On Stability Conditions of Operator Semigroups

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Abstract

The principal aim of this thesis is to prove stability results for semigroups of operators. This study is based on two central operator theoretical results, namely the Arendt–Batty–Lyubich–Vũ theorem and the Katznelson–Tzafriri theorem. Both theorems help provide a connection between the spectral properties of semigroups of operators and their asymptotic behaviour.

We shall give an extension of the ABLV theorem for certain unbounded representations that have a regular norm-function. Then we will characterise those C_0-semigroups whose norm-function is topologically regular. In addition, we shall prove a Katznelson–Tzafriri type theorem for discrete one-parameter semigroups in Hilbert spaces.

Contents

Chapter 1

Introduction

1.1 Asymptotic behaviour of operator semigroups

Consider the well-posed abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \dot{u}(t) & = Au(t) \quad (0 \leq t), \\ u(0) & = x, \end{cases}$$

where A is a densely defined, closed operator acting on a complex Banach space $\mathfrak{X}$ and $x \in \mathfrak{X}$. We say that the family of bounded linear operators $(T(s))_{s \geq 0}$ acting on $\mathfrak{X}$ is a C_0-semigroup if $T(0) = I$, $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and the mapping $s \mapsto T(s)x$ is continuous for every $x \in \mathfrak{X}$. Then the (classical and mild) solutions of the equation (ACP) are derived from a C_0-semigroup $(T(s))_{s \geq 0}$. The operator semigroup $(T(s))_{s \geq 0}$ uniquely determines the operator A, which is called the *infinitesimal generator* of $(T(s))_{s \geq 0}$; that is,

$$Ax = \lim_{h \downarrow 0} \frac{T(h)x - x}{h}$$

when the limit exists. The set of vectors for which the above limit does exists is called the *domain* of A and it shall be denoted by $\mathcal{D}(A)$. For a comprehensive study on C_0-semigroups we refer the reader to Engel–Nagel's monograph [20] and [2].

One of the most studied parts of the theory of linear operator semigroups is the asymptotic behaviour of the semigroup. We say that a C_0-semigroup $(T(s))_{s \geq 0}$ is stable if $T(s)$ tends to zero, if $s \to \infty$ in the strong operator topology. A systematic account of the stability theory of linear operator semigroups is presented in [54], [2] and [13]. The uniform and weak stability properties of the semigroup can be also described in terms of

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the operator norm topology and the weak operator topology. On recent developments of weak stability theorems we refer the reader to the articles [18] and [19]. A detailed overview of these results can also be found in [61].

1.1.1 The ABLV theorem

In 1988, W. Arendt and C.J.K. Batty [1], and Y. Lyubich and Q. P. Vũ [50], independently and simultaneously, proved a famous stability result on bounded continuous one-parameter semigroups of operators. Let us consider a C_0-semigroup T of bounded linear operators on a complex Banach space $\mathcal{X}$. We will denote the algebra of bounded linear operators acting on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$. The generator of T shall be denoted by A, and $\sigma(A)$, $\sigma_{\mathrm{p}}(A^*)$ will stand for the spectrum of A and the point spectrum of its adjoint A^*, respectively. We say that T is bounded if $\sup_{t \in \mathbb{R}_+} \|T(t)\| < \infty$. Next, we will present the well-known ABLV theorem.

Theorem 1.1.1. *Let T be a bounded C_0-semigroup on $\mathcal{X}$ having the generator A, and suppose that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_{\mathrm{p}}(A^*)$ is empty. Then $\lim_{s \to \infty} \|T(s)x\| = 0$ for every $x \in \mathcal{X}$.*

In the Hilbert space setting, B. Szőkefalvi-Nagy and C. Foias [63] showed earlier that if T is a completely non-unitary contraction and the peripheral spectrum of T has zero Lebesgue measure, then the operator T is stable (i.e. $T^n \to 0$ strongly). This result remains valid in the continuous one-parameter case as well, due to a result by L. Kérchy and J. Neerven [33].

The ABLV theorem has been investigated by many authors and quite a few generalizations of the theorem have been proved for bounded and unbounded representations of suitable, locally compact abelian semigroups (see [4], [8], [6], [36], [37]).

The first abstract version of the theorem was given by Batty and Vũ [8]. Let G be a locally compact group and let S be a σ-compact subsemigroup of G, with non-empty interior and with $S - S = G$. Let S^* and S_{u}^* stand for the space of non-zero continuous bounded homomorphisms of S into $\mathbb{C}$, and of S into the unit circle of $\mathbb{C}$, respectively. Next, consider S with the restriction of a Haar measure on G. We can define for $f \in L^1(S)$ and $\chi \in S^*$ the integral

$$\widehat{f}(\chi) := \int_S f(s)\chi(s)\, ds.$$

Let us assume that the functions $\widehat{f}$ ($f \in L^1(S)$) separate the points of S^* from each other and from 0. By a representation T of S, we mean a strongly continuous semigroup homomorphism $T: S \to \mathcal{L}(\mathcal{X})$. For any bounded representation T of S and f in $L^1(S)$, we shall define

$$\widehat{f}(T) := \int_S f(s)T(s)\,ds.$$

In their paper [8] Batty and Vũ introduced the spectrum of T in the following way: the spectrum $\mathrm{Sp}(T)$ of T is the set of all characters χ in S^* such that $|\widehat{f}(\chi)| \leq \|\widehat{f}(T)\|$ holds for every $f \in L^1(S)$. The *unitary spectrum* of T is $\mathrm{Sp}_{\mathrm{u}}(T) := \mathrm{Sp}(T) \cap S_{\mathrm{u}}^*$. The unitary point spectrum $P_{\mathrm{u}}\sigma(T^*)$ of T^* is the set of all $\chi \in S_{\mathrm{u}}^*$, for which there exists a non-zero $\phi \in \mathcal{X}^*$ such that $T^*(s)\phi = \chi(s)\phi$ ($s \in S$). After these preliminaries, the following was proved in [8].

Theorem 1.1.2. *Let T be a bounded representation of S such that $\mathrm{Sp}_{\mathrm{u}}(T)$ is countable and $P_{\mathrm{u}}\sigma(T^*)$ is empty. Then $T(s) \to 0$ strongly (through S).*

We note that first Vũ [66] proved a weighted version of the ABLV theorem. Later, C.J.K. Batty and S. Yeates [6], [67] gave a detailed study on the spectral theory and stability of non-quasianalytic representations. A more recent article [3] presents a very general result in the spirit of [6], but it uses a different approach.

1.2 Representations and regularity

Operators with regular norm-sequences were characterized by L. Kérchy and V. Müller (see [38], [43]). These operators lead to useful generalizations of classical theorems in operator theory. They were used in [40], [41] to establish a satisfactory symbolic calculus for generalized Toeplitz operators and to prove results for their elementary and reflexive hyperplanes. In the papers by G. Cassier [11], [12] the concept of regularity was also used to derive new results on similarity problems. Discrete representations with regular norm-function were studied in [36], [37], [39] in connection with unbounded versions of the Arendt–Batty–Lyubich–Vũ stability theorem and the Katznelson–Tzafriri theorem.

In this thesis, we shall extend the method, which appears in Kérchy's papers ([36], [37]) on discrete abelian semigroups, to topological semigroups; that is, we shall prove stability results for unbounded representations having a regular norm-function. The stability results we derived are quite similar to [6], but the main differences lie in the

way we express the stability of the semigroup and its norm-conditions. Our proof follows the usual scheme and relies on results concerning isometric representations proved in [8]. The originality of our approach appears in the usage of almost convergence defining regular norm-behaviour and in the systematic study of representations with this property. This study is a joint work with L. Kérchy that was published in [42].

In the following chapter we will give a brief summary of the concepts and results of amenability. The most important tools for us are invariant means which are used to define regularity and Følner's condition which characterises amenability (Theorem 2.1.5). After this, we will prove that every representation with regular norm-behaviour determines a unique character on S. This fact makes it possible for us to introduce the peripheral spectrum of such a representation. At the end of Chapter 2 we will present and prove an extension of the ABLV theorem.

1.2.1 Topological regularity

In Chapter 3 we shall study a slightly different concept of regularity. We recall that regularity is defined by means of invariant means, but we were able to use also a (generally) strictly smaller class of means, the topologically invariant means, to define another kind of regularity property of the norm function of representations. We should mention here that counterparts of the main results in Chapter 2 remain valid with this approach. (In fact, the main results like the existence of the limit functional and the associated isometric representation can be proved in a similar way.) In discrete semigroups these two concepts are the same because the classes of these means coincide in the discrete case. We shall give a detailed exposition of the second alternative on the real half-line (see [46]). We will also give a characterization of C_0-semigroups with topologically regular norm-function in the spirit of [43].

Our hope is that this result will help provide a better understanding of the regularity property on the real half line.

1.3 The Katznelson–Tzafriri theorem

In this thesis we will also study another type of stability result. The Katznelson–Tzafriri theorem is a general operator theoretical result which is related to the ABLV theorem. (The interested reader should consult [23] to learn more about this connection.) Now,

let $A(\mathbb{T})$ denote the set of continuous functions on the unit circle $\mathbb{T}$ whose Fourier series is absolutely convergent, and let $A^+(\mathbb{T})$ be the set of functions in $A(\mathbb{T})$ whose Fourier coefficients with negative indices vanish. The algebra $A(\mathbb{T})$ is a Banach algebra with the norm $\|f\| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|$ (where $f \in A(\mathbb{T})$ and $\{\widehat{f}(n)\}_{n=-\infty}^{\infty}$ are the Fourier coefficients of f). We say that an $f \in A^+(\mathbb{T})$ is of spectral synthesis with respect to a closed set $E \subseteq \mathbb{T}$ if there exists a sequence $\{f_n\}_n \subset A(\mathbb{T})$ such that each f_n vanishes in a neighbourhood of E and $\lim_{n\to\infty} \|f_n - f\| = 0$. The Katznelson–Tzafriri theorem is the following.

Theorem 1.3.1. *Let T be a power-bounded operator on the Banach space $\mathfrak{X}$, and let $f \in A^+(\mathbb{T})$, which is of spectral synthesis with respect to the peripheral spectrum of T. Then*

$$\lim_{n\to\infty} \|T^n f(T)\| = 0.$$

The important special case of $f(z) = z - 1$ was studied earlier by Esterle [22]. He proved that $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. A simple corollary of this statement is the classical result of Gelfand that an operator T with $\sigma(T) = \{1\}$ and $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$ is the identity. For further details on this statement, see [68]. Esterle's result was later obtained independently and generalized by Y. Katznelson and L. Tzafriri. Using this result, they gave a new proof and an extension of the zero–two law in [34].

In Chapter 4 we shall prove that the assumption made in the Katznelson–Tzafriri theorem can be weakened in Hilbert spaces, and we shall provide a complete characterization of the condition $\lim_{n\to\infty} \|T^n Q\| = 0$ whenever Q commutes with T. These results are based on [47].

We note that many extensions of the Katznelson–Tzafriri theorem were proved in the discrete case as well as in the continuous case using different methods. For details, see [8], [24], [33], [51], [52], [65] and related survey articles [4], [13]. However, we should also remark that all former extensions of the Katznelson–Tzafriri theorem are related to bounded functional calculi of T or elements of the Banach algebra generated by T.

Chapter 2

Representations with regular norm-functions

In Section 2.1 we shall introduce the concept of almost convergence (defined for sequences in [48]) in terms of invariant means, which will be needed to define the limit functional in Section 2.3 for unbounded representations satisfying a certain regularity condition. It turns out that this limit functional is intrinsic to the representation itself and independent of any choice of gauge function involved in the regularity condition. This allows us to define the peripheral spectrum in Section 2.4 along with various spectral notions of representations with a regular norm-function. In the bounded case the definition coincides with the one introduced by Batty and Vũ [8], and Lyubich [49]. In Section 2.5 we will generalize a well-known method associating an isometric representation to a representation with a regular norm-function. With these results at hand we will give, in Section 2.6, a new extension of the ABLV theorem.

2.1 Almost convergence on semigroups

Consider an abelian semigroup $(S, +)$. For any $s \in S$ and $\omega \subseteq S$ set $\omega \ominus s := \{s' \in S : s + s' \in \omega\}$. The translation of a function $f \colon S \to \mathbb{C}$ by $s \in S$ is the mapping $f_s \colon S \to \mathbb{C}$ defined by $f_s(s') := f(s + s')$ $(s' \in S)$. Let (S, Ω, λ) be a σ-finite measure space on S such that (i) $\omega \ominus s \in \Omega$ whenever $\omega \in \Omega$ and $s \in S$, and (ii) $\lambda(\omega) = 0$ $(\omega \in \Omega)$ implies $\lambda(\omega \ominus s) = 0$ for all $s \in S$. Obviously, $S = \mathbb{R}_+^n$ with the Lebesgue measure and $S = \mathbb{Z}_+^n$ with the counting measure satisfy these conditions. Let $L^\infty(\lambda)$ denote the Banach space

of all essentially bounded, complex-valued, measurable functions with the usual norm.

Lemma 2.1.1. *Under the conditions* (i) *and* (ii), $\tau_s\colon L^\infty(\lambda) \to L^\infty(\lambda), f \mapsto f_s$ *is a well-defined linear mapping with* $\|\tau_s\| = \|\tau_s \mathbf{1}\| = 1$, *for every* $s \in S$.

Proof. Let χ_ω be the characteristic function of $\omega \in \Omega$. Since χ_ω is measurable we can infer from (i) that $(\chi_\omega)_s = \chi_{\omega \ominus s}$ is also measurable. Using the usual approximation method we get that f_s is measurable for any $f \in L^\infty(\lambda)$. To see that the translation is well defined on $L^\infty(\lambda)$ suppose that $f_1 = f_2$ a.e.; then by (ii) we have $(f_1)_s = (f_2)_s$ a.e.. Obviously, $\|f_s\|_\infty \le \|f\|_\infty$. Since $\|\mathbf{1}\|_\infty = 1$ and $\tau_s \mathbf{1} = \mathbf{1}$, we obtain $\|\tau_s\| = \|\tau_s \mathbf{1}\| = 1$. $\qquad\square$

The set $\tilde{M}(S, \lambda) := \{m \in L^\infty(\lambda)^* : \|m\| = m(\mathbf{1}) = 1\}$ is called the set of *means* on S with respect to λ. The Hahn–Banach theorem implies that $\tilde{M}(S, \lambda) \ne \emptyset$. A functional $m \in \tilde{M}(S, \lambda)$ is called an *invariant mean* if $m(f_s) = m(f)$ holds for every $f \in L^\infty(\lambda)$ and $s \in S$; the set of all invariant means with respect to λ shall be denoted by $M(S, \lambda)$. It is well known that there are invariant means on abelian groups, and this can be readily extended to semigroups. For the sake of clarity we shall now sketch the proof.

Proposition 2.1.2. $M(S, \lambda)$ *is not empty.*

Proof. Since $\tilde{M}(S, \lambda)$ is a convex, weak-$*$ closed subset of the unit ball of $L^\infty(\lambda)^*$, it follows that $\tilde{M}(S, \lambda)$ is a weak-$*$ compact set. Taking an $m \in \tilde{M}(S, \lambda)$, for any $s \in S$ we have $m \circ \tau_s \in \tilde{M}(S, \lambda)$, and $T(s)\colon \tilde{M}(S, \lambda) \to \tilde{M}(S, \lambda), m \mapsto m \circ \tau_s$ is an affine, weak-$*$ continuous mapping. The Markov–Kakutani theorem (see e.g. [14, theorem V.10.1]) implies the existence of a common fixed point $m_0 \in \tilde{M}(S, \lambda)$. Then it immediately follows that m_0 is an invariant mean. $\qquad\square$

Now consider a locally compact, Hausdorff abelian group G. Let S be a closed subsemigroup of G with non-empty interior S° such that $S - S = G$ and $S \cap (-S) = \{0\}$. By definition, for any $s_1, s_2 \in S$, $s_1 \preceq s_2$ if $s_2 - s_1 \in S$. In this way we obtain an inductive partial ordering on S, hence S is a directed set. We say that a function $f\colon S \to \mathbb{C}$ tends to 0 at infinity: $\lim_s f(s) = 0$, if for every $\varepsilon > 0$ there exists an $s_0 \in S$ such that $|f(s)| < \varepsilon$ whenever $s_0 \preceq s$. Let μ denote the restriction of the Haar measure $\tilde{\mu}$ on G to S. It is clear that the conditions (i) and (ii) of Lemma 1 are satisfied for μ; indeed, $\omega \ominus s = (\omega - s) \cap S$. The (invariant) means with respect to μ are simply called (invariant) means on S; that is $L^\infty(S) := L^\infty(\mu), \tilde{M}(S) := \tilde{M}(S, \mu), \mathcal{M}(S) := M(S, \mu)$.

Throughout this thesis a function f defined on S will also be treated as a function on G that is zero outside S.

Definition 2.1.3. A net $\{K_\lambda\}_{\lambda \in \Lambda}$ of compact subsets of G with nonempty interior is a *strong Følner net* for G if

(i) $K_{\lambda_1} \subseteq K_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,

(ii) $G = \bigcup K_\lambda^\circ$,

(iii) $\tilde{\mu}((x + K_\lambda) \,\triangle\, K_\lambda)/\tilde{\mu}(K_\lambda) \to 0$ (as $\lambda \to \infty$) uniformly when x runs through compact sets. (Here and in the sequel $\triangle$ stands for the symmetric difference.)

A net $\{K_\lambda\}_{\lambda \in \Lambda}$ of compact subsets of G with nonempty interior is called a *Følner net* for G if (iii) is satisfied.

Example 2.1.4. We can give some simple examples here.

- if $G = \mathbb{Z}$, the sets $\{0, ..., n\}$ ($n \in \mathbb{N}$) (or $\{-n, ..., n\}$) form a Følner sequence for $\mathbb{Z}$;

- the intervals $[0, n]$ ($n \in \mathbb{N}$) (or $[-n, n]$) provides a Følner sequence for $\mathbb{R}$.

We recall that an arbitrary locally compact group G is called *amenable* if there exists an invariant mean on G. The following theorem is a useful and deep characterization of amenable groups (for details of its proof, see [56, Theorem 4.16]).

Theorem 2.1.5. *A locally compact group G is amenable if and only if there exists a strong Følner net for G. If G is σ-compact, then G is amenable if and only if there exists a strong Følner sequence for G.*

We shall assume in the sequel that G is σ-compact. Since G is abelian by our assumption, the Markov–Kakutani theorem implies that G is amenable (see the proof of Proposition 2.1.2), thus using the characterization theorem we obtain the existence of a strong Følner sequence $\{K_n\}_{n \in \mathbb{N}}$ on G. We shall translate this sequence to the interior of S, preserving property (iii), by means of the following two lemmas.

Lemma 2.1.6. *If $\{K_n\}_n$ is a Følner sequence for G then $\{K_n + s_n\}_n$ is also a Følner sequence for G, for every sequence $\{s_n\}_{n=1}^\infty$ in G.*

Proof. Using the translation invariance of the Haar measure, for any pair of measurable sets B_1, B_2 and $g \in G$, we have $\tilde{\mu}((B_1 + g) \,\triangle\, (B_2 + g)) = \tilde{\mu}(B_1 \,\triangle\, B_2)$; hence $\{K_n + s_n\}_n$ is also a Følner sequence. $\square$

Lemma 2.1.7. *Let K be a compact set in G. Then there exists an $s \in S$ such that $K + s \subseteq S^\circ$.*

Proof. Since by the assumption $G = S - S$, for any $g \in G$ and $s_0 \in S^\circ$ we have $g - s_0 = s' - s''$ with some $s', s'' \in S$. Since $s_0 + s' \in S^\circ$ clearly holds, we find that $G = S^\circ - S$ is satisfied. The family $\{S^\circ - s : s \in S\}$ forms an open covering of K and so there exist $s_1, \ldots, s_n \in S$ such that $K \subseteq \bigcup_{i=1}^n (S^\circ - s_i)$. Setting $s = \sum_{i=1}^n s_i$ the relation $s_i \preceq s$ implies $S^\circ - s_i \subseteq S^\circ - s$ for every $1 \le i \le n$. It follows that $K \subseteq S^\circ - s$, that is $K + s \subseteq S^\circ$. $\qquad\square$

An immediate consequence of Lemma 2.1.6 and Lemma 2.1.7 is that *there must be a Følner sequence in the interior of S.*

We shall need yet another result on topological semigroups that will be used in this chapter. Recall that a function f on S is called locally bounded if it is bounded on the compact subsets of S. Let $C_c(S)$ denote the set of continuous functions with compact support in S.

Lemma 2.1.8. *Let $K \subseteq S^\circ$ be a compact set of G. If f is a measurable, locally bounded function on S and $g \in L^\infty(S)$ then the convolution*

$$(f * g)(s) := \int_K f(s + t)g(t)\, d\mu(t)$$

is continuous on S.

Proof. Assuming $f \in C_c(S)$, we can easily verify that f is uniformly continuous on S; that is, for every $\varepsilon > 0$, there exists an open set U in G containing 0 such that $|f(s') - f(s)| \le \varepsilon$ whenever $s' - s \in U$ and $s', s \in S$. Thus we may infer that $f * g$ is continuous.

Suppose now that f is any locally bounded, measurable function. We can choose a compact neighbourhood K_1 of 0 in G such that $K + K_1 \subseteq S^\circ$. Indeed, every $s \in K$ is an inner point of S hence there exists an open neighbourhood V_s of 0 (in G) such that $s + V_s \subseteq S^\circ$. Furthermore, choosing open neighbourhoods W_s of 0 such that $W_s + W_s \subseteq V_s$, the family $\{s + W_s : s \in K\}$ forms an open covering of K and therefore has a finite subcovering $s_1 + W_{s_1}, ..., s_n + W_{s_n}$. Let us form the open set $W := \bigcap_{i=1}^n W_{s_i}$; then $K + W \subseteq (\bigcup_{i=1}^n (s_i + W_{s_i})) + W = \bigcup_{i=1}^n (s_i + W_{s_i} + W) \subseteq \bigcup_{i=1}^n (s_i + V_{s_i}) \subseteq S^\circ$. Since G is a locally compact Hausdorff space there is a compact neighbourhood $K_1 \subseteq W$ of 0, hence $K_2 := K + K_1 \subseteq S^\circ$. It is clear that K_2 is also compact.

