Ph.D. Thesis

Combinatorial completely 0-simple semigroups and free spectra

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Introduction

The notion of "term" – more specifically that of "word" – has always played an important role in algebraic studies. Terms are, for example, Boolean expressions, polynomials with integer coefficients, a sequence of variables and inverses of variables. If we consider all evaluations of a term over an algebra then we get a function – for an n-ary term we have an n-ary function – over the algebra. These functions are called *term operations.* For example, the Boolean functions are the term operations of Boolean expressions over the two element Boolean algebra. We consider two terms *equivalent* over an algebra if there does not exist an assignment of values to variables such that the value of the two terms are different, i.e., they determine the same term operation over the algebra. The number of term operations of an algebra is an important atribute of the algebra. For example, it is known that every function, i.e., every Boolean function is a term operation of the two element Boolean algebra. Similarly, it is well known that if we consider a finite field and we are allowed to use all elments as constants, i.e., new 0-ary operations in addition to the basic operations then every function is a term operation of this algebra. It is obvious that every function is a term operation over a k-element algebra if and only if there are k^{k^n} many n-ary term operations.

A class of algebras satisfies an identity if the terms at the two side of the identity are equivalent over every algebras of the class, shortly, the two terms are equivalent over the class. A class defined by identities is called a *variety.* Varieties are, for example, the class of all groups, the class of Abelian groups, the class of Boolean algebras, the class of rings, etc. It is known that for every variety $\mathcal{V}$ and all set X there exists the so called *free algebra in $\mathcal{V}$ generated by X* and it is unique up to isomorphism. It is the most general algebra in the sence that every algera in $\mathcal{V}$ generated by X is a homomorphic image of the free algebra. A model of the free algebra generated by X in $\mathcal{V}$ is $\mathbf{F}_{\mathcal{V}}(X)$, its elements are the equivalence classes of the terms over $\mathcal{V}$ with the basic operations induced by the basic operations of the terms in a natural way. If $|X| = n$ $(n \in \mathbb{N}_0)$ then we write $\mathbf{F}_{\mathcal{V}}(n)$ instead of $\mathbf{F}_{\mathcal{V}}(X)$. Consequently, $|\mathbf{F}_{\mathcal{V}}(n)|$ is the number of pairwise non-equivalent terms. We are interested in such varieties $\mathcal{V}$, where $\mathbf{F}_{\mathcal{V}}(n)$ is finite for every n, i.e., every finitely generated algebra is finite in $\mathcal{V}$. The free spectrum of a variety $\mathcal{V}$ is the sequence of cardinalities $|\mathbf{F}_{\mathcal{V}}(n)|$, $n = 0, 1, 2, \ldots$. For example, the free spectrum of Boolean algebras is $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$.

Within the above bounds on the free spectrum what are the possible numbers? The so called gap theorems try to answer this question. For example, Theorem 12.2 in [HM] says: Let $\mathcal{V}$ be a finitely generated variety. Then either $|\mathbf{F}_{\mathcal{V}}(n)| \leq cn^k$ for some finite constants c and k, or else $|\mathbf{F}_{\mathcal{V}}(n)| \geq 2^{n-k}$ for some positive integer k and for all n.

If the variety $\mathcal{V}$ is finitely generated, i.e., $\mathcal{V}$ is generated by a finite algebra $\mathbf{A}$ then it is easy to see that $|\mathbf{F}_{\mathcal{V}}(n)|$ is the number of term operations over $\mathbf{A}$. Specifically , if $\mathbf{A}$ is a k-element algebra then $|\mathbf{F}_{\mathcal{V}}(n)| \leq k^{k^n}$. On the other hand, if $k \geq 2$ then $|\mathbf{F}_{\mathcal{V}}(n)| \geq n$. Another approach is used by J. Berman in [Be], where he gave the characterization of possible free spectra of varieties generated by a simple algerba using tame congruence theory.

For finitely generated varieties there is a strong connection between structural properties of the generating algebra and the free spectra of the variety. If $\mathbf{G}$ is a finite group then the size of the n-generated relatively free group in the variety generated by $\mathbf{G}$ is exponential in n if $\mathbf{G}$ is nilpotent and doubly exponential if $\mathbf{G}$ is not nilpotent ([Hi] and [Ne]).

