MONOIDAL INTERVALS

Ph.D. Dissertation

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Dedicated to the memory of my parents

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Chapter 1

Introduction

The story of monoidal intervals does not have a long history, but it is related to a central theme in universal algebra: composition of operations. Sets of operations that are closed under composition naturally arise in logic (e.g., in the propositional calculus of two-valued or many-valued logics), in algebra (e.g., in studying word problems), and in computer science (e.g., in the synthesis of automata). E. L. Post [Pos41] started to investigate composition-closed sets of operations on a 2-element set (that is, composition-closed sets of truth functions) in order to understand all possible propositional calculi in 2-valued logic. P. Hall [Hal58] was lead to the concept of a **clone**, which can be defined as a composition-closed set of operations containing all projection operations, by studying the word problem for various classes of groups.

For each set A, the clones on A form a complete lattice $\mathbb{CL}_A$ with respect to set-theoretic inclusion. Post's result mentioned in the preceding paragraph is a complete description of all members of the clone lattice $\mathbb{CL}_{\{0,1\}}$. It turns out that if A has two elements, then there are $\aleph_0$ clones on A. The situation changes dramatically when A has more than two elements. In [JM59] Ju. I. Janov and A. A. Mučnik proved that on a finite set A with more than two elements there are $2^{\aleph_0}$ clones. Moreover, as the following results of A. A. Bulatov show, the structure of the clone lattice is rather complicated; namely, for any finite set A,

- if $|A| \geqslant 3$ then the subsemigroup lattice of the absolutely free one-generated semigroup can be embedded into the clone lattice $\mathbb{CL}_A$ ([Bul92]);
- if $|A| \geqslant 4$ then any direct product of countably many finite lattices can be embedded into the clone lattice $\mathbb{CL}_A$ ([Bul94]).

Next we explain how the study of monoidal intervals may help understanding the structure of the clone lattice.

Let A be a set. For arbitrary clone $\mathcal{C}$ on A the set of unary operations in $\mathcal{C}$ is clearly a transformation monoid on A. Furthermore, it is not hard to show (see Á. Szendrei [Sze86], Proposition 3.1) that for arbitrary transformation monoid M on A the clones in which the set of unary operations is M form an interval $\operatorname{Int}(M)$ in the clone lattice $\mathbb{CL}_A$. Such an interval is called a **monoidal interval**. If A is finite, then there are only finitely many transformation monoids on A. Hence the monoidal intervals $\mathrm{Int}(M)$ partition the clone lattice $\mathbb{CL}_A$ into finitely many blocks. Since $\mathbb{CL}_A$ has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one might expect that 'for most M' the monoidal interval $\mathrm{Int}(M)$ contains uncountably many clones. This expectation is justified by the fact (cf. [Dor]) that if $|A| = 3$, then more than half of the monoidal intervals have cardinality $2^{\aleph_0}$. Nevertheless, it turns out that for many interesting transformation monoids M the interval $\mathrm{Int}(M)$ is countable. So, studying these intervals may lead to a better understanding of some parts of the clone lattice $\mathbb{CL}_A$.

The monoidal intervals are also related to the following unsolved problem on the congruences of the clone lattice: If A is a finite set with more than two elements, does $\mathbb{CL}_A$ have a nontrivial congruence? The relationship is revealed by a result of A. A. Krokhin [Kro01b] proving that any proper congruence of $\mathbb{CL}_A$ is a subrelation of the equivalence relation whose equivalence classes are the monoidal intervals. We note that for the case when A has only two elements, the congruences of $\mathbb{CL}_A$ have been determined by Krokhin–Semigrodskikh [KS01], using Post's description of $\mathbb{CL}_A$.

The problem of classifying all monoids on a finite set A according to the cardinalities of the corresponding monoidal intervals was first raised by Á. Szendrei [Sze86]. For the case when A is a two-element set Post's description of the clone lattice provides a complete solution to this problem: there are three finite and three infinite intervals. For the case when A is a finite set with more than two elements, and hence the clone lattice has cardinality $2^{\aleph_0}$, I. G. Rosenberg and N. Sauer in [RS] observed that each monoidal interval in $\mathbb{CL}_A$ either has cardinality $2^{\aleph_0}$ or is countable (see also M. Pinsker [Pin08]). Thus, Szendrei's problem can be refined as follows (see A. A. Krokhin [Kro97b]): for which transformation monoids does the corresponding monoidal interval have cardinality

- 1,
- finite but greater than 1,
- $\aleph_0$,
- $2^{\aleph_0}$?

We conclude this section with