Let us take an arbitrary $\varepsilon > 0$, and fix a $y \in S$. Since f is in $L^1(S, \chi_{K_2 + y} d\mu)$, there exists an $h \in C_c(S)$ such that $\int_{K_2 + y} |f(t) - h(t)|\, d\mu(t) < \varepsilon$. (Notice that the restricted

Haar measure μ is regular on the σ-compact S; see [57, p. 238].) We already know that $h * g$ is continuous. Hence we can choose a neighbourhood U of 0 in G such that $U \subseteq K_1$ and $|(h * g)(y) - (h * g)(v)| < \varepsilon$ is true whenever $v \in (y + U) \cap S$. Then

$$
\begin{aligned}
|(f * g)(y) - (f * g)(v)| \ &\leq\ \int_K |f_y - h_y||g|\,d\mu + |(h * g)(y) - (h * g)(v)| \\
&\quad + \int_K |h_v - f_v||g|\,d\mu \\
&<\ \int_{K+y} |f - h||g|\,d\mu + \varepsilon + \int_{K+v} |h - f||g|\,d\mu \\
&\leq\ 2\|g\|_\infty \int_{y+K_2} |f - h|\,d\mu + \varepsilon \\
&<\ 2\varepsilon\|g\|_\infty + \varepsilon,
\end{aligned}
$$

and the lemma follows. $\qquad\square$

Our intention is to prove stability results and discuss the asymptotic properties of operator semigroups. In most cases we shall use weaker notions of the usual convergence of orbits, which requires invariant means. First we will proceed with a study of almost convergent functions.

Definition 2.1.9. A function $f \in L^\infty(S)$ is called *almost convergent* if the set $\{m(f) : m \in \mathcal{M}(S)\}$ is a singleton. We shall use the notation a-lim $f = c$ whenever $m(f) = c$ for all $m \in \mathcal{M}(S)$.

A useful property of almost convergence is described in the following proposition.

Proposition 2.1.10. *If $f \in L^\infty(S)$ is almost convergent with a-lim $f = c$ and $\{K_n\}_n$ is a Følner sequence on S then*

$$
\lim_{n \to \infty} \frac{1}{\mu(K_n)} \int_{K_n} f_y \, d\mu = c
$$

uniformly with respect to $y \in S$.

Proof. Let us assume that there exist an $\varepsilon > 0$, a strictly increasing sequence $\{n_k\}_k$ of positive integers, and a sequence $\{y_k\}_k \in S$ such that

$$
\left| \frac{1}{\mu(K_{n_k})} \int_{K_{n_k}} f_{y_k} \, d\mu \ -\ c \right| > \varepsilon
$$

is true for every $k \in \mathbb{N}$.

Applying Lemma 2.1.6 to the subsequence $\{K_{n_k}\}_k$ we find that $\{C_k := K_{n_k} + y_{n_k}\}_k$ is also a Følner sequence. For every $k \in \mathbb{N}$, let us consider the linear functional φ_k on $L^\infty(S)$ defined by

$$\varphi_k(g) := \frac{1}{\mu(C_k)} \int_{C_k} g \, d\mu \,, \quad g \in L^\infty(S).$$

Obviously, $\|\varphi_k\| = \varphi_k(\mathbf{1}) = 1$ is valid. By the Banach–Alaoglu theorem there exists a weak-$*$ cluster point m_0 of the sequence $\{\varphi_k\}_k$. Clearly, $\|m_0\| = m_0(\mathbf{1}) = 1$. Furthermore, for every $g \in L^\infty(S)$ and $y \in S$, we have

$$
\begin{aligned}
|\varphi_k(g) - \varphi_k(g_y)| &= \frac{1}{\mu(C_k)} \left| \int_{C_k} g \, d\mu - \int_{C_k + y} g \, d\mu \right| \\
&\leq \frac{\mu((C_k + y) \triangle C_k)}{\mu(C_k)} \|g\|_\infty \to 0, \quad \text{as } k \to \infty,
\end{aligned}
$$

whence $m_0(g) = m_0(g_y)$ follows; that is, m_0 is an invariant mean.

Since $|m_0(f) - c| \geq \varepsilon$ by our assumption we conclude that f cannot almost converge to c. $\qquad\square$

Remark 2.1.11. This result suggests that almost convergence has some natural connection with abstract ergodic theory. The interested reader should see H.A. Dye's paper [17] on abstract weak mixing and almost convergence.

We shall also introduce a stronger form of almost convergence.

Definition 2.1.12. We say that a function $f \in L^\infty(S)$ *almost converges in the strong sense* to a complex number c, if a-lim $|f - c| = 0$. In that case we shall use the notation: as-lim $f = c$.

In the lemma below we shall prove a multiplicative property of as-lim. This statement is analogous to [36, Lemma 1].

Lemma 2.1.13. *For every $f \in L^\infty(S)$, the following statements are equivalent:*

(i) *as-lim $f = c$,*

(ii) *$m(fg) = cm(g)$ for every $g \in L^\infty(S)$ and $m \in \mathcal{M}(S)$.*

Proof. (i) $\Rightarrow$ (ii): Every invariant mean $m \in \mathcal{M}(S)$ is a positive functional on $L^\infty(S)$. Given any $h \in L^\infty(S)$, let α be a complex number with the properties $|\alpha| = 1$ and $|m(h)| = \alpha m(h)$. Since $|h| - \operatorname{Re}(\alpha h) \geq 0$, it follows that $m(|h|) \geq m(\operatorname{Re}(\alpha h)) = \operatorname{Re} m(\alpha h) = |m(h)|$. Therefore

$$|m(fg) - cm(g)| = |m(fg - cg)| \leq m(|fg - cg|) \leq \|g\|_\infty m(|f - c\mathbf{1}|) = 0.$$

(ii) $\Rightarrow$ (i): Let $g(s) := |f(s) - c|(f(s) - c)^{-1}$ if $f(s) \neq c$ and 0 otherwise. Then $g \in L^\infty(S)$, and $m(|f - c\mathbf{1}|) = m((f - c)g) = m(fg) - cm(g) = 0.$ $\qquad\square$

It is easy to check that convergence implies almost convergence in the strong sense. Let us consider the function $f \in L^\infty(\mathbb{R}_+)$, where $f(s) := 1$ if $n \leq s < n + 1$ and n is even, while $f(s) := -1$ otherwise. Since $f + f_1$ is identically zero, we infer that a-lim $f = 0$, but $\lim_{s\to\infty} f(s)$ does not exists. It is also clear that f does not converge in the strong sense. Let us now take the function $g \in L^\infty(\mathbb{R}_+)$, where $g(s) := 1$ if $2^n \leq s < 2^n + 1$ $(n \in \mathbb{N})$, while $g(s) := 0$ otherwise. For any $a \in \mathbb{R}_+$ and $N \in \mathbb{N}$, let $G_{a,N} := N^{-1} \sum_{k=1}^{N} g_{a+k}$. Since $m(G_{a,N}) = m(g)$ holds for every invariant mean $m \in M(\mathbb{R}_+)$, and since $\|G_{a,N}\|_\infty$ can be arbitrarily small, we conclude that as-lim $g = 0$. On the other hand, $\lim_{s\to\infty} g(s)$ does not exist.

2.2 Existence of the limit functional

The regular norm-behaviour of representations will be defined in terms of gauge functions.

Definition 2.2.1. We say that $p\colon S \to (0, \infty)$ is a *gauge function* if it is measurable and, for every $s \in S$, $p_s/p \in L^\infty(S)$ almost converges in strong sense to a positive number $c_p(s)$. The function c_p is called the *limit functional* of the gauge function p.

If $p\colon S \to (0, \infty)$ is a gauge function then, by Lemma 2.1.13 the equation

$$\frac{p_{s_1+s_2}}{p} = \frac{(p_{s_1})_{s_2}}{p_{s_2}} \cdot \frac{p_{s_2}}{p}$$

implies that $c_p(s_1 + s_2) = c_p(s_1)c_p(s_2)$ $(s_1, s_2 \in S)$; that is, the limit functional c_p of p is multiplicative.

Next we shall prove a basic property of the limit functional which seems to be important for deriving results later on.

Lemma 2.2.2. *Let p be a gauge function with $p(s) \geq 1$ for $s \in S$. Then $c_p(s) \geq 1$ for every $s \in S$.*

Proof. We can see from Proposition 2.1.10 that, for every $s \in S$,

$$\frac{1}{\mu(K_n)} \int_{K_n} (p_s/p)_y \, d\mu \ \to\ c_p(s) \qquad (n \to \infty),$$

uniformly with respect to y, where $\{K_n\}_n$ is a Følner sequence in S. Assume that $c_p(s_0) < 1$ for some $s_0 \in S$, and let us choose $c_p(s_0) < \delta < 1$. There exists an $n_0 \in \mathbb{N}$ such that $\mu(K_{n_0})^{-1} \int_{K_{n_0}} (p_{s_0}/p)_y \, d\mu \leq \delta$ is true for every $y \in S$. Now the Jensen inequality implies that

$$
\begin{aligned}
\frac{1}{\mu(K_{n_0})} \int_{K_{n_0}} (\log p_{s_0+y} - \log p_y) \, d\mu &\leq \log \left(\frac{1}{\mu(K_{n_0})} \int_{K_{n_0}} (p_{s_0}/p)_y \, d\mu \right) \\
&\leq \log \delta < 0 \qquad (y \in S).
\end{aligned}
$$

Summing the inequalities obtained with the choices $y = 0, s_0, 2s_0, ..., (m-1)s_0$ we find that

$$
\frac{1}{\mu(K_{n_0})} \int_{K_{n_0}} \log p_{ms_0} \, d\mu \leq \frac{1}{\mu(K_{n_0})} \int_{K_{n_0}} \log p \, d\mu + m \log \delta
$$

is true for every $m \in \mathbb{N}$. This is a contradiction because the left-hand side of the inequality must be nonnegative, since $p \geq 1$, but the right-hand side is negative for large m. $\qquad\square$

We recall that the non-zero, continuous homomorphisms of S into the multiplicative semigroup of $\mathbb{C}$ are called the *characters of S*.

Corollary 2.2.3. *Let χ be a character of S such that $c_p \leq |\chi| \leq p$. Then $|\chi| = c_p$.*

Proof. It is clear that $\tilde{p} = p/|\chi|$ is a gauge function with $c_{\tilde{p}} = c_p/|\chi|$. Since $\tilde{p} \geq 1$, Lemma 2.2.2 tells us that $c_{\tilde{p}} \geq 1$, and so $c_p \geq |\chi|$. $\qquad\square$

Throughout this thesis $\mathfrak{X}$ will stands for a complex Banach space and $\mathcal{L}(\mathfrak{X})$ will be the algebra of all bounded linear operators on $\mathfrak{X}$. A representation ρ of S on the Banach space $\mathfrak{X}$ is a mapping $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ such that

- $\rho(0) = I$,

- $\rho(s+t) = \rho(s)\rho(t) \ (s, t \in S)$,

- $s \mapsto \rho(s)x$ is continuous for every $x \in \mathfrak{X}$.

Proposition 2.2.4. *Let $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ be a bounded representation. Then, for every $x \in \mathfrak{X}$, $\|\rho(s)x\|$ almost converges to zero if and only if $\lim_s \|\rho(s)x\| = 0$.*

Proof. Assuming a-$\lim_s \|\rho(s)x\| = 0$ with $x \in \mathfrak{X}$, by Proposition 2.1.10 we get that, for any $\varepsilon > 0$, $\|\rho(s_0)x\| \leq \varepsilon M^{-1}$ holds for some $s_0 \in S$, where $M := \sup\{\|\rho(s)\| : s \in S\}$. Then $\|\rho(s+s_0)x\| \leq M\varepsilon M^{-1} = \varepsilon$ is true for every $s \in S$, and so $\lim_s \|\rho(s)x\| = 0$. The converse implication is trivial. $\qquad\square$

Now we will introduce the concept of regularity.

Definition 2.2.5. The representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is of *regular norm behaviour with respect to the gauge function p* or has *p-regular norm-function* if $\|\rho(s)\| \leq p(s)$ holds for every $s \in S$, and a-$\lim_s \|\rho(s)\|/p(s) = 0$ is not true.

There is a connection between the spectral radius function $r(\rho(s))$ of a representation ρ with p-regular norm function and the limit functional $c_p(s)$. Namely, the following inequality always holds (see [37, Proposition 8]):

Lemma 2.2.6. *The inequality $c_p(s) \leq r(\rho(s))$ is true for every $s \in S$.*

Proof. Let $m \in \mathcal{M}(S)$ be an invariant mean such that $m(\|\rho(s)/p(s)\|) \neq 0$. Since the inequality

$$\frac{p(s+t)}{p(s)} \frac{\|\rho(s+t)\|}{p(s+t)} \leq \|\rho(t)\| \frac{\|\rho(s)\|}{p(s)} \quad (s,t \in S)$$

holds, we may infer by the positivity of m and Lemma 2.1.13 that

$$m\left(\frac{p(s+t)}{p(s)}\right) m\left(\frac{\|\rho(s+t)\|}{p(s+t)}\right) \leq \|\rho(t)\| m\left(\frac{\|\rho(s)\|}{p(s)}\right).$$

Hence

$$c_p(t) \leq \|\rho(t)\|$$

is satisfied for every $t \in S$. Since $c_p^n(t) = c_p(nt) \leq \|\rho(nt)\| \leq \|\rho(t)^n\|$ if $n \in \mathbb{N}$, applying Gelfand's spectral radius formula, we find that $c_p(t) \leq r(\rho(t))$. $\qquad\square$

As we are mostly interested in unbounded representations of S which are bounded on compact sets, the following assumption seems natural:

Assumption 2.2.7. *We shall assume throughout this thesis that for any gauge function p we have $p \geq 1$ and that p is locally bounded on S.*

Now we will show that the limit functional $c_p(s)$ is a character of the semigroup S.

Theorem 2.2.8. *Let p be a gauge function on S which satisfies Assumption 2.2.7 and let us assume that there exists a representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ with a p-regular norm-function. Then the limit functional c_p of p is a positive character of S.*

Proof. We already know that $c_p(s' + s'') = c_p(s')c_p(s'')$, $s', s'' \in S$, so c_p is a homomorphism. It remains to show that c_p is continuous. First, applying Proposition 2.1.10 and Lemma 2.1.8, we see that c_p is a measurable function because it is a sequential limit of continuous functions.

The Principle of Uniform Boundedness tells us that the function $\|\rho(s)\|$ is bounded on compact sets, thus Lemma 2.2.6 implies that the funcion $c_p(s)$ is also locally bounded on S. Let us choose a compact set $K \subseteq S^\circ$ such that $\mu(K) > 0$. Then $\alpha := \int_K c_p(t) d\mu(t) \in (0, \infty)$. Given any $y \in S$, we have

$$\int_K c_p(y + t)\, d\mu(t) = \int_K c_p(y)\, c_p(t)\, d\mu(t) = c_p(y) \int_K c_p(t)\, d\mu(t),$$

whence $c_p(y) = \alpha^{-1} \int_K c_p(y + t)\, d\mu(t)$ follows. Then we can apply Lemma 2.1.8 to deduce the continuity of the limit functional c_p. $\qquad \square$

We have now arrived at the main result of the section.

Theorem 2.2.9. *Let G be a locally compact, σ-compact, Hausdorff abelian group with a closed subsemigroup S such that $S - S = G$, $S \cap (-S) = \{0\}$ and $S^\circ \neq \emptyset$. If the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ is of regular norm-behaviour with respect to the gauge functions p and q which satisfy Assumption 2.2.7, then*

$$c_p = c_q.$$

Proof. We can see from Lemma 2.2.6 that

$$c_p(s) \leq r(\rho(s)) \leq \|\rho(s)\| \leq q(s) \quad (s \in S),$$

and $c_p^{-1} q$ is a gauge function by Theorem 2.2.8. Thus by Lemma 2.2.2 we have

$$1 \leq c_{c_p^{-1} q} = c_p^{-1}\, c_q,$$

so $c_p \leq c_q$. In a similar way we find that $c_p \geq c_q$, thus $c_p = c_q$. $\qquad \square$

The above theorem leads to the following definition.

Definition 2.2.10. The function $c_\rho := c_p$ is called the *limit functional of the representation ρ with p-regular norm-function.*

It was already shown in [36] that for $S = \mathbb{Z}_+$ the limit functional $c_\rho(n)$ is equal to $r(\rho(n))$ $(n \in \mathbb{Z}_+)$. The analogous statement concerning C_0-semigroups is valid. To prove this, we need to state a simple lemma. We say that a representation $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is *quasinilpotent* if $r(T(s)) = 0$ holds for every $s > 0$. (Notice that in the quasinilpotent case $r(T(s))$ is not continuous, since $r(T(0)) = 1$, and so it is not a character.)

Lemma 2.2.11. *If the representation $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is not quasinilpotent then $r(T(s))$ is a character of $\mathbb{R}_+$.*

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Proof. As this statement is well known, we will only sketch a proof. The submultiplicativity $r(T(s + t)) \leq r(T(s))r(T(t))$ $(s, t \in \mathbb{R}_+)$ and $r(T(ns)) = r(T(s))^n$ $(n \in \mathbb{N}, s \in \mathbb{R}_+)$ yield that $r(T(s)) > 0$ for every $s \in \mathbb{R}_+$. Furthermore, it can be easily checked that $\omega_0 := \lim_{s \to \infty} s^{-1} \log \|T(s)\| \in \mathbb{R}$ exists; see e.g. [20, p. 251]. Thus $r(T(s)) = \lim_{n \to \infty} \|T(ns)\|^{\frac{1}{n}} = \exp(s \lim_{n \to \infty} (ns)^{-1} \log \|T(ns)\|) = \exp(\omega_0 s)$ is true for every $s \in \mathbb{R}_+$. $\qquad\qquad\square$

Proposition 2.2.12. *If the representation* $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ *has regular norm-behaviour (with respect to a gauge function p which satisfies Assumption 2.2.7), then* $c_T(s) = r(T(s))$ $(s \in \mathbb{R}_+)$ *holds.*

Proof. We know from Lemma 2.2.6 that $r(T(s)) \geq c_T(s) > 0$ $(s \in \mathbb{R}_+)$, and so $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is not quasinilpotent. Hence Lemma 2.2.11 implies that $r(T(s))$ is a character of $\mathbb{R}_+$. Applying Corollary 2.2.3 we find that $r(T(s)) = c_T(s)$ $(s \in \mathbb{R}_+)$. $\qquad\square$

However, the following example shows that the spectral radius function and the limit functional can be different.

Example 2.2.13. Let us choose the semigroup $\mathbb{R}_+^2$ and the weight function $w(x, y) := e^{x(1-y)} + 1$. Next, consider the Banach space C_w of continuous functions $f \colon \mathbb{R}_+^2 \to \mathbb{C}$ which satisfies the conditions $f|\partial\mathbb{R}_+^2 \equiv 0$ and $\|f\|_w := \sup_{\mathbb{R}_+^2} |f(x, y)|w(x, y) < \infty$. The strongly continuous representation $T \colon \mathbb{R}_+^2 \mapsto \mathcal{L}(C_w)$ is defined by

$$(T(s, t)f)(x, y) := \begin{cases} f(x - s, y - t) & \text{if } (x - s, y - t) \in \mathbb{R}_+^2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{(x,y) \to \infty} w(x, y) = 1$ it follows that, for every $(s, t) \in \mathbb{R}_+^2$,

$$\lim_{(x,y) \to \infty} w_{(s,t)}(x, y)/w(x, y) = 1,$$

hence w is a gauge function (which satisfies Assumption 2.2.7) with $c_w \equiv 1$. Since w is submultiplicative, we have $\|T(s, t)\| \leq w(s, t)$ $((s, t) \in \mathbb{R}_+^2)$. Taking a sequence $\{f_n\}_n$ in C_w satisfying the condition $0 \leq f_n(x, y) \leq f_n(1/n, 1/n) = 1$, $(x, y) \in \mathbb{R}_+^2$, where the support of f_n is included in the disc centered at $(1/n, 1/n)$ and of radius $1/(2n)$, we find that $\lim_n \|f_n\|_w = 2$ and $\lim_n \|T(s, t)f_n\|_w = w(s, t)$, whence $\frac{1}{2}w(s, t) \leq \|T(s, t)\|$ follows $((s, t) \in \mathbb{R}_+^2)$. This implies that $T(s, t)$ has a regular norm-function with respect to w. We can see from the spectral radius formula that $r(T(s, 0)) = e^s$.

2.3 Spectra of representations

Let $S^\sharp$, S^* denote the characters and bounded characters of S, respectively. We note that for any bounded character χ the inequality $|\chi(s)| \le 1$ holds for every $s \in S$. We recall that $C_c(S)$ stands for the set of continuous functions with compact support in S. The Fourier transform of a function $f \in C_c(S)$ with respect to the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ is given by

$$\widehat{f}(\rho) := \int_S f(s)\rho(s)\,d\mu(s).$$

The integral exists pointwise: $\widehat{f}(\rho)x = \int_S f(s)\rho(s)x\,d\mu(s)$ $(x \in \mathcal{X})$ in the Bochner sense (see e.g. [27, Chapter 7.5]). Since the representation ρ is strongly continuous, the Uniform Boundedness Principle implies that $\|\rho(s)\|$ is bounded on the support of f, hence $\widehat{f}(\rho) \in \mathcal{L}(\mathcal{X})$. The characters of S are one-dimensional representations, hence the formula for $\widehat{f}(\chi)$ also makes sense when $\chi \in S^\sharp$.

Since S is a locally compact Hausdorff space it is not hard to see by the Urysohn lemma that for each $\chi \in S^\sharp$ there exists an $f \in C_c(S)$ such that $\widehat{f}(\chi) \ne 0$. Moreover, for all distinct $\chi_1, \chi_2 \in S^\sharp$ we can find an $f \in C_c(S)$ such that $\widehat{f}(\chi_1) \ne \widehat{f}(\chi_2)$. In other words the functions $\widehat{f}$ separate the points of $S^\sharp$ from each other and from zero.

We shall define the spectrum for unbounded representations related to Lyubich's δ-spectrum [49] and Kérchy's algebraic and balanced spectra [37]. In the discrete case the algebraic spectrum of the representation ρ was defined in the following way (see [37]). Let $\mathcal{A}_\rho$ denote the Banach algebra generated by the set $\{\rho(s) : s \in S\}$, then $\sigma_{\mathrm{a}}(\rho) := \{h \circ \rho : h \in \Sigma(\mathcal{A}_\rho)\}$, where $\Sigma(\mathcal{A}_\rho)$ is the Gelfand spectrum of the commutative Banach algebra $\mathcal{A}_\rho$. It is easy to see that $\sigma_{\mathrm{a}}(\rho)$ is equal to the set of characters $\chi \in S^\sharp$ which satisfy the condition $\left| \sum_{s \in \mathcal{F}} a_s \chi(s) \right| \le \left\| \sum_{s \in \mathcal{F}} a_s \rho(s) \right\|$ for all finite subsets $\mathcal{F}$ of S and $a_s \in \mathbb{C}$. In the continuous case one is led to the following extensions of the concept of algebraic and balanced spectra.