A term is called *essentially n-ary* if it depends on all of its variables. Denote by $p_n(\mathbf{A})$ the number of essentially n-ary term operations over the algebra $\mathbf{A}$. The $p_n(\mathbf{A})$ $(n \in \mathbb{N}_0)$ sequence is called the p_n *sequence* of the algebra $\mathbf{A}$. There is a strong connection between the p_n sequence of an algebra $\mathbf{A}$ and the free spectrum of the variety $\mathcal{V}$ generated by the algebra $\mathbf{A}$, because $|\mathbf{F}_{\mathcal{V}}(n)| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$ holds for all $n \in \mathbb{N}_0$. The properities of p_n sequences of semigroups were examined in the following papers. In [CDR1] and [CR] a full description of finite semigroups is presented for which the p_n sequence is bounded by a polynomial (which means there exists an integer k, such that $p_n \leq n^k$). Semigroups with bounded p_n sequences are described first in terms of identities and then structurally as nilpotent extensions of semilattices, Boolean groups and rectangular bands in [CDR2].

Csaba Szabó and the author systematically analyzed the free spectra of the semigroup varieties and the present Ph.D. thesis contains these results. Completely 0-simple semigroups are basic blocks for semigroups, similarly, as simple groups are building blocks for groups. There are nine varieties generated by combinatorial completely 0-simple semigroups, we estimate the free spectra for all of them.

In order to make our arguments more expressive, we assign certain graphs to semigroup terms in the following way. In the Rees-representation the elements of a combinatorial completely 0-simple semigroup are represented as pairs of elements, therefore we assign a bipartite graph to terms of such an algebra. The connected components of these bipartite graphs induce partitions on the vertex sets, and we use the number of these partitions to estimate the free spectrum of the so called five element combinatorial Brandt semigroup. Only in one out of the nine varieties generated by comibnatorial completely 0-simple semigroups we assign directed graphs instead of biparted graphs to semigroup terms, in which case we estimate the number of closed Eurlerian walks in directed graphs. Our results are summarized in Table 1, where $f(n) \sim g(n)$ denotes that $\lim_{n\to\infty} f(n)/g(n) = 1$ holds and $f(n) \sim_{\log} g(n)$ means that $\log f(n) \sim \log g(n)$.

The results of this thesis are published in [KSz1], [KSz2] and [KSz3]. Inspired by these papers the following results were obtained by others. S. Seif [Se] proved that, for a variety $\mathcal{V}$ generated by a non-orthodox monoid, $\log |\mathbf{F}_{\mathcal{V}}(n)|$ is exponential (as a function of n). I. Dolinka [Do] gave a Higman-Neuman type condition if the free spectrum of a class of semigroups is not log-exponential. In [PW], [PSz] and in [Pl] p_n sequences of band varieties are examined.

Preliminaries

A semigroup is called 0-*simple* if it has a zero element 0, is different from the two-element semigroup with zero multiplication, and its only ideals are $\{0\}$ and the whole semigroup. Note that the term simple has different meanings in universal algebra and semigroup theory. In universal algebra, an algebra is called simple if it has no nontrivial congruences. In semigroup theory, a semigroup of this kind is called *congruence-free*, and a semigroup is termed *simple* if it has no proper ideals. The notion of a 0-simple semigroup is the analogue of the latter notion within the class of semigroups with zero. Notice that a group considered as a semigroup has no proper ideal, hence every group with 0 adjoined is 0-simple, but not all 0-simple semigroups arise in this way. Every finite 0-simple semigroup is completely 0-simple, so every congruence-free semigroup is completely 0-simple within the class of finite semigroups with zero, different from the two-element null-semigroup. A semigroup is called *combinatorial* if it does not contain a nontrivial group as a subsemigroup.

We use the following representation of combinatorial completely 0-simple semigroups from [Ree]. Let Λ and I be nonempty sets, and let $M = (m_{\lambda,i})$ be a Λ by I matrix with 0 and 1 entries such that each row and column contains at least one nonzero element.

We define an operation on the set $(I \times \Lambda) \cup \{0\}$ as follows:

$$(i,\lambda)(j,\mu) = \begin{cases} (i,\mu) & \text{if } m_{\lambda,j} = 1 \\ 0 & \text{if } m_{\lambda,j} = 0, \end{cases}$$
$$(i,\lambda)0 = 0(i,\lambda) = 00 = 0.$$

This way, we get a combinatorial semigroup which is called the combinatorial Rees matrix semigroup. We can interpret a Rees matrix semigroup as a semigroup of matrices. Assign the $I \times \Lambda$ type zero matrix to the 0 element and to an element (i, λ) the $I \times \Lambda$ type matix with exactly one non-zero element in the ith row and λth column. It is easy to check that this set of matrices with the operation $A \circ B = AMB$, where the later is the usual matrix product, is a semigroup isomorphic to the original semigroup. The matrix M is called a *sandwich matrix*. The importance of the combinatorial Rees matrix semigroups is the following: the combinatorial completely 0-simple semigroups are exactly the combinatorial Rees matrix semigroups up to isomorphism.

