# MINIMAL CLONES

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## 1. Introduction

**1.1. Minimal clones.** A *(concrete) clone* is a collection $\mathcal{C}$ of finitary operations on a set that is closed under composition of functions and contains all projections. An *(abstract) clone* is a heterogeneous algebra that captures the compositional structure of concrete clones. The notion of a subclone, clone homomorphism and factor clone can be defined in a natural way, and the isomorphism theorems can be proved for abstract clones. Every concrete clone can be regarded as an abstract clone, and every abstract clone is isomorphic to a concrete clone, so in the following we will not always make a sharp distinction between concrete and abstract clones.

If $\mathbb{A} = (A; F)$ is an algebra, then the set of its term functions, denoted by $\mathrm{Clo}\,\mathbb{A}$, is a clone on $A$, called the *clone of the algebra* $\mathbb{A}$. This is the smallest clone containing $F$, therefore we say that $F$ *generates* $\mathrm{Clo}\,\mathbb{A}$, and we write $[F] = \mathrm{Clo}\,\mathbb{A}$. Clearly, every clone arises as the clone of an algebra: we just need to pick a generating set for the clone, and let these be the basic operations of the algebra.

A *representation* of an abstract clone $\mathcal{C}$ is (the image of) a clone homomorphism from $\mathcal{C}$ to the clone of operations on some set. The representations of $\mathcal{C}$ form a variety (which is determined only up to term equivalence). On the other hand, a clone can be assigned to any variety, namely the clone of the countably generated free algebra of the variety. These assignments are inverses of each other (up to term equivalence and clone isomorphism), thus we can say that abstract clones are the same as varieties up to term equivalence. The elements of $\mathcal{C}^{(n)}$, the $n$-ary part of $\mathcal{C}$, may be identified with the elements of the $n$-generated free algebra of the variety corresponding to $\mathcal{C}$. Projections correspond to variables under this identification, therefore we will use the notation $x_1, \ldots, x_n$ for the $n$-ary projections. In the binary case we will also use $x$ and $y$ instead of $x_1$ and $x_2$, and $x, y, z$ will stand for the three ternary projections.

All clones on a given set $A$ form a lattice with respect to inclusion; the smallest element of this lattice is the *trivial clone*, the clone of all projections on $A$ (denoted by $\mathcal{I}_A$), while the greatest element is the clone of all finitary operations on $A$. *Minimal clones* are the atoms of the clone lattice, i.e. a clone is minimal if its only proper subclone is the trivial clone. On finite sets there are finitely many minimal clones, and every clone contains a minimal one. Clearly, a nontrivial clone is minimal iff it is generated by any of its nontrivial elements. Therefore all minimal clones are one-generated, thus they arise as clones of algebras with a single basic operation. We usually define a minimal clone by a generating function.

It is convenient to choose a function of the least possible arity as a generator of a minimal clone. These generators are called *minimal functions*: $f$ is a minimal function iff $[f]$ is a minimal clone and there is no nontrivial function in $[f]$ whose arity is less than the arity of $f$. A minimal function must be of one of five types according to the following theorem of I. G. Rosenberg.

**Theorem 1.1** [Ros]. *Let $f$ be a nontrivial operation of minimum arity in a minimal clone. Then $f$ satisfies one of the following conditions:*

(I) $f$ *is unary, and* $f^2(x) = f(x)$ *or* $f^p(x) = x$ *for some prime $p$;*
(II) $f$ *is a binary idempotent operation, i.e.* $f(x, x) = x$;
(III) $f$ *is a ternary majority operation, i.e.* $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;
(IV) $f(x, y, z) = x + y + z$, *where $+$ is a Boolean group operation;*
(V) $f$ *is a semiprojection, i.e. there exists an $i$ $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ whenever the values of $x_1, \ldots, x_n$ are not pairwise distinct.*

In cases (I) and (IV) the conditions ensure the minimality of $f$, while in the other three cases they do not, and a general characterization seems to be far beyond reach. There are numerous partial results that describe minimal clones under certain restrictions. We will quote some of these results later.

Trivial and minimal abstract clones can be defined naturally as well (though we cannot speak about atoms of clone lattices in this context), and Rosenberg's theorem holds almost verbatim

$$\text{(semilattices) } \mathcal{SL} : (xy)\,z = x\,(yz)\,, xy = yx$$
$$\text{(rectangular bands) } \mathcal{RB} : (xy)\,z = x\,(yz)\,, xyz = xz$$
$$\text{(right normal bands) } \mathcal{RNB} : (xy)\,z = x\,(yz)\,, xyz = yxz$$
$$\text{(right regular bands) } \mathcal{RRB} : (xy)\,z = x\,(yz)\,, xyx = yx$$
$$\mathcal{B} : x\,(yx) = (xy)\,x = (xy)\,y = (xy)\,(yx) = x\,(xy) = xy$$
$$\mathcal{D} : x\,(yx) = (xy)\,x = (xy)\,y = (xy)\,(yx) = xy,$$
$$x \cdot \overleftarrow{x \cdot y_1 \cdot \ldots \cdot y_n} = x \ \ (n = 1, 2, \ldots)$$
$$\mathcal{D} \cap \mathcal{A} : x\,(yz) = xy, xy^2 = xy$$
$$\text{(right semilattices) } \mathcal{RSL} : x\,(yz) = xy, xy^2 = xy, (xy)\,z = (xz)\,y$$
$$\text{(}p\text{-cyclic groupoids) } \mathcal{C}_p : x\,(yz) = xy, xy^p = x, \ (xy)\,z = (xz)\,y$$

TABLE 1. Some groupoid varieties with minimal clones

in the abstract case. An algebra $\mathbb{A}$ has a minimal clone iff the clone of HSP $\mathbb{A}$ is minimal, thus both concrete and abstract minimal clones can be described by varieties having a minimal clone.

**1.2. Examples.** The simplest examples of minimal clones of type (II), i.e. groupoids with a minimal clone, are semilattices and rectangular bands. We give the defining identities of some more groupoid varieties with a minimal clone in Table 1. To save parentheses we write $\overleftarrow{x_1 \cdot \ldots \cdot x_n}$ for the left-associated product $(\cdots((x_1 x_2)\,x_3)\cdots)\,x_n$, and similarly $\overrightarrow{x_1 \cdot \ldots \cdot x_n}$ for the right-associated product $x_1\,(\cdots(x_{n-2}\,(x_{n-1}x_n))\cdots)$. We abbreviate $\overleftarrow{x \cdot y \cdot \ldots \cdot y}$ to $xy^n$ (where $n$ is the number of $y$'s appearing in the product). Analogously $^n xy$ stands for $\overrightarrow{x \cdot \ldots \cdot x \cdot y}$. We have omitted the identity $xx = x$ everywhere, but of course these are all idempotent varieties.