Definition 2.3.1. The *algebraic spectrum* of the representation ρ is

$$\sigma_{\mathrm{a}}(\rho) := \left\{ \chi \in S^\sharp : |\widehat{f}(\chi)| \le \|\widehat{f}(\rho)\| \text{ for all } f \in C_c(S) \right\}.$$

The *balanced spectrum* is defined by

$$\sigma_{\mathrm{b}}(\rho) := \sigma_{\mathrm{a}}(\rho) \cap S_b^\sharp,$$

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where $S_b^\sharp := \{\ \chi \in S^\sharp : \chi(s) \neq 0 \text{ for all } s \in S\}$.

The *spectrum of ρ* with regular norm-function is

$$\sigma(\rho) := \{\chi \in \sigma_{\mathrm{a}}(\rho) : |\chi| \leq c_\rho\},$$

where c_ρ denotes the limit functional of ρ.

Now we are going to show that the balanced spectrum $\sigma_{\mathrm{b}}(\rho)$ is contained in the spectrum $\sigma(\rho)$. To do this, we need a lemma.

Lemma 2.3.2. *If the norm-function of the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is continuous at $s_0 \in S$, then $|\chi(s_0)| \leq \|\rho(s_0)\|$ holds for every $\chi \in \sigma_{\mathrm{a}}(\rho)$.*

Proof. Let us take a character $\chi \in \sigma_{\mathrm{a}}(\rho)$ and a positive $\varepsilon > 0$. Since the functions $\|\rho(s)\|$ and $\chi(s)$ are continuous at s_0, there exists an open set $V \subseteq S$ containing s_0 such that $|\|\rho(s)\| - \|\rho(s_0)\|| < \varepsilon$ and $|\chi(s) - \chi(s_0)| < \varepsilon$ are valid for every $s \in V$. By the Urysohn lemma we can find a positive function $f_0 \in C_c(S)$ such that its support is contained in V and $\int_S f_0\, d\mu = 1$. Then the inequalities

$$\begin{aligned}
\|\widehat{f_0}(\rho)x\| &\leq \int_V f_0(s)\|\rho(s)x\|\, d\mu(s) \leq \int_V f_0(s)(\|\rho(s_0)\| + \varepsilon)\|x\|\, d\mu(s) \\
&= (\|\rho(s_0)\| + \varepsilon)\|x\| \quad (x \in \mathcal{X})
\end{aligned}$$

imply $\|\widehat{f_0}(\rho)\| \leq \|\rho(s_0)\| + \varepsilon$. Taking into account the fact that $|\widehat{f_0}(\chi) - \chi(s_0)| < \varepsilon$ also holds, we may infer that

$$|\chi(s_0)| - \varepsilon < |\widehat{f_0}(\chi)| \leq \|\widehat{f_0}(\rho)\| \leq \|\rho(s_0)\| + \varepsilon.$$

Since ε can be arbitrarily small, we conclude that $|\chi(s_0)| \leq \|\rho(s_0)\|$. $\qquad\square$

Proposition 2.3.3. *If $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is a representation with regular norm-function, then $\sigma_{\mathrm{b}}(\rho) \subseteq \sigma(\rho)$.*

Proof. Let $S_c(\rho)$ stand for the set of points $s \in S$ where $\|\rho(s)\|$ is continuous. The norm-function $\|\rho(s)\|$, being the least upper bound of the continuous functions $\|\rho(s)x\|$ ($x \in \mathcal{X}, \|x\| = 1$), is lower semicontinuous. It follows that $S \setminus S_c$ is of first Baire category, and so $S_c(\rho)$ is dense in S. (See e.g. [21, p. 87].)

Let us take a character $\chi \in \sigma_{\mathrm{b}}(\rho)$. We know from Lemma 2.3.2 that $|\chi(s)| \leq \|\rho(s)\|$ for every $s \in S_c(\rho)$. Let us consider the representation $\tilde{\rho} := \chi^{-1}\rho$. It is clear that $\|\tilde{\rho}(s)\| \geq 1$ for every $s \in S_c(\rho)$. Let us assume that $\|\tilde{\rho}(s_0)\| < 1$ holds for some $s_0 \in S$. Taking an open set V in S° with compact closure, the number $\alpha := \sup\{\|\rho(s)\| : s \in V\}$

is finite. For any $n \in \mathbb{N}$ and any $s \in V$, we have $\|\tilde{\rho}(ns_0 + s)\| \leq \|\tilde{\rho}(s_0)\|^n \alpha$. If $n_0 \in \mathbb{N}$ is sufficiently large, then $\|\tilde{\rho}(s_0)\|^{n_0} \alpha < 1$, and so $\|\tilde{\rho}(n_0 s_0 + s)\| < 1$ is true for every $s \in V$. However, since $(n_0 s_0 + V) \cap S_c(\rho)$ is non-empty we arrive at a contradiction. Thus $\|\tilde{\rho}(s)\| \geq 1$ and so $|\chi(s)| \leq \|\rho(s)\|$ holds for all $s \in S$.

Assume ρ has p-regular norm function. Since $|\chi(s)| \leq \|\rho(s)\| \leq p(s)$ $(s \in S)$, we have that $\tilde{p}(s) \geq 1$ $(s \in S)$ is true for the gauge function $\tilde{p} = |\chi|^{-1} p$. Lemma 2.2.2 tells us that $c_{\tilde{p}} \geq 1$, and the relations $c_{\tilde{p}} = |\chi|^{-1} c_p = |\chi|^{-1} c_\rho$ imply $|\chi| \leq c_\rho$, and so $\chi \in \sigma(\rho)$. $\qquad\square$

Remark 2.3.4. For any $\chi \in \sigma_{\mathrm{b}}(\rho)$ we know that $|\chi(s)|^n = |\chi(ns)| \leq \|\rho(ns)\| \leq \|\rho(s)^n\|$ $(s \in S, n \in \mathbb{N})$, whence it follows that $|\chi(s)| \leq r(\rho(s))$, with an arbitrary representation ρ.

Remark 2.3.5. If $S = \mathbb{R}_+^n$, then it can be easily verified that every character $\chi \in (\mathbb{R}_+^n)^\sharp$ is non-vanishing, thus $\sigma_{\mathrm{a}}(\rho)$, $\sigma_{\mathrm{b}}(\rho)$ and $\sigma(\rho)$ coincide in this case. (Indeed, let $\{e_i\}_{i=1}^n$ be the standard basis in $\mathbb{R}_+^n$. If $\chi(t) = 0$ holds for some $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$, then the equation $\chi(t) = \prod_{i=1}^n \chi(e_i)^{t_i}$ tells us that $\chi(e_j) = 0$ must be true for some $1 \leq j \leq n$. Hence χ vanishes on a dense set. Since $\chi(0) = 1$, it follows that χ is not continuous.)

Remark 2.3.6. Let T be a bounded, linear operator on $\mathcal{X}$. We denote the representation induced by T by $\rho_T \colon \mathbb{Z}_+ \to \mathcal{L}(\mathcal{X})$. Then it can be shown that $\sigma_{\mathrm{a}}(\rho_T) = \sigma_{\mathrm{b}}(\rho_T) = \widehat{\sigma(T)}$, where $\widehat{\sigma(T)}$ denotes the polynomially convex hull of the spectrum of T [55, Theorem 2.10.3].

Actually, a geometrically similar result can be verified for C_0-semigroups as well; for its proof we refer the reader to Proposition 3.2.3.

The most important spectrum that we shall need in the following is the peripheral spectrum.

Definition 2.3.7. The *peripheral spectrum* of the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ with regular norm-function is defined by

$$\sigma_{\mathrm{per}}(\rho) := \{\, \chi \in \sigma(\rho) : |\chi(s)| = c_\rho(s) \text{ for all } s \in S \,\}.$$

Next, the point spectrum will be defined.

Definition 2.3.8. The *point spectrum* of the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ is

$$\sigma_{\mathrm{p}}(\rho) := \{\chi \in S^\sharp : \text{ there exists a } 0 \neq x \in \mathcal{X} \text{ with } \rho(s)x = \chi(s)x \text{ for all } s \in S\}.$$

The adjoint $\rho^*(s) := \rho(s)^*$ $(s \in S)$ of ρ is not necessarily strongly continuous, hence the spectrum of ρ^* cannot be defined in general. However, there is no difficulty defining $\sigma_{\mathrm{p}}(\rho^*)$ in an analogous way to $\sigma_{\mathrm{p}}(\rho)$.

The multiplication of the representation ρ (of regular norm-function) by a non-vanishing character τ yields a representation of a regular norm-function whose various kinds of spectra can be naturally derived from the spectra of ρ.

Lemma 2.3.9. *If $\tau \in S_b^\sharp$ then $\sigma_{\mathrm{a}}(\tau\rho) = \tau\sigma_{\mathrm{a}}(\rho)$, $\sigma_{\mathrm{b}}(\tau\rho) = \tau\sigma_{\mathrm{b}}(\rho)$, $\sigma(\tau\rho) = \tau\sigma(\rho)$, $\sigma_{\mathrm{per}}(\tau\rho) = \tau\sigma_{\mathrm{per}}(\rho)$, $\sigma_{\mathrm{p}}(\tau\rho) = \tau\sigma_{\mathrm{p}}(\rho)$ and $\sigma_{\mathrm{p}}(\tau\rho^*) = \tau\sigma_{\mathrm{p}}(\rho^*)$.*

Proof. Consider the bijection $\Psi \colon C_c(S) \to C_c(S)$, $\Psi(f) := f\tau$. Since $(\Psi(f))\widehat{\,}(\chi) = \widehat{f}(\tau\chi)$ and $(\Psi(f))\widehat{\,}(\rho) = \widehat{f}(\tau\rho)$ hold for every $f \in C_c(S)$, we may infer that $\sigma_{\mathrm{a}}(\tau\rho) = \tau\sigma_{\mathrm{a}}(\rho)$, whence $\sigma_{\mathrm{b}}(\tau\rho) = \tau\sigma_{\mathrm{b}}(\rho)$ readily follows. In view of $c_{\tau\rho} = \tau c_\rho$ we deduce that $\sigma(\tau\rho) = \tau\sigma(\rho)$ and $\sigma_{\mathrm{per}}(\tau\rho) = \tau\sigma_{\mathrm{per}}(\rho)$. Finally, the equalities $\sigma_{\mathrm{p}}(\tau\rho) = \tau\sigma_{\mathrm{p}}(\rho)$ and $\sigma_{\mathrm{p}}(\tau\rho^*) = \tau\sigma_{\mathrm{p}}(\rho^*)$ are immediate consequences of the definition. $\qquad\square$

Let us equip $S^\sharp$ with the compact-open topology, so that convergence of a net in $S^\sharp$ means uniform convergence on compact subsets of S. The closed set S^* of bounded characters can be identified with the Gelfand spectrum $\Sigma(L^1(S))$ of the abelian Banach algebra $L^1(S)$. Indeed, $L^1(S)$ with the multiplication $(f * g)(s) = \int_S f(s - t)g(t)\,d\mu(t)$ is a closed subalgebra of the abelian Banach algebra $L^1(G)$. Adapting the proof of [58, Theorem 1.2.2] to a semigroup setting tells us that the mapping

$$\Lambda \colon S^* \to \Sigma(L^1(S)), \quad \chi \mapsto h_\chi, \quad \text{where } h_\chi(f) = \int_S f\chi = \widehat{f}(\chi),$$

is a bijection. We recall that $\Sigma(L^1(S))$ is a locally compact Hausdorff space with the Gelfand topology, induced by the weak-$*$ topology of the dual space of $L^1(S)$. Notice also that if $\check{f} \in C_0(\Sigma(L^1(S)))$ is the Gelfand transform of $f \in L^1(S)$, then $\check{f}(h_\chi) = h_\chi(f) = \widehat{f}(\chi)$ holds for every $\chi \in S^*$.

Proposition 2.3.10. *The mapping $\Lambda \colon S^* \to \Sigma(L^1(S))$, $\chi \mapsto h_\chi$ is a homeomorphism, and so S^* is locally compact in the compact-open topology.*

Proof. For the sake of completeness we shall sketch a proof. The continuity of Λ is an immediate consequence of the fact that $C_c(S)$ is dense in $L^1(S)$ and that $\|h\| = 1$ is true for every $h \in \Sigma(L^1(S))$.

Let us now assume that the net $\{h_\nu = h_{\chi_\nu}\}_{\nu \in \mathcal{N}}$ converges to $h = h_\chi$; that is, $\lim_\nu h_\nu(g) = h(g)$ holds for every $g \in L^1(S)$. Let us fix a function $f \in L^1(S)$ such that $h(f) \neq 0$, and let us consider the equations

$$h_\nu(f)\chi_\nu(t) = h_\nu(f_{-t}) \quad (\nu \in \mathcal{N}, \ t \in S)$$

and

$$h(f)\chi(t) = h(f_{-t}) \quad (t \in S).$$

(For their validity we refer to the proof given in [58, Theorem 1.2.2].) Since the mapping $\varphi \colon S \to L^1(S)$, $t \mapsto f_{-t}$ is continuous, we can readily deduce that $\{\chi_\nu\}_\nu$ converges to χ uniformly on compact sets. $\qquad\square$

Taking any $\tau \in S_b^\sharp$, the multiplication $M_\tau \colon S^\sharp \to S^\sharp$, $\chi \mapsto \tau\chi$ is clearly a homeomorphism, and so τS^* is also a locally compact Hausdorff space with the compact-open topology.

Now we shall turn to the question of the spectrum of a representation.

Proposition 2.3.11. *Let $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ be a representation of regular norm-behaviour. The spectrum $\sigma(\rho)$ of ρ is a locally compact Hausdorff space with the compact-open topology, and so is its closed subset $\sigma_{\mathrm{per}}(\rho)$.*

Proof. It is not hard to see that the algebraic spectrum $\sigma_{\mathrm{a}}(\rho)$ is closed in $S^\sharp$. As the limit functional c_ρ belongs to $S_b^\sharp$, it follows from Proposition 2.3.10 that $c_\rho S^*$ is locally compact. Then the closed subset $\sigma(\rho) = \sigma_{\mathrm{a}}(\rho) \cap c_\rho S^*$ of $c_\rho S^*$ is also locally compact. $\qquad\square$

Remark 2.3.12. Let us assume that the representation $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ is bounded: $\alpha := \sup\{\|\rho(s)\| : s \in S\} < \infty$. If $\|\rho(s_0)\| < 1$ holds for some $s_0 \in S$, then the inequalities $\|\rho(ns_0 + s)\| \leq \|\rho(s_0)\|^n \alpha$ $(n \in \mathbb{N})$ show that $\lim_s \|\rho(s)\| = 0$, i.e. ρ is uniformly stable.

Assuming that $\|\rho(s)\| \geq 1$ is true for every $s \in S$, we can see that ρ is of regular norm behaviour with respect to the gauge function $p(s) := \alpha$ $(s \in S)$. The limit functional c_ρ of ρ is clearly the constant $\mathbf{1}$ function. Thus $\sigma_{\mathrm{per}}(\rho)$ coincides with the *unitary spectrum* $\sigma_{\mathrm{u}}(\rho) := \{\chi \in \sigma(\rho) : |\chi| = 1\}$ of ρ. It can be easily verified that $\sigma_{\mathrm{a}}(\rho) = \sigma(\rho)$ is also true. Indeed, if $|\chi(s_0)| > 1$ holds for some $\chi \in S^\sharp$ and $s_0 \in S$, then $\lim_n |\chi(ns_0)| = \infty$, and by the Urysohn lemma we find that $\sup\{|\widehat{f}(\chi)| : 0 \leq f \in C_c(S), \ \|f\|_1 = 1\} = \infty$. Since $\sup\{\|\widehat{f}(\rho)\| : 0 \leq f \in C_c(S), \ \|f\|_1 = 1\} \leq \alpha$, we obtain that χ cannot belong to $\sigma_{\mathrm{a}}(\rho)$.

The Fourier transforms $\widehat{f}(\chi)$ and $\widehat{f}(\rho)$ are clearly defined for every $f \in L^1(S)$ and $\chi \in S^*$. Taking into account the fact that $C_c(S)$ forms a dense subset of $L^1(S)$, we

may conclude that if $|\widehat{f}(\chi)| \leq \|\widehat{f}(\rho)\|$ holds for every $f \in C_c(S)$ then so does follow for every $f \in L^1(S)$. Thus $\sigma(\rho)$ coincides with the spectrum introduced by Batty and Vũ for bounded representations in [8]. We recall also that this concept is an adaptation of the finite L-spectrum and the Arveson spectrum, defined for group representations, to a semigroup setting (see [49] and [15]).

Remark 2.3.13. Let $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ be a representation of regular norm-behaviour. Since $c_\rho \in S_b^\sharp$, the representation $\tilde{\rho} := c_\rho^{-1}\rho \colon S \to \mathcal{L}(\mathfrak{X})$ is also of regular norm-behaviour and $c_{\tilde{\rho}} = 1$. Clearly $|\chi| = 1$ holds for the characters in $\sigma_{\mathrm{per}}(\tilde{\rho})$. Hence every $\chi \in \sigma_{\mathrm{per}}(\tilde{\rho})$ can be uniquely extended to a character $\tilde{\chi}$ of the extension group G. We conclude that $\sigma_{\mathrm{per}}(\tilde{\rho})$ can be identified with the unitary spectrum $\mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})$ introduced in [6], namely $\sigma_{\mathrm{per}}(\tilde{\rho}) = \{\tilde{\chi}|S : \tilde{\chi} \in \mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})\}$. Therefore $\sigma_{\mathrm{per}}(\rho) = \{c_\rho(\tilde{\chi}|S) : \tilde{\chi} \in \mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})\}$ is true by Lemma 2.3.9.

2.4 Regularity and isometric representations

The key ingredient in proving the stability of ρ is the transmission of the conditions to a related isometric representation. The method is well known for bounded representations and we can extend it using the regularity condition. This kind of associated representation is included in the following theorem.

Theorem 2.4.1. *For any representation $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ with p-regular norm-function, there exists an isometric representation $\psi \colon S \to \mathcal{L}(\mathcal{Y})$ on a Banach space $\mathcal{Y}$ and a contractive transformation $Q \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$ such that:*

(i) $\ker Q = \{x \in \mathfrak{X} : \mathrm{a}\text{-}\lim_s \|\rho(s)x\|/p(s) = 0\}$, *and $\mathrm{ran}\, Q$ is dense in $\mathcal{Y}$,*

(ii) $Q\rho(s) = c_\rho(s)\,\psi(s)Q$ *holds for every $s \in S$,*

(iii) *for every operator $C \in \{\rho(S)\}'$, there exists a unique operator $D \in \{\psi(S)\}'$ such that $QC = DQ$; furthermore, the mapping $\gamma \colon \{\rho(S)\}' \mapsto \{\psi(S)\}', C \mapsto D$ is a contractive algebra-homomorphism,*

(iv) $\sigma(\rho) \supseteq c_\rho\sigma(\psi), \quad \sigma_{\mathrm{per}}(\rho) \supseteq c_\rho\sigma_{\mathrm{per}}(\psi), \quad \sigma_{\mathrm{p}}(\rho^*) \supseteq c_\rho\sigma_{\mathrm{p}}(\psi^*)$.

Proof. (i): Define a seminorm on $\mathfrak{X}$ in the following way:

$$\ell(x) := \sup\left\{m\left(\frac{\|\rho(\cdot)x\|}{p}\right) : m \in \mathcal{M}(S)\right\}.$$

One sees immediately that $\ell(x) \le \|x\|$, and $\ell(x) = 0$ if and only if a-$\lim_s \|\rho(s)x\|/p(s) = 0$. Let $\mathcal{Y}$ be the completion of the quotient space $\mathcal{X}/\ker \ell$ with norm $\|x + \ker \ell\| := \ell(x)$, and let $Q \colon \mathcal{X} \to \mathcal{Y}$ denote the natural embedding. Obviously, ran Q is dense.

(ii): Applying Lemma 2.1.13, for any $x \in \mathcal{X}$ and $t \in S$ we have

$$
\begin{aligned}
\|Q\rho(t)x\| = \ell(\rho(t)\,x) &= \sup_{m \in \mathcal{M}(S)} m\left(\frac{\|\rho(\cdot + t)x\|}{p} \right) \\
&= \sup_{m \in \mathcal{M}(S)} m\left(\frac{\|\rho(\cdot + t)x\|}{p_t} \frac{p_t}{p} \right) \\
&= c_\rho(t) \sup_{m \in \mathcal{M}(S)} m\left(\frac{\|\rho(\cdot)x\|}{p} \right) \\
&= c_\rho(t)\,\ell(x) = c_\rho(t)\,\|Qx\|.
\end{aligned}
$$

Hence there exists a unique isometry $\psi(t) \in \mathcal{L}(\mathcal{Y})$ such that $Q\rho(t) = c_\rho(t)\,\psi(t)Q$. This intertwining relation readily yields that $\psi(s + t) = \psi(s)\psi(t)$, so ψ is a homomorphism.

The inequality

$$
\begin{aligned}
\|c_\rho(s)\psi(s)\,Qx - c_\rho(t)\psi(t)\,Qx\| &= \|Q\rho(s)x - Q\rho(t)x\| \\
&\le \|Q\|\|\rho(s)x - \rho(t)x\| \quad (x \in \mathcal{X})
\end{aligned}
$$

shows that the function $c_\rho(t)\psi(t)y$ is continuous in t for every y in the dense range of Q, and then so is the function $\psi(t)y$. Taking into account the fact that ψ is isometric, we may infer that it is strongly continuous. Thus ψ is an isometric representation.

(iii): Taking any $C \in \{\rho(S)\}'$, we have that

$$
\|QCx\| = \sup_{m \in \mathcal{M}(S)} m\left(\frac{\|C\rho(\cdot)x\|}{p} \right) \le \|C\|\ell(x) = \|C\|\|Qx\|,
$$

so there exists a unique $D \in \mathcal{L}(\mathcal{Y})$ such that $QC = DQ$. Clearly $\|D\| \le \|C\|$, and it is easy to check that $D \in \{\psi(S)\}'$ and that γ is an algebra homomorphism.

