/\theta)| \geq 7 \cdot 2^{n-7} > 4 \cdot 2^{n-7} = 2^{n-5} = 2^{(n-3)-2}$, Theorem 4.3.6 ensures us that $|\mathrm{Con}(L/\theta)| = 2^{n-4} = 2^{(n-3)-1}$, thus

$L/\theta \cong \mathcal{C}_{n-3}$. But Case (iii).3 shows us that, in this case, $|\mathrm{Con}(L)| \neq 7 \cdot 2^{n-6}$, so we have a contradiction to the current assumption that $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$. □

Corollary 4.3.8.

(i) $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-1}$.

(ii) $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $n \geq 4$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-2}$.

(iii) $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ *iff* $n \geq 5$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-5} \times (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$.

(iv) $|\mathrm{Con}(L)| = 2^{n-3}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-3}$.

(v) $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-6} \times (\mathcal{C}_2^2 \dotplus \mathcal{C}_2^2)$.

Proof. The converse implications are trivial, and the direct implications follow from Theorems 4.3.6 and 4.3.7. □

Summary

In this dissertation, our goal was to get a better understanding of the structure of some lattices and some related lattices. We described slim rectangular lattices by permutations, and we also counted these lattices. We searched for minimum-sized generating sets of the lattices of quasiorders. Also, we characterized lattices with many congruences. While counting these congruences, we described the structure of the congruence lattices, too.

Following the introductory Chapter 1, Chapter 2 is about slim rectangular lattices and is based on [33]. An element of a lattice is *join-irreducible* if it has exactly one lower cover. A finite lattice L is *slim*, if $\mathrm{Ji}\, L$, the set of the join-irreducible elements of L, is included in the union of two chains of L. Slim lattices are *planar*, that is, they possess planar diagrams. By our convention, the lattice properties of a planar lattice diagram D are those of the lattice determined by D. A semimodular (lattice) diagram D is *rectangular* if both its *left boundary chain*, denoted by $\mathrm{C}_{\mathrm{l}}(D)$, and its *right boundary chain*, $\mathrm{C}_{\mathrm{r}}(D)$, have exactly one doubly irreducible element, and these two elements, called the *corners* of D, are complementary. Rectangular lattices are those that have rectangular diagrams.

A minimal non-chain region of a planar lattice diagram D is called a *cell*. A four-element cell is a 4-*cell*. A diagram is a 4-*cell diagram* if all of its cells are 4-cells. It was proved in Grätzer and Knapp [61, Lemmas 4 and 5] that D is a slim semimodular diagram iff it is a 4-cell diagram and no two distinct 4-cells have the same bottom. Two prime intervals of a slim semimodular diagram D are *consecutive* if they are opposite sides of a 4-cell. The consecutiveness of two prime intervals in a slim semimodular lattice L does not depend on the planar

diagram chosen. Maximal sequences of consecutive prime intervals form a *trajectory*. In other words, a trajectory is a class of the equivalence relation generated by consecutiveness. By Czédli and Schmidt [43, Lemma 2.8], if T is a trajectory of a slim semimodular diagram D, then T contains exactly one prime interval of $\mathrm{C_l}(D)$, and the same holds for $\mathrm{C_r}(D)$. Going from left to right, T does not branch out. First T goes up (possibly in zero steps), then it may turn to the lower right, and finally it goes down (possibly, in zero steps).

We denote the set of the similarity classes of *slim rectangular diagrams* of length n and that of *slim semimodular diagrams* of length n by the acronyms $\mathrm{SRectD}(n)$ and $\mathrm{SSmodD}(n)$, respectively. Similarly, the set of the isomorphism classes of *slim rectangular lattices* of length n, that of *slim semimodular lattices* of length n are denoted by $\mathrm{SRectL}(n)$ and $\mathrm{SSmodL}(n)$.

There are several known tools for examining semimodular lattices; one of them is describing these lattices by permutations. For a slim rectangular diagram D of length n, let $\mathrm{C_l}(D) = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $\mathrm{C_r}(D) = \{0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1\}$. Following Czédli and Schmidt [45], the *permutation* $\pi = \pi_D \in S_n$ is defined by the rule $\pi(i) = j$ iff $[c_{i-1}, c_i]$ and $[d_{j-1}, d_j]$ belong to the same trajectory. Czédli and Schmidt proved in [45] that the map $\mathrm{SSmodD}(n) \to S_n$, defined by $D \mapsto \pi_D$, is a bijection.

In Chapter 2, we described the permutations belonging to slim rectangular lattices.

Definition 2.3.1. A permutation $\pi \in S_n$ is called *rectangular* if it satisfies the following three properties.

(i) For all i and j, if $\pi^{-1}(1) < i < j \leq n$, then $\pi(i) < \pi(j)$.

(ii) For all i and j, if $\pi(1) < i < j \leq n$, then $\pi^{-1}(i) < \pi^{-1}(j)$.

(iii) $\pi(n) < \pi(1)$.

Proposition 2.3.3. *A slim, semimodular, planar diagram D of length $n \geq 2$ is rectangular if and only if $\pi = \pi_D \in S_n$ is rectangular.*

With the help of this description, we gave formulas for the numbers of slim rectangular diagrams and slim rectangular lattices.

Proposition 2.4.2. *For $2 \leq n \in \mathbb{N}$, the number of (the similarity classes of) slim rectangular diagrams of length n is*

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\leq n\\ a,b\in\mathbb{N}}} \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,.$$

Let $\mathrm{Invl}(k) = \{\pi \in S_k : \pi = \pi^{-1}\}$ denote the set of *involutions* acting on the set $\{1,\ldots,k\}$. For $k \in \mathbb{N}$, the number of involutions in S_k is $|\mathrm{Invl}(k)| = \sum_{j=0}^{\lfloor k/2\rfloor} \binom{k}{k-2j} \cdot (2j-1)!!$.

Proposition 2.4.4. *For $2 \leq n \in \mathbb{N}$, the number of (the isomorphism classes of) slim rectangular lattices of length n is*

$$|\mathrm{SRectL}(n)| = \frac{1}{2} \cdot \left(|\mathrm{SRectD}(n)| + \sum_{a=1}^{\lfloor n/2\rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)|\right).$$

Based on the formulas, we were able to give asymptotic results, in which $e \approx 2.71828$.

Proposition 2.5.1. *The number of (the similarity classes of) slim rectangular diagrams of length n is asymptotically $(n-2)!\cdot e^2$, that is, $|\mathrm{SRectD}(n)| \sim (n-2)!\cdot e^2$.*

This led to the main result of Chapter 2.

