Orthodox semigroups and semidirect products

thesis of PhD dissertation

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A mathematical modelling of partial symmetries appearing in the nature may be done by partial bijections. The partial bijections of a given set constitutes a semigroup. Inverse semigroups are abstract counterparts of subsemigroups of such semigroups closed under taking inverses. Semilattices (inverse semigroups containing only idempotents) and groups (inverse semigroups with a unique idempotent) play an important role in investigations of inverse semigroups. A semilattice can be assigned to every inverse semigroup very naturally, because the idempotents of an inverse semigroup form a semilattice. In the sequel we denote the set of idempotents of an arbitrary semigroup S by $E(S)$. Furthermore, inverse semigroups always have a smallest congruence, denoted by σ, such that the factor semigroup modulo this congruence is a group. This group is called the greatest group homomorphic image of the inverse semigroup. Another possibility of assigning a group to an inverse semigroup arises if the semigroup is a monoid. Namely, if M is an arbitrary monoid, the so-called $\mathcal{H}$-class of the identity forms a subgroup, which is denoted by $U(M)$ in the sequel. On the other hand, there are several ways to produce inverse semigroups from semilattices and groups. One of these ways is taking a semidirect product of a semilattice by a group.

The aim of this dissertation is to generalize some results connecting inverse semigroups and semidirect products to a wider class of semigroups, namely, to the class of orthodox semigroups. The idempotents of an orthodox semigroup need not commute, so they form a band instead of a semilattice. This is why we try to construct orthodox semigroups by taking semidirect products of bands by groups. In the sequel, we call a semidirect product of a band by a group simply a semidirect product. Of course, if we deal with inverse semigroups, then by a semidirect product, we mean a semidirect product of a semilattice by a group. If B is a band, and G is a group acting on B then we denote by $B * G$ the semidirect product of B by G.

Preliminaries - inverse semigroups

The notion of an E-unitary inverse semigroup has been introduced by D. B. McAlister. These semigroups play an important role in the theory of inverse semigroups. One of the reasons is that several natural examples of inverse semigroups are E-unitary (i. e. free inverse semigroups). Another reason is that every inverse semigroup is an idempotent separating homomorphic image of an E-unitary inverse semigroup as the following theorem of D. B. McAlister reveals.

Theorem 2.5. [McA1] *Every inverse semigroup has an E-unitary cover.*

The semidirect product-like structure of E-unitary inverse semigroups was described by D. B. McAlister in the aforementioned article. By making use of this result, L. O'Carroll has proved the following theorem.

Theorem 2.6. [OCa] *Every E-unitary inverse semigroup is embeddable into a semidirect product of a semilattice by a group.*

As a consequence of the previous two theorems, every inverse semigroup is an idempotent separating homomorphic image of a subsemigroup of a semidirect product.

S. Y. Chen and H. S. Hsieh, by generalizing the notion of factorizibility of rings, have introduced the notion of a factorizable inverse monoid, and have proved the following theorem.

Theorem 2.7. [CH] *Every inverse semigroup is embeddable into a factorizable inverse monoid.*

D. B. McAlister [McA2] has proved indirectly that factorizable inverse monoids are exactly the (idempotent separating) homomorphic images of monoid semidirect products. This result together with the previous one implies that every inverse semigroup is embeddable into an idempotent separating homomorphic image of a semidirect product. Such embeddings always give rise to E-unitary covers. The following theorem by D. B. McAlister and N. R. Reilly shows that this way every E-unitary cover arises.

Theorem 2.8. [MR] *Let S be an inverse semigroup, and let $\iota\colon S \to M$ be an embedding of S into a factorizable inverse monoid M. Then the subsemigroup*

$$\{(s, g) \in S \times U(M) : s\iota \leq g\})$$

of the direct product $S \times U(M)$ is an E-unitary cover of S. Conversely, every E-unitary cover of S is isomorphic to a semigroup of this kind.