(iv): Since $\widehat{f}(\rho) \in \{\rho(S)\}'$ and $Q\widehat{f}(\rho) = \widehat{f}(c_\rho\psi)Q$, we have $\gamma(\widehat{f}(\rho)) = \widehat{f}(c_\rho\psi)$ ($f \in C_c(S)$). Recalling the fact that the mapping $\gamma \colon \{\rho(S)\}' \to \{\psi(S)\}'$ is contractive by (iii), we may infer that $\|\widehat{f}(c_\rho\psi)\| \le \|\widehat{f}(\rho)\|$, whence $c_\rho\sigma_{\mathrm{a}}(\psi) = \sigma_{\mathrm{a}}(c_\rho\psi) \subseteq \sigma_{\mathrm{a}}(\rho)$ follows (see Lemma 2.3.9). As the limit functional of the isometric representation ψ is clearly the constant $\mathbf{1}$ function, we conclude that $c_\rho\sigma_{\mathrm{per}}(\psi) \subseteq \sigma_{\mathrm{per}}(\rho)$. Finally, the equations $\ker Q^* = \{0\}$ and $\rho^*(s)Q^* = c_\rho(s)\,Q^*\psi^*(s)$ ($s \in S$) readily imply that $\sigma_p(\rho^*) \supseteq c_\rho\sigma_p(\psi^*)$. $\qquad \square$

Now we will deduce the following corollary.

Corollary 2.4.2. *If $\rho\colon S \to \mathcal{L}(\mathfrak{X})$ is a representation with p-regular norm-function and* a-$\lim_s \|\rho(s)x\|/p(s) = 0$ *does not hold for some $x \in \mathfrak{X}$, then $\sigma_{\mathrm{per}}(\rho)$ is not empty.*

Proof. The conditions ensure that the associated isometric representation ψ of Theorem 2.4.1 acts on a non-zero Banach space $\mathcal{Y}$. Thus $\sigma_{\mathrm{per}}(\psi)$ is non-empty by Corollary 3.3 in [8]. Since $c_\rho\sigma_{\mathrm{per}}(\psi)$ is included in $\sigma_{\mathrm{per}}(\rho)$, we infer that $\sigma_{\mathrm{per}}(\rho)$ is not empty. $\qquad\square$

2.5 The stability theorem

Now we are in a position to prove a stability result in terms of almost convergence for representations with a regular norm-function.

Theorem 2.5.1. *Let $\rho\colon S \to \mathcal{L}(\mathfrak{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi : |\chi| = c_\rho\}$ is empty then*

$$\text{a-}\lim_s \frac{\|\rho(s)x\|}{p(s)} = 0$$

holds for all $x \in \mathfrak{X}$.

Proof. Supposing it is not the case, then by Theorem 2.4.1 the associated isometric representation $\psi\colon S \to \mathcal{L}(\mathcal{Y})$ acts on a non-zero Banach space $\mathcal{Y}$. Hence, by Corollary 3.3 in [8], $\sigma_{\mathrm{per}}(\psi)$ is not empty. The relation $\sigma_{\mathrm{per}}(\rho) \supseteq c_\rho\sigma_{\mathrm{per}}(\psi)$ implies that $\sigma_{\mathrm{per}}(\psi)$ is also countable. Thus $\sigma_{\mathrm{per}}(\psi)$ contains, by Proposition 4.1 in [8], an eigenvalue χ of ψ^*. Applying Theorem 2.4.1 again, we conclude that $c_\rho\chi$ must belong to $\sigma_{\mathrm{p}}(\rho^*) \cap \sigma_{\mathrm{per}}(\rho)$, which is a contradiction. $\qquad\square$

Applying Proposition 2.1.10 we obtain the following corollary.

Corollary 2.5.2. *Let $\rho\colon S \to \mathcal{L}(\mathfrak{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi : |\chi| = c_\rho\}$ is empty then*

$$\lim_{i\to\infty} \frac{1}{\mu(K_i)} \int_{K_i} \frac{\|\rho(s)x\|}{p(s)} \, d\mu(s) = 0$$

is true for all $x \in \mathfrak{X}$, where $\{K_i\}_i$ is any Følner sequence.

In view of Proposition 2.2.4 and the Remark 2.3.12, Theorem 2.5.1 is a generalization of the stability result Theorem 1.1.2 for bounded representations. From Remark 2.3.13, the spectral conditions of Theorem 2.5.1 are essentially the same as those in the main result of Theorem 3.2 in [6]. The differences lie in the norm-condition on ρ and in the nature of convergence expressing stability.

Chapter 3

C_0-semigroups and regularity

In this chapter we shall introduce a slightly different notion of regularity from that presented in Chapter 2. Our aim is to characterize this kind of regularity for continuous one-parameter operator semigroups. To do this, first we will outline some basic properties of topologically invariant means which are known from the theory of amenable groups. Then, after making some preliminary remarks, we will define the class of C_0-semigroups with topologically regular norm-function. It turns out that this type of regularity can be described in terms of an integral condition. Next we will introduce the regularity constant which plays a crucial role in the characterization. We will show that topological regularity of a representation is equivalent to the positivity of the regularity constant.

3.1 Topologically invariant means

Let $\mathbb{R}_+$ denote the additive semigroup of nonnegative real numbers. We use the notation $L^\infty(\mathbb{R}_+)$ for the Banach space of essentially bounded, Lebesgue measurable functions with the usual $\|\cdot\|_\infty$ norm. We recall that a *mean* m is a continuous linear functional on $L^\infty(\mathbb{R}_+)$ which satisfies $\|m\| = m(\mathbf{1}) = 1$ (here $\mathbf{1}$ denotes the constant 1 function on $\mathbb{R}_+$). We say that a mean m is an *invariant mean* if $m(f) = m(f_s)$ for any $f \in L^\infty(\mathbb{R}_+)$ and $s \in \mathbb{R}_+$, where the function $f_s \colon \mathbb{R}_+ \to \mathbb{C}$ is defined by $f_s(t) := f(s+t)$ $(t \in \mathbb{R}_+)$. Let $\mathcal{M}(\mathbb{R}_+)$ stand for the set of all invariant means. Furthermore, let $\mathcal{G}$ denote the set of non-negative, measurable functions g on $\mathbb{R}_+$ which satisfy the condition $\int_0^\infty g(s)\,ds = 1$.

For any $f \in L^\infty(\mathbb{R}_+)$ and $g \in \mathcal{G}$, let us consider the convolution $f * g \in L^\infty(\mathbb{R}_+)$, defined by $(f * g)(y) = \int_0^\infty f(s+y)g(s)\,ds$.

Definition 3.1.1. We say that a mean m is *topologically invariant* if $m(f * g) = m(f)$ holds for every $f \in L^\infty(\mathbb{R}_+)$ and any $g \in \mathcal{G}$.

These kinds of means were originally introduced by A. Hulanicki for locally compact Hausdorff groups (see [56, p. 9]). For convenience, we shall use the notation $\mathcal{M}_t$ for the set of topologically invariant means. In our first statement we will show that $\mathcal{M}_t(\mathbb{R}_+)$ is not empty. This is well-known for groups and we will simply reproduce the proof in the semigroup context for the sake of completeness.

Proposition 3.1.2. $\mathcal{M}_t(\mathbb{R}_+)$ *is not empty.*

Proof. Let us take the functionals $\varphi_n(f) := \frac{1}{n}\int_0^n f(s)\,ds$ $(n \in \mathbb{N})$ on $L^\infty(\mathbb{R}_+)$. Then it follows that $\|\varphi_n\| = \varphi_n(\mathbf{1}) = 1$ for any $n \in \mathbb{N}$. Since the unit ball of the dual space $L^\infty(\mathbb{R}_+)^*$ is weak-$*$ compact, the sequence $\{\varphi_n\}_n$ has a weak-$*$ cluster point. We will now show that the cluster points are topologically invariant means. Let us have an $f \in L^\infty(\mathbb{R}_+)$ and $g \in \mathcal{G}$. Choosing some $\varepsilon \in (0,1)$, we can find a $t_0 \in \mathbb{R}_+$ such that $\int_{t_0}^\infty g(s)\,ds < \varepsilon$. Since

$$
\begin{aligned}
\varphi_n(f * g) - \varphi_n(f) &= \frac{1}{n}\int_0^n \left(\int_0^\infty (f(s+t)g(t) - f(s)g(t))\,dt\right) ds \\
&= \int_0^\infty \left(\frac{1}{n}\int_0^n (f(s+t)g(t) - f(s)g(t))\,ds\right) dt \\
&= \int_0^{t_0} \frac{1}{n}\left(\int_{[t,t+n]\setminus[0,n]} f(s)\,ds - \int_{[0,n]\setminus[t,t+n]} f(s)\,ds\right) g(t)\,dt \\
&\quad + \int_{t_0}^\infty \left(\frac{1}{n}\int_0^n (f(s+t) - f(s))\,ds\right) g(t)\,dt,
\end{aligned}
$$

we may infer that

$$
\begin{aligned}
|\varphi_n(f * g) - \varphi_n(f)| &\leq \|f\|_\infty \cdot |[0,n]\triangle[t_0, t_0+n]| \cdot n^{-1}\int_0^{t_0} g(t)\,dt \\
&\quad + 2\|f\|_\infty \int_{t_0}^\infty g(t)\,dt \\
&\leq \|f\|_\infty 2t_0 n^{-1} + 2\|f\|_\infty \varepsilon \leq 3\|f\|_\infty \varepsilon,
\end{aligned}
$$

provided n is sufficiently large (here $|A|$ denotes the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}_+$). Since $\varepsilon > 0$ was arbitrary, we have $\lim_n |\varphi_n(f * g) - \varphi_n(f)| = 0$. Thus any weak-$*$ cluster point of $\{\varphi_n\}_n$ is a topologically invariant mean. $\qquad\square$

A little reasoning shows that every topologically invariant mean m is a (translation) invariant mean. Indeed, for any fixed $f \in L^\infty(\mathbb{R}_+)$ and $y \in \mathbb{R}_+$, let us choose a function $g \in \mathcal{G}$ such that the support of g is included in $y + \mathbb{R}_+$. We have

$$m(f) = m(f * g) = m\left(\int_y^\infty f(\,\cdot\,+s)g(s)\,ds\right) = m(f_y * g_y) = m(f_y)$$

because g_y is in $\mathcal{G}$.

In the following we shall construct a function $f_0 \in L^\infty(\mathbb{R}_+)$ such that every topologically invariant mean vanishes on f_0, but $m(f_0) = 1$ holds with an appropriate invariant mean m. Thus $\emptyset \neq \mathcal{M}_t(\mathbb{R}_+)$, $\mathcal{M}_t(\mathbb{R}_+) \subset \mathcal{M}(\mathbb{R}_+)$ and $\mathcal{M}_t(\mathbb{R}_+) \neq \mathcal{M}(\mathbb{R}_+)$. First observe that if $f \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, then $m(f) = 0$ holds for every $m \in \mathcal{M}_t(\mathbb{R}_+)$. Indeed, denoting the characteristic function of $[0, n]$ by χ_n, the asymptotic relation $\lim_n \|f * \frac{1}{n}\chi_n\|_\infty = 0$ implies that $\lim_n m(f * \frac{1}{n}\chi_n) = 0$ holds for every mean m. If $m \in \mathcal{M}_t(\mathbb{R}_+)$ then $m(f * \frac{1}{n}\chi_n) = m(f)$ $(n \in \mathbb{N})$, and so $m(f) = 0$. Let $r_1, r_2, \ldots$ be an enumeration of the nonnegative rational numbers. Setting $\Omega = \bigcup_{n=1}^\infty (r_n - 2^{-n}, r_n + 2^{-n})$, let f_0 be the characteristic function of $\Omega \cap \mathbb{R}_+$. Since $f_0 \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, we get that $m(f_0) = 0$ for every $m \in \mathcal{M}_t(\mathbb{R}_+)$.

The following statement is valid as well.

Proposition 3.1.3. *There exists an invariant mean m on $L^\infty(\mathbb{R}_+)$ such that $m(f_0) = 1$ for the function f_0 defined above.*

Proof. Let W denote the linear span of the functions $(f_0)_s$ $(s \in \mathbb{R}_+)$ and the constant $\mathbf{1}$. Let m_0 be the linear functional on W, defined by $m_0(f) := \sum_{k=0}^n \alpha_k$ whenever $f = \alpha_0 \cdot \mathbf{1} + \sum_{k=1}^n \alpha_k (f_0)_{s_k}$ $(\alpha_k \in \mathbb{C}, s_k \in \mathbb{R}_+, n \in \mathbb{N})$. Since $(f_0)_{s_k}$ is the characteristic function of the dense, open set $\Omega_k := (\Omega - s_k) \cap \mathbb{R}_+$ in $\mathbb{R}_+$, and since $\bigcap_{k=1}^n \Omega_k$ is also a dense open subset of $\mathbb{R}_+$, we may infer that $|m_0(f)| \leq \|f\|_\infty$. Noting the linearity property, we find that m_0 is well-defined. Furthermore, W and m_0 are clearly translation invariant.

Now let $\mathcal{N}$ denote the set of all norm-preserving extensions of m_0 from W to $L^\infty(\mathbb{R}_+)$. Then $\mathcal{N}$ is a non-empty, convex, weak-$*$ compact set in the dual space $L^\infty(\mathbb{R}_+)^*$. For any $s \in \mathbb{R}_+$, let $T(s) \colon \mathcal{N} \to \mathcal{N}$ be defined by $(T(s)m)(f) := m(f_s)$ $(f \in L^\infty(\mathbb{R}_+))$. Since

the commuting mappings $T(s)$ $(s \in \mathbb{R}_+)$ are affine and weak-$*$ continuous, the Markov–Kakutani theorem implies the existence of a common fixed point $m \in \mathcal{N}$. Whence it follows that m is an invariant mean and $m(f_0) = m_0(f_0) = 1$. $\qquad\qquad\qquad\square$

Generally, if G is a non-compact, non-discrete and locally compact group which is amenable, the sets $\mathcal{M}_t(G)$ and $\mathcal{M}(G)$ are different. For details, we suggest [56, p. 277], see also [26], [59], [62].

For any compact set $K \subseteq \mathbb{R}_+$ of positive measure, we introduce the mean φ_K on $L^\infty(\mathbb{R}_+)$, defined by $\varphi_K(f) := \frac{1}{|K|} \int_K f(s)\, ds$. We recall that a sequence of compact sets $\{K_n\}_n \subseteq \mathbb{R}_+$ with non-empty interiors is a *Følner sequence* if

$$|(s+K_n) \bigtriangleup K_n|/|K_n| \to 0 \quad (n \to \infty)$$

uniformly in $s \in K$, for any compact set $K \subseteq \mathbb{R}_+$. The following statement describes the set $\mathcal{M}_t(\mathbb{R}_+)$. Its counterpart for groups is due to C. Chou. We sketch a proof, which follows the lines of [56, p. 138], for the sake of completeness.

Theorem 3.1.4. *The set $\mathcal{M}_t(\mathbb{R}_+)$ is the weak-$*$ closure of the convex hull of the set of all weak-$*$ cluster points of the sequences*

$$\{\varphi_{K_n+s_n}\}_{n \in \mathbb{N}} \quad (\{s_n\}_n \in \mathbb{R}_+^{\mathbb{N}}),$$

where $\{K_n\}_n$ is an arbitrarily given fixed Følner sequence in $\mathbb{R}_+$.

Proof. We can see from the proof of Proposition 3.1.2 that every such weak-$*$ cluster point is a topologically invariant mean, and then so are the means in the weak-$*$ closure of the convex hull.

To prove the converse statement, let us denote the weak-$*$ closure of the convex hull of the above cluster points by $\mathfrak{L}_t$. Suppose that there is a functional $\phi \in \mathcal{M}_t(\mathbb{R}_+) \setminus \mathfrak{L}_t$. By the Hahn–Banach theorem there exists a function $f \in L^\infty(\mathbb{R}_+)$, a real number c and an $\varepsilon > 0$ such that

$$\mathrm{Re}\ m(f) \leq c < c + \varepsilon \leq \mathrm{Re}\ \phi(f) \quad \text{for all } m \in \mathfrak{L}_t.$$

Every mean is a positive functional which implies $\mathrm{Re}\ m(f) = m(\mathrm{Re}\ f)$ and $\mathrm{Re}\ \phi(f) = \phi(\mathrm{Re}\ f)$, thus we may suppose that f is real-valued. Since the functional ϕ is topologically invariant, and since ϕ is monotone increasing as a mean, it follows that

$$\phi(f) = \frac{1}{|K_n|}\phi(f * \chi_{K_n}) \leq \sup_s \frac{1}{|K_n|} \int_{K_n} f(s+t)\, dt \leq \|f\|_\infty < \infty.$$

For every $n \in \mathbb{N}$ choose an $s_n \in \mathbb{R}_+$ such that

$$\sup_s \frac{1}{|K_n|} \int_{K_n} f(s+t)\, dt - \frac{1}{|K_n|} \int_{K_n} f(s_n + t)\, dt < \varepsilon/2.$$

Let m_0 denote a weak-$*$ cluster point of the sequence $\{\varphi_{K_n + s_n}\}$, where clearly $m_0 \in \mathfrak{L}_t$. Thus, for some large n, $|\varphi_{K_n + s_n}(f) - m_0(f)| < \varepsilon/2$, and so the following inequalities are satisfied

$$\frac{1}{|K_n|} \int_{K_n + s_n} f(t)\, dt < c + \varepsilon/2 \leq \phi(f) - \varepsilon/2 \leq \frac{1}{|K_n|} \int_{K_n} f(s_n + t)\, dt.$$

This is a contradiction, hence the theorem must be valid. $\square$

Definition 3.1.5. A function $f \in L^\infty(\mathbb{R}_+)$ is called *topologically almost convergent* if the set $\{m(f) : m \in \mathcal{M}_t(\mathbb{R}_+)\}$ is a singleton. We shall use the notation at-lim$f = c$ whenever $m(f) = c$ for all $m \in \mathcal{M}_t(\mathbb{R}_+)$.

Definition 3.1.6. We say that a function $f \in L^\infty(\mathbb{R}_+)$ is *topologically almost convergent in the strong sense* to a complex number c if at-lim$|f - c| = 0$.

We note that at-lim$|f - c| = 0$ if and only if $m(fg) = c\, m(g)$ for every $g \in L^\infty(\mathbb{R}_+)$ and $m \in \mathcal{M}_t(\mathbb{R}_+)$. The proof follows the same argument as the proof of Lemma 2.1.13, which played an important role in the previous chapter.

The following characterization will be very useful for us. (Notice that its counterpart in $\ell^\infty(\mathbb{Z}_+)$ is the classical characterization of almost convergent sequences by Lorentz [49].)

Proposition 3.1.7. *An $f \in L^\infty(\mathbb{R}_+)$ is topologically almost convergent to c if and only if*

$$\lim_{n \to \infty} \frac{1}{|K_n|} \int_{K_n} f_y(s)\, ds = c$$

uniformly with respect to $y \in \mathbb{R}_+$, where $\{K_n\}_n$ is an arbitrarily chosen Følner sequence.

Proof. Let us assume that there exists an $\varepsilon > 0$, a strictly increasing sequence $\{n_k\}_k$ of positive integers, and a sequence $\{y_k\}_k \in \mathbb{R}_+$ such that

$$\left| \frac{1}{|K_{n_k}|} \int_{K_{n_k}} f_{y_k}(s)\, ds - c \right| > \varepsilon$$

is true for every $k \in \mathbb{N}$. Then $\{C_k := K_{n_k} + y_k\}_k$ is also a Følner sequence. For every $k \in \mathbb{N}$, let us consider the linear functional φ_k on $L^\infty(\mathbb{R}_+)$ defined by

$$\varphi_k(h) := \frac{1}{|C_k|} \int_{C_k} h(s)\, ds\,, \quad h \in L^\infty(\mathbb{R}_+).$$

We know from Theorem 3.1.4 that any weak-$*$ cluster point m_0 of the sequence $\{\varphi_k\}_k$ is in $\mathcal{M}_{\mathrm{t}}(\mathbb{R}_+)$. Since $|m_0(f) - c| \geq \varepsilon$ by our assumption, we conclude that f cannot topologically almost converge to c. This proves the necessity of the integral condition.

The converse is a direct corollary of Theorem 3.1.4. $\qquad\qquad\square$

We recall from Section 2.1 that a function $f \in L^\infty(\mathbb{R}_+)$ almost converges to $c \in \mathbb{C}$, that is, a-$\lim f = c$, if $m(f) = c$ for all invariant means $m \in \mathcal{M}$. It is clear that a-$\lim f = c$ implies at-$\lim f = c$, but the converse does not follow (see Proposition 3.1.3). The advantage of considering topologically almost convergent functions instead of almost convergent functions appears in the complete characterization given in Proposition 3.1.7.

Remark 3.1.8. We note that the results of this section remain true in a more general framework. Let us consider a locally compact, σ-compact, Hausdorff abelian group $(G, +)$, and let $(S, +)$ be a closed subsemigroup of G with non-empty interior S° such that $S - S = G$ and $S \cap (-S) = \{0\}$. Since these groups are amenable, we can find a Følner sequence in G, and even in S° (see [56, Theorem 4.16] and [42]). Chou's theorem works in this situation as well, hence $\mathcal{M}_{\mathrm{t}}(S) \neq \emptyset$ and the above integral characterization is also true.

3.1.1 An immediate application

In the following, we will give an application of (topologically) almost convergent functions.

Let $\mathfrak{X}$ be a complex Banach space and pick an $x \in \mathfrak{X}$. Let us form the closure of the subspace spanned by the vectors $\{T(s)x : s \in \mathbb{R}_+\}$, denoted by $\mathfrak{X}_x$, where T is a bounded C_0-semigroup. Obviously, $\mathfrak{X}_x$ is T-invariant hence the restriction of T to $\mathfrak{X}_x$, $T_x := T_{|\mathfrak{X}_x}$ is a well defined C_0-semigroup whose generator is denoted by A_x.