Theorem 2.5.2. *The number of (the isomorphism classes of) slim rectangular lattices of length n is asymptotically $(n-2)! \cdot e^2/2$, that is,*

$$\lim_{n\to\infty} \frac{|\mathrm{SRectL}(n)|}{(n-2)! \cdot e^2/2} = 1.$$

In Chapter 3, we aimed to determine a minimum-sized generating set of the lattice of quasiorders, also of the lattice of transitive relations. This chapter was based on [37] and [72].

A *quasiorder* is a reflexive and transitive relation. Quasiorders on a set A form a complete lattice $\mathrm{Quo}(A)$. So do the transitive relations on A; their complete lattice is denoted by Tran(A). Similarly, Equ(A) will stand for the lattice of all equivalences on A.

For a subset X of Equ(A), Quo(A), or Tran(A), we say that X generates the complete lattice in question if the only complete sublattice including X is the whole lattice itself. For $k \in \mathbb{N} := \{1, 2, 3, \dots\}$, we say that a complete lattice L is *k-generated* if it can be generated by a k-element subset X. If a complete lattice is generated by a four-element subset $X = \{x_1, x_2, x_3, x_4\}$ such that $x_1 < x_2$ but both $\{x_1, x_3, x_4\}$ and $\{x_2, x_3, x_4\}$ are antichains, then we say that this lattice is *$(1 + 1 + 2)$-generated.*

All sets in this chapter were assumed to be of accessible cardinalities. A cardinal κ is *accessible* if it is finite, or it is infinite and for every $\lambda \leq \kappa$,

- either $\lambda \leq 2^\mu$ for some cardinal $\mu < \lambda$,
- or there is a set I of cardinals such that $\lambda \leq \sum_{\mu \in I} \mu$, $|I| < \lambda$, and $\mu < \lambda$ for all $\mu \in I$.

ZFC has a model in which all cardinals are accessible, hence the scope of many of our results includes all sets in an appropriate model of set theory.

It was known by Strietz [83] and [84], Zádori [87], and Czédli [7] that the complete lattice Equ(A) of all equivalences is four-generated, provided the size $|A|$ of A is an accessible cardinal and $|A| \geq 2$. Also, Equ(A) cannot be generated by less than four elements if $|A| \geq 4$. We know from Chajda and Czédli [4] and Takách [85] that Quo(A) is six-generated as a complete lattice, provided that $|A|$ is accessible. Actually, we know from Dolgos [48] for $2 \leq |A| \leq \aleph_0$ that the complete lattice Quo(A) is five-generated.

We extended Dolgos' result in two ways. The first one is short and states more (about all sets A where $|A|$ is accessible) than the second one, but it is based heavily on Czédli's quite involved and long constructions from [7] and [9]. This justifies the second way: we gave an easier, more understandable and self-contained construction for a five-element generating set of Quo(A) if $|A| \leq 2^{\aleph_0}$, based on Dolgos' work.

Theorem 3.3.1. *Let A be a set with at least three elements.*

(i) *If $|A|$ is an accessible cardinal, then* $\mathrm{Quo}(A)$ *is five-generated as a complete lattice.*

(ii) *If $\aleph_0 \leq |A| \leq 2^{\aleph_0}$, then* $\mathrm{Quo}(A)$ *is five-generated as a complete lattice.*

Following this result, Czédli proved in [19] that the complete lattice $\mathrm{Quo}(A)$ is four-generated for $|A| = \{\aleph_0\} \cup (\mathbb{N} \setminus \{1, 4, 6, 8, 10\})$. It is also shown in [19] that the complete lattice $\mathrm{Quo}(A)$ cannot be generated by less than four elements, provided $|A| \geq 3$. Concerning transitive relations, Dolgos [48] has shown that the complete lattice $\mathrm{Tran}(A)$ is eight-generated for $2 \leq |A| \leq \aleph_0$.

So our second goal in Chapter 3 was to show, in a concise but not self-contained way, that $\mathrm{Quo}(A)$ is four-generated if $|A| \neq 4$ and $|A|$ is an accessible cardinal. Furthermore, we proved that $\mathrm{Quo}(A)$ is $(1+1+2)$-generated in many (however not all) cases. We also improved the earlier results on the generating sets of $\mathrm{Tran}(A)$.

Theorem 3.4.9. *Let A be a non-singleton set. Then the following statements hold.*

- *If $|A| \neq 4$ and $|A|$ is an accessible cardinal, then the complete lattice* $\mathrm{Quo}(A)$ *is four-generated.*
- *If $|A| \geq 13$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the complete lattice* $\mathrm{Quo}(A)$ *is $(1+1+2)$-generated.*
- *If $13 \leq |A| \leq \aleph_0$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the lattice* $\mathrm{Quo}(A)$ (*not a complete one now*) *contains a $(1+1+2)$-generated sublattice that includes all atoms of* $\mathrm{Quo}(A)$.

Theorem 3.4.12. *If $3 \leq |A|$ and $|A|$ is an accessible cardinal, then* $\mathrm{Tran}(A)$ *is six-generated as a complete lattice*

Chapter 4 deals with the problem that given a natural number n, find the n-element finite lattices with the most, second-most, third-most, etc. congruences; also, give the diagram of the lattice of their congruences. This chapter is based on [77].

By Czédli and Mureşan [40], the set of all the congruences of an infinite lattice can be of any size between 2 and the cardinality of the lattice, or it can have the same cardinality as the lattice's subsets. But the situation is quite different for finite lattices. To formulate our results, the following lattice operations and notations are needed.

Let L and M be lattices. If L has a largest element 1^L and M has a smallest element 0^M, then the *glued sum* of L and M, denoted by $L \dot{+} M$, is obtained from L and M by identifying 1^L with 0^M and stacking M on top of L. If L and M are nontrivial bounded lattices, then the *horizontal sum* of L and M, denoted by $L \boxplus M$, is obtained from L and M by identifying their bottom elements 0^L and 0^M, identifying their top elements 1^L and 1^M, and letting every element of $L \setminus \{0^L, 1^L\}$ be incomparable to every element of $M \setminus \{0^M, 1^M\}$ in $L \boxplus M$. For any $n \in \mathbb{N}$, we denote the n-element chain by $\mathcal{C}_n$. As usual, $\mathcal{N}_5$ denotes the five-element nonmodular lattice $\mathcal{C}_3 \boxplus \mathcal{C}_4$.