The varieties $\mathcal{SL}$ and $\mathcal{RB}$ are selfdual; the duals of right normal bands, right regular bands, right semilattices are left normal bands ($\mathcal{LNB}$), left regular bands ($\mathcal{LRB}$), left semilattices ($\mathcal{LSL}$), respectively. (The variety $\mathcal{A}$ is defined by the identity $x\,(y\,(zu)) = x\,((yz)\,u)$; we will need it later, for the study of almost associative operations.) Figure 1 shows the meet-semilattice generated by these varieties and their duals ($\mathcal{LZ}$ and $\mathcal{RZ}$ denote the variety of left and right zero semigroups, and the bottom element is the variety of one-element groupoids).

The minimality of the clone of $\mathcal{B}$ and $\mathcal{D}$ is proved in [LP]; J. Płonka introduced $p$-cyclic groupoids in [Pł2], and he showed that $\mathrm{Clo}\,\mathcal{C}_p$ is minimal iff $p$ is a prime [Pł1].

Affine spaces provide further examples of binary minimal clones. An *affine space* is an algebra whose base set is a vector space, and its clone is the full idempotent reduct of the clone of that vector space. The clone of an affine space is minimal iff the base field is isomorphic to $\mathbb{Z}_p$ for some prime number $p$. If $p = 2$, then this clone is of type (IV); if $p > 2$, then the clone is of type (II). In the following affine spaces are always meant to be affine spaces over $\mathbb{Z}_p$ (for an arbitrary prime $p$).

There are much less examples of minimal clones of type (III). The simplest ones are those containing just one nontrivial ternary operation. The clone generated by the median function $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ on any lattice is an example of such a clone [PK].

There is no minimal clone with exactly two majority functions (see Theorem 3.4), so the next simplest examples are those that contain three majority functions. The dual discriminator function [FP] on any set defined by

$$d\,(a, b, c) = \begin{cases} a & \text{if } a = b \\ c & \text{if } a \neq b \end{cases}$$

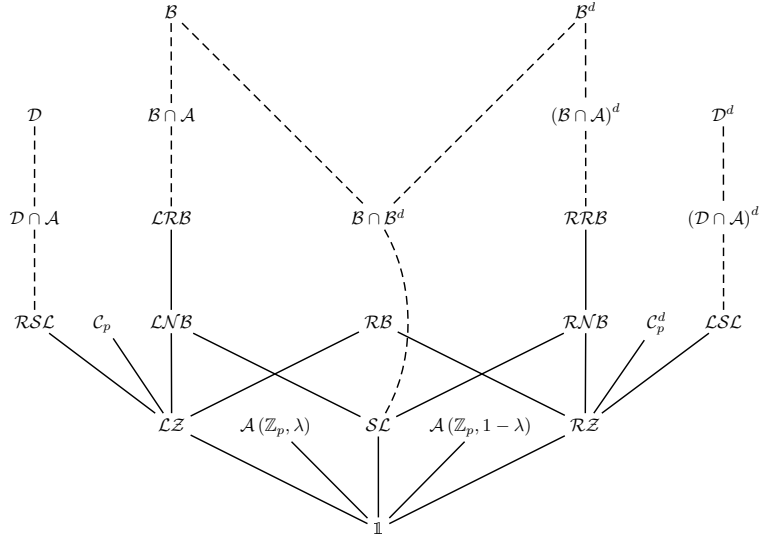generates such a clone (cf. [CsG]).

FIGURE 1. Some groupoid varieties with minimal clones

**1.3. Characterizations.** It seems to be a very hard problem to characterize minimal clones in full generality, but there are some results that describe minimal clones or minimal functions under certain assumptions. We mention some of these results; we formulate precisely only the theorems that we will need in the sequel.

One of the most natural approaches is to restrict the size of the underlying set of a concrete clone. E. Post determined all clones on the two-element set [Po]; seven of them are minimal. Minimal clones on the three-element set were described by B. Csákány [Cs1]; we quote the result for type (III) below. For the four-element set minimal clones of type (II) were described by B. Szczepara [Szcz]. A nontrivial semiprojection on a four-element set has to be of arity 3 or 4, and the latter case was settled in [JQ]. We are going to describe minimal majority functions on the four-element set in Theorem 2.6; the case of ternary semiprojections remains open.

**Theorem 1.2** [Cs1]**.** *There are twelve minimal majority functions on $\{1, 2, 3\}$ up to isomorphism, and they belong to three minimal clones containing 1, 3 and 8 majority operations respectively. These clones are generated by the majority operations $m_1$, $m_2$, $m_3$ that are defined for $\{a_1, a_2, a_3\} = \{1, 2, 3\}$ by*

$$m_1(a_1, a_2, a_3) = 1;$$
$$m_2(a_1, a_2, a_3) = a_1;$$
$$m_3(a_1, a_2, a_3) = a_{i+1} \text{ if } a_i = 2 \text{ (subscripts taken modulo 3).}$$

Based on this theorem, B. Csákány obtained a characterization of minimal majority operations which are *conservative*, i.e. which preserve all subsets of the underlying set [Cs2]. Conservative binary minimal operations are also determined in [Cs2]; for semiprojections there are only partial results [JQ].

One may restrict the size of the clone instead of the underlying set as well. There is a result in this direction by L. Lévai and P. P. Pálfy; they described binary minimal clones with at most seven binary operations [LP]. (The cases 5 and 7 are actually due to J. Dudek and J. Gałuszka, cf. [Du, DG].) In Theorem 3.7 we characterize minimal majority clones with at most seven ternary operations.

Another possibility is to look for minimal functions satisfying certain identities. Probably the most natural problem of this kind is to characterize semigroups with a minimal clone. This problem was solved by M. B. Szendrei; she determined all bands whose subclone lattice is a chain [SzM] (see also [P³]).

**Theorem 1.3** [P[3],SzM]**.** *A semigroup with a minimal clone is either a left regular band, a right regular band or a rectangular band.*

In Theorems 5.7 and 5.8 we generalize this theorem by characterizing minimal clones generated by almost associative binary operations for two different interpretations of the term 'almost associative'.

Á. Szendrei and K. Kearnes investigated minimal clones generated by an operation that commutes with itself [KSz]. In the binary case this commutativity property is equivalent to the so-called entropic or medial law $(xy)(zu) = (xz)(yu)$, and the result is the following.

**Theorem 1.4** [KSz]**.** *Let $\mathbb{A}$ be an entropic groupoid with a minimal clone. Then $\mathbb{A}$ or its dual is an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid for some prime p.*

We show in Theorem 4.5 that we get the same list of minimal clones if we assume only distributivity (which is weaker than entropicity for idempotent groupoids). We also characterize groupoids satisfying the identity $x(yz) = xy$ and having a minimal clone (cf. Lemma 4.3).

Finally, let us quote a result of K. Kearnes describing abelian algebras with a minimal clone [Kea]. In Theorem 4.8 we generalize this theorem to weakly abelian algebras.