Theorem 3.1.9. *Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ be a bounded C_0-semigroup with the generator A. Then the following assertions are equivalent for any $x \in \mathfrak{X}$:*

(i) $\sigma_{\mathrm{p}}(A_x^*) \cap i\mathbb{R} = \emptyset$,

(ii) $\langle x^*, e^{i\lambda s} T(s)x \rangle$ *topologically almost converges to zero for every $\lambda \in \mathbb{R}$ and $x^* \in \mathfrak{X}_x^*$,*

(iii) $\langle x^*, e^{i\lambda s}\, T(s)x \rangle$ *almost converges to zero for every* $\lambda \in \mathbb{R}$ *and* $x^* \in \mathcal{X}_x^*$,

(iv) $\inf \left\{ \left\| \int_{\mathbb{R}_+} e^{i\lambda s} g(s)T(s)x\, ds \right\| : g \in L^1(\mathbb{R}_+),\ \int_{\mathbb{R}_+} g = 1 \right\} = 0$ *for every* $\lambda \in \mathbb{R}$.

Proof. (i) $\Rightarrow$ (ii): Let us assume that (ii) is not satisfied. Then there exists an $x^* \in \mathcal{X}_x^*$, $\lambda \in \mathbb{R}$ and $m \in \mathcal{M}_t(\mathbb{R}_+)$ such that $m(\langle x^*, e^{i\lambda s} T(s)\, x \rangle) \neq 0$. Using Theorem 3.1.4 we can find a sequence $\{s_n\}_{n=1}^{\infty} \in \mathbb{R}_+^{\mathbb{N}}$ and a $0 \neq c \in \mathbb{C}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \int_{s_n}^{s_n+n} \langle x^*, e^{i\lambda s}T(s)x \rangle\, ds = c.$$

Let us define for each $n \in \mathbb{N}$ the linear functional z_n^* on $\mathcal{X}_x$ where

$$\langle z_n^*, y \rangle := \frac{1}{n} \int_{s_n}^{s_n+n} \langle x^*, e^{i\lambda s}T(s)y \rangle\, ds, \qquad y \in \mathcal{X}_x.$$

Since $\sup_n \|z_n^*\| < \infty$, the Banach–Alaoglu theorem implies the existence of a weak-$*$ cluster point of the sequence $\{z_n^*\}_{n=1}^{\infty}$ denoted by z^*. We find that z^* is non-zero on $\mathcal{X}_x$ because $\langle z^*, x \rangle = c$ and $T_x^*(t)z^* = e^{-i\lambda s}z^*$ holds for every $t \in \mathbb{R}_+$. Indeed, if $y \in \mathcal{X}_x$,

$$
\begin{aligned}
|\langle z_n^*, e^{i\lambda t}\, T(t)y - y \rangle| &= \left| \frac{1}{n} \int_{s_n}^{s_n+n} \langle x^*, e^{i\lambda(s+t)}T(s+t)y - e^{i\lambda s}T(s)y \rangle\, ds \right| \\
&\leq \|x^*\| \sup_s \|e^{i\lambda s}T(s)y\| \frac{|(t + s_n + [0,n]) \triangle (s_n + [0,n])|}{n} \\
&\to 0 \quad (n \to \infty),
\end{aligned}
$$

which implies $T_x^*(t)z^* = e^{-i\lambda t}z^*$ for every $t \in \mathbb{R}_+$. However, this means that $A_x^* z^* = -i\lambda z^*$, which is a contradiction, so the implication is true.

(ii) $\Rightarrow$ (i): Let us choose an $i\lambda \in \sigma_{\mathrm{p}}(A_x^*)$ with corresponding eigenvector $0 \neq x^* \in \mathcal{D}(A_x^*)$. Then

$$(3.1) \qquad\qquad e^{-i\lambda t}T_x^*(t)x^* = x^* \quad \text{ for all } t \in \mathbb{R}_+$$

holds (see [20, Chapter II.2 and Theorem IV.3.6]) and there exists an $s \in \mathbb{R}_+$ with the property $\langle x^*, T(s)\, x \rangle \neq 0$. For any $m \in \mathcal{M}_t(\mathbb{R}_+) \subseteq \mathcal{M}(\mathbb{R}_+)$, we have

$$
\begin{aligned}
0 = m(\langle x^*, e^{-i\lambda t}T(t)x \rangle) &= m(\langle x^*, e^{-i\lambda(t+s)}T(t+s)x \rangle) \\
&= e^{-i\lambda s}\, m(\langle e^{-i\lambda t}T_x^*(t)x^*, T(s)x \rangle).
\end{aligned}
$$

But from (3.1) we have $m(\langle e^{-i\lambda t}T_x^*(t)x^*, T(s)x \rangle) = m(\langle x^*, T(s)\, x \rangle \cdot \mathbf{1}) \neq 0$. This is a contradiction, so $\sigma_{\mathrm{p}}(A_x^*) \cap i\mathbb{R} = \emptyset$.

(ii) $\Leftrightarrow$ (iii) We can easily see that the function $\psi_\lambda(s) := e^{i\lambda s}T(s)x$ $(\lambda \in \mathbb{R})$ is a bounded uniformly continuous $\mathcal{X}$-valued function on $\mathbb{R}_+$, i.e. $\psi_\lambda \in BUC(\mathbb{R}_+, \mathcal{X})$. On the other

hand, the left translation semigroup $S(s) : BUC(\mathbb{R}_+, \mathbb{C}) \to BUC(\mathbb{R}_+, \mathbb{C})$, $(S(s)f)(t) :=$ $f(t + s)$ $(s, t \in \mathbb{R}_+)$ is a C_0-semigroup on $BUC(\mathbb{R}_+, \mathbb{C})$. Hence, for any $m \in \mathcal{M}(\mathbb{R}_+)$ and $g \in \mathcal{G} \subset L^1(\mathbb{R}_+)$, we may infer that

$$m(f) = \int_0^\infty g(s)\, m(S(s)f)\, ds = m\left(\int_0^\infty g(s)\, S(s)f\, ds\right)$$

(see [49, Theorem–Definition p. 2]). This means that the set of the topologically invariant means and the set of the invariant means restricted to the closed subspace $BUC(\mathbb{R}_+, \mathbb{C})$ of $L^\infty(\mathbb{R}_+)$ are the same. We conclude that $\{m(\langle x^*, \psi_\lambda \rangle) : m \in \mathcal{M}_t(\mathbb{R}_+)\} = \{m(\langle x^*, \psi_\lambda \rangle) : m \in \mathcal{M}(\mathbb{R}_+)\}$ for any $x^* \in \mathcal{X}_x^*$.

(ii) $\Rightarrow$ (iv): Fix a $\lambda \in \mathbb{R}$. By (ii), $m(\langle x^*, e^{i\lambda t}T(t)x \rangle) = 0$ is valid for every $m \in \mathcal{M}_t(\mathbb{R}_+)$ and $x^* \in \mathcal{X}_x^*$. We can see from Theorem 3.1.7 that

$$\lim_{n \to \infty} \left\langle x^*, \frac{1}{n}\int_0^n e^{i\lambda s}T(s)x\, ds \right\rangle = 0,$$

because $\{[0, n]\}_{n=1}^\infty$ is a Følner sequence on $\mathbb{R}_+$. This implies that $n^{-1}\int_0^n e^{i\lambda s}T(s)x\, ds$ converges to 0 as well (see [44, II. Theorem 1.5]). Since $n^{-1}\chi_{[0,n]} \in \mathcal{G}$, the implication (ii) $\Rightarrow$ (iv) is verified.

(iv) $\Rightarrow$ (ii): Choosing an $m \in \mathcal{M}_t(\mathbb{R}_+)$, we readily see that $m(g * f) = m(f)$ for any $g \in L^1(\mathbb{R}_+)$ such that $\int_{\mathbb{R}_+} g(t)\, dt = 1$. Hence the following straightforward inequalities hold, if $x^* \in \mathcal{X}_x^*$, $\lambda \in \mathbb{R}$ and $g \in L^1(\mathbb{R}_+)$:

$$
\begin{aligned}
|m(\langle x^*, e^{i\lambda s}T(s)x \rangle)| &= \left| m\left(\int_0^\infty g(t)\langle x^*, e^{i\lambda(s+t)}T(s+t)x \rangle\, dt\right) \right| \\
&\leq \|x^*\| \sup_{s \in \mathbb{R}_+} \left\| T(s)\int_0^\infty e^{i\lambda t}g(t)T(t)x\, dt \right\| \\
&\leq \|x^*\| \sup_{s \in \mathbb{R}_+} \|T(s)\| \left\| \int_0^\infty e^{i\lambda t}g(t)T(t)x\, dt \right\|.
\end{aligned}
$$

We conclude that $m(\langle x^*, e^{i\lambda s}T(s)x \rangle) = 0$ for any $m \in \mathcal{M}_t(\mathbb{R}_+)$, thus $\langle x^*, e^{i\lambda s}T(s)x \rangle$ topologically almost converges to 0. $\qquad\square$

Remark 3.1.10. The above result is partly well-known [5, Proposition 3.2]) and goes back to classical operator ergodic results, see [44, Theorem II.1.3]; however, to the best of our knowledge, the originality of this approach seems to be in the use of (topologically) almost convergent functions.

3.2 C_0-semigroups and topological regularity

We would like to introduce an upper bound for the norm-function of C_0-semigroups that is well behaved. To do this, we will introduce the following type of gauge functions.

Definition 3.2.1. We say that $p\colon \mathbb{R}_+ \to [1, \infty)$ is a *topological gauge function* if (i) it is measurable, (ii) it is locally bounded (i.e. bounded on compact sets) (iii) for every compact subset K of $\mathbb{R}_+$, $\sup\{p(t+s)/p(t) : t \in \mathbb{R}_+, s \in K\} < \infty$, and (iv) for every $s \in \mathbb{R}_+$, p_s/p topologically almost converges in the strong sense to a positive real number $c_p(s)$. The function c_p is called the *limit functional* of the gauge function p. Here the set of topological gauge functions shall be denoted by $\mathcal{P}_t$.

Let $\mathfrak{X}$ denote a complex Banach space and let $\mathcal{L}(\mathfrak{X})$ stand for the set of all bounded linear operators on $\mathfrak{X}$. Also, let $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ be a strongly continuous, one-parameter semigroup, that is a C_0-semigroup.

Definition 3.2.2. The C_0-semigroup $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ has *regular norm behaviour with respect to the topological gauge function p* or has a *p-regular norm-function* if (i) $\|T(s)\| \le p(s)$ holds for every $s \in \mathbb{R}_+$, and (ii) at-$\lim_s \|T(s)\|/p(s) = 0$ does not hold.

We note that the results in Chapter 2 remain valid with this definition of regularity. In fact, the main ingredients in that chapter, e.g. the limit functional and the associated isometric representation, can be introduced in a similar way using $\mathcal{M}_t(\mathbb{R}_+)$ instead of $\mathcal{M}$. It can be also shown that if T has a p-regular norm-function, $p \in \mathcal{P}_t$, then the corresponding limit functional $c_T = c_p$ is equal to the spectral radius function $r(T(s))$ of T (see the proof of Proposition 2.2.12). Furthermore, it is well known that $r(T(s)) = e^{\omega_0(T)s}$ ($s \in \mathbb{R}_+$), where $\omega_0(T) = \lim_{s\to\infty} (\log \|T(s)\|)/s$ is the exponential growth bound of T (see [20, p. 251]).

Now to get a complete adaptation of the main result in Chapter 2, we shall identify the spectrum of an arbitrary C_0-semigroup $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ with an appropriate hull of the spectrum of the infinitesimal generator A of T. We will assume that $w_0(T) > -\infty$, which is the case when T has regular norm-behaviour.

Let $\mathbb{R}_+^\sharp$ denote the set of all characters of $\mathbb{R}_+$, that is the set of all continuous representations of $\mathbb{R}_+$ on $\mathbb{C}$. We note that $\mathbb{R}_+^\sharp = \{\chi_\lambda : \lambda \in \mathbb{C}\}$, where $\chi_\lambda(s) = e^{\lambda s}$ ($s \in \mathbb{R}_+$). In the general semigroup setting the spectrum of T is defined as the set of

characters $\chi \in \mathbb{R}_+^\sharp$ satisfying the condition

$$\left| \int_0^\infty f(s)\chi(s)\,ds \right| \leq \|\widehat{f}(T)\|$$

for every $f \in C_c(\mathbb{R}_+)$, the set of continuous functions with compact support on $\mathbb{R}_+$. Here

$$\widehat{f}(T)x := \int_0^\infty f(s)\,T(s)x\,ds \quad (x \in \mathfrak{X})$$

is the Fourier transform of f with respect to T. With the special form of the characters in $\mathbb{R}_+^\sharp$ we can identify the spectrum of T as a subset of $\mathbb{C}$, namely

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\widehat{f}(\lambda)| \leq \|\widehat{f}(T)\| \text{ for every } f \in C_c(\mathbb{R}_+)\},$$

where $\widehat{f}(\lambda) = \int_0^\infty e^{\lambda s} f(s)\,ds$ ($\lambda \in \mathbb{C}$) is the Laplace transform of f. The function $\widehat{f}$ being analytic on $\mathbb{C}$, the set $\sigma(T)$ is closed. Moreover, since the characters are non-vanishing, $\mathrm{Re}\,\lambda \leq \omega_0(T)$ holds for every $\lambda \in \sigma(T)$ (see e.g. the proof of Proposition 2.3.3). The peripheral spectrum of T is defined as

$$\sigma_{\mathrm{per}}(T) := \{\lambda \in \sigma(T) : \mathrm{Re}\,\lambda = \omega_0(T)\}.$$

Notice that these definitions correspond to the ones given in Chapter 2 for regular representations of general semigroups.

Now let us consider the infinitesimal generator A of T, where it is known that A is a densely defined closed operator. Applying the equations

$$T(t)x - x = A \int_0^t T(s)x\,ds \quad (t \in \mathbb{R}_+, x \in \mathfrak{X})$$

and

$$T(t)x - x = \int_0^t T(s)Ax\,ds \quad (t \in \mathbb{R}_+, x \in \mathcal{D}(A))$$

(see e.g. the proof of Theorem 34.4 in [45]), it is straightforward to show that

$$\widehat{f}(\sigma(A)) \subseteq \sigma(\widehat{f}(T)) \text{ for every } f \in C_c(\mathbb{R}_+).$$

Thus the closed set $\sigma(A)$ is contained in $\sigma(T)$. Next, let $\rho_\infty(A)$ stand for the open component of $\mathbb{C} \setminus \sigma(A)$ containing the right half-plane $\{z \in \mathbb{C} : \mathrm{Re}\,z > \omega_0(T)\}$.

Proposition 3.2.3. *With the previous notation, we have*

$$\sigma(T) = \mathbb{C} \setminus \rho_\infty(A).$$

Proof. Replacing $T(s)$ by $e^{-(\omega_0(T)+1)s}T(s)$ ($s \in \mathbb{R}_+$), we may assume that $\omega_0(T) = -1$. Let Ω be a component of $\mathbb{C} \setminus \sigma(A)$, other than $\rho_\infty(A)$. If Ω is bounded, then the Maximum Principle applied to $\widehat{f}$ ($f \in C_c(\mathbb{R}_+)$) tells us that $\Omega \subseteq \sigma(T)$. If Ω is unbounded, then we can consider the bounded open set $\Omega_0 = \{1/z : z \in \Omega\}$ and to the function $F(z) = \widehat{f}(1/z)$ ($z \in \Omega_0^- \setminus \{0\}$). Applying the Maximum Principle to the function $g_n(z)F(z)$, where $g_n(re^{i\varphi}) := \sqrt[n]{r}e^{i\varphi/n}$ ($r \in [0, \infty), \varphi \in (0, 2\pi)$), and letting n tend to infinity, we again obtain that $\Omega \subseteq \sigma(T)$.

Since $(f * g)\widehat{\,}(T) = \widehat{f}(T)\widehat{g}(T)$ holds for every $f, g \in C_c(\mathbb{R}_+)$, the norm closure $\mathcal{A}_T$ of the Fourier transforms $\{\widehat{f}(T) : f \in C_c(\mathbb{R}_+)\}$ forms a commutative Banach algebra. Fixing a point $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\,\lambda_0 > \omega_0(T)$, the formula

$$R(\lambda_0, A) = \int_0^\infty e^{-\lambda_0 s}T(s)\,ds = \lim_{t\to\infty} \int_0^t e^{-\lambda_0 s}T(s)\,ds$$

for the resolvent $R(\lambda_0, A) = (\lambda_0 I - A)^{-1}$ (see e.g. Theorem 34.4 in [45]) shows that $R(\lambda_0, A) \in \mathcal{A}_T$. Next, let $\lambda \in \rho_\infty(A)$ be arbitrary. Taking an appropriate polygon joining λ_0 with λ in $\rho_\infty(A)$, and considering Taylor series expansions at the vertices, we get that $R(\lambda, A) \in \mathcal{A}_T$ also holds.

Choosing an arbitrary $z \in \sigma(T)$, there exists a unique contractive algebra homomorphism $h_z \colon \mathcal{A}_T \to \mathbb{C}$ satisfying the condition $h_z(\widehat{f}(T)) = \widehat{f}(z)$ ($f \in C_c(\mathbb{R}_+)$). It readily follows that $h_z(R(\lambda_0, A)) = (\lambda_0 - z)^{-1}$. Setting $w := h_z(R(\lambda, A))$, the resolvent equation

$$R(\lambda_0, A) - R(\lambda, A) = (\lambda - \lambda_0)R(\lambda_0, A)R(\lambda, A)$$

implies that

$$(\lambda_0 - z)^{-1} - w = (\lambda - \lambda_0)(\lambda_0 - z)^{-1}w,$$

whence $w(\lambda - z) = 1$, and it follows that $z \neq \lambda$.

We conclude that $\rho_\infty(A)$ and $\sigma(T)$ are disjoint. Thus $\sigma(T) = \mathbb{C} \setminus \rho_\infty(A)$. $\qquad \square$

Remark 3.2.4. From the proof of the inclusion $\sigma(T) \subset \mathbb{C} \setminus \rho_\infty(A)$, one can see that the above statement also holds if $\omega_0(T) = -\infty$ because then $\sigma(A) = \emptyset$.

Corollary 3.2.5. *Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup and let A stand for the generator of T. Then*

$$\sigma(T) \cap (\omega_0(T) + i\mathbb{R}) = \sigma(A) \cap (\omega_0(T) + i\mathbb{R});$$

that is, the peripheral spectrum of T and that of the generator A are the same.

In this context, we can reformulate Theorem 2.5.1, which is a generalisation of the celebrated Arendt–Batty–Lyubich–Vũ theorem.

Theorem 3.2.6. *Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup with a p-regular norm-function, $p \in \mathcal{P}_t$. Let A stand for the generator of T. If $\sigma(A) \cap (\omega_0(T) + \mathrm{i}\mathbb{R})$ is countable and $\sigma_p(A^*) \cap (\omega_0(T) + \mathrm{i}\mathbb{R})$ is empty then*

$$\lim_{n \to \infty} \sup_{t \geq 0} \frac{1}{n} \int_t^{t+n} \frac{\|T(s)x\|}{p(s)} \, ds = 0$$

is true for all $x \in \mathcal{X}$.

3.3 The regularity constant

Let us take a C_0-semigroup T which is not quasinilpotent; that is, $r(T(s)) > 0$ for some (and then for all) $s \in \mathbb{R}_+$. Then let us introduce the *regularity constant* κ_T as it was done in the discrete case (see [43]):

$$\kappa_T := \inf_{n \in \mathbb{N}} \sup_{s \in \mathbb{R}_+} \left(\frac{1}{n} \int_s^{s+n} r(T(t))^{-1} \|T(t)\| \, dt \right) \left(\sup_{s \leq y \leq s+n} r(T(y))^{-1} \|T(y)\| \right)^{-1}.$$

Evidently, we have $0 \leq \kappa_T \leq 1$. The regularity constant makes it possible for us to give a description of semigroups whose norm-function exhibits a regular behaviour.

Our main result is the following.

Theorem 3.3.1. *Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup. Then the next conditions are equivalent:*

(i) *T has a p-regular norm-function with a topological gauge function $p \in \mathcal{P}_t$,*

(ii) *T has a p-regular norm-function with a continuous gauge $p \in \mathcal{P}_t$,*

(iii) *$\|T(s)\| \geq 1$ for every $s \in \mathbb{R}_+$ and $\kappa_T > 0$.*

First we will show that it is always possible to find a continuous gauge function.

Proposition 3.3.2. *If T has a p-regular norm-function, $p \in \mathcal{P}_t$, then there exists a continuous gauge $q \in \mathcal{P}_t$ such that T has regular norm behaviour with respect to q.*

Proof. Step 1. To get some control over the jumps of the gauge function, in the first step we will construct a gauge that is lower semicontinuous.

For every $k \in \mathbb{Z}_+$, applying Lusin's theorem and the regularity of the Lebesgue measure, we can choose an open set $E_k \subseteq (k, k+1)$ such that $|E_k| < 2^{-k}$ and p is a continuous function on $(k, k+1) \setminus E_k$.

By our assumption on p, the constant $G_p := \sup\{p(s+t)/p(t) : t \in \mathbb{R}_+, s \in [0, 1]\}$ is finite; in fact $G_p \in [1, \infty)$. Let us define the function $v \colon \mathbb{R}_+ \to [1, \infty)$ in the following way. For every $k \in \mathbb{Z}_+$, let

$$v(s) := p(s) \quad \text{if } s \in (k, k+1) \setminus E_k,$$

and let

$$v(s) := \inf \{G_p p(t) : t \in [k, s) \setminus E_k\} \quad \text{if } s \in E_k.$$

Finally $v(k) := \liminf_{s \to k} v(s)$ for $k \in \mathbb{Z}_+$.

Note that if $s \in E_k$ ($k \in \mathbb{Z}_+$), then for any $t \in [k, s) \setminus E_k (\neq \emptyset)$ we have $p(s) \leq G_p p(t)$; hence $p(s) \leq v(s)$ holds for all $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. A simple calculation shows also that the locally bounded function v is lower semicontinuous, and so it is measurable. Since the norm-function $\|T(s)\|$ ($s \in \mathbb{R}_+$) is also lower semicontinuous we may infer that $\|T(k)\| \leq \liminf_{s \to k} \|T(s)\| \leq \liminf_{s \to k} v(s) = v(k)$ for $k \in \mathbb{Z}_+$. Thus $\|T(s)\| \leq v(s)$ holds for all $s \in \mathbb{R}_+$. Next, since the set

$$E := \{s \in \mathbb{R}_+ : p(s) \neq v(s)\} \subseteq \bigcup_{k \in \mathbb{Z}_k} (E_k \cup \{k\})$$

is of finite measure, the function

$$g(s) := \frac{\|T(s)\|}{p(s)} - \frac{\|T(s)\|}{v(s)} \qquad (s \in \mathbb{R}_+)$$

belongs to the class $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, and so $m(g) = 0$ for every $m \in \mathcal{M}_t(\mathbb{R}_+)$. Therefore at-$\lim_s \|T(s)\|/p(s) \neq 0$ implies at-$\lim_s \|T(s)\|/v(s) \neq 0$.