Using these notations, Freese [50] and Czédli [21] determined the largest and second largest numbers of congruences. Namely, if L is a finite lattice with n elements, then $|\mathrm{Con}(L)| \le 2^{n-1}$, also, $|\mathrm{Con}(L)| = 2^{n-1}$ iff $L \cong \mathcal{C}_n$. In other words, a finite lattice can have at most as many congruences as the chain with the same number of elements has. Furthermore, if $|\mathrm{Con}(L)| < 2^{n-1}$, then $|\mathrm{Con}(L)| \le 2^{n-2}$, moreover, $|\mathrm{Con}(L)| = 2^{n-2}$ iff $L \cong \mathcal{C}_k \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_{n-k-2}$ for some $k \in [1, n-3]$. That means the second largest possible number of congruences is witnessed by a glued sum of two chains with the four-element Boolean algebra. Following the line of Czédli's proof, we obtained the next result about the lattices with the third, fourth and fifth largest possible numbers of congruences.

Theorem 4.3.7. *Let L be a finite lattice with n elements.*

(i) *If $|\mathrm{Con}(L)| < 2^{n-2}$, then $n \ge 5$, $|\mathrm{Con}(L)| \le 5 \cdot 2^{n-5} = 2^{n-3} + 2^{n-5}$, and: $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ iff $L \cong \mathcal{C}_k \dot{+} \mathcal{N}_5 \dot{+} \mathcal{C}_{n-k-3}$ for some $k \in [1, n-4]$.*

(ii) *If $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, then $|\mathrm{Con}(L)| \le 2^{n-3}$, and: $|\mathrm{Con}(L)| = 2^{n-3}$ iff either $n \ge 6$ and $L \cong \mathcal{C}_k \dot{+} (\mathcal{C}_2 \times \mathcal{C}_3) \dot{+} \mathcal{C}_{n-k-4}$ for some $k \in [1, n-5]$, or $n \ge 7$ and $L \cong \mathcal{C}_k \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_m \dot{+} \mathcal{C}_2^2 \dot{+} \mathcal{C}_{n-k-m-4}$ for some $k, m \in \mathbb{N}$ such that $k + m \le n - 5$.*

(iii) *If* $|\mathrm{Con}(L)| < 2^{n-3}$, *then* $|\mathrm{Con}(L)| \leq 7 \cdot 2^{n-6} = 2^{n-4} + 2^{n-5} + 2^{n-6}$, *and:* $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and, for some* $k \in [1, n-5]$, $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ *or* $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$.

Combining the earlier theorems with ours, we summarized the results on the lattices of the congruences of a finite lattice with the most, second-most, third-most, etc. congruences.

Corollary 4.3.8.

(i) $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-1}$.

(ii) $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $n \geq 4$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-2}$.

(iii) $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ *iff* $n \geq 5$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-5} \times (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$.

(iv) $|\mathrm{Con}(L)| = 2^{n-3}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-3}$.

(v) $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-6} \times (\mathcal{C}_2^2 \dotplus \mathcal{C}_2^2)$.

This dissertation is based on four of the author's papers. These publications are the following:

1. G. Czédli, T. Dékány, G. Gyenizse and J. Kulin: *The number of slim rectangular lattices.* Algebra Universalis **75/1** (2016), 33–50.

2. G. Czédli and J. Kulin: *A concise approach to small generating sets of lattices of quasiorders and transitive relations.* Acta Sci. Math. (Szeged) **83** (2017), 3–12.

3. J. Kulin: *Quasiorder lattices are five-generated.* Discussiones Mathematicae - General Algebra and Applications **36** (1) (2016), 59–70.

4. C. Mureşan and J. Kulin: *On the largest numbers of congruences of finite lattices.* Order **37** (2020), 445–460.

Another publication of the author not used in this dissertation:

- T. Dékány, G. Gyenizse and J. Kulin: *Permutations assigned to slim rectangular lattices*. Acta Sci. Math. (Szeged) **82** (2016), 19–28.

Összefoglaló (Summary in Hungarian)

E disszertációban az volt a célunk, hogy jobban megértsük bizonyos hálók és bizonyos kísérőhálók szerkezetét. A sovány téglalapszerű hálókat permutációk segítségével jellemeztük, és megadtuk e hálók számát. A kvázirendezések hálóinak minimális elemszámú generálóhalmazait kerestük. Továbbá jellemeztük a „sok" kongruenciával rendelkező hálókat, valamint megadtuk a kongruenciák számának néhány lehetséges legnagyobb értékét és a jellemzett hálók kongruenciahálóit is.

A bevezető 1. fejezetet követően a 2. fejezet sovány téglalapszerű hálókkal foglalkozik, és a [34] cikkünket dolgozza fel. Egy háló valamely elemét *egyesítés-irreducibilisnek* nevezzük, ha pontosan egy elemet fed. Egy L véges háló *sovány*, ha az egyesítés-irreducibilis elemek $\mathrm{Ji}\, L$ halmaza lefedhető két lánccal. A sovány hálók *síkbarajzolhatóak*, azaz van olyan diagramjuk, amely síkgráf. Amikor azt mondjuk, hogy egy D síkba rajzolt hálódiagram rendelkezik a Φ hálótulajdonsággal, azt úgy értjük, hogy a D által meghatározott háló rendelkezik a Φ tulajdonsággal. Egy D féligmoduláris hálódiagram *téglalapszerű*, ha a *bal határlánca*, amelyet $\mathrm{C_l}(D)$ jelöl, és a *jobb határlánca*, amelyet $\mathrm{C_r}(D)$ jelöl, pontosan egy-egy duplán irreducibilis elemet tartalmaz, és ez a két elem, melyeket D *sarkainak* hívunk, egymás komplementuma. A téglalapszerű diagrammal rendelkező hálókat nevezzük téglalapszerű hálóknak.

Egy síkba rajzolt D hálódiagram minimális, nem lánc tartományait *celláknak* hívjuk, a négyelemű cellákat pedig 4-*celláknak*. Egy diagram 4-*cella diagram*, ha az összes cellája 4-cella. Grätzer és Knapp [61, Lemma 4 és 5] bizonyította, hogy

D akkor és csak akkor sovány féligmoduláris diagram, ha 4-cella diagram, és két különböző 4-cellájának nem lehet ugyanaz az alsó eleme. Egy D sovány téglalapszerű diagram két prímintervalluma *egymásutáni*, ha egy 4-cella szemközti oldalai. Az, hogy egy L sovány féligmoduláris hálóban két prímintervallum egymásutáni-e, nem függ a diagram síkba rajzolásától. Egymásutáni prímintervallumok maximális sorozata *trajektóriát* alkot. Másképp fogalmazva, az egymásutániság által generált ekvivalenciareláció osztályait trajektóriáknak nevezzük. Czédli és Schmidt [43, Lemma 2.8] munkája alapján tudjuk, hogy egy sovány téglalapszerű D diagram bármely T trajektóriája $\mathrm{C}_\mathrm{l}(D)$-nek pontosan egy prímintervallumát tartalmazza, és ugyanez elmondható $\mathrm{C}_\mathrm{r}(D)$-re is. A T trajektória balról jobbra tart, nem ágazik ketté. Először felfelé halad (lehetséges, hogy nulla lépésben), majd lefelé fordulhat, és végül lefelé halad (lehetséges, hogy nulla lépésben).