**Theorem 1.5** [Kea]**.** *If a minimal clone has a nontrivial abelian representation, then it is either unary, or the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

## 2. Minimal majority clones on the four-element set

Our goal in this chapter is to determine the minimal majority functions on the four-element set. This is a finite task, since it is possible to test in finitely many steps whether a function is minimal or not, and there are finitely many majority operations on a finite set. However, the four-element set is already very big from this point of view. There is only one majority operation on the two-element set, and $3^6 = 729$ on the three-element set, while on the four-element set we have $4^{24} = 281\,474\,976\,710\,656$ majority functions. Thus it seems hopeless to test them one by one, even with the help of a computer.

**2.1. Minimal majority functions on finite sets.** We reduce the number of functions to be checked by proving that on a finite set every minimal majority clone can be generated by a function satisfying a certain identity.

**Theorem 2.1** [Wa1]**.** *Let $f$ be a majority function on a finite set. Then there exists a majority function $g \in [f]$ which satisfies the following identity.*

$$(2.1) \qquad g\big(\,g(x,y,z),\,g(y,z,x),\,g(z,x,y)\,\big) = g(x,y,z)$$

Next we describe how the validity of identity (2.1) can be seen in the operation table of a majority operation. We need the following notation for this. Let us put $\langle abc \rangle = \{(a,b,c), (b,c,a), (c,a,b)\}$, and we will use the symbol $f|_{\langle abc \rangle} \equiv u$ to mean that $f(a,b,c) = f(b,c,a) = f(c,a,b) = u$, and $f|_{\langle abc \rangle} = p$ to mean that $f(a,b,c) = a, f(b,c,a) = b, f(c,a,b) = c$. (Here 'p' stands for 'projection': $f|_{\langle abc \rangle} = p$ means that $f$ agrees with the first projection on the set $\langle abc \rangle$. If both $f|_{\langle abc \rangle} = p$ and $f|_{\langle bac \rangle} = p$ hold, then $f|_{\{a,b,c\}}$ looks like a first projection – except that it is a majority function. Similarly, $f|_{\langle abc \rangle} \equiv u \equiv f|_{\langle bac \rangle}$ means that $f$ is as constant on $\{a,b,c\}$ as a majority function can be.)

**Lemma 2.2** [Wa1]**.** *Let $f$ be a majority function on a set $A$ satisfying (2.1), and let $a, b, c$ be pairwise distinct elements of $A$. Let $u = f(a,b,c)$, $v = f(b,c,a)$, $w = f(c,a,b)$. Then $|\{u,v,w\}| \neq 2$, and if $u, v, w$ are pairwise different, then $f|_{\langle uvw \rangle} = p$.*

This lemma reduces the number of functions to consider to about 60 million. The next theorem shows that we can say a bit more, if we suppose that $f$ is a minimal function, leaving about 4 million functions.

| | $M_1$ | $M_2$ | | | $M_3$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1,2,3)$ | 4 | 4 | 2 | 3 | 3 | 3 | 4 | 3 | 4 | 4 | 3 | 4 |
| $(2,3,1)$ | 4 | 2 | 3 | 4 | 3 | 4 | 3 | 3 | 4 | 3 | 4 | 4 |
| $(3,1,2)$ | 4 | 3 | 4 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 3 |
| $(2,1,3)$ | 4 | 2 | 4 | 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 | 3 |
| $(1,3,2)$ | 4 | 4 | 3 | 2 | 4 | 4 | 4 | 3 | 3 | 3 | 3 | 4 |
| $(3,2,1)$ | 4 | 3 | 2 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | 4 | 3 |
| $\{1,2,4\}$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $\{1,3,4\}$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| $(4,2,3)$ | 4 | 4 | 2 | 3 | 3 | 3 | 4 | 3 | 4 | 4 | 3 | 4 |
| $(2,3,4)$ | 4 | 2 | 3 | 4 | 3 | 4 | 3 | 3 | 4 | 3 | 4 | 4 |
| $(3,4,2)$ | 4 | 3 | 4 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 3 |
| $(2,4,3)$ | 4 | 2 | 4 | 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 | 3 |
| $(4,3,2)$ | 4 | 4 | 3 | 2 | 4 | 4 | 4 | 3 | 3 | 3 | 3 | 4 |
| $(3,2,4)$ | 4 | 3 | 2 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | 4 | 3 |

TABLE 2. Nonconservative minimal majority functions on the four-element set

**Theorem 2.3** [Wa1]. *Let $f$ be a minimal majority function on a set $A$ satisfying (2.1), and let $a, b, c$ be pairwise distinct elements of $A$. If $u = f(a, b, c)$, $v = f(b, c, a)$, $w = f(c, a, b)$ are pairwise different, then $f|_{\langle uvw \rangle} = p$ and also $f|_{\langle vuw \rangle} = p$.*

**2.2. The four-element case.** Next we show that in the four-element case we can assume even more regularity about our minimal function.

**Lemma 2.4** [Wa1]. *Let $f$ be a minimal majority function on the four-element set $A = \{a, b, c, d\}$ satisfying (2.1). If $f(\langle abc \rangle) \subseteq \{a, b, c\}$ then either $f|_{\langle abc \rangle} = p$ and $f|_{\langle bac \rangle} = p$ or $f|_{\langle abc \rangle} \equiv u$ and $f|_{\langle bac \rangle} \equiv v$ for some $u, v \in A$.*

After Theorem 2.3 and Lemma 2.4, we are left with about a million functions, and this number can be reduced further by taking into account the possible symmetries (isomorphisms). Our strategy is to show that certain local patterns in the operation table imply that the function is not minimal. It turns out that only a few functions can avoid all of these patterns. The details are quite tedious, and things get harder and harder as we get closer and closer to the minimal functions. In the process we make use of Theorem 1.2 and the description of conservative minimal majority operations (see [Cs2]). The result is the following (see Table 2 for the definition of the functions $M_1, M_2, M_3$).

**Theorem 2.5** [Wa1]. *If $f$ is a nonconservative minimal majority function on $A = \{1, 2, 3, 4\}$ satisfying (2.1), then $f$ is isomorphic to $M_1$, $M_2$, $M_3$ or $M_3 (y, x, z)$.*

**2.3. The minimal clones.** Only three nonconservative functions remained up to isomorphism and permutation of variables that have a chance to be minimal. They are indeed minimal; actually their clones are isomorphic to the three minimal majority clones on the three-element set. (Let us recall that the conservative case is settled in [Cs2].)

**Theorem 2.6** [Wa1]. *If $f$ is a minimal majority function on the set $\{1, 2, 3, 4\}$, then $f$ is either conservative, or isomorphic to one of the twelve majority functions shown in Table 2. These functions belong to three minimal clones containing $1, 3$ and $8$ majority operations respectively, as shown in the table. Moreover, the clone generated by $M_i$ is isomorphic to $[m_i]$ (cf. Theorem 1.2) for $i = 1, 2, 3$. (The middle two rows of the table mean that if $\{a, b, c\}$ equals $\{1, 2, 4\}$ or $\{1, 3, 4\}$, then the value of the functions on $(a, b, c)$ is $4$.)*

There are 4, 12, 24 majority operations on $A = \{1, 2, 3, 4\}$ that are isomorphic to $M_1$, $M_2$, $M_3$ respectively, so there are $4 + 12 + 24 = 40$ nonconservative majority minimal clones on a four-element set. These clones contain $4 \cdot 1 + 12 \cdot 3 + 24 \cdot 8 = 232$ majority operations, hence there

| | conservative | nonconservative | all |
|---|---|---|---|
| minimal functions | 32646 | 232 | 32 878 |
| minimal functions up to isomorphism | 1653 | 12 | 1665 |
| minimal clones | 2401 | 40 | 2441 |
| minimal clones up to algebra isomorphism | 126 | 3 | 129 |
| minimal clones up to clone isomorphism | 123 | 3 | 124 |

TABLE 3. The number of minimal majority functions and clones on the four-element set

are 232 nonconservative minimal majority operations on $A$, and they fall into $1 + 3 + 8 = 12$ isomorphism classes.