D. B. McAlister [McA2] has introduced the notion of almost factorizable inverse semigroups as well (under the name of covering semigroups), and he has shown that almost factorizable inverse semigroups are just the idempotent separating homomorphic images of semidirect products. The connection between almost factorizable inverse semigroups and factorizable inverse monoids is revealed in the following theorem by M. V. Lawson.

Theorem 2.12. [La2] *If M is a factorizable inverse monoid then $M \setminus U(M)$ is an almost factorizable inverse semigroup. Conversely, every almost factorizable inverse semigroup arises in this way from a factorizable inverse monoid.*

Preliminaries - orthodox semigroups

Some of the results formulated for inverse semigroups were generalized in the 1980's. The following theorem, generalizing Theorem 2.5, have been proved independently by M. B. Szendrei and K. Takizawa.

Theorem 2.13. [Sze1],[Ta] *Every orthodox semigroup has an E-unitary cover.*

We say that an E-unitary regular semigroup S is *embeddable* if it is embeddable into a semidirect product of a band by a group. If the band part of the semidirect product can be chosen from the variety generated by the band of idempotents of S, then we call S *closely embeddable*. The following theorem due to B. Billhardt shows that Theorem 2.6 cannot be generalized for the whole class of orthodox semigroups.

Theorem 2.14. [Bi] *There exists an E-unitary regular semigroup that is not embeddable into a semidirect product of a band by a group.*

M. B. Szendrei has given an equivalent condition of embeddability, and by making use of this condition, she has proved the following theorems.

Theorem 2.15. [Sze2] *Every E-unitary regular semigroup having a regular band of idempotents is closely embeddable.*

Theorem 2.16. [Sze3] *The idempotent separating homomorphic images of bifree orthodox semigroups are closely embeddable. Consequently, every orthodox semigroup has a closely embeddable E-unitary cover.*

The last theorem can be seen as a common generalization of Theorems 2.5 and 2.6.

E-unitary covers over group varieties

In this section of the dissertation we sharpen the results of Theorem 2.14 by showing that, given a non-trivial group variety $\mathbf{V}$ distinct from the variety of all groups, there exists an E-unitary regular semigroup such that its greatest group homomorphic image is in $\mathbf{V}$, but it has no embeddable E-unitary cover over $\mathbf{V}$.

Our construction requires the notion of graph-semigroups, which can be defined by making use of generalized Cayley graphs of groups and of band varieties.

Theorem 3.1. *Every E-unitary regular semigroup is isomorphic to a graph-semigroup.*

By applying the previous theorem, one can construct a wide class of E-unitary covers of an E-unitary regular semigroup from group homomorphisms.

Theorem 3.3. *Let S be an E-unitary regular semigroup. Then every group G together with a surjective homomorphism $\varphi\colon G \to S/\sigma$ determines an E-unitary cover of S over G. Conversely, every E-unitary cover of S contains a regular subsemigroup which is also an E-unitary cover of S, and which is isomorphic to an E-unitary cover arising from a group G and a homomorphism φ in this way.*

By making use of the previous theorem and the embeddability condition by B. Billhardt, one can prove the following theorem.

Theorem 3.8. *For every non-trivial group variety $\mathbf{V}$ that is different from the variety of all groups, there exists an E-unitary regular semigroup such that its greatest group homomorphic image is in $\mathbf{V}$, but it has no embeddable E-unitary cover over $\mathbf{V}$.*

Almost factorizable orthodox semigroups

We say that an orthodox monoid M is *factorizable* if for every $s \in M$, there exists $e \in E(M)$ and $u \in U(M)$ such that $s = eu$. The following theorem can be proved in the same way as for inverse monoids.