Az n hosszúságú *sovány téglalapszerű diagramok* hasonlóság-osztályainak halmazát $\mathrm{SRectD}(n)$-nel jelöljük, az n hosszúságú *sovány féligmoduláris diagramok* hasonlóság-osztályainak halmazát pedig $\mathrm{SSmodD}(n)$-nel. Hasonlóan, az n hosszúságú *sovány téglalapszerű hálók* izomorfia-osztályainak halmazát $\mathrm{SRectL}(n)$ jelöli, az n hosszúságú *sovány féligmoduláris hálók* izomorfia-osztályainak halmazát pedig $\mathrm{SSmodL}(n)$.

Féligmoduláris hálók tanulmányozásához több ismert eszköz is a rendelkezésünkre áll, ezek egyike e hálók permutációkkal történő leírása. Egy n hosszúságú sovány téglalapszerű D diagram esetén legyen $\mathrm{C}_\mathrm{l}(D) = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ és $\mathrm{C}_\mathrm{r}(D) = \{0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1\}$. A Czédli és Schmidt [45] által leírt felépítést követve a $\pi = \pi_D \in S_n$ *permutációt* a következő szabállyal definiáljuk: $\pi(i) = j$, ha $[c_{i-1}, c_i]$ és $[d_{j-1}, d_j]$ ugyanahhoz a trajektóriához tartozik. Czédli és Schmidt bebizonyította [45]-ben, hogy a $D \mapsto \pi_D$ hozzárendeléssel definiált $\mathrm{SSmodD}(n) \to S_n$ leképezés bijekció.

A disszertáció 2. fejezetében leírtuk a sovány téglalapszerű hálókhoz tartozó permutációkat.

2.3.1. Definíció. Egy $\pi \in S_n$ permutációt *téglalapszerűnek* nevezünk, ha rendelkezik a következő három tulajdonsággal.

(i) Minden i és j esetén, ha $\pi^{-1}(1) < i < j \leq n$, akkor $\pi(i) < \pi(j)$.

(ii) Minden i és j esetén, ha $\pi(1) < i < j \leq n$, akkor $\pi^{-1}(i) < \pi^{-1}(j)$.

(iii) $\pi(n) < \pi(1)$.

2.3.3. Állítás. *Egy sovány, féligmoduláris, síkba rajzolt, n (≥ 2) hosszúságú D diagram akkor és csak akkor téglalapszerű, ha a $\pi = \pi_D \in S_n$ permutáció téglalapszerű.*

Ennek a jellemzésnek a segítségével formulákat tudtunk adni a sovány négyszögletes diagramok és a sovány négyszögletes hálók számára.

2.4.2. Állítás. *Az n (≥ 2) hosszúságú sovány téglalapszerű diagramok (hasonlóság-osztályainak) száma*

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b\leq n\\ a,b\in\mathbb{N}}} \binom{n-a-1}{b-1}\binom{n-b-1}{a-1}(n-a-b)!\,.$$

Legyen $\mathrm{Invl}(k) = \{\pi \in S_k : \pi = \pi^{-1}\}$ az S_k-beli *involúciók* halmaza. Ismert, hogy az S_k-beli involúciók száma $|\mathrm{Invl}(k)| = \sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{k-2j}\cdot(2j-1)!!$ (tetszőleges $k \in \mathbb{N}$ esetén).

2.4.4. Állítás. *Az n (≥ 2) hosszúságú sovány téglalapszerű hálók (izomorfia-osztályainak) száma*

$$|\mathrm{SRectL}(n)| = \frac{1}{2}\cdot\left(|\mathrm{SRectD}(n)| + \sum_{a=1}^{\lfloor n/2\rfloor}\binom{n-a-1}{a-1}\cdot|\mathrm{Invl}(n-2a)|\right).$$

A formulák alapján aszimptotikus eredményeink is születtek, ezekben e az Euler-féle számot jelöli ($e \approx 2.71828$).

2.5.1. Állítás. *Az n hosszúságú sovány téglalapszerű diagramok (hasonlóság-osztályainak) száma aszimptotikusan $(n-2)!\cdot e^2$, vagyis $|\mathrm{SRectD}(n)| \sim (n-2)!\cdot e^2$.*

A disszertáció 2. fejezetének fő eredménye a következő.

2.5.2. Tétel. *Az n hosszúságú sovány téglalapszerű hálók (izomorfia-osztályainak) száma aszimptotikusan $(n-2)! \cdot e^2/2$, azaz*

$$\lim_{n\to\infty} \frac{|\mathrm{SRectL}(n)|}{(n-2)! \cdot e^2/2} = 1.$$

A 3. fejezetben a célunk az volt, hogy meghatározzuk, legkevesebb hány elemmel generálható a kvázirendezések hálója, valamint a tranzitív relációk hálója. Ez a fejezet a [37] és [72] cikkeinken alapul.

Egy reflexív és tranzitív relációt *kvázirendezésnek* nevezünk. Egy A halmaz kvázirendezései teljes hálót alkotnak, melyet $\mathrm{Quo}(A)$-val jelölünk. Az A halmaz tranzitív relációi szintén teljes hálót alkotnak, ezt a hálót $\mathrm{Tran}(A)$ jelöli. Hasonlóan, $\mathrm{Equ}(A)$-val jelöljük az A ekvivalenciarelációi által alkotott teljes hálót.

Az $\mathrm{Equ}(A)$, $\mathrm{Quo}(A)$, illetve $\mathrm{Tran}(A)$ háló egy X részhalmaza esetén azt mondjuk, hogy X generálja az adott teljes hálót, ha az egyetlen teljes, X-et tartalmazó részháló maga az egész háló. Egy L teljes hálót k *generáltnak* nevezünk valamely $k \in \mathbb{N} := \{1, 2, 3, \dots\}$ egész számra, ha a hálót generálja egy k elemű részhalmaza. Ha egy teljes hálót generál olyan négyelemű $X = \{x_1, x_2, x_3, x_4\}$ részhalmaza, ahol $x_1 < x_2$, viszont $\{x_1, x_3, x_4\}$ és $\{x_2, x_3, x_4\}$ is antilánc, akkor azt mondjuk, hogy ez a háló $(1 + 1 + 2)$-*generált.*

Az ebben a fejezetben előforduló összes halmazról feltettük, hogy elérhető számosságú. Egy κ számosság *elérhető*, ha vagy véges, vagy pedig végtelen, és bármely $\lambda \le \kappa$ számosság esetén

- vagy $\lambda \le 2^\mu$ valamely $\mu < \lambda$ számosságra,
- vagy létezik számosságok olyan I halmaza, amelyre $\lambda \le \sum_{\mu\in I} \mu$, $|I| < \lambda$ és $\mu < \lambda$ minden $\mu \in I$ esetén.