There are $7^4 = 2401$ conservative minimal majority clones on $A$; this follows easily from the description given in [Cs2]. It is harder to count the conservative functions and clones up to isomorphism. Here we can use the correspondence between clones and varieties, and we take advantage of the fact, that algebras with a majority operation generate congruence distributive varieties. The numerical results are summarized in Table 3.

## 3. Minimal clones with few majority functions

In this chapter we describe minimal clones of type (III) with at most seven ternary operations. A unique property of clones generated by a majority operation is that the minimality of such a clone depends only on its ternary functions. We denote the ternary part of $\mathcal{C}$ by $\mathcal{C}^{(3)}$, and we regard it as an algebra with one quaternary operation (the composition of ternary functions) and three constants (the projections). If $\mathcal{C}$ is generated by a majority function, then $\mathcal{C}$ is minimal iff $\mathcal{C}^{(3)}$ has no proper nontrivial subalgebras.

**3.1. Symmetries of minimal majority functions.** In this section we prove a general theorem about the symmetries of the majority functions in a minimal clone. We say that a majority operation is *cyclically symmetric*, if it is invariant under cyclic permutations of its variables, and we say that it is *totally symmetric*, if it is invariant under all permutations of its variables.

Let us introduce the following three binary operations on ternary operations [!]:

$$f * g = f\left(g\left(x,y,z\right), g\left(y,z,x\right), g\left(z,x,y\right)\right);$$
$$f \bullet g = f\left(g\left(x,y,z\right), y, z\right);$$
$$f \odot g = f\left(x, g\left(x,y,z\right), g\left(x,z,y\right)\right).$$

The next theorem extends Theorem 2.1 (note that the left hand side of (2.1) is $g * g$). Concerning the operation $\bullet$ see also Lemma 4.4 of [HM].

**Theorem 3.1** [Wa4]**.** *The operations $*$, $\bullet$ and $\odot$ are associative, and if $\mathcal{C}$ is a clone generated by a majority function, then $\mathcal{C}^{(3)} \setminus \mathcal{I}$ is closed under them. Therefore if $\mathcal{C}^{(3)}$ is finite, then it contains a nontrivial idempotent element for each of these operations.*

The above theorem is the basis for the main result of this section, which is an analogue of a theorem of J. Dudek and J. Gałuszka concerning minimal clones containing only commutative nontrivial binary operations [DG].

**Theorem 3.2** [Wa4]**.** *Let $\mathcal{C}$ be a majority minimal clone with finitely many ternary operations. If every nontrivial ternary operation in $\mathcal{C}$ is cyclically symmetric, then $\mathcal{C}$ contains only one nontrivial ternary operation.*

**3.2. Minimal clones with at most four majority operations.** If $\mathcal{C}$ is a majority clone with just one majority operation, then the majority rule and the clone axioms completely determine the structure of $\mathcal{C}^{(3)}$, and it is clear that in this case $\mathcal{C}$ is minimal. For example, $[m_1]$ is such a clone, so we have the following theorem.

**Theorem 3.3** [Wa4]. *If $\mathcal{C}$ is a minimal clone with one majority operation, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$.*

If $f$ is the unique majority operation in such a clone, then every nontrivial ternary superposition of $f$ yields $f$ itself. In particular, $f$ is totally symmetric, and satisfies $f\left(f\left(x,y,z\right),y,z\right) = f\left(x,y,z\right)$. It is easy to check that this identity together with the total symmetry ensures that $f$ does not generate any nontrivial ternary operation other than $f$, hence the clones described in the above theorem are exactly the clones of the subvarieties of the variety $\mathcal{M}_1$ defined by the following identities:

$$f\left(x,y,z\right) = f\left(y,z,x\right) = f\left(y,x,z\right) = f\left(f\left(x,y,z\right),y,z\right), \ \ f\left(x,x,y\right) = x.$$

This variety has infinitely many subvarieties, therefore there are infinitely many nonisomorphic minimal clones with just one majority operation. To see this, we construct a subdirectly irreducible (in fact, simple) algebra $\mathbb{A}_n \in \mathcal{M}_1$ of size $n$ for every $n > 6$. Let $\mathbb{A}_n = (\{1,2,\ldots,n\}\,;f)$, where $f$ is a totally symmetric majority operation defined for $1 \le a < b < c \le n$ by

$$f\left(a,b,c\right) = \begin{cases} a & \text{if } \left\lceil\frac{a+c}{2}\right\rceil < b < c; \\ b & \text{if } b = \left\lfloor\frac{a+c}{2}\right\rfloor \text{ or } b = \left\lceil\frac{a+c}{2}\right\rceil; \\ c & \text{if } a < b < \left\lfloor\frac{a+c}{2}\right\rfloor. \end{cases}$$

It can be shown that $\mathbb{A}_n$ is simple if $n > 6$. Since $\mathcal{M}_1$ is congruence distributive, $\mathbb{A}_m \notin \mathrm{HSP}(\mathbb{A}_n)$ if $m > n$ by Jónsson's lemma, hence the subvarieties $\mathrm{HSP}(\mathbb{A}_n)$ are all different, and the clones $\mathrm{Clo}\,\mathbb{A}_n$ are pairwise nonisomorphic.

The case of two majority operations is easily settled with the help of Theorem 3.2.

**Theorem 3.4** [Wa4]. *There is no minimal clone with exactly two majority operations.*

Dual discriminator functions generate minimal clones with three majority operations. The next theorem shows that this is the only example up to isomorphism of the ternary part of the clone.

**Theorem 3.5** [Wa4]. *If $\mathcal{C}$ is a minimal clone with three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_2]^{(3)}$.*

The previous theorem can be formulated in terms of algebras and varieties as follows. Let $\mathcal{M}_2$ be the variety defined by the three-variable identities satisfied by $(\{1,2,3\}\,;m_2)$. If $f$ is a majority operation on a set $A$, then $(A;f)$ is term equivalent to an element of $\mathcal{M}_2 \setminus \mathcal{M}_1$ iff $[f]$ is a minimal clone with exactly three majority operations. The variety $\mathcal{M}_2$ has infinitely many subvarieties that are not contained in $\mathcal{M}_1$, therefore there are infinitely many nonisomorphic minimal clones with three majority operations. Indeed, if $d_A$ is the dual discriminator function on a set $A$ with at least three elements, then $(A; d_A\left(z,y,x\right)) \in \mathcal{M}_2 \setminus \mathcal{M}_1$, and by Jónsson's lemma we have $(B; d_B\left(z,y,x\right)) \notin \mathrm{HSP}\left(A; d_A\left(z,y,x\right)\right)$ if $A$ is finite and $|A| < |B|$.