N. Reilly described the lattice of varieties generated by completely 0-simple semigroups in [Rei]. In particular, he proved that there are exactly 9 combinatorial varieties among them (see 1st column of Table 1). Each of these varieties can be generated by one finite 0-simple semigroup (see 2nd column of Table 1), but the choice of which is in general not unique.

In [SSz1] and [SSz2] Cs. Szabó and S. Seif have investigated the term equivalence problem over completely 0-simple semigroups. Their analysis distinguishes two cases depending on whether the sandwich matrix M of the semigroup (see the 3rd column in Table 1) is a so called 1-block matrix or not. Furthermore, the order of variables has an important role in the evaluation of a semigroup term.

Variety	generating semigroup	sandwich matrix (M)	free spectra of variety
$\mathcal{SL}$	$\mathbf{Y}$	$[1]$	$2^n - 1$
$\mathcal{LNB}$	$\mathbf{L}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$n2^{n-1}$
$\mathcal{RNB}$	$\mathbf{R}$	$\begin{bmatrix}1 & 1\end{bmatrix}$	$n2^{n-1}$
$\mathcal{NB}$	$\mathbf{N}$	$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$	$n(n+1)2^{n-2}$
$\mathcal{B}$	$\mathbf{B_2}$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{LNB}_2$	$\mathbf{L_2}$	$\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{RNB}_2$	$\mathbf{R_2}$	$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{NB}_2$	$\mathbf{N_2}$	$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix}$	$\sim_{\log} n^{2n}$
$\mathcal{A}$	$\mathbf{A_2}$	$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$	$\sim n^2 2^{n^2}$

Table 1

In the Rees-representation the elements of a combinatorial completely 0-simple semigroup are represented as pairs of elements, therefore we assign a bipartite graph to terms of such an algebra. The bipartite graph $G(t)$ assigned to the n-ary term $t = t(x_1, x_2, \ldots, x_n)$ is defined as follows. Let the top vertices of $G(t)$ be $u_1, \ldots, u_n$ and the bottom vertices $v_1, \ldots, v_n$. There is an edge between v_i and u_j if x_j follows x_i somewhere in t, that is, if $x_i x_j$ is a subterm of t. The *components* of the graph in Figure 1 are the

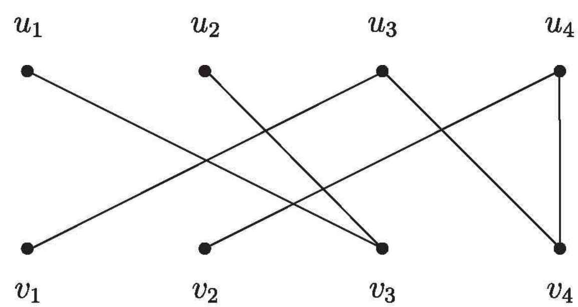

Figure 1:
$t = x_1x_3x_2x_4x_4x_3x_1x_3x_2$

sets $\{u_1, u_2, v_3\}$ and $\{u_3, u_4, v_1, v_2, v_4\}$ of vertices . Note that the notion of component is not used in the usual way as it contains only vertices and no edges. The components of a bipartite graph *induce a pair of partitions* on the sets of vertices. For example in the Figure 1 they induce the partition $\{\{u_1, u_2\}, \{u_3, u_4\}\}$ on the top vertices and $\{\{v_1, v_2, v_4\}, \{v_3\}\}$ on the bottom vertices.

$\mathbf{A_2}$ is the only semigroup in Table 1 whose sandwich matrix is not a 1-block matrix. In this case we can assign a directed graph to a term in a natural way. The directed graph contains as many vertices as there are variables in the term. If $x_i x_j$ is a subterm then we draw an edge from the vertex corresponding to x_i to the vertex corresponding to x_j.

Free spectra of varieties generated by combinatorial completely 0-simple semigroups

The varieties in the first four rows of Table 1 are varieties of bands (i.e., semigroups in which every element is idempotent), their names are: variety of semilattices, left normal bands, right normal bands, and normal bands, respectively. The free spectra of these varieties are well known.