Given any $k \in \mathbb{Z}_+$, a brief consideration of the possible cases makes it clear that $v(t+s)/v(t) \leq G_p$ holds, whenever $k < t \leq t + s < k + 1$. As for the definition of v in the integer case, we can extend the validity of this inequality to the case when $k \leq t \leq t + s \leq k + 1$. If $k \leq t < k + 1 < t + s$ and $0 < s \leq 1$, then

$$\frac{v(s+t)}{v(t)} = \frac{v(s+t)}{v(k+1)} \frac{v(k+1)}{v(t)} \leq G_p^2.$$

Thus for any $t \in \mathbb{R}_+$, $s \in [0, 1]$ and $n \in \mathbb{N}$, we have

$$\frac{v(t+ns)}{v(t)} = \prod_{i=1}^{n} \frac{v(t+is)}{v(t+(i-1)s)} \leq G_p^{2n}.$$

We may conclude that $\sup\{v(t + s)/v(t) : t \in \mathbb{R}_+, s \in K\}$ is finite for any compact set $K \subseteq \mathbb{R}_+$. Furthermore, for every $s \in \mathbb{R}_+$ the functions $|v_s v^{-1} - c_p(s)|$ and $|p_s p^{-1} - c_p(s)|$ can differ only on the set $E \cup ((E - s) \cap \mathbb{R}_+)$ of finite measure, and so at-$\lim|p_s p^{-1} - c_p(s)| = 0$ implies that at-$\lim|v_s v^{-1} - c_p(s)| = 0$.

Therefore v is a lower semicontinuous topological gauge function (with $c_v = c_p$) and T has a v-regular norm-function.

Step 2. Next we can construct a piecewise continuous gauge function p adjusted to T.

Let us have an arbitrary $\eta \in (0, 1)$. Since v is lower semicontinuous, for every $t \in \mathbb{R}_+$, the set $U_t := \{s \in \mathbb{R}_+ : v(s) > \eta v(t)\}$ is open in $\mathbb{R}_+$; clearly $t \in U_t$. We can introduce the extended real number $\tau(t) := \sup\{s \in (t, \infty) : (t, s) \subseteq U_t\} \in (t, \infty]$. If $\tau(t)$ is finite, then $v(\tau(t)) \leq \eta v(t)$. As $v \geq 1$, it follows that for every $t \in \mathbb{R}_+$ there exists an $n \in \mathbb{N}$ such that $\tau^n(t) = \infty$.

Given a $k \in \mathbb{Z}_+$ there exists a unique $n_k \in \mathbb{N}$ such that $\tau^{n_k - 1}(k) < k + 1 \leq \tau^{n_k}(k) \in [k + 1, \infty]$. Next, let us introduce the notation $t_{k,i} := \tau^i(k)$ for $0 \leq i < n_k$, and $t_{k,n_k} := k + 1$. For every $0 \leq i < n_k$, we have

$$\eta v(t_{k,i}) \leq v(t) \leq G_v v(t_{k,i}) \text{ for all } t \in [t_{k,i}, t_{k,i+1}).$$

By Lusin's theorem, there exists a continuous function

$$g_{k,i} \colon [t_{k,i}, t_{k,i+1}) \to [\eta v(t_{k,i}), G_v v(t_{k,i})],$$

which differs from the restriction of v to this interval in a set $E_{k,i}$ of measure less than $(2n_k)^{-1} 2^{-k}$. Selecting $t_{k,i} < \hat{t}_{k,i} < \check{t}_{k,i} < t_{k,i+1}$ such that $\max(\hat{t}_{k,i} - t_{k,i}, t_{k,i+1} - \check{t}_{k,i}) < (4n_k)^{-1} 2^{-k}$, we set $h_{k,i}(t) := g_{k,i}(\hat{t}_{k,i})$ if $t_{k,i} \leq t \leq \hat{t}_{k,i}$, $h_{k,i}(t) := g_{k,i}(t)$ if $\hat{t}_{k,i} \leq t \leq \check{t}_{k,i}$ and $h_{k,i}(t) := g_{k,i}(\check{t}_{k,i})$ if $\check{t}_{k,i} \leq t < t_{k,i+1}$.

Note that the continuous function $h_{k,i}$ differs from the restriction of v in a set $H_{k,i}$ of measure less than $n_k^{-1} 2^{-k}$; furthermore the range of $h_{k,i}$ is contained in the range of $g_{k,i}$.

Let us define the piecewise continuous function h on $\mathbb{R}_+$ so that h coincides with $h_{k,i}$ on the interval $[t_{k,i}, t_{k,i+1})$ for all $k \in \mathbb{Z}_+$ and $0 \leq i < n_k$. It is clear that h differs from v on a set of finite measure. Furthermore, if $t, s \in [k, k + 1)$ and $t < s$ ($k \in \mathbb{Z}_+$), then $t \in [t_{k,i}, t_{k,i+1})$ and $s \in [t_{k,j}, t_{k,j+1})$ for some $0 \leq i \leq j < n_k$, and so

$$\frac{h(s)}{h(t)} \leq \frac{G_v v(t_{k,j})}{\eta v(t_{k,i})} \leq \frac{G_v \eta^{j-i} v(t_{k,i})}{\eta v(t_{k,i})} \leq \frac{G_v}{\eta}.$$

If $t \in [t_{k,i}, t_{k,i+1}) \subseteq [k, k + 1)$ and $s = k + 1$, then

$$\frac{h(s)}{h(t)} \leq \frac{G_v v(k + 1)}{\eta v(t_{k,i})} \leq \frac{G_v^2}{\eta}.$$

We find as in Step 1 that $\sup\{h(t+s)/h(t) : t \in \mathbb{R}_+, s \in K\}$ is finite for any compact subset K of $\mathbb{R}_+$.

The piecewise continuous function $w := \eta^{-1} G_v h$ preserves the latter property. Furthermore, for any $t \in [t_{k,i}, t_{k,i+1})$ $(k \in \mathbb{Z}_+, 0 \le i < n_k)$, we have that

$$w(t) = \eta^{-1} G_v h(t) \ge \eta^{-1} G_v \eta v(t_{k,i}) \ge \eta^{-1} G_v \eta G_v^{-1} v(t) = v(t);$$

that is $w \ge v$. Taking into account the fact that w differs from $\eta^{-1} G_v v$ on a set of finite measure, we may deduce as in Step 1 that $w \in \mathcal{P}_t$ and T has a w-regular norm-function.

Step 3. We complete the proof by associating a continuous topological gauge function q with T.

The function w has countable discontinuity points, and we may assume that it has infinitely many. We place these points into the strictly increasing sequence $\{t_n\}_{n \in \mathbb{N}}$; clearly $0 < t_1$ and $\lim_{n \to \infty} t_n = \infty$. For every $n \in \mathbb{N}$, let δ_n denote the minimum of the positive numbers $\{3^{-1}(t_j - t_{j-1})\}_{j=1}^{n+1}$ (here $t_0 := 0$). We know that w is continuous to the right of t_n and that is has a left limit $w(t_n - 0) := \lim_{t \to t_n - 0} w(t)$. Let $\mathcal{N}_+$ stand for the set of positive integers n, where $w(t_n - 0) < w(t_n)$, and $\mathcal{N}_- := \mathbb{N} \setminus \mathcal{N}_+$. For any $n \in \mathcal{N}_+$, we can find a real number r_n such that $0 < t_n - r_n < \min(\delta_n, 2^{-n})$ and w is constant on the interval $[r_n, t_n)$. The function q is defined on $[r_n, t_n]$ so that it is linear and $q(r_n) := w(r_n)$, $q(t_n) := w(t_n)$. If $n \in \mathcal{N}_-$, then we can choose an r_n such that $0 < r_n - t_n < \min(\delta_n, 2^{-n})$ and w is constant on the interval $[t_n, r_n]$. The function q is linear on $[t_n, r_n]$ and satisfies the conditions $q(t_n) = w(t_n - 0)$ and $q(r_n) = w(r_n)$. Next, for every $t \in \mathbb{R}_+ \setminus ((\bigcup_{n \in \mathcal{N}_+}[r_n, t_n]) \cup (\bigcup_{n \in \mathcal{N}_-}[t_n, r_n]))$, we will set $q(t) := w(t)$.

It immediately follows that q is a continuous function, $q \ge w$, and q differs from w on a set of positive measure. Now let $t, s \in \mathbb{R}_+$ be arbitrary and satisfy $0 < s - t \le 1$. If $t \in (r_j, t_j), s \in (r_k, t_k)$ $(j, k \in \mathcal{N}_+, j \le k)$, then

$$\frac{q(s)}{q(t)} \le \frac{q(t_k)}{q(r_j)} = \frac{w(t_k)}{w(r_j)} \le G_w^2$$

since $t_k - r_j \le 2$. The reader can readily check that $q(s)/q(t) \le G_w^2$ holds in other possible cases too. We get as before that $q \in \mathcal{P}_t$ and T has a q-regular norm-function. $\square$

3.4   One consequence of regularity

Now we will prove that the regularity of the norm-function implies the positivity of the regularity constant.

Proposition 3.4.1. *Let* $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ *be a* C_0-*semigroup with a* p-*regular norm function, where* $p \in \mathcal{P}_\mathrm{t}$ *is continuous. Then* $\|T(s)\| \geq 1$ *for every* $s \in \mathbb{R}_+$ *and the regularity constant* κ_T *is positive.*

Proof. Let T be a C_0-semigroup with p-regular norm function, where $p \in \mathcal{P}_\mathrm{t}$ is continuous. Changing $T(s)$ to $r(T(s))^{-1}T(s)$, we may assume that $c_p(s) = c_T(s) = r(T(s)) = 1$ ($s \in \mathbb{R}_+$). If $\|T(s_0)\| < 1$ for some $s_0 \in (0, \infty)$, then $\lim_s \|T(s)\| = 0$, which contradicts the assumption at-$\lim_s \|T(s)\|/p(s) \neq 0$. Therefore $\|T(s)\| \geq 1$ holds for all $s \in \mathbb{R}_+$.

Supposing that $\kappa_T = 0$ we need to show that it leads to a contradiction. For any $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$ let us introduce the notation

$$A(s, n) := \frac{1}{n} \int_s^{s+n} \|T(t)\| \, dt, \qquad B(s, n) := \sup\{\|T(t)\| : t \in [s, s+n]\}$$

and

$$C(s, n) := \frac{1}{n} \int_s^{s+n} \frac{\|T(t)\|}{p(t)} \, dt, \qquad D(s, n) := \frac{1}{n} \int_s^{s+n} \left| \frac{p(t+1)}{p(t)} - 1 \right| dt.$$

We infer by Proposition 3.1.7 that $\tilde{\alpha} := \limsup_n \sup\{C(s, n) : s \in \mathbb{R}_+\} \in (0, 1]$; let us give an arbitrary $\alpha \in (0, \tilde{\alpha})$. Let us choose a positive ε which satisfies the condition

$$(3.2) \qquad 4G_p^2 \varepsilon < \frac{\alpha}{3},$$

where $G_p := \sup\{p(t+s)/p(t) : t \in \mathbb{R}_+, \ s \in [0, 1]\}$. Since $\kappa_T = 0$, it follows that there exists $k \in \mathbb{N}$ such that

$$(3.3) \qquad \frac{A(s, k)}{B(s, k)} \leq \varepsilon \quad \text{for all } s \in \mathbb{R}_+.$$

Let us give a positive η and a positive integer N satisfying the conditions

$$(3.4) \qquad \sqrt{2\eta} < \min\left(\frac{\alpha}{3k}, \ \sqrt[k+1]{2} - 1, 1 - \frac{1}{\sqrt[k+1]{2}} \right)$$

and

$$(3.5) \qquad 2N^{-1} < \frac{\alpha}{3}.$$

Since $\alpha < \tilde{\alpha}$ and $\lim_n \sup\{D(s, n) : s \in \mathbb{R}_+\} = 0$ by Proposition 3.1.7, we can find $l \in \mathbb{N}$ and $s \in \mathbb{R}_+$ such that

$$(3.6) \qquad l/k > N, \quad C(s, l) > \alpha \ \text{ and } \ D(s, l) < \eta.$$

The integer l can be written in the form $l = nk + r$, where $n := [l/k] \geq N$ and $r \in \mathbb{Z}_+$, $r < k$. Applying (3.5) and (3.6), we have

$$\alpha < \frac{k}{l} \sum_{j=0}^{n-1} C(s + jk, k) + \frac{r}{l} C(s + nk, r)$$

$$(3.7) \qquad \leq \frac{2k}{l} + \frac{k}{l} \sum_{j=1}^{n-1} C(s + jk, k)$$

$$< \frac{\alpha}{3} + \frac{k}{l} \sum_{j=1}^{n-1} C(s + jk, k).$$

From (3.6) it also follows that

$$(3.8) \qquad \sum_{j=1}^{n-1} D(s + jk - 1, k + 1) \leq 2lD(s, l)(k + 1)^{-1} < 2\eta(k + 1)^{-1} l.$$

Next, let $\mathcal{N}_+$ (respectively, $\mathcal{N}_-$) stand for the set of those positive integers j which satisfy the conditions $j < n$ and $D(s + jk - 1, k + 1) \geq \sqrt{2\eta}(k + 1)^{-1}$ (respectively, $j < n$ and $D(s + jk - 1, k + 1) < \sqrt{2\eta}(k + 1)^{-1}$). We get from (3.8) that

$$(\operatorname{card} \mathcal{N}_+)\sqrt{2\eta}(k + 1)^{-1} \leq \sum_{j \in \mathcal{N}_+} D(s + jk - 1, k + 1) < 2\eta(k + 1)^{-1} l$$

whence

$$(3.9) \qquad \operatorname{card} \mathcal{N}_+ < l\sqrt{2\eta}$$

follows.

Fixing an index $j \in \mathcal{N}_-$, we define ω_+ (respectively, ω_-) as the set of points t in the interval $[s + jk - 1, s + (j + 1)k]$ which satisfy the condition $|p(t + 1)p(t)^{-1} - 1| \geq \sqrt{2\eta}$ (respectively, $|p(t + 1)p(t)^{-1} - 1| < \sqrt{2\eta}$). The inequality

$$\sqrt{2\eta}|\omega_+| \leq (k + 1)D(s + jk - 1, k + 1) < \sqrt{2\eta}$$

implies $|\omega_+| < 1$; thus there exists $s_0 \in (s + jk - 1, s + jk)$ such that $s_0 + i \in \omega_-$ holds for every non-negative integer i with $i \leq k$. For any $t \in \omega_-$ we have by (3.4) that

$$\frac{1}{\sqrt[k+1]{2}} < 1 - \sqrt{2\eta} < \frac{p(t + 1)}{p(t)} < 1 + \sqrt{2\eta} < \sqrt[k+1]{2}.$$

Hence for every integer i with $1 \leq i \leq k + 1$ we get

$$2^{-1} \leq 2^{-\frac{i}{k+1}} \leq \frac{p(s_0 + i)}{p(s_0)} = \prod_{v=0}^{i-1} \frac{p(s_0 + v + 1)}{p(s_0 + v)} \leq 2^{\frac{i}{k+1}} \leq 2.$$

Thus

$$(3.10) \qquad 2^{-1}p(s_0) \leq p(s_0 + i) \leq 2p(s_0) \qquad \text{whenever } 0 \leq i \leq k + 1.$$

Given any $t_1, t_2 \in [s + jk, s + (j+1)k]$ there exist integers $i_1, i_2 \in [0, k]$ and real numbers $a_1, a_2 \in [0, 1)$ such that $t_1 = s_0 + i_1 + a_1$ and $t_2 = s_0 + i_2 + a_2$. Applying (3.10) we find that

$$(3.11) \qquad \begin{aligned} \frac{p(t_1)}{p(t_2)} &= \frac{p(s_0 + i_1 + a_1)}{p(s_0 + i_1)} \frac{p(s_0 + i_1)}{p(s_0 + i_2 + 1)} \frac{p(s_0 + i_2 + 1)}{p(s_0 + i_2 + a_2)} \\ &\leq 4G_p^2. \end{aligned}$$

Assuming that p takes its maximum on the interval $[s + jk, s + (j+1)k]$ at t^*, we infer by (3.11), (3.3) and (3.2) that

$$(3.12) \qquad \begin{aligned} C(s + jk, k) &\leq \frac{1}{k} \int_{s+jk}^{s+(j+1)k} \frac{\|T(t)\|}{B(s + jk, k)} \frac{p(t^*)}{p(t)} \, dt \\ &\leq 4G_p^2 \frac{A(s + jk, k)}{B(s + jk, k)} \leq 4G_p^2 \varepsilon < \frac{\alpha}{3}. \end{aligned}$$

Finally, by the use of (3.7), (3.12), (3.9) and (3.4) we may conclude that

$$\begin{aligned} \alpha \; &< \; \frac{\alpha}{3} + \frac{k}{l} \sum_{j \in \mathcal{N}_+} C(s + jk, k) + \frac{k}{l} \sum_{j \in \mathcal{N}_-} C(s + jk, k) \\ &\leq \; \frac{\alpha}{3} + \frac{k}{l}(\text{card } \mathcal{N}_+) + \frac{k}{l}(\text{card } \mathcal{N}_-)\frac{\alpha}{3} \\ &\leq \; \frac{\alpha}{3} + \frac{k}{l} l \sqrt{2\eta} + \frac{k}{l} n \frac{\alpha}{3} \\ &< \; \frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3} = \alpha, \end{aligned}$$

which is a contradiction. $\qquad\qquad\square$

3.5 The converse statement

We will begin with an observation which will be useful in the proof of the sufficiency of our regularity condition.

Lemma 3.5.1. *Let* $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ *be a non-quasinilpotent* C_0-*semigroup such that* $\kappa_T > 0$. *Then for every* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_+$ *there exists an* $x_0 > x$ *such that*

$$\frac{1}{n} \int_{x_0}^{x_0+n} \frac{r(T(s))^{-1}\|T(s)\|}{\sup_{y \in [x_0, x_0+n]} \|r(T(y))^{-1}T(y)\|} \, ds \geq \frac{\kappa_T}{2}.$$

Proof. We shall use an indirect argument: suppose that the statement is not true for some n and x. By the definition of κ_T, we can choose $m \in \mathbb{N}$ and $s_m \in \mathbb{R}_+$ such that $m^{-1}(xn^{-1} + 1) < \kappa_T/4$ and

$$I := \frac{1}{mn} \int_{s_m}^{s_m+mn} \frac{\|r(T(s))^{-1}T(s)\|}{\sup_{y \in [s_m, s_m+mn]} \|r(T(y))^{-1}T(y)\|} \, ds > \frac{3\kappa_T}{4}.$$

Let l stand for the integer part of $\max(0, (x - s_m)/n)$. If $k > l$ then $s_m + kn > x$. Thus

$$
\begin{aligned}
I &\leq \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{n} \int_{s_m+kn}^{s_m+(k+1)n} \frac{\|r(T(s))^{-1}T(s)\|}{\sup_{y \in [s_m+kn, s_m+(k+1)n]} \|r(T(y))^{-1}T(y)\|} \, ds \\
&\leq \frac{l+1}{m} + \frac{m-1-l}{m} \frac{\kappa_T}{2} \\
&\leq \frac{xn^{-1}+1}{m} + \frac{\kappa_T}{2} \\
&< \frac{\kappa_T}{4} + \frac{\kappa_T}{2} = \frac{3\kappa_T}{4}.
\end{aligned}
$$

This is a contradiction, so the lemma must be true. $\qquad\square$

We are now in a position to prove the remaining implication.

Proposition 3.5.2. *Let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup such that $\|T(s)\| \geq 1$ for every $s \in \mathbb{R}_+$ and $\kappa_T > 0$. Then T has a p-regular norm-function with some $p \in \mathcal{P}_{\mathrm{t}}$.*

Proof. Rescaling the semigroup, we may suppose again that $e^{\omega_0(T)s} = r(T(s)) = 1$ $(s \in \mathbb{R}_+)$. This means that for every $\omega \in (0, \infty)$ there exists an $M_\omega \in [1, \infty)$ such that $\|T(s)\| \leq M_\omega e^{\omega s}$ for all $s \in \mathbb{R}_+$. Then for every $n \in \mathbb{N}$ we can choose a $k_n \in \mathbb{N}$ such that $k_n > n$ and $\|T(t)\| \leq e^{t/n}$ whenever $t \in [k_n, \infty)$. Note that for any $s \in \mathbb{R}_+$

$$(3.13) \qquad \|T(t)\| \leq \|T(s)\| \|T(t-s)\| \leq \|T(s)\| e^{(t-s)/n} \text{ whenever } t \in [s + k_n, \infty).$$

Next, define an $l \in \mathbb{N}$ with the condition

$$(3.14) \qquad\qquad\qquad\qquad l > 4/\kappa_T.$$

We shall again use the notation $A(s, n)$, $B(s, n)$ introduced in the proof of Proposition 3.4.1. From the definition of κ_T we can choose an $s_1 \in \mathbb{R}_+$ such that $A(s_1, (l+1)k_1)/B(s_1, (l+1)k_1) \geq \kappa_T/2$. Assuming that $\{s_i\}_{i=1}^{n}$ $(n \in \mathbb{N})$ has been already defined, by Lemma 3.5.1 we can find an $s_{n+1} \in \mathbb{R}_+$ which satisfies the conditions

$$(3.15) \qquad\qquad s_{n+1} > s_n + n(l+1)(k_n + k_{n+1}) + n(n+1)$$

and $A(s_{n+1}, (l+1)k_{n+1})/B(s_{n+1}, (l+1)k_{n+1}) \geq \kappa_T/2$. By induction, we obtain a sequence $\{s_n\}_{n=1}^{\infty}$ such that $s_{n+1} > s_n + (l+1)k_n$ and

(3.16) $\qquad A(s_n, (l+1)k_n)/B(s_n, (l+1)k_n) \geq \kappa_T/2$ for all $n \in \mathbb{N}$.