A halmazelmélet ZFC axiómarendszerének létezik olyan modellje, amelyben minden számosság elérhető, tehát több eredményünk érvényes minden halmazra a halmazelmélet egy megfelelő modelljében.

Strietz [83] és [84], Zádori [87] és Czédli [7] eredményei alapján tudjuk, hogy az ekvivalenciarelációk $\mathrm{Equ}(A)$ teljes hálója négygenerált, feltéve, hogy $|A|$ egy elérhető számosság és $|A| \geq 2$. Továbbá $\mathrm{Equ}(A)$ nem generálható kevesebb, mint négy elemmel, ha $|A| \geq 4$. Chajda és Czédli [4] és Takách [85] megmutatta, hogy $\mathrm{Quo}(A)$ mint teljes háló hatgenerált, ha A elérhető számosságú. Dolgos [48] munkájából tudjuk, hogy $2 \leq |A| \leq \aleph_0$ esetén a $\mathrm{Quo}(A)$ teljes háló ötgenerált.

Dolgos eredményét kétféle módon általánosítottuk. Az első módszer rövid és a másodiknál többet ad (négygeneráltságot elérhető $|A|$ esetére), de nagyban támaszkodik Czédli [7], [9] bonyolultan és hosszan bizonyított eredményeire. Ez indokolja második módszerünket, amellyel Dolgos módszerét továbbfejlesztve külső hivatkozás nélkül bizonyítottuk $\mathrm{Quo}(A)$ ötgeneráltságát $|A| \leq 2^{\aleph_0}$ esetén.

3.3.1. Tétel. *Legyen A legalább háromelemű halmaz.*

(i) *Ha $|A|$ elérhető számosság, akkor* $\mathrm{Quo}(A)$ *mint teljes háló ötgenerált.*

(ii) *Ha $\aleph_0 \leq |A| \leq 2^{\aleph_0}$, akkor* $\mathrm{Quo}(A)$ *mint teljes háló ötgenerált.*

Ezt az eredményünket követően Czédli bebizonyította a [19] cikkében, hogy a $\mathrm{Quo}(A)$ teljes háló négygenerált abban az esetben, ha $|A| = \{\aleph_0\} \cup (\mathbb{N} \setminus \{1, 4, 6, 8, 10\})$. Azt is megmutatta [19]-ben, hogy a $\mathrm{Quo}(A)$ teljes háló nem generálható négynél kevesebb elemmel, feltéve, hogy $|A| \geq 3$. A tranzitív relációkat tekintve Dolgos [48] megmutatta, hogy a $\mathrm{Tran}(A)$ teljes háló nyolcgenerált $2 \leq |A| \leq \aleph_0$ esetén.

Tehát ezután a második célunk a 3. fejezetben az volt, hogy tömör bizonyítást adjunk arra, hogy $\mathrm{Quo}(A)$ négygenerált, ha $|A| \neq 4$ és $|A|$ elérhető számosság; ezen tömör bizonyításunk viszont számos korábbi eredményre építkezik. Továbbá több esetben is igazoltuk (bár nem az összesben), hogy $\mathrm{Quo}(A)$ $(1 + 1 + 2)$-generált. Javítottunk a $\mathrm{Tran}(A)$ generátorhalmazairól szóló korábbi eredményeken is.

3.4.9. Tétel. *Legyen A legalább kételemű halmaz. Ekkor teljesülnek a következők.*

- *Ha $|A| \neq 4$ és $|A|$ elérhető számosság, akkor a* $\mathrm{Quo}(A)$ *teljes háló négygenerált.*
- *Ha $|A| \geq 13$, valamint vagy $|A|$ páratlan, vagy $|A| \geq 56$ páros, akkor a* $\mathrm{Quo}(A)$ *teljes háló* $(1 + 1 + 2)$*-generált.*

- *Ha* $13 \leq |A| \leq \aleph_0$ *és vagy* $|A|$ *páratlan, vagy* $|A| \geq 56$ *páros, akkor a* $\mathrm{Quo}(A)$ *hálónak van olyan (itt most nem teljes értelemben)* $(1+1+2)$*-generált részhálója, amely* $\mathrm{Quo}(A)$ *összes atomját tartalmazza.*

3.4.12. Tétel. *Ha* $3 \leq |A|$ *és* $|A|$ *elérhető számosság, akkor* $\mathrm{Tran}(A)$ *mint teljes háló hatgenerált.*

A 4. fejezetben azzal a problémával foglalkoztunk, hogy adott n természetes szám esetén mely n elemű véges hálóknak van a legtöbb, második legtöbb, harmadik legtöbb, stb. kongruenciája; továbbá azzal, hogy az ilyen hálók kongruenciahálóinak milyen a szerkezete. Ezek az eredmények a [77] cikkünkben jelentek meg.

Czédli és Mureşan [40] munkája alapján végtelen hálók kongruenciáinak számossága tetszőleges értéket felvehet 2 és a háló számossága között, vagy a kongruenciák számossága megegyezhet a háló részhalmazainak számosságával. De véges hálók esetén egészen más a helyzet. Az eredményeink ismertetéséhez a következő hálóműveletekre és jelölésekre van szükség.

Legyen L és M háló. Ha L-nek van legnagyobb eleme, amelyet 1^L jelöl, és M-nek van legkisebb eleme, 0^M, akkor L és M *ragasztott összege*, amelyet $L \dotplus M$-mel jelölünk, az a háló, amelyet úgy kapunk L-ből és M-ből, hogy 1^L-et azonosítjuk 0^M-mel, és M-et L „tetejére ragasztjuk". Ha L és M nemtriviális korlátos hálók, akkor L és M *vízszintes összege*, amelyet $L \boxplus M$-mel jelölünk, az a háló, amelyet úgy kapunk L-ből és M-ből, hogy azonosítjuk egymással a 0^L és 0^M legkisebb elemeiket, azonosítjuk egymással az 1^L és 1^M legnagyobb elemeiket, és $L \setminus \{0^L, 1^L\}$ egyik eleme sem lesz összehasonlítható $M \setminus \{0^M, 1^M\}$ egyik elemével sem $L \boxplus M$-ben. Tetszőleges $n \in \mathbb{N}$ esetén az n-elemű láncot $\mathcal{C}_n$-nel jelöljük. Szokás szerint $\mathcal{N}_5$ jelöli az ötelemű nem moduláris $\mathcal{C}_3 \boxplus \mathcal{C}_4$ hálót.