Theorem 4.1. *Let M be an orthodox monoid. Then the following are equivalent:*

(i) *M is factorizable,*

(ii) *M is an idempotent separating homomorphic image of a semidirect product of a band monoid by a group,*

(iii) *M is a homomorphic image of a semidirect product of a band by a group.*

If S is a semigroup and $s \in S$ then multiplying each element by s on the right, and also on the left, we obtain two transformations of S. The associativity of the multiplication of S determines a link between these two transformations. A generalization of these pairs leads to the notion of the translational hull of S, which consists of certain linked pairs of transformations of S. The translational hull of a semigroup S is a monoid, and we denote its group of units by $\Sigma(S)$. We say that an orthodox semigroup S is *almost factorizable* if for every $s \in S$, there exists $e \in E(S)$ and $(\lambda, \rho) \in \Sigma(S)$ such that $s = e\rho$. An orthodox semigroup S is called *weakly coverable* if it is a homomorphic image of a semidirect product.

Theorem 4.3. *An orthodox semigroup is almost factorizable if and only if it is an idempotent separating homomorphic image of a semidirect product of a band by a group.*

In case of inverse semigroups, homomorphic images of semidirect products are the same as idempotent separating homomorphic images. As the following statement shows, this fails for orthodox semigroups.

Statement 4.5. *There exists a weakly coverable combinatorial completely* 0*-simple orthodox semigroup, which is not almost factorizable.*

The previous statement shows that the investigation of weak coverability of orthodox semigroups is more complicated than that of inverse semigroups. However, it is easy to see that the greatest inverse semigroup homomorphic image of each weakly coverable inverse semigroup is necessarily almost factorizable. In a special subclass being close to inverse semigroups this condition is sufficient.

Theorem 4.10. *A generalized inverse semigroup is weakly coverable if and only if its greatest inverse semigroup homomorphic image is almost factorizable.*

The following statement shows that the previous condition is not sufficient for characterizing weak coverability of orthodox semigroups in general.

Statement 4.12. *There exists an orthodox semigroup which is not weakly coverable, but whose greatest inverse semigroup homomorphic image is almost factorizable.*

Embedding orthodox semigroups

In this section of the dissertation, by making use of Theorem 2.16, we prove that every orthodox semigroup is embeddable into an almost factorizable orthodox semigroup. Let S be an orthodox semigroup, and let T be an E-unitary cover of S such that T is an idempotent pure homomorphic image of a bifree orthodox semigroup. Denote by α an idempotent separating congruence of T such that $T/\alpha \cong S$. By the proof of Theorem 2.16, T is embeddable into a semidirect product $\mathcal{B} * G$ where the band $\mathcal{B}$ is in the variety generated by the band of idempotents of T. Investigating the previous embedding, one can show that the congruence α extends to $\mathcal{B} * G$. This extension need not be idempotent separating, however, by factoring out $\mathcal{B} * G$ by an appropriate idempotent pure congruence, we obtain a semidirect product $B * G$ such that T is embeddable into $B * G$. Furthermore, α extends to an idempotent separating congruence of $B * G$. With such an argument, we can prove the following theorem.

Theorem 5.9. *Every orthodox semigroup is embeddable into an almost factorizable orthodox semigroup.*

Let S be an orthodox semigroup, and let $\iota\colon S \to F$ be an embedding of S into an almost factorizable orthodox semigroup F. Furthermore, let $\varphi\colon B * G \to F$ be a surjective idempotent separating homomorphism from a semidirect product $B * G$ onto F. Then the subsemigroup $\{(e, g) \in B * G : (e, g)\varphi \in S\iota\}$ of $B * G$ is an E-unitary cover of S. We say that an E-unitary cover T of S *arises from an embedding into an almost factorizable orthodox semigroup* if it is isomorphic to an E-unitary cover of this kind. Notice that during the proof of the previous theorem, we have shown that certain E-unitary covers arise from embeddings into almost factorizable orthodox semigroups. If the band of idempotents of the orthodox semigroup is regular, this statement is proved by the author to be true for all covers.

Theorem 5.11. [Ha3] *Every E-unitary cover of an orthodox semigroup with regular band of idempotents arises from an embedding into an almost factorizable orthodox semigroup.*

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