**Theorem 3.6** [Wa4]. *There is no minimal clone with exactly four majority operations.*

Summarizing the last four theorems we get the main result of this chapter.

**Theorem 3.7** [Wa4]. *There is no minimal clone with exactly two or four majority operations. If $\mathcal{C}$ is a minimal clone with one or three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$ or $[m_2]^{(3)}$, respectively (cf. Theorem 1.2).*

## 4. Minimal clones with weakly abelian representations

The main result of this chapter is a generalization of Theorem 1.5 using a weaker term condition. Let us first recall the definition of four variants of abelianness (cf. [KK]). For an algebra $\mathbb{A}$ let $\mathcal{M}(\mathbb{A})$ denote the set of $2 \times 2$ matrices of the form $\begin{pmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{pmatrix}$ where $t$ is a polynomial of $\mathbb{A}$ of arity $n + m$ and $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$. We say that the algebra $\mathbb{A}$ is

(1) *weakly abelian*, if $\begin{pmatrix} u & u \\ u & v \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $u = v$;
(2) *abelian*, if $\begin{pmatrix} u & u \\ v & w \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $v = w$;
(3) *rectangular*, if $\begin{pmatrix} u & v \\ w & u \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $u = v = w$;
(4) *strongly abelian*, if it is both abelian and rectangular.

It was proved in [Kea] that minimal clones of type (III) and (V) do not have nontrivial abelian representations, and the proof actually shows that they do not have nontrivial weakly abelian representations either. Every representation of a minimal clone of type (I) or (IV) is clearly abelian, therefore we only need to consider weakly abelian groupoids with a minimal clone.

### 4.1. Weak abelianness and distributivity.
In the theory of quasigroups a different notion of 'weak abelianness' is defined by the identities

$$(4.1) \qquad (xx)(yz) = (xy)(xz), \qquad (yz)(xx) = (yx)(zx),$$

and a groupoid is called 'abelian' (or medial, or entropic) if $(xy)(zu) = (xz)(yu)$ holds (see [Kep]). To avoid confusion with the universal algebraic definitions, we will use the word *entropic* in the latter case. Minimal clones are always idempotent, and in this case the identities (4.1) are equivalent to the *distributive laws*:

$$x(yz) = (xy)(xz), \qquad (yz)x = (yx)(zx).$$

Any idempotent abelian groupoid is entropic [Kea], and one might expect that idempotent weakly abelian groupoids are distributive. We do not know if this is true or not, but for our present purposes the weaker properties stated in the next lemma are sufficient.

**Lemma 4.1** [Wa2]. *Every idempotent weakly abelian groupoid satisfies the following identities:*

(i) $(xy)(xz) = (x(yz))((xy)(xz))$;
(ii) $(yx)(zx) = ((yx)(zx))((yz)x)$;
(iii) $(xy)x = x(yx)$.

To make the connection between distributivity and weak abelianness more explicit, we will define a relation $\sim$ on our groupoid by $a \sim b$ iff $ab = a$. Identity (ii) says that idempotent weakly abelian groupoids are right distributive 'modulo $\sim$'. This does not make perfect sense yet, since $\sim$ may not be a congruence, maybe not even an equivalence relation. Our strategy will be to reduce the problem to the case when $\sim$ is a congruence relation. As a preparation, we first show that assuming that the clone of the groupoid is minimal, we can conclude that it satisfies at least one-sided distributivity.

**Lemma 4.2** [Wa2]. *A weakly abelian groupoid with a minimal clone must satisfy at least one of the distributive laws.*

### 4.2. Left distributive weakly abelian groupoids with minimal clones.
Let $\mathbb{A}$ be a weakly abelian groupoid with a minimal clone. By Lemma 4.2 we can suppose without loss of generality that $\mathbb{A}$ is left distributive. First we prove that if $\sim$ is not a congruence relation, then our groupoid must be a $p$-cyclic groupoid. We use the following lemma, which describes binary minimal operations satisfying a certain identity.

**Lemma 4.3** [Wa2]. *If a groupoid has a minimal clone and satisfies the identity $x(yz) = xy$, then it belongs to the variety $\mathcal{D} \cap \mathcal{A}$ or $\mathcal{C}_p$ for some prime $p$.*

**Theorem 4.4** [Wa2]. *If $\mathbb{A}$ is a weakly abelian left distributive groupoid with a minimal clone such that the relation $\sim$ defined by $a \sim b \Leftrightarrow ab = a$ is not a congruence, then $\mathbb{A}$ is a p-cyclic groupoid for some prime p.*

Now we can suppose that $\mathbb{A}$ is a left distributive weakly abelian groupoid with a minimal clone, and $\sim$ is a congruence of $\mathbb{A}$. The corresponding factor groupoid $\mathbb{A}/\sim$ is distributive (right distributivity holds because $\mathbb{A}$ satisfies identity (ii) of Lemma 4.1). Furthermore, $\mathbb{A}/\sim$ has a minimal or trivial clone. To describe this factor groupoid, we need to characterize distributive groupoids with a minimal clone. It turns out that the distributive and entropic properties are equivalent for groupoids with a minimal clone, hence we get the same list of groupoids as in Theorem 1.4.

**Theorem 4.5** [Wa2]. *Every distributive groupoid having a minimal clone is entropic, therefore such a groupoid must be (the dual of) an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid for some prime p.*

Using the list of entropic groupoids with a minimal clone, we can prove that $\mathbb{A}$ itself is entropic, too. The key observation in passing form $\mathbb{A}/\sim$ to $\mathbb{A}$ is that by the definition of $\sim$ we have for any terms $t_1, t_2$

$$\mathbb{A}/\sim \models t_1 = t_2 \iff \mathbb{A} \models t_1 t_2 = t_1.$$

**Theorem 4.6** [Wa2]. *If $\mathbb{A}$ is a weakly abelian left distributive groupoid with a minimal clone such that the relation $\sim$ defined by $a \sim b \Leftrightarrow ab = a$ is a congruence, then $\mathbb{A}$ is entropic.*

Combining Theorems 4.4 and 4.6 with Theorem 1.4 we get the following result; we just need to observe that nontrivial left (right) normal bands and nontrivial left (right) semilattices cannot be weakly abelian.

**Theorem 4.7** [Wa2]. *A left distributive weakly abelian groupoid with a minimal clone is either a rectangular band, an affine space or (the dual of) a p-cyclic groupoid for some prime p.*

**4.3. Minimal clones with term conditions.** Only minimal clones of types (I), (II) and (IV) can have nontrivial weakly abelian representations, and in case of types (I) and (IV) all representations are abelian. A weakly abelian groupoid with a minimal clone is left or right distributive by Lemma 4.2, therefore we can apply Theorem 4.7 (after dualizing if necessary) to see that such a groupoid must be a rectangular band, an affine space or (the dual of) a $p$-cyclic groupoid. This list does not contain any new items compared to Theorem 1.5, thus the two abelianness concepts coincide at the level of abstract minimal clones.