Now the function $p \colon \mathbb{R}_+ \to [1, \infty)$ is defined in the following way. If $s_1 > 0$ then $p(t) := \sup\{\|T(s)\| : s \in [0, s_1)\}$ for every $t \in [0, s_1)$. Furthermore, for every $n \in \mathbb{N}$ we will set

(3.17) $\qquad p(t) := \begin{cases} B(s_n, (l+1)k_n) & \text{if } t \in [s_n, s_n + lk_n], \\ B(s_n, (l+1)k_n)e^{(t-s_n-lk_n)/n} & \text{if } t \in [s_n + lk_n, s_{n+1}). \end{cases}$

The locally bounded, measurable function p clearly dominates the norm function of T on the intervals $[0, s_1)$ and $[s_n, s_n + (l+1)k_n]$ $(n \in \mathbb{N})$. On the other hand, if $t \in (s_n + (l+1)k_n, s_{n+1})$ $(n \in \mathbb{N})$ then by (3.13) and (3.17) we have

$$\|T(t)\| \leq \|T(s_n + lk_n)\|e^{(t-s_n-lk_n)/n} \leq p(t).$$

Thus $\|T(t)\| \leq p(t)$ holds for all $t \in \mathbb{R}_+$. Applying the relations

$$A(s_n, lk_n) = \frac{l+1}{l}A(s_n, (l+1)k_n) - \frac{1}{l}A(s_n + lk_n, k_n)$$

and $B(s_n, (l+1)k_n) \geq B(s_n + lk_n, k_n)$, by (3.17), (3.16) and (3.14) we find that

$$\begin{aligned} \frac{1}{lk_n} \int_{s_n}^{s_n + lk_n} \frac{\|T(t)\|}{p(t)} \, dt &= \frac{A(s_n, lk_n)}{B(s_n, (l+1)k_n)} \\ &\geq \frac{A(s_n, (l+1)k_n)}{B(s_n, (l+1)k_n)} - \frac{1}{l}\frac{A(s_n + lk_n, k_n)}{B(s_n + lk_n, k_n)} \\ &\geq \frac{\kappa_T}{2} - \frac{1}{l} > \frac{\kappa_T}{4} \end{aligned}$$

is true for every $n \in \mathbb{N}$. Since $\{[0, lk_n]\}_n$ is a Følner sequence, we may infer from Proposition 3.1.7 that at-$\lim_s \|T(s)\|/p(s) = 0$ does not hold.

Now let $t, s \in \mathbb{R}_+$ be arbitrary. If $s_n \leq t \leq t + s < s_{n+1}$ holds for some $n \in \mathbb{N}$, then it immediately follows that $1 \leq p(t+s)/p(s) \leq e^{s/n}$, hence

(3.18) $\qquad \left| \frac{p(t+s)}{p(s)} - 1 \right| \leq e^{s/n} - 1.$

Let us next assume that

$$s_{n+1} - n \leq t < s_{n+1} \leq t + s \leq s_{n+1} + n$$

is valid for some $n \in \mathbb{N}$. Since $s_{n+1} - n > s_n + (l+1)k_n$ by (3.15), it follows from (3.17) that

$$(3.19) \quad p(t) = Q_n \exp\left(\frac{t - s_n - lk_n}{n}\right) \geq Q_n \exp\left(\frac{s_{n+1} - n - s_n - lk_n}{n}\right),$$

where $Q_n := B(s_n, (l+1)k_n)$. On the other hand, $s_{n+1} > s_n + lk_n + k_{n+1}$ is also true by (3.15), and so we infer by (3.13) that

$$
\begin{aligned}
\|T(u)\| &\leq \|T(s_n + lk_n)\| \exp\left(\frac{u - s_n - lk_n}{n+1}\right) \\
&\leq Q_n \exp\left(\frac{s_{n+1} + (l+1)k_{n+1} - s_n - lk_n}{n+1}\right)
\end{aligned}
$$

holds for every $u \in [s_{n+1}, s_{n+1} + (l+1)k_{n+1}]$. Thus

$$(3.20) \quad p(t+s) = Q_{n+1} \leq Q_n \exp\left(\frac{s_{n+1} + (l+1)k_{n+1} - s_n - lk_n}{n+1}\right).$$

Applying (3.15), (3.19) and (3.20), a short computation shows that $p(t+s) \leq p(t)$, hence

$$(3.21) \quad \left|\frac{p(t+s)}{p(s)} - 1\right| \leq 1.$$

Taking into account the fact that $s_{n+1} - s_n > n^2$ tends to infinity, the relations (3.18) and (3.21) imply that $\sup\{p(t+s)/p(t) : t \in \mathbb{R}_+, s \in K\}$ is finite for every compact subset K of $\mathbb{R}_+$. Furthermore, in view of Proposition 3.1.7 we may conclude that

$$\text{at-}\lim_t \left|\frac{p(t+s)}{p(t)} - 1\right| = 0 \quad \text{for every } s \in \mathbb{R}_+.$$

Therefore p is a topological gauge function and the C_0-semigroup T has regular norm behaviour with respect to p. $\qquad\square$

3.6 An example for a gauge function

Now we will show that condition (iii) is not a consequence of other properties in the definition of a topological gauge function. We will construct a locally bounded, measurable function $p\colon \mathbb{R}_+ \mapsto [1, \infty)$ such that for every $s \in \mathbb{R}_+$ p_s/p is bounded and at-$\lim_t |p(s+t)/p(t) - 1| = 0$, but $\sup_{s\in[0,1]} \sup_{t\in\mathbb{R}_+} p(s+t)/p(t)$ is infinite. Next, let us define a p in the following way:

$$p(t) := \begin{cases} 1 + \frac{1}{n}(t - 2^n) & \text{if } 2^n \le t < 2^n + n^2, \\ 1 & \text{if } 2^n + n^2 \le t < 2^n + n^2 + \frac{1}{n}, \\ 1 + n^2(t - 2^n - n^2 - \frac{1}{n}) & \text{if } 2^n + n^2 + \frac{1}{n} \le t \le 2^n + n^2 + \frac{2}{n}, \\ 1 & \text{if } 2^n + n^2 + \frac{2}{n} < t \le 2^{n+1}. \end{cases}$$

Evidently, the mapping $t \mapsto p(t + s)/p(t)$ is bounded for every $s \in \mathbb{R}_+$. Moreover, $\lim_{t \to \infty} p(t + s)/p(t) = 1$ for every $s \in [0, 1]$ except for the set $\cup_{n=1}^{\infty}[2^n + n^2 - 1, 2^n + n^2 + 1]$ which has a zero upper Banach density. Now it is easy to verify that at-$\lim_t |p(s + t)/p(t) - 1| = 0$, so $c_p(s) = 1$ ($s \in \mathbb{R}_+$). On the other hand,

$$\frac{p_{\frac{1}{n}}(2^n + n^2 + \frac{1}{n})}{p(2^n + n^2 + \frac{1}{n})} = n;$$

that is, $\sup_{s \in [0,1]} \sup_{t \in \mathbb{R}_+} p(s + t)/p(t) = \infty$.

This example suggests the following question.

Question 3.6.1. Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup. Let $p \colon \mathbb{R}_+ \to [1, \infty)$ be a measurable, locally bounded function such that p_s/p is bounded and topologically almost converges in the strong sense to a positive real number $c_p(s)$ for every $s \in \mathbb{R}_+$. Next, assume that $\|T(s)\| \le p(s)$ holds for every $s \in \mathbb{R}_+$, and that at-$\lim_s \|T(s)\|/p(s) = 0$ does not hold. Does it then follow that T has a topologically regular norm-function?

Chapter 4

A Katznelson–Tzafriri type result

4.1 Introduction

Let us consider a complex Banach space $\mathfrak{X}$ and let T be a power-bounded operator on $\mathfrak{X}$, i.e. $\sup_{n \geq 1} \|T^n\| < \infty$ holds. Then it is a simple matter to give a characterization of the norm stability of T (that is $\lim_{n \to \infty} \|T^n\| = 0$) via the spectral radius formula. Indeed, $r(T) < 1$ if and only if $\|T^n\|$ $(n \to \infty)$ tends to zero, or equivalently $\|T^n\|$ tends to zero exponentially; that is, there exists a $C > 0$ and $\lambda \in (0, 1)$ such that $\|T^n\| \leq C\lambda^n$ for every $n \in \mathbb{N}$. However, it is more interesting to give necessary and sufficient conditions for the uniform convergence

$$\lim_{n \to \infty} \|T^n Q\| = 0$$

with some bounded operator Q, as this is far from trivial. The case $Q = T - I$, which is of great importance because of its role in the uniform zero–two law (cf. [34]) and its connection with the Gelfand–Hille theorem (see [68]), was first characterized by J. Esterle [22]. Moreover, if the sequence $\|T^n - T^{n+1}\|$ tends to zero as $n \to \infty$, the convergence may be far from exponential. On the quantitative behaviour of this sequence we refer the reader to [55, Chapter 4] and [22].

We recall that the celebrated Katznelson–Tzafriri theorem asserts that if T is a power-bounded operator acting on a Banach space $\mathfrak{X}$, and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a power series with absolute convergent coefficients, which is of spectral synthesis with respect to the peripheral spectrum of T, then $\lim_{n \to \infty} \|T^n f(T)\| = 0$ (see [34]). In a Hilbert space setting a richer functional calculus can be defined for contractions due to von Neumann's inequality. We recall that the disk algebra $A(\mathbb{D})$ is the set of all holomorphic functions

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in the unit disc which extend continuously to the unit circle. Actually, the following was proved in [23].

Theorem 4.1.1. *Let T be a contraction in a Hilbert space and let $f \in A(\mathbb{D})$. Then $f(\lambda) = 0$ for every $\lambda \in \sigma(T) \cap \mathbb{T}$ if and only if $\lim_{n \to \infty} \|T^n f(T)\| = 0$.*

On the other hand, contractions even admit a more general H^∞ calculus on the unit disc. This fact was exploited by H. Bercovici, who proved the following ([9]).

Theorem 4.1.2. *Let T be a completely nonunitary contraction on a Hilbert space, and let f be a bounded holomorphic function on the unit disc. Then $\lim_{n \to \infty} \|T^n f(T)\| = 0$ holds if $\lim_{r \to 1} f(re^{i\theta}) = 0$ for every $e^{i\theta} \in \sigma(T) \cap \mathbb{T}$.*

An example shows that the converse implication of the theorem is not true (see [9]). The statement of Theorem 4.1.1 was recently extended by S. Mustafayev, who considered the Banach algebra $\mathcal{A}_T$ generated by a contraction T. In fact, he proved in [52, Theorem 3.7] that if T is a contraction on a Hilbert space such that its peripheral spectrum $\sigma(T) \cap \mathbb{T}$ has zero Lebesgue measure, then the Gelfand transform of $R \in \mathcal{A}(T)$ vanishes on $\sigma(T) \cap \mathbb{T}$ if and only if $\lim_{n \to \infty} \|T^n R\| = 0$.

Here we shall prove that the assumption of the Katznelson–Tzafriri theorem can be weakened in Hilbert spaces, and we shall present a characterization of the condition $\lim_{n \to \infty} \|T^n Q\| = 0$ if Q commutes with T, via an ergodic condition. We shall also prove that, if $f \in A^+(\mathbb{T})$ and $Q = f(T)$, the given condition is equivalent to the vanishing of f on the peripheral spectrum of T.

The proof shall partly follow Vũ's method ([64], [65]); that is, we will first verify convergence in the strong operator topology by reducing the problem to isometries. After that we shall complete the proof using some aspects of an ultrapower approach.

4.2 Preliminaries and the main result

Let $\mathcal{L}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Let $\sigma(T)$ stand for the spectrum of $T \in \mathcal{L}(\mathcal{X})$, and let I be the identity operator on $\mathcal{X}$. Next, let $A^+(\mathbb{T})$ denote the set of sums of power series on the unit circle $\mathbb{T}$ whose coefficients are absolutely convergent. Then, for each power-bounded operator T, a bounded functional calculus naturally arises which is defined by $f(T) := \sum_{k=0}^\infty a_k T^k \in \mathcal{L}(\mathcal{X})$, where $f(z) = \sum_{k=0}^\infty a_k z^k$ and $\sum_{k=0}^\infty |a_k| < \infty$.

Our main result will now be presented.

Theorem 4.2.1. *Let T be a power-bounded operator on a Hilbert space $\mathcal{H}$. If $Q \in \mathcal{L}(\mathcal{H})$ and $TQ = QT$, then the following statements are equivalent:*

(i) $\displaystyle \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k Q \right\| = 0$ *for every* $\lambda \in \sigma(T) \cap \mathbb{T}$,

(ii) $\lim_{n\to\infty} \|T^n Q\| = 0$.

Moreover, if $Q = f(T)$ for some $f \in A^+(\mathbb{T})$, then (i) and (ii) are equivalent to

(iii) $f(\lambda) = 0$ *for every* $\lambda \in \sigma(T) \cap \mathbb{T}$.

First we will prove a lemma which leads us to the ergodic condition of the above theorem.

Lemma 4.2.2. *Let T be a power-bounded operator on a complex Banach space $\mathcal{X}$ and let $f \in A^+(\mathbb{T})$. Then, for every $\lambda \in \mathbb{T}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda)I \right) \right\| = 0.$$

Proof. Using the Taylor expansion $f(z) = \sum_{m=0}^{\infty} a_m z^m$ of f, and changing the order of summation, we have

$$
\begin{aligned}
\frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda)I \right) &= \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(\sum_{m=0}^{\infty} a_m (T^m - \lambda^m I) \right) \\
&= \sum_{m=0}^{\infty} a_m \lambda^m \left(\frac{1}{n} \sum_{k=0}^{n-1} (\lambda^{-(k+m)} T^{k+m} - \lambda^{-k} T^k) \right).
\end{aligned}
$$

Next, pick an $\varepsilon > 0$ and let us choose an index $M \in \mathbb{N}$ such that $\sum_{m=M}^{\infty} |a_m| \leq \varepsilon$. Then we may infer that (with $\triangle$ denoting the symmetric difference of two sets)

$$
\begin{aligned}
\left\| \frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda)I \right) \right\| &\leq \sum_{m=0}^{M-1} |a_m| \left\| \frac{1}{n} \sum_{k=0}^{n-1} (\lambda^{-(k+m)} T^{k+m} - \lambda^{-k} T^k) \right\| \\
&\quad + \sum_{m=M}^{\infty} |a_m| \left\| \frac{1}{n} \sum_{k=0}^{n-1} (\lambda^{-(k+m)} T^{k+m} - \lambda^{-k} T^k) \right\| \\
&\leq \|f\|_1 L \max_{1 \leq m \leq M} \frac{\operatorname{card}(\{1, ..., n\} \triangle \{m+1, ..., m+n\})}{n} \\
&\quad + 2L\varepsilon \\
&\leq \|f\|_1 2LMn^{-1} + 2L\varepsilon < 4L\varepsilon
\end{aligned}
$$

provided n is sufficiently large, where $\|f\|_1 := \sum_{m=0}^{\infty} |a_m|$ and $L := \sup\{\|T^k\| : k = 0, 1, 2, \ldots\}$. Since ε was arbitrarily chosen, the lemma is proved. $\qquad\square$

The uniform ergodic theorem tells us that $\frac{1}{n} \sum_{k=0}^{n-1} T^k$ tends to zero in norm if and only if 1 is in the resolvent set of T (cf. [44, Theorem 2.7]). With this result, the following corollary of the above lemma is straightforward.

Corollary 4.2.3. *Let T be a power-bounded operator on a Banach space $\mathfrak{X}$ and $f \in A^+(\mathbb{T})$. Then, for each $\lambda \in \sigma(T) \cap \mathbb{T}$,*

$$f(\lambda) = 0 \ \ \text{if and only if} \ \ \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k f(T) \right\| = 0.$$

In the following lemma we shall prove the theorem for an arbitrary isometry. Recall that Douglas's extension theorem ([16]) states that any isometry V on a Banach space $\mathfrak{X}$ can be extended to a surjective isometry on a larger space. This result was already well known in the Hilbert space setting (cf [32]). Later, C.J.K. Batty and S. Yeates gave a different construction in [7] which preserves the structure of the original space in most cases (for instance, when $\mathfrak{X}$ is a Hilbert space or superreflexive). In addition, their construction makes it possible for one to define an extension of the commutant of V; they showed that there exists a unital isometric algebra homomorphism from the commutant of V into the commutant of its surjective extension [7, Proposition 3.5]. Then, under this homomorphism, the spectrum of an element of the commutant contains the spectrum of its image. (The reader should see Bercovici's paper [10] for a similar construction in the Hilbert space case.) Below, we shall make use of these results.

Lemma 4.2.4. *Let V be an isometry on a Hilbert space $\mathcal{H}$. Suppose that $Q \in \mathcal{L}(\mathcal{H})$ and $QV = VQ$. If*
$$\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} V^k Q \right\| = 0$$
holds for every $\lambda \in \sigma(V) \cap \mathbb{T}$, then $Q = 0$.

Proof. Applying the extension theorem of Batty and Yeates, throughout the proof we can assume, without loss of generality, that V is unitary.

It can easily be seen that $VQ^* = Q^*V$, hence the C^*-algebra $\mathcal{A}$ generated by V and QQ^* is commutative. Now let Σ stand for the Gelfand spectrum of $\mathcal{A}$ and let

$\Gamma_{\mathcal{A}}\colon \mathcal{A} \to C(\Sigma)$, $A \mapsto \widehat{A}$ be the Gelfand transform of $\mathcal{A}$. Then the ergodic condition $\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} \widehat{V}^k \widehat{QQ^*} \right\|_\infty = 0$ is satisfied for every λ in the range of $\widehat{V}$ because $\operatorname{ran} \widehat{V} = \sigma(V)$.

Next, let us assume that the non-negative function $\widehat{QQ^*} \in C(\Sigma)$ takes its maximum q_0 at $h_0 \in \Sigma$. Choosing $\lambda_0 := \widehat{V}(h_0)$, we get for any $n \in \mathbb{N}$ that

$$\frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda_0^{-k} \widehat{V}^k \widehat{QQ^*} \right\|_\infty \geq \frac{1}{n} \left| \sum_{k=0}^{n-1} \lambda_0^{-k} \widehat{V}(h_0)^k \, \widehat{QQ^*}(h_0) \right| = q_0.$$

This means that $q_0 = 0$ must hold, so $\widehat{QQ^*}(h) = 0$ for every $h \in \Sigma$. Hence $\sigma(QQ^*) = \{0\}$ and $QQ^* = 0$. This readily implies that $Q = 0$. $\qquad\square$

Let J be an infinite set, and $\mathcal{U}$ let be a free ultrafilter on J. Now consider the Banach spaces $\ell^\infty(J, \mathcal{X})$ of all bounded functions from J to $\mathcal{X}$, and $c_0(J, \mathcal{X}; \mathcal{U})$ of all bounded functions from J to $\mathcal{X}$ which converge to zero through the ultrafilter. The quotient space $\ell^\infty(J, \mathcal{X})/c_0(J, \mathcal{X}; \mathcal{U})$ is an ultrapower of the Banach space $\mathcal{X}$, which we shall denote by $\mathcal{X}_{\mathcal{U}}$. Recall that if $\mathcal{X}$ is Hilbert space then $\mathcal{X}_{\mathcal{U}}$ is also a Hilbert space. In fact, the parallelogram identity is preserved by taking the ultrapower and the polarization identity tells us that the inner product on $\mathcal{X}_{\mathcal{U}}$ is given by

$$\langle \bar{x}, \bar{y} \rangle_{\mathcal{U}} = \mathcal{U}\text{-}\lim_n \langle x_n, y_n \rangle,$$

where $\bar{x}$ and $\bar{y}$ denote the equivalence classes of $\{x_n\}_n$ and $\{y_n\}_n$ in $\mathcal{X}_{\mathcal{U}}$, respectively.

For each $T \in \mathcal{L}(\mathcal{X})$, the ultrapower $T_{\mathcal{U}}$ of T is defined by the formula

$$T_{\mathcal{U}}(\{x_n\} + c_0(J, \mathcal{X}; \mathcal{U})) := \{Tx_n\} + c_0(J, \mathcal{X}; \mathcal{U}).$$

Note that the mapping $T \mapsto T_{\mathcal{U}}$ is an isometric unital algebra homomorphism from $\mathcal{L}(\mathcal{X})$ into $\mathcal{L}(\mathcal{X}_{\mathcal{U}})$ such that $\sigma(T) = \sigma(T_{\mathcal{U}})$ (cf. [27], [60, Theorem V.1.4]).

Proof of Theorem 4.2.1. The implication (ii) $\implies$ (i) is evident. To prove (i) $\implies$ (ii), let us introduce a new semi-inner product on $\mathcal{H}$ by

$$\langle x, y \rangle_T := m(\{\langle T^n x, T^n y \rangle\}_n) \quad (x, y \in \mathcal{H}),$$

where m denotes a Banach limit. Forming quotient space and completion result a Hilbert space $\mathcal{H}_T$, where T acts as an isometry V. If X denotes the canonical embedding of $\mathcal{H}$ into $\mathcal{H}_T$, we get that $VX = XT$. In addition, the intertwining transformation X naturally induces a contractive unital algebra homomorphism between the commutants

of T and V. That is, if A is in $\{T\}'$, the commutant of T, then there exists a unique operator $B \in \{V\}'$ such that $XA = BX$ and $\|B\| \leq \|A\|$. It can be also shown that the mapping

$$\gamma_T \colon \{T\}' \to \{V\}', \quad A \mapsto B$$

is a contractive unital algebra homomorphism and hence $\sigma(B) \subseteq \sigma(A)$ ([35]). Then (i) implies that V and $\gamma_T(Q)$ satisfy the ergodic condition

$$\lim_{n \to \infty} n^{-1} \left\| \sum_{k=0}^{n-1} \lambda^{-k} V^k \gamma_T(Q) \right\| = 0$$

for every $\lambda \in \sigma(V) \cap \mathbb{T}$. We can apply now Lemma 2.4 to this and then conclude that $\gamma_T(Q) = 0$. Thus $m(\{\|QT^n x\|^2\}_n) = m(\{\|T^n Qx\|^2\}_n) = 0$ for every $x \in \mathcal{H}$, and hence $\inf_{n \geq 0} \|T^n Qx\| = 0$ by the positivity of m. Therefore, condition (i) in Theorem 4.2.1 implies that $\inf_{n \geq 0} \|T^n Qx\| = \lim_{n \to \infty} \|T^n Qx\| = 0$ for every $x \in \mathcal{H}$.