Freese [50] és Czédli [21] meghatározta a kongruenciák lehetséges legnagyobb és második legnagyobb számát, eredményeiket a következőképpen fogalmazhatjuk meg az előző jelöléseket használva. Ha L véges n elemű háló, akkor $|\mathrm{Con}(L)| \leq 2^{n-1}$, valamint $|\mathrm{Con}(L)| = 2^{n-1}$ akkor és csak akkor, ha $L \cong \mathcal{C}_n$. Más szóval egy véges hálónak legfeljebb annyi kongruenciája lehet, mint az azonos elemszámú lánc kongruenciáinak a száma. Továbbá, ha $|\mathrm{Con}(L)| < 2^{n-1}$, akkor $|\mathrm{Con}(L)| \leq 2^{n-2}$, valamint $|\mathrm{Con}(L)| = 2^{n-2}$ akkor és csak akkor, ha $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-2}$ valamely $k \in [1, n-3]$ esetén. Azaz a lehetséges második legtöbb kongruenciával pontosan azok a hálók rendelkeznek, amelyek két lánc és a négyelemű Boole-algebra ragasztott összegei. A Czédli által adott bizonyítást követve, a következő eredményt kaptuk hálók kongruenciáinak harmadik, negyedik és ötödik lehetséges legnagyobb számáról.

4.3.7. Tétel. *Legyen L véges n elemű háló.*

(i) *Ha $|\mathrm{Con}(L)| < 2^{n-2}$, akkor $n \geq 5$, $|\mathrm{Con}(L)| \leq 5 \cdot 2^{n-5} = 2^{n-3} + 2^{n-5}$, és $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ akkor és csak akkor, ha $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$ valamely $k \in [1, n-4]$ esetén.*

(ii) *Ha $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, akkor $|\mathrm{Con}(L)| \leq 2^{n-3}$, és $|\mathrm{Con}(L)| = 2^{n-3}$ akkor és csak akkor, ha vagy $n \geq 6$ és $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$ valamely $k \in [1, n-5]$ esetén, vagy $n \geq 7$ és $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-4}$ valamely $k, m \in \mathbb{N}$ esetén, ahol $k + m \leq n - 5$.*

(iii) *Ha $|\mathrm{Con}(L)| < 2^{n-3}$, akkor $|\mathrm{Con}(L)| \leq 7 \cdot 2^{n-6} = 2^{n-4} + 2^{n-5} + 2^{n-6}$, és $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ akkor és csak akkor, ha $n \geq 6$, és $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ vagy $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$ valamely $k \in [1, n-5]$-re.*

A korábbi eredményeket és a sajátjainkat összegezve adódik a legtöbb, második legtöbb, harmadik legtöbb, stb. kongruenciával rendelkező véges hálók kongruenciahálóinak alábbi jellemzése.

4.3.8. Következmény.

(i) *$|\mathrm{Con}(L)| = 2^{n-1}$ akkor és csak akkor, ha $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-1}$.*

(ii) *$|\mathrm{Con}(L)| = 2^{n-2}$ akkor és csak akkor, ha $n \geq 4$ és $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-2}$.*

(iii) *$|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ akkor és csak akkor, ha $n \geq 5$ és $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-5} \times (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$.*

(iv) *$|\mathrm{Con}(L)| = 2^{n-3}$ akkor és csak akkor, ha $n \geq 6$ és $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-3}$.*

(v) $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *akkor és csak akkor, ha* $n \geq 6$ *és* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-6} \times (\mathcal{C}_2^2 \dot{+} \mathcal{C}_2^2)$.

A disszertáció a szerző publikáció közül négyen alapul, ezek a következők:

1. G. Czédli, T. Dékány, G. Gyenizse, J. Kulin: *The number of slim rectangular lattices.* Algebra Universalis **75/1** (2016), 33–50.
2. G. Czédli, J. Kulin: *A concise approach to small generating sets of lattices of quasiorders and transitive relations.* Acta Sci. Math. (Szeged) **83** (2017), 3–12.
3. J. Kulin: *Quasiorder lattices are five-generated.* Discussiones Mathematicae - General Algebra and Applications **36** (1) (2016), 59–70.
4. C. Mureşan, J. Kulin: *On the largest numbers of congruences of finite lattices.* Order **37** (2020), 445–460.

A szerző további, a disszertációban fel nem használt publikációja:

- T. Dékány, G. Gyenizse, J. Kulin: *Permutations assigned to slim rectangular lattices.* Acta Sci. Math. (Szeged) **82** (2016), 19–28.

Acknowledgments

First and foremost, I would like to thank all my supervisors for their hard work. Gábor Czédli has been guiding me for the longest period, most of my scientific results originated from him or are connected to him. Among many things, I thank him for asking me the right questions and for continuously planning my progression. I knew I could ask him a question anytime and I would get a thorough and detailed answer. I thank Kamilla Kátai-Urbán for her suggestions on the composition of this dissertation, for her careful commenting on multiple drafts, and for the deadlines she set. Also, for showing me around the very first time I visited the Bolyai Institute, and for keeping up her caring attitude ever since. I thank László Zádori for the mathematical problems he introduced to me and for bringing the corresponding topics closer to me. And I am thankful to my MSc thesis supervisor, Csaba Szabó, for pointing me in this direction which led to where I am today.

I am truly lucky to be able to work in a great atmosphere, I appreciate the collaborations, the fun and all the help and understanding I received from my colleagues throughout the years.

Growing up, my mother was my best maths teacher, available 24 hours a day, seven days a week. I am very thankful for my family and my friends, for their love and support.

Bibliography

[1] H. Abels: *The gallery distance of flags.* Order **8** (1991), 77–92.

[2] D. Ahmed and G. Czédli: *(1+1+2)-generated lattices of quasiorders.* Acta Sci. Math. (Szeged) **87** (2021), 415–427.

[3] M. Bóna: *Combinatorics of Permutations.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL (2004).

[4] I. Chajda and G. Czédli: *How to generate the involution lattice of quasiorders?* Studia Sci. Math. Hungar. **32** (1996), 415–427.