**Theorem 4.8** [Wa2]. *If a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation. Therefore such a clone must be a unary clone, the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

Unary algebras, rectangular bands and affine spaces are abelian. This fact together with the following lemma yields an interesting homogeneity property for weakly abelian representations.

**Lemma 4.9** [Wa2]. *Every p-cyclic groupoid is weakly abelian.*

**Theorem 4.10** [Wa2]. *If a minimal clone has a nontrivial weakly abelian representation, then all representations are weakly abelian.*

We conclude with a theorem about rectangular and strongly abelian representations of minimal clones. A nontrivial affine space or $p$-cyclic groupoid cannot be rectangular, but unary algebras and rectangular bands are all strongly abelian. Thus these two term conditions are equivalent for groupoids with minimal clones.

**Theorem 4.11** [Wa2]. *If a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian. Such a clone must be unary or the clone of rectangular bands.*

## 5. Almost associative operations generating a minimal clone

In this chapter we generalize Theorem 1.3 by characterizing minimal clones generated by almost associative binary operations. To explain what we mean by this, we need a way to measure how far a binary operation is from being associative. First we discuss two such measures of associativity, and then we describe binary minimal operations that are close to being associative according to these two measures.

**5.1. Measuring associativity.** One way to measure associativity is to count the *nonassociative triples* in the groupoid; this number is called the *index of nonassociativity*, and is denoted by $ns$. Formally, we have $ns(\mathbb{A}) = |\{(a,b,c) \in A^3 : (ab)c \neq a(bc)\}|$. This notion was studied in [Cl1, Cl2, DK, KT1, Szá]. Clearly $\mathbb{A}$ is a semigroup iff $ns(\mathbb{A}) = 0$, and it is natural to say that the multiplication of $\mathbb{A}$ is almost associative if $ns(\mathbb{A}) = 1$. Such groupoids are called *Szász-Hájek groupoids* (SH-groupoids for short). SH-groupoids were investigated in [Há1, Há2] and [KT3–KT6] in much detail.

Clearly, a subgroupoid of an SH-groupoid $\mathbb{A}$ with nonassociative triple $(a, b, c)$ is an SH-groupoid or a semigroup, depending on whether it contains $a$, $b$ and $c$ or not. Specially, $\mathbb{A}$ is generated by $\{a, b, c\}$ iff all proper subgroupoids of $\mathbb{A}$ are semigroups. Such a groupoid is called a *minimal SH-groupoid.*

Another way of measuring associativity is possible by considering the identities implied by associativity, and somehow counting how many of these are (not) satisfied. To make this more precise, let us say that $B$ is a *bracketing*, if $B$ is a groupoid term, and each variable occurs exactly once in $B$. If these variables are $x_1, x_2, \ldots, x_n$ and they appear in this order (as we suppose most of the time), then $B$ is nothing else but a way to put brackets into the product $x_1 \cdot \ldots \cdot x_n$ such that the order of the $n-1$ multiplications is well determined. We express this fact by writing $B = B(x_1, \ldots, x_n)$.

The number of bracketings of the product $x_1 \cdot \ldots \cdot x_n$ is $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$, the $(n-1)$st Catalan number. In a semigroup all of these $C_{n-1}$ many terms induce the same term function, but in an arbitrary groupoid they may induce more than one term function. Intuitively, the more term functions of this kind there are, the less associative the multiplication is. Therefore we define the *associative spectrum* of a groupoid $\mathbb{A}$ to be the sequence $s_{\mathbb{A}}(1), s_{\mathbb{A}}(2), \ldots, s_{\mathbb{A}}(n), \ldots$, where $s_{\mathbb{A}}(n)$ is the number of different term functions on $\mathbb{A}$ arising from bracketings of $x_1 \cdot \ldots \cdot x_n$. Thus the associative spectrum gives (only quantitative) information about identities of the form $B_1(x_1, \ldots, x_n) = B_2(x_1, \ldots, x_n)$ satisfied by the groupoid. The associative spectrum was introduced and investigated in [CsW].

Clearly, $s_{\mathbb{A}}(1) = s_{\mathbb{A}}(2) = 1$ for every groupoid $\mathbb{A}$, and $s_{\mathbb{A}}(3) = 1$ iff $\mathbb{A}$ is a semigroup. In the latter case $s_{\mathbb{A}}(n) = 1$ for all $n$ by the general law of associativity. The smallest possible spectrum for a nonassociative multiplication is $1, 1, 2, 1, 1, \ldots$, so we could say that a binary operation is almost associative if its spectrum is this sequence. However, there is no groupoid having a minimal clone with this spectrum (not even an idempotent one) as we will see later. Therefore we have to be more generous: in Theorem 5.7 we determine groupoids with a minimal clone satisfying $s(4) < 5 = C_3$.

The two ways of measuring associativity introduced here do not seem to be closely related. For example, the three-element groupoid of Theorem 5.9 is an SH-groupoid, with the largest possible associative spectrum: $s(n) = C_{n-1}$ for every $n$.

Let us mention finally that there is a third possibility to measure associativity with the help of the Hamming distance of multiplication tables. This yields the notion of the *semigroup distance* of a groupoid. Groupoids with small semigroup distance and connections between the semigroup distance and the index of nonassociativity were studied in [KT2].

**5.2. Minimal clones with small spectrum.** In this section we describe nonassociative binary operations generating a minimal clone that have a relatively small associative spectrum. The first three theorems show that the spectrum of such an operation cannot be too small.

**Theorem 5.1** [Wa3]. *If an idempotent groupoid satisfies the identity*

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} = \overrightarrow{x_1 \cdot \ldots \cdot x_n}$$

*for some $n \geq 3$, then it is a semigroup.*

**Theorem 5.2** [Wa3]. *An idempotent groupoid satisfying the following two identities for some $n \geq 3$ must be a semigroup.*

$$x_0 \cdot \overleftarrow{x_1 \cdot \ldots \cdot x_n} = x_0 \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n}$$
$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} \cdot x_0 = \overrightarrow{x_1 \cdot \ldots \cdot x_n} \cdot x_0$$

**Theorem 5.3** [Wa3]. *If a groupoid has a minimal clone and satisfies*

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} = x_1 \cdot \overleftarrow{x_2 \cdot \ldots \cdot x_n}$$

*for some $n \geq 3$, then it is a semigroup.*

Let us now turn to the investigation of four-variable 'associativity conditions'. There are five bracketings of a product of four variables:

$$\begin{aligned} B_1 &= x\,(y\,(zu))\,; \\ B_2 &= x\,((yz)\,u)\,; \\ B_3 &= (xy)\,(zu)\,; \\ B_4 &= ((xy)\,z)\,u; \\ B_5 &= (x\,(yz))\,u. \end{aligned}$$