Now let $\alpha := \inf_{n \geq 0} \|T^n Q\|$ and $L := \sup_{n \geq 0} \|T^n\|$. Choose a sequence of unit vectors $\{x_k\}_{k=0}^\infty$ in $\mathcal{H}$ such that $\|T^k Qx_k\| \geq \alpha/2$ holds for every $k \in \mathbb{N}$. Then, of course, $\|T^n Qx_k\| \geq \alpha/(2L)$ for every $n \leq k$. Taking a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we can define a unit vector $\bar{x} = \{x_k\}_k + c_0(\mathbb{N}, \mathcal{H}; \mathcal{U})$ in the ultrapower space $\mathcal{H}_\mathcal{U}$. Then, for any $n \in \mathbb{N}$, we have

$$\|T_\mathcal{U}^n Q_\mathcal{U} \bar{x}\| = \|\{T^n Qx_k\}_k + c_0(\mathbb{N}, \mathcal{H}; \mathcal{U})\| = \mathcal{U}\text{-}\lim_k \|T^n Qx_k\|$$
$$\geq \liminf_k \|T^n Qx_k\| \geq \frac{\alpha}{2L}.$$

However, applying the previous conclusion to $T_\mathcal{U}$ and $Q_\mathcal{U}$ which satisfy (i), we find that $T_\mathcal{U}^n Q_\mathcal{U}$ tends to zero strongly. This means that α must be zero, hence $\lim_{n \to \infty} \|T^n Q\| = 0$ also holds which is we intended to show.

For (i) $\iff$ (iii), see Corollary 4.2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We should mention here that condition (i) implies the convergence

$$\lim_{n \to \infty} \|T^n Q\| = 0$$

but $\lim_{n \to \infty} \|T_{|(\operatorname{ran} Q)^-}^n\| = 0$ does not hold in general (here $T_{|(\operatorname{ran} Q)^-}$ denotes the restriction of T to $(\operatorname{ran} Q)^-$, the closure of the range of Q); see, for instance, [33, Example 1.8]. Actually, the structure of the orbits can be quite complicated on $(\operatorname{ran} Q)^-$, they may decay arbitrarily slowly due to a result by V. Müller [53]. More precisely, if $\sigma(T_{|(\operatorname{ran} Q)^-}) \cap \mathbb{T} \neq \emptyset$ (i.e. $\|T_{|(\operatorname{ran} Q)^-}^n\| \not\to 0$) then for every positive sequence $\{a_n\}_{n=1}^\infty \in c_0(\mathbb{N})$ we can find a vector $h \in (\operatorname{ran} Q)^-$ such that $\|T^n h\| \geq a_n$ for every $n \in \mathbb{N}$.

Remark 4.2.5. We note that the local version of Theorem 4.2.1 does not hold. In fact, let us consider the Hilbert space $\ell^2(\mathbb{N})$, and let T be the right forward shift on $\ell^2(\mathbb{N})$ and let Q be the identity operator. Choosing the unit vector $e_0 := (1, 0, 0, \ldots)$, it can be easily seen that

$$\lim_{n \to} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^k T^k e_0 \right\| = 0$$

for every $\lambda \in \mathbb{T}$, but $1 = \|T^n e_0\| \not\to 0$ if $n \to \infty$.

Remark 4.2.6. Note that Lemma 4.2.4 appears to be crucial in the proof of the above theorem. We do not know whether a similar statement could be verified in more general spaces. The extension theorem given by Batty and Yeates preserves the structure of superreflexive spaces, hence it seems to be natural to ask whether Lemma 4.2.4 can be generalized to superreflexive spaces. It is an open problem whether the ergodic result Theorem 4.2.1 is true in L^p-spaces ($1 < p < \infty$), or in superreflexive spaces.

Chapter 5

Summary

In essence, this dissertation is a study on the stability properties of strongly continuous operator semigroups. I provide generalizations of well-known stability results including the Arendt–Batty–Lyubich–Vũ theorem and the Katznelson–Tzafriri Theorem. Furthermore, I give a characterization of C_0-semigroups whose norm function is topologically regular. These results are based on three recent papers [42], [46] and [47].

5.1 Representations of locally compact commutative semigroups

This part of the thesis is based on a joint paper with László Kérchy.

Consider a locally compact, Hausdorff abelian group $(G; +)$. Let S be a closed subsemigroup of G with non-empty interior S° such that $S - S = G$ and $S \cap (-S) = \{0\}$. Let $S^\sharp$ stand for the set of characters of S. In Chapter 2, by using invariant means on S, we introduced the concept of gauge functions, representations with regular norm-function, and we also studied the properties of limit functionals. We proved the following.

Theorem. *Let p be a gauge function on S and let us assume that there exists a representation $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ with a p-regular norm-function. Then the limit functional c_p of p is a positive character of S.*

Theorem. *If the representation $\rho \colon S \to \mathcal{L}(\mathfrak{X})$ has regular norm-behaviour with respect to the gauge functions p and q, then*

$$c_p = c_q.$$

The above theorem leads to the following definition. The function $c_\rho := c_p$ is called the *limit functional of the representation ρ with p-regular norm-function.*

When $S = \mathbb{R}_+$, we proved that if $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ is a representation that has a regular norm-function then the spectral radius function of T is equal to c_T.

We recall that $C_c(S)$ stands for the set of continuous functions with compact support in S. The Fourier transform of a function $f \in C_c(S)$ with respect to the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ is given by

$$\widehat{f}(\rho) := \int_S f(s)\rho(s)\,d\mu(s).$$

We can similarly define $\widehat{f}(\chi)$ when $\chi \in S^\sharp$, since the characters of S are one-dimensional representations. In Chapter 2 we introduced different kinds of spectra of unbounded representations and studied connections among them, which provide an interplay with already known concepts.

The *peripheral spectrum* of the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ with regular norm-function is defined by

$$\sigma_{per}(\rho) := \{\, \chi \in \sigma(\rho) : |\chi| = c_\rho \text{ and } |\chi(s)| = c_\rho(s) \text{ for all } s \in S \,\}.$$

Let the *point spectrum* of the representation $\rho \colon S \to \mathcal{L}(\mathcal{X})$ be the set

$$\sigma_p(\rho) := \left\{ \chi \in S^\sharp : \text{ there exists } 0 \neq x \in \mathcal{X} \text{ with } \rho(s)x = \chi(s)x \text{ for all } s \in S \right\}.$$

The following statement is one of the main results of this dissertation, which is an Arendt–Batty–Lyubich–Vũ type theorem for representations having a regular norm-function. Instead of the usual stability of orbits, we shall apply a slightly weaker concept of stability that may be described by using almost convergence.

Theorem. *Let $\rho \colon S \to \mathcal{L}(\mathcal{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ is empty, then $\|\rho(s)x\|/p(s)$ almost converges to zero for all $x \in \mathcal{X}$.*

A corollary of the theorem is the following.

Corollary. *Let $\rho \colon S \to \mathcal{L}(\mathcal{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ is empty, then*

$$\lim_{i \to \infty} \frac{1}{\mu(K_i)} \int_{K_i} \frac{\|\rho(s)x\|}{p(s)}\,d\mu(s) = 0$$

is true for all $x \in \mathcal{X}$, where $\{K_i\}_i$ is any Følner sequence.

5.2 C_0-semigroups and topological regularity

In Chapter 3 we focused on the representations of the real half line. Here, similar to the previous chapter, we introduced C_0-semigroups with a topological norm-function. This can be achieved using the set of topologically invariant means instead of the translation invariant means.

The main result of this chapter provides a characterization of C_0-semigroups whose norm-function is topologically regular. The statement is similar to the characterization proved by Kérchy and Müller [38], [43] in the discrete case. Let us assume that $(T(s))_{s \geq 0}$ is a C_0-semigroup and the spectral radius of $T(s)$ is positive, i.e. $r(T(s)) > 0$, for every $s \in \mathbb{R}_+$. Let us define

$$\kappa_T := \inf_{n \in \mathbb{N}} \sup_{s \in \mathbb{R}_+} \left[\left(\frac{1}{n} \int_s^{s+n} r(T(t))^{-1} \|T(t)\| \, dt \right) \left(\sup_{s \leq y \leq s+n} r(T(y))^{-1} \|T(y)\| \right)^{-1} \right].$$

Let $\mathcal{P}_t$ denote the set of topological gauge functions with respect to T. The following theorem is the main result here.

Theorem. *Let $T \colon \mathbb{R}_+ \to \mathcal{L}(\mathfrak{X})$ be a C_0-semigroup. Then the following conditions are equivalent:*

  (i) *T has a p-regular norm-function with a topological gauge function $p \in \mathcal{P}_t$,*

 (ii) *T has a p-regular norm-function with a continuous gauge $p \in \mathcal{P}_t$,*

(iii) *$\|T(s)\| \geq 1$ for every $s \in \mathbb{R}_+$ and $\kappa_T > 0$.*

5.3 A Katznelson–Tzafriri type theorem in Hilbert spaces

The Katznelson–Tzafriri theorem is an operator-theoretic result which is related to the ABLV theorem. When $S = \mathbb{Z}_+$, we can present an extension of the result in the Hilbert space setting. Our aim is to characterize, via an ergodic condition, the norm convergence $\lim_{n \to \infty} \|T^n Q\| = 0$ when T is a power-bounded operator on a Hilbert space and Q commutes with T.

Let I be the identity operator on $\mathfrak{X}$. If $f \in A^+(\mathbb{T})$ and T is power-bounded operator, then a bounded functional calculus naturally arises which can be defined by $f(T) := \sum_{k=0}^{\infty} \widehat{f}(k) T^k \in \mathcal{L}(\mathfrak{X})$, where $f(\lambda) = \sum_{k=0}^{\infty} \widehat{f}(k) \lambda^k$ and $\sum_{k=0}^{\infty} |\widehat{f}(k)| < \infty$.

In Chapter 4, our starting point is an observation which leads us to introduce the ergodic condition in our generalization.

Lemma. *Let T be a power-bounded operator on a complex Banach space $\mathcal{X}$ and let $f \in A^+(\mathbb{T})$. Then, for every $\lambda \in \mathbb{T}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda)I \right) \right\| = 0.$$

An important corollary of this lemma is the following statement which provides a link between a scalar condition and an operator ergodic condition.

Corollary. *Let T be a power-bounded operator on a Banach space $\mathcal{X}$ and $f \in A^+(\mathbb{T})$. Then, for each $\lambda \in \sigma(T) \cap \mathbb{T}$,*

$$f(\lambda) = 0 \ \ if \ and \ only \ if \ \ \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k f(T) \right\| = 0.$$

The main result of Chapter 4 which applies to Hilbert spaces is the following:

Theorem. *Let T be a power-bounded operator on a Hilbert space $\mathcal{H}$. If $Q \in \mathcal{L}(\mathcal{H})$ and $TQ = QT$, then the following statements are equivalent:*

(i) $\displaystyle \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k Q \right\| = 0$ *for every $\lambda \in \sigma(T) \cap \mathbb{T}$,*

(ii) $\lim_{n\to\infty} \|T^n Q\| = 0$.

Moreover, if $Q = f(T)$ for some $f \in A^+(\mathbb{T})$, then (i) *and* (ii) *are equivalent to*

(iii) $f(\lambda) = 0$ *for every $\lambda \in \sigma(T) \cap \mathbb{T}$.*

Many extensions of the Katznelson–Tzafriri were proved in the discrete case as well as in the continuous case; see [8], [24], [33], [51], [52], [65] and [4], [13]. However, we recall that all former extensions of the Katznelson–Tzafriri theorem are related to bounded functional calculi of T or elements of the Banach algebra generated by T.

It remains an open problem whether this ergodic statement remains valid in more general spaces; for instance, in L^p-spaces ($1 < p < \infty$) or in superreflexive Banach spaces. It is also an open problem whether one can prove a similar statement for C_0-semigroups or more general representations.

Chapter 6

Összefoglalás

Disszertációmban erősen folytonos operátorfélcsoportok stabilitását vizsgáltam. Két nevezetes stabilitási tétel az Arendt–Batty–Lyubich–Vũ és a Katznelson–Tzafriri tétel egy-egy kiterjesztését adtam meg, illetve karakterizáltam azokat a C_0-félcsoportokat is, amelyek normafüggvénye ún. topologikus regularitást mutat. A disszertációmban szereplő eredmények [42], [46] és [47] publikációkon alapulnak.

6.1 Lokálisan kompakt, kommutatív félcsoportok reprezentációi

A disszertációm ezen része Kérchy Lászlóval közös cikken alapul.

Tegyük fel, hogy $(G; +)$ egy lokálisan kompakt, σ-kompakt, kommutatív csoport. Jelölje S egy olyan zárt részfélcsoportját G-nek, amelynek S° belseje nemüres, $S - S = G$ és $S \cap (-S) = \{0\}$ teljesülnek. A továbbiakban jelölje S karaktereinek a halmazát $S^\sharp$. A 2. fejezetben az S-en található invariáns közepeket használva bevezettük a normalizáló függvény fogalmát, értelmeztük a reguláris norma viselkedésű reprezentációkat, illetve vizsgáltuk a hozzájuk tartozó limeszfunkcionálok tulajdonságait.

Tétel. *Legyen p normalizáló függvény S-en és tegyük fel, hogy létezik egy p-reguláris normafüggvényű $\rho \colon S \to \mathcal{L}(\mathcal{X})$ reprezentáció. Ekkor a p normalizáló függvény c_p limeszfunkcionálja S egy pozitív karaktere.*

Tétel. *Ha a $\rho \colon S \to \mathcal{L}(\mathcal{X})$ reprezentáció reguláris norma viselkedésű a p és q normalizáló függvényekre nézve, akkor*

$$c_p = c_q.$$

Az előző tétel lehetővé teszi a következő definiciót. A $c_\rho := c_p$ függvényt, amely a normalizáló függvény megválasztásától független, a p-reguláris normafüggvényű ρ *reprezentáció limeszfunkcionáljának* nevezzük. Beláttuk az $S = \mathbb{R}_+$ esetben, hogy egy $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ reguláris norma viselkedésű reprezentáció spektrálsugárfüggvénye megegyezik T limeszfunkcionáljával, c_T-vel.

Legyen $f \in C_c(S)$ egy kompakt tartójú, folytonos függvény S-en. Az f függvény $\rho : S \to \mathcal{L}(\mathcal{X})$ reprezentáció szerinti Fourier transzformáltján az

$$\widehat{f}(\rho) := \int_S f(s)\rho(s)\, d\mu(s)$$

korlátos operátort értjük. Hasonlóan értelmezhetjük $\widehat{f}(\chi)$-t tetszőleges $\chi \in S^\sharp$-re is, hiszen S karakterei egy-dimenziós reprezentációk.

A 2. fejezetben bevezettük a nemkorlátos reprezentációk különféle spektrumfogalmait, részletesen megvizsgáltuk a köztük lévő kapcsolatot, illetve, hogy milyen rokonságban állnak már ismert spektrumokkal.

A ρ reguláris normafüggvényű reprezentáció *periferális spektrumán* a

$$\sigma(\rho)_{\text{per}} := \left\{ \chi \in S^\sharp : |\chi| = c_\rho \text{ és } |\widehat{f}(\chi)| \leq \|\widehat{f}(\rho)\| \text{ minden } f \in C_c(S)\text{-re} \right\}$$

halmazt értjük. Általában egy $\rho : S \to \mathcal{L}(\mathcal{X})$ reprezentáció pontspektruma a

$$\sigma_{\text{p}}(\rho) := \left\{ \chi \in S^\sharp : \text{ létezik } 0 \neq x \in \mathcal{X} \text{ hogy } \rho(s)x = \chi(s)x \text{ minden } s \in S\text{-re} \right\}$$

halmaz.

A következő állítás a disszertáció egyik fő eredménye, amely egy Arendt–Batty–Lyubich–Vũ típusú tétel reguláris normafüggvényű reprezentációkra. A pályák szokásos stabilitása helyett egy gyengébb típusú stabilitást kapunk eredményül, amelyet az ún. *majdnem konvergencia* fogalmával írhatunk le.

Tétel. *Legyen a $\rho : S \to \mathcal{L}(\mathcal{X})$ reprezentáció p-reguláris normafüggvényű. Ha $\sigma_{\text{per}}(\rho)$ megszámlálható és $\sigma_{\text{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ üres, akkor $\|\rho(s)x\|/p(s)$ majdnem konvergál zéróba minden $x \in \mathcal{X}$-re.*

A tétel következménye az alábbi állítás.

Következmény. *Legyen a* $\rho\colon S \to \mathcal{L}(\mathcal{X})$ *reprezentáció p-reguláris norma-viselkedésű. Ha* $\sigma_{\mathrm{per}}(\rho)$ *megszámlálható és* $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ *üres, akkor*

$$\lim_{i\to\infty} \frac{1}{\mu(K_i)} \int_{K_i} \frac{\|\rho(s)x\|}{p(s)}\, d\mu(s) = 0$$

igaz minden $x \in \mathcal{X}$*-re, ahol* $\{K_i\}_i$ *tetszőleges Følner sorozat.*

6.2 C_0-félcsoportok és topologikus regularitás

A disszertáció 3. fejezetében a valós félegyenes reprezentációit vizsgáltuk meg részletesebben. Itt az előző fejezet mintájára értelmeztük azokat a C_0-félcsoportokat, amelyek normafüggvénye topologikusan reguláris. Ez hasonlóan történt, mint a korábbi regularitás esetében; a fő különbség, hogy definíciónkban az invariáns közepek halmaza helyett egy szűkebb halmazt használtunk, a topologikusan invariáns közepeket.

A 3. fejezet fő eredménye megadja a topologikusan reguláris normafüggvénnyel rendelkező C_0-félcsoportok karakterizációját, hasonlóan a Kérchytől és Müllertől származó eredményhez [38], [43] a diszkrét esetben. Tegyük fel, hogy $(T(s))_{s\geq 0}$ egy olyan C_0-félcsoport, ahol $T(s)$ spektrálsugara pozitív, $r(T(s)) > 0$, minden $s \in \mathbb{R}_+$-re. Legyen

$$\kappa_T := \inf_{n\in\mathbb{N}} \sup_{s\in\mathbb{R}_+} \left[\left(\frac{1}{n} \int_s^{s+n} r(T(t))^{-1}\|T(t)\|\, dt \right) \left(\sup_{s\leq y\leq s+n} r(T(y))^{-1}\|T(y)\| \right)^{-1} \right].$$

Jelölje a T-re nézve topologikus normalizáló függvények halmazát $\mathcal{P}_{\mathrm{t}}$. A következő tétel itt a fő eredményünk.

Tétel. *Tekintsünk egy* $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ C_0*-félcsoportot. Ekkor az alábbi állítások ekvivalensek:*

(i) *létezik olyan* $p \in \mathcal{P}_{\mathrm{t}}$, *amelyre* T *normafüggvénye p-reguláris;*

(ii) *létezik olyan folytonos* $p \in \mathcal{P}_{\mathrm{t}}$, *amelyre* T *normafüggvénye p-reguláris;*

(iii) $\|T(s)\| \geq 1$ *minden* $s \in \mathbb{R}_+$*-re és* $\kappa_T > 0$.

6.3 Katznelson–Tzafriri típusú tétel Hilbert tereken

Az ABLV tétellel rokonságot mutató operátorelméleti állítás a Katznelson–Tzafriri tétel. Az $S = \mathbb{Z}_+$ esetben megadtuk a tétel egy kiterjesztését Hilbert tereken. Célunk az volt, hogy karaketrizáljuk a $\lim_{n \to \infty} \|T^n Q\| = 0$ konvergenciát, ha T Hilbert térbeli hatványkorlátos operátor és Q kommutál T-vel.

Jelölje I az identikus leképezést $\mathcal{X}$-en. Ha $f \in A^+(\mathbb{T})$ és $T \in \mathcal{L}(\mathcal{X})$ hatványkorlátos operátor, akkor értelmezhető $f(T) := \sum_{k=0}^{\infty} \widehat{f}(k) T^k \in \mathcal{L}(\mathcal{X})$, ahol $f(\lambda) = \sum_{k=0}^{\infty} \widehat{f}(k) \lambda^k$ és $\sum_{k=0}^{\infty} |\widehat{f}(k)| < \infty$. A 4. fejezet kiindulási pontja az alábbi észrevétel, amelyet egy lemmában fogalmazunk meg.

Lemma. *Legyen T hatványkorlátos operátor az $\mathcal{X}$ komplex Banach téren és legyen $f \in A^+(\mathbb{T})$. Ekkor minden $\lambda \in \mathbb{T}$ esetén a következő teljesül*

$$\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda) I \right) \right\| = 0.$$

A lemma fontos következménye az alábbi állítás.

Következmény. *Legyen T hatványkorlátos operátor az $\mathcal{X}$ Banach téren és $f \in A^+(\mathbb{T})$. Ekkor tetszőleges $\lambda \in \sigma(T) \cap \mathbb{T}$ esetén,*

$$f(\lambda) = 0 \quad akkor \ és \ csak \ akkor, \ ha \quad \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k f(T) \right\| = 0.$$

A 4. fejezet fő eredménye, amely Hilbert terekre vonatkozik a következő:

Tétel. *Legyen T hatványkorlátos operátor a $\mathcal{H}$ Hilbert téren. Ha $Q \in \mathcal{L}(\mathcal{H})$ és $TQ = QT$, akkor a következő állítások ekvivalensek:*

(i) $\lim_{n \to \infty} \dfrac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k Q \right\| = 0$ *minden $\lambda \in \sigma(T) \cap \mathbb{T}$-re,*

(ii) $\lim_{n \to \infty} \|T^n Q\| = 0.$

Továbbá, ha $Q = f(T)$ valamely $f \in A^+(\mathbb{T})$-re, akkor (i) és (ii) ekvivalens az alábbival

(iii) $f(\lambda) = 0$ *minden $\lambda \in \sigma(T) \cap \mathbb{T}$-re.*

A Katznelson–Tzafriri tételnek számos általánosítása született diszkrét és folytonos reprezentációkra, ld. [8], [24], [33], [51], [52], [65] és [13], [4]. Megjegyezzük azonban, hogy ezek az általánosítások mind T egy függvénykalkulusához kapcsolódnak vagy a T által generált Banach algebrához.

Nyitott kérdés maradt, hogy az előző ergodikus állításunk érvényes-e L^p-terekben $(1 < p < \infty)$, vagy általánosabban, szuperreflexív Banach terekben. További nyitott probléma, hogy milyen hasonló állítás bizonyítható C_0-félcsoportokra, illetve általánosabb reprezentációkra.

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