[5] S. Chowla, I.N. Herstein and W.K. Moore: *On recursions connected with symmetric groups I.* Canadian J. Math. **3** (1951), 328–334.

[6] G. Czédli: *On properties of rings that can be characterized by infinite lattice identities.* Studia Sci. Math. Hungar. **16** (1981), 45–60.

[7] G. Czédli: *Four-generated large equivalence lattices.* Acta Sci. Math. (Szeged) **62** (1996), 47–69.

[8] G. Czédli: *Lattice generation of small equivalences of a countable set.* Order **13** (1996), 11–16.

[9] G. Czédli: *(1+1+2)-generated equivalence lattices.* J. Algebra **221** (1999), 439–462.

[10] G. Czédli: *Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices.* Algebra Universalis **67** (2012), 313–345.

[11] G. Czédli: *The matrix of a slim semimodular lattice*. Order **29** (2012), 85–103.

[12] G. Czédli: *Coordinatization of finite join-distributive lattices*. Algebra Universalis **71/4** (2014), 385–404.

[13] G. Czédli: *Patch extensions and trajectory colorings of slim rectangular lattices*. Algebra Universalis **72/2** (2014), 125–154.

[14] G. Czédli: *Representing a monotone map by principal lattice congruences*. Acta Mathematica Hungarica **147** (2015), 12–18.

[15] G. Czédli: *An independence theorem for ordered sets of principal congruences and automorphism groups of bounded lattices*. Acta Sci. Math. (Szeged) **82** (2016), 3–18.

[16] G. Czédli: *The asymptotic number of planar, slim, semimodular lattice diagrams*. Order **33**, 2 (2016), 231–237.

[17] G. Czédli: *The ordered set of principal congruences of a countable lattice*. Algebra Universalis **75** (2016), 351–380.

[18] G. Czédli: *Complete congruence lattices of two related modular lattices*. Algebra Universalis **78** (2017), 251–289.

[19] G. Czédli: *Four-generated quasiorder lattices and their atoms in a four-generated sublattice*. Communications in Algebra **45/9** (2017), 4037–4049.

[20] G. Czédli: *Representing some families of monotone maps by principal lattice congruences*. Algebra Universalis **77** (2017), 51–77.

[21] G. Czédli: *A note on finite lattices with many congruences*. Acta Universitatis Matthiae Belii, Series Mathematics Online (2018), 22–28.

[22] G. Czédli: *Cometic functors and representing order-preserving maps by principal lattice congruences*. Algebra Universalis **79:59** (2018), 1–32.

[23] G. Czédli: *Finite semilattices with many congruences*. Order **36**, Issue 2 (2019), 233–247.

[24] G. Czédli: *Lattices with many congruences are planar.* Algebra Universalis **80:16** (2019)

[25] G. Czédli: *Four-generated direct powers of partition lattices and authentication.* Publicationes Mathematicae (Debrecen) **99** (2021), 447–472.

[26] G. Czédli: *Lamps in slim rectangular planar semimodular lattices.* Acta Sci. Math. (Szeged) **87** (2021), 381–413.

[27] G. Czédli: *Cyclic congruences of slim semimodular lattices and non-finite axiomatizability of some finite structures.* Archivum Mathematicum **58/1** (2022), 15–33.

[28] G. Czédli: *Revisiting Faigle geometries from a perspective of semimodular lattices.* Discussiones Mathematicae - General Algebra and Application **43** (2023), 207–222.

[29] G. Czédli: *Generating some large filters of quasiorder lattices.* Acta Sci. Math. (Szeged) (2024) DOI: 10.1007/s44146-024-00139-5

[30] G. Czédli: *Minimum-sized generating sets of the direct powers of free distributive lattices.* CUBO, A Mathematical Journal **26/2** (2024), 217–237.

[31] G. Czédli: *Generating Boolean lattices by few elements and exchanging session keys.* arXiv:2303.10790

[32] G. Czédli: *Sperner theorems for unrelated copies of posets and generating distributive lattices.* https://tinyurl.com/czg-spmgd (earlier version: arXiv:2308.15625)

[33] G. Czédli, T. Dékány, G. Gyenizse and J. Kulin: *The number of slim rectangular lattices.* Algebra Universalis **75/1** (2016), 33–50.

[34] G. Czédli, T. Dékány, L. Ozsvárt, N. Szakács and B. Udvari: *On the number of slim, semimodular lattices.* Mathematica Slovaca **66** (2016), 5–18.

[35] G. Czédli and G. Grätzer: *Notes on planar semimodular lattices. VII. Resections of planar semimodular lattices.* Order **30** (2013), 847–858.

[36] G. Czédli and G. Grätzer: *Planar Semimodular Lattices: Structure and Diagrams.* Chapter 3 in G. Grätzer and F. Wehrung (editors): Lattice Theory: Special Topics and Applications I. Birkhäuser, Cham (2014), 91–130.

[37] G. Czédli and J. Kulin: *A concise approach to small generating sets of lattices of quasiorders and transitive relations.* Acta Sci. Math. (Szeged) **83** (2017), 3–12.

[38] G. Czédli and Á. Kurusa: *A convex combinatorial property of compact sets in the plane and its roots in lattice theory.* Categories and General Algebraic Structures with Applications **11** 2019), 57–92.

[39] G. Czédli and A. Molkhasi: *Absolute retracts for finite distributive lattices and slim semimodular lattices.* Order **40** (2023), 127–148.

[40] G. Czédli and C. Mureşan: *On principal congruences and the number of congruences of a lattice with more ideals than filters.* Acta Sci. Math. (Szeged) **85** (2019), 363–380.

[41] G. Czédli and L. Oluoch: *Four-element generating sets of partition lattices and their direct products.* Acta Sci. Math. (Szeged) **86** (2020), 405–448.

[42] G. Czédli, L. Ozsvárt and B. Udvari: *How many ways can two composition series intersect?* Discrete Mathematics **312** (2012), 3523–3536.

[43] G. Czédli and E.T. Schmidt: *The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices.* Algebra Universalis **66** (2011), 69–79.

[44] G. Czédli and E.T. Schmidt: *Slim semimodular lattices. I. A visual approach.* Order **29** (2012), 481–497.

[45] G. Czédli and E.T. Schmidt: *Composition series in groups and the structure of slim semimodular lattices.* Acta Sci. Math. (Szeged) **79** (2013), 369–390.

[46] G. Czédli and E.T. Schmidt: *Slim semimodular lattices. II. A description by patchwork systems.* Order **30** (2013), 689–721.

[47] T. Dékány, G. Gyenizse and J. Kulin: *Permutations assigned to slim rectangular lattices.* Acta Sci. Math. (Szeged) **82** (2016), 19–28.