The variety $\mathcal{B}$ is generated by $\mathbf{B_2}$, the so called five element combinatorial Brandt semigroup. We proved in [KSz1] that we can assign a pair of partitions of the same size to a term t, such that $t^{\mathbf{B_2}}$ is an essentially n-ary term operation and the corresponding bipartite graph $G(t)$ induces these partitions on the sets of vertices. We get an asymptotic formula for the p_n sequence of $\mathbf{B_2}$ by estimating the number of induced partitions.

Theorem 4.8. [KSz1] *Let $p_n = p_n(\mathbf{B_2})$ denote the number of essentially n-ary term operations of $\mathbf{B_2}$. Then*

$$\log p_n \sim 2n \log n.$$

Using the previous result we get the following theorem for the free spectrum of $\mathcal{B}$.

Theorem 4.9. [KSz1]

$$\log |\mathbf{F}_{\mathcal{B}}(n)| \sim 2n \log n.$$

The following proposition describes the connections between the p_n sequences of the $\mathbf{N_2}$, $\mathbf{L_2}$, $\mathbf{R_2}$ semigroups and the $p_n(\mathbf{B_2})$ sequence.

Proposition 4.10. [KSz3] *The number of essentially n-ary term operations for the algebras $\mathbf{N_2}$, $\mathbf{L_2}$, $\mathbf{R_2}$ satisfy the following inequalities:*

$$p_n(\mathbf{B_2}) \leq p_n(\mathbf{L_2}) = p_n(\mathbf{R_2}) \leq p_n(\mathbf{N_2}) \leq n^2 p_n(\mathbf{B_2}).$$

Using the previous two results we have obtained the following theorem for the free spectra.

Theorem 4.11. [KSz3] *Let $\mathcal{V}$ denote one of the varieties $\mathcal{LNB}_2$, $\mathcal{RNB}_2$, $\mathcal{NB}_2$. Then*

$$\log |\mathbf{F}_{\mathcal{V}}(n)| \sim 2n \log n.$$

There is a strong connection between essentially n-ary term operations over $\mathbf{A_2}$ and directed graphs with n vertices containing a closed Eulerian walk, so we used the next proposition.

Proposition 4.13. [KSz2] *Let $D(n)$ denote the number of directed graphs on n vertices with a closed Eulerian walk. Then $D(n) = o(2^{n^2})$.*

Now we give an asymptotic estimation for the free spectrum of $\mathcal{A}$.

Theorem 4.14. [KSz2]

$$|\mathbf{F}_{\mathcal{A}}(n)| \sim n^2 2^{n^2}.$$

The 4th column of Table 1 contains the summary of our results, that is we gave approximations of the free spectra of all the varieties generated by combinatorial completely 0-simple semigroups.

The approximations of the free spectra of the varieties $\mathcal{B}$, $\mathcal{LNB}_2$, $\mathcal{RNB}_2$ and $\mathcal{NB}_2$ coincide, and the relations $\mathcal{B} < \mathcal{LNB}_2 < \mathcal{NB}_2$ and $\mathcal{B} < \mathcal{RNB}_2 < \mathcal{NB}_2$ hold. We can give better bounds for the p_n sequences via analizing the size of the partitions induced on the sets of vertices of the bipartite graph. In the following proposition we give better lower and upper bounds for $p_n(\mathbf{B_2})$.

Proposition 4.16. [KSz3] *The following inequalities hold for the*

p_n sequence of $\mathbf{B_2}$

$$\sum_{k=1}^{n} k^2 S(n,k)^2 + 2\sum_{k=1}^{n-1} n(k-1)S(n-1,k)^2 + \sum_{k=1}^{n-1} n(n-1)S(n-2,k)^2 \leq$$

$$\leq p_n(\mathbf{B_2}) \leq$$

$$\leq \sum_{k=1}^{n} k^2 k! S(n,k)^2 + 2\sum_{k=1}^{n-1} n(k-1)(k-1)! S(n-1,k)S(n,k) +$$

$$+ \sum_{k=1}^{n-1} n(n-1)k! S(n-1,k)^2.$$

We get similar bounds for the p_n sequence of $\mathbf{L_2}$, $\mathbf{R_2}$ and $\mathbf{N_2}$ too, where term equivalence depends on the choice of the first and last variables, thus we get different coefficients.

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