Specializing the previous three theorems to $n = 4$ we can see that many of the possible $\binom{5}{2}$ identities cannot be satisfied by a nonassociative groupoid with a minimal clone. It turns out that if a groupoid $\mathbb{A}$ has a minimal clone, and $1 < s_{\mathbb{A}}(4) < 5$ holds for its spectrum, then $s_{\mathbb{A}}(4) = 4$, and $\mathbb{A}$ satisfies either $B_1 = B_2$ or its dual, but not both. Thus the right notion of almost associativity seems to be that $\mathbb{A}$ or its dual belongs to the variety $\mathcal{A}$ defined by $x\,(y\,(zu)) = x\,((yz)\,u)$. If $\text{Clo}\,\mathbb{A}$ is minimal, then $\mathcal{V} = \text{HSP}\,\mathbb{A}$ has a minimal clone too, therefore we can apply Theorem 1.3 to describe the semigroups in this variety. The following lemma shows how we can use this to derive information about $\mathcal{V}$ itself.

**Lemma 5.4** [Wa3]. *Let $\mathcal{V}$ be a subvariety of $\mathcal{A}$, and let $\mathcal{W}$ be the intersection of $\mathcal{V}$ and the variety of semigroups. If an identity $t_1 = t_2$ holds in $\mathcal{W}$, then $xt_1 = xt_2$ holds in $\mathcal{V}$ (where $x$ is an arbitrary variable).*

The next lemma is based on the method of minimal monoids [KSz].

**Lemma 5.5** [Wa3]. *Suppose that $\mathbb{A}$ is a groupoid with a minimal clone, and $M$ is a subset of $\text{Clo}^{(2)}(\mathbb{A})$ containing the first projection and at least one nontrivial element, such that for all $f, g, h \in M$ we have*

(i) $f\,(g, h) = g$;
(ii) $f\,(g, h^d) = f\,(g, e_2) \in M$.

*Then $\mathbb{A}$ or its dual belongs to the variety $\mathcal{D}$ or $\mathcal{C}_p$ for some prime number $p$.*

These two lemmas allow us to determine which algebras of the variety $\mathcal{A}$ have a minimal clone.

**Theorem 5.6** [Wa3]. *Let $\mathcal{V} \subseteq \mathcal{A}$ be a variety with a minimal clone. Then $\mathcal{V}$ or its dual is a subvariety of $\mathcal{B}$, $\mathcal{C}_p$, $\mathcal{D}$ or $\mathcal{RB}$ for some prime $p$.*

The main result of this section is the following theorem, which characterizes groupoids with a minimal clone that are almost semigroups in the 'spectral' sense.

**Theorem 5.7** [Wa3]. *For any groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) $\mathbb{A}$ *has a minimal clone and* $1 < s_{\mathbb{A}}(4) < 5$;

(ii) $\mathbb{A}$ *is not a semigroup and* $\mathbb{A}$ *or its dual belongs to one of the varieties* $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ *or* $\mathcal{D} \cap \mathcal{A}$ *for some prime* $p$.

*If these conditions are fulfilled, then we have* $s_{\mathbb{A}}(n) = 2^{n-2}$ *for* $n \geq 2$.

**5.3. Szász-Hájek groupoids with a minimal clone.** In this section we determine binary operations generating a minimal clone that are almost associative in the 'index' sense, i.e. SH-groupoids with a minimal clone.

**Theorem 5.8** [Wa3]. *For any Szász-Hájek groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) $\mathbb{A}$ *has a minimal clone;*

(ii) $\mathbb{A}$ *or its dual belongs to the variety* $\mathcal{B}$.

Finally we determine minimal SH-groupoids in the varieties $\mathcal{B}$ and $\mathcal{B}^d$. In [KT3-KT6] the project of characterizing minimal SH-groupoids was begun, but completed only for certain types. However, these types of groupoids do not have minimal clones (except for one groupoid), so the next theorem gives new minimal SH-groupoids.

**Theorem 5.9** [Wa3]. *Every minimal SH-groupoid having a minimal clone is isomorphic or dually isomorphic to one of the following ten groupoids.*

| · | a | b | c | d | f | g | h | i |
|---|---|---|---|---|---|---|---|---|
| a | a | d | f | d | f | g | d | g |
| b | h | b | c | h | i | i | h | i |
| c | c | c | c | c | c | c | c | c |
| d | d | d | g | d | g | g | d | g |
| f | f | f | f | f | f | f | f | f |
| g | g | g | g | g | g | g | g | g |
| h | h | h | i | h | i | i | h | i |
| i | i | i | i | i | i | i | i | i |

| · | a | b | c | d | f | g |
|---|---|---|---|---|---|---|
| a | a | d | f | d | f | g |
| b | d | b | c | d | g | g |
| c | c | c | c | c | c | c |
| d | d | d | g | d | g | g |
| f | f | f | f | f | f | f |
| g | g | g | g | g | g | g |

| · | a | b | c | e | f |
|---|---|---|---|---|---|
| a | a | a | c | f | f |
| b | b | b | e | e | e |
| c | c | c | c | c | c |
| e | e | e | e | e | e |
| f | f | f | f | f | f |

| · | a | b | c | e | g |
|---|---|---|---|---|---|
| a | a | a | g | e | g |
| b | a | b | e | e | g |
| c | c | c | c | c | c |
| e | e | e | e | e | e |
| g | g | g | g | g | g |

| · | a | b | c | e |
|---|---|---|---|---|
| a | a | a | c | e |
| b | b | b | e | e |
| c | c | c | c | c |
| e | e | e | e | e |

| · | a | b | c | d | f | g | h |
|---|---|---|---|---|---|---|---|
| a | a | d | f | d | f | g | d |
| b | h | b | c | h | g | g | h |
| c | c | c | c | c | c | c | c |
| d | d | d | g | d | g | g | d |
| f | f | f | f | f | f | f | f |
| g | g | g | g | g | g | g | g |
| h | h | h | g | h | g | g | h |

| · | a | b | c | e | f | g |
|---|---|---|---|---|---|---|
| a | a | a | g | f | f | g |
| b | a | b | e | e | f | g |
| c | c | c | c | c | c | c |
| e | e | e | e | e | e | e |
| f | f | f | f | f | f | f |
| g | g | g | g | g | g | g |

| · | a | b | c | d | f | h |
|---|---|---|---|---|---|---|
| a | a | d | f | d | f | d |
| b | h | b | c | h | h | h |
| c | c | c | c | c | c | c |
| d | d | d | d | d | d | d |
| f | f | f | f | f | f | f |
| h | h | h | h | h | h | h |

| · | a | b | c | d | f |
|---|---|---|---|---|---|
| a | a | d | f | d | f |
| b | d | b | c | d | d |
| c | c | c | c | c | c |
| d | d | d | d | d | d |
| f | f | f | f | f | f |

| · | a | b | c |
|---|---|---|---|
| a | a | a | c |
| b | b | b | b |
| c | c | c | c |

Let us note that the class of groupoids found in Theorem 5.7 is disjoint from the class described in Theorem 5.8, i.e. there is no groupoid with a minimal clone that is almost associative in both the 'spectral' and the 'index' sense.