[48] T. Dolgos: *Generating equivalence and quasiorder lattices over finite sets* (in Hungarian). BSc Thesis, University of Szeged (2015).

[49] M. Erné, J. Heitzig and J. Reinhold: *On the number of distributive lattices.* Electron. J. Combin. **9/1** (2002), Research Paper 24, 23 pp.

[50] R. Freese: *Computing Congruence Lattices of Finite Lattices.* Proc. Amer. Math. Soc. **125** (1997), 3457–3463.

[51] R. Giuntini, C. Mureşan and F. Paoli: *PBZ*–Lattices: Ordinal and Horizontal Sums.* In D. Fazio, A. Ledda, F. Paoli (editors): Algebraic Perspectives on Substructural Logics. Trends in Logic **55**. Springer, Cham (2021), 73–105.

[52] G. Grätzer: *General Lattice Theory.* Birkhäuser Akademie–Verlag, Basel–Boston–Berlin (1978).

[53] G. Grätzer: *Universal Algebra.* Second Edition, Springer Science+Business Media, LLC, New York (2008).

[54] G. Grätzer: *Lattice Theory: Foundation.* Birkhäuser, Basel (2011).

[55] G. Grätzer: *Notes on planar semimodular lattices. VI. On the structure theorem of planar semimodular lattices.* Algebra Universalis **69** (2013), 301–304.

[56] G. Grätzer: *The order of principal congruences of a bounded lattice.* Algebra Universalis **70** (2013), 95–105.

[57] G. Grätzer: *The Congruences of a Finite Lattice. A Proof-by-Picture Approach.* Second edition, Birkhäuser–Springer, Cham (2016).

[58] G. Grätzer: *Homomorphisms and principal congruences of bounded lattices. I. Isotone maps of principal congruences.* Acta Sci. Math. (Szeged) **82** (2016), 353–360.

[59] G. Grätzer: *Homomorphisms and principal congruences of bounded lattices. II. Sketching the proof for sublattices.* Algebra Universalis **78** (2017), 291–295.

[60] G. Grätzer: *Homomorphisms and principal congruences of bounded lattices. III. The independence theorem.* Algebra Universalis **78** (2017), 297–301.

[61] G. Grätzer and E. Knapp: *Notes on planar semimodular lattices. I. Construction.* Acta Sci. Math. (Szeged) **73** (2007), 445–462.

[62] G. Grätzer and E. Knapp: *Notes on planar semimodular lattices. III. Rectangular lattices.* Acta Sci. Math. (Szeged) **75** (2009), 29–48.

[63] G. Grätzer and H. Lakser: *Homomorphisms of distributive lattices as restrictions of congruences.* Canad. J. Math. **38** (1986), 1122–1134.

[64] G. Grätzer and H. Lakser: *Some preliminary results on the set of principal congruences of a finite lattice.* Algebra Universalis **79:21** (2018).

[65] G. Grätzer, H. Lakser and E. T. Schmidt: *Congruence lattices of finite semimodular lattices.* Canad. Math. Bull. **41** (1998), 290–297.

[66] G. Grätzer and J.B. Nation: *A new look at the Jordan–Hölder theorem for semimodular lattices.* Algebra Universalis **64** (2010), 309–311.

[67] G. Grätzer and R.W. Quackenbush: *The variety generated by planar modular lattices.* Algebra Universalis **63** (2010), 187–201.

[68] G. Grätzer and E.T. Schmidt: *On congruence lattices of lattices.* Acta Math. Sci. Hungar. **13** (1962), 179–185.

[69] G. Grätzer and E.T. Schmidt: *A lattice construction and congruence-preserving extensions.* Acta Math. Hungar. **66** (1995), 275–288.

[70] J. Heitzig and J. Reinhold: *Counting finite lattices.* Algebra Universalis **48** (2002), 43–53.

[71] D. Kelly and I. Rival: *Planar lattices.* Canad. J. Math. **27** (1975), 636–665.

[72] J. Kulin: *Quasiorder lattices are five-generated.* Discussiones Mathematicae - General Algebra and Applications **36** (1) (2016), 59–70.

[73] K. Kuratowski: *Sur l'état actuel de l'axiomatique de la théorie des ensembles.* Ann. Soc. Polon. Math. **3** (1925), 146–147.

[74] A. Levy: *Basic Set Theory.* Springer-Verlag, Berlin—Heidelberg—New York (1979).

[75] C. Mureşan: *On the cardinalities of the sets of congruences, ideals and filters of a lattice.* Analele Universităţii Bucureşti. Seria Informatică. Proceedings of the Workshop Days of Computer Science (DACS) 2015 LXII, affiliated workshop of the 11th edition of the conference Computability in Europe, University of Bucharest, Bucharest, Romania. (2015), 55–68.

[76] C. Mureşan: *Cancelling congruences of lattices, while keeping their filters and ideals.* South American Journal of Logic **6/1** (2020), 105–120.

[77] C. Mureşan and J. Kulin: *On the largest numbers of congruences of finite lattices.* Order **37** (2020), 445–460.

[78] M.M. Pawar and B.N. Waphare: *Enumeration of nonisomorphic lattices with equal number of elements and edges.* Indian J. Math. **45** (2003), 315–323.

[79] M. Ploščica: *Uncountable critical points for congruence lattices.* Algebra Universalis **76** (2016), 415–429.

[80] P. Růžička: *Free trees and the optimal bound in Wehrung's theorem.* Fund. Math. **198** (2008), 217–228.

[81] E.T. Schmidt: *A Survey on Congruence Lattice Representations.* Teubner–Texte zur Mathematik, Leipzig (1982).

[82] R.P. Stanley: *Supersolvable lattices.* Algebra Universalis **2** (1972), 197–217.

[83] H. Strietz: *Finite partition lattices are four-generated.* Proc. Lattice Th. Conf. Ulm (1975), 257–259.

[84] H. Strietz: *Über Erzeugendenmengen endlicher Partitionverbände.* Studia Sci. Math. Hungar. **12** (1977), 1–17.

[85] G. Takách: *Three-generated quasiorder lattices.* Discussiones Matematicae, Algebra and Stochastic Methods **16** (1996), 81–98.

[86] F. Wehrung: *A solution to Dilworth's congruence lattice problem.* Adv. Math. **216** (2007), 610–625.

[87] L. Zádori: *Generation of finite partition lattices.* Lectures in Universal Algebra, Colloquia Math. Soc. J. Bolyai **43**, Proc. Conf. Szeged (1983), 573–586, North Holland, Amsterdam—Oxford—New York (1986).