## References

[Cl1]    A. C. Climescu, *Études sur la théorie des systèmes multiplicatifs uniformes I. L'indice de non-associativité,* Bull. École Polytech. Jassy **2** (1947), 347–371. (French)

[Cl2]    A. C. Climescu, *L'indépendance des conditions d'associativité,* Bull. Inst. Polytech. Jassy **1** (1955), 1–9. (Romanian)

[Cs1]    B. Csákány, *All minimal clones on the three-element set,* Acta Cybernet. **6** (1983), no. 3, 227–238.

[Cs2]    B. Csákány, *On conservative minimal operations,* Lectures in Universal Algebra (Szeged, 1983), Colloq. Math. Soc. János Bolyai, **43**, North-Holland, Amsterdam, 1986, 49–60.

[CsG]    B. Csákány, T. Gavalcová, *Finite homogeneous algebras I,* Acta Sci. Math. (Szeged) **42** (1980), no. 1-2, 57–65.

[CsW]    B. Csákány, T. Waldhauser, *Associative spectra of binary operations,* Mult.-Valued Log. **5** (2000), no. 3, 175–200.

[DK]    A. Drápal, T. Kepka, *Sets of associative triples,* Europ. J. Combinatorics **6** (1985), 227–231.

[Du]    J. Dudek, *The unique minimal clone with three essentially binary operations,* Algebra Universalis **27** (1990), no. 2, 261–269.

[DG]    J. Dudek, J. Gałuszka, *Theorems of idempotent commutative groupoids,* Algebra Colloq. **12** (2005), no. 1, 11–30

[FP]    E. Fried, A. F. Pixley, *The dual discriminator function in universal algebra,* Acta Sci. Math. (Szeged) **41** (1979), no. 1-2, 83–100.

[Há1]    P. Hájek, *Die Szászschen Gruppoide,* Mat.-Fys. Časopis Sloven. Akad. Vied **15** (1965) no. 1, 15–42. (German)

[Há2]    P. Hájek, *Berichtigung zu meiner arbeit "Die Szászschen Gruppoide",* Mat.-Fys. Časopis Sloven. Akad. Vied **15** (1965) no. 4, 331. (German)

[HM]    D. Hobby, R. McKenzie, *The structure of finite algebras,* Contemporary Mathematics **76**, American Mathematical Society, Providence, RI, 1988.

[JQ]    J. Ježek, R. W. Quackenbush, *Minimal clones of conservative functions,* Internat. J. Algebra Comput. **5** (1995), no. 6, 615–630.

[Kea]    K. A. Kearnes, *Minimal clones with abelian representations,* Acta Sci. Math. (Szeged) **61** (1995), no. 1-4, 59–76.

[KK]    K. Kearnes, E. W. Kiss, *Finite algebras of finite complexity,* Discrete Math. **207** (1999), no. 1-3, 89–135.

[KSz]    K. A. Kearnes, Á. Szendrei, *The classification of commutative minimal clones,* Discuss. Math. Algebra Stochastic Methods **19** (1999), no. 1, 147–178.

[Kep]    T. Kepka, *The structure of weakly abelian quasigroups,* Czechoslovak Math. J. **28(103)** (1978), no 2, 181–188.

[KT1]    T. Kepka, M. Trch, *Groupoids and the associative law I. (Associative triples),* Acta Univ. Carol. Math. Phys. **33** (1992), no. 1, 69–86.

[KT2]    T. Kepka, M. Trch, *Groupoids and the associative law II. (Groupoids with small semigroup distance),* Acta Univ. Carol. Math. Phys. **34** (1993), no. 1, 67–83.

[KT3]    T. Kepka, M. Trch, *Groupoids and the associative law III. ( Szász-Hájek groupoids),* Acta Univ. Carol. Math. Phys. **36** (1995), no. 1, 17–30.

[KT4]    T. Kepka, M. Trch, *Groupoids and the associative law IV. (Szász-Hájek groupoids of type $(a, b, a)$),* Acta Univ. Carol. Math. Phys. **35** (1994), no. 1, 31–42.

[KT5]    T. Kepka, M. Trch, *Groupoids and the associative law V. (Szász-Hájek groupoids of type $(a, a, b)$),* Acta Univ. Carol. Math. Phys. **36** (1995), no. 1, 31–44.

[KT6]    T. Kepka, M. Trch, *Groupoids and the associative law VI. (Szász-Hájek groupoids of type $(a, b, c)$),* Acta Univ. Carol. Math. Phys. **38** (1997), no. 1, 13–22.

[LP]    L. Lévai, P. P. Pálfy, *On binary minimal clones,* Acta Cybernet. **12** (1996), no. 3, 279–294.

[P³]    P. P. Pálfy, *Minimal clones,* Preprint of the Math. Inst. Hungarian Acad. Sci. 27/1984.

[Pł1]    J. Płonka, *On groups in which idempotent reducts form a chain,* Colloq. Math. **29** (1974), 87–91.

[Pł2]    J. Płonka, *On k-cyclic groupoids,* Math. Japon. **30** (1985), no. 3, 371–382.

[Po]    E. Post, *The two-valued iterative systems of mathematical logic,* Annals of Mathematics Studies, no. 5, Princeton University Press, Princeton, 1941.

[PK]    R. Pöschel, L. A. Kalužnin, *Funktionen- und Relationenalgebren,* Mathematische Monographien, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979. (German)

[Ros]    I. G. Rosenberg, *Minimal clones I. The five types,* Lectures in Universal Algebra (Szeged, 1983), Colloq. Math. Soc. János Bolyai, **43**, North-Holland, Amsterdam, 1986, 405–427.

[Szá]    G. Szász, *Die Unabhängigkeit der Assoziativitätsbedingungen,* Acta Sci. Math. (Szeged) **15** (1953), 20–28. (German)

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[Szcz]    B. Szczepara, *Minimal clones generated by groupoids,* Ph.D. Thesis, Université de Montréal, 1995.

[SzM]     M. B. Szendrei *On closed sets of term functions on bands,* Semigroups (Proc. Conf., Math. Res. Inst., Oberwolfach, 1978), pp. 156–181, Lecture Notes in Math., 855, Springer, Berlin, 1981.

[Wa1]     T. Waldhauser, *Minimal clones generated by majority operations,* Algebra Universalis **44** (2000), no. 1-2, 15–26.

[Wa2]     T. Waldhauser, *Minimal clones with weakly abelian representations,* Acta Sci. Math. (Szeged) **69** (2003), no. 3-4, 505–521.

[Wa3]     T. Waldhauser, *Almost associative operations generating a minimal clone,* Discuss. Math. Gen. Algebra Appl. **26** (2006), no. 1, 45–73

[Wa4]     T. Waldhauser, *Minimal clones with few majority operations,* accepted for publication in Acta Sci. Math. (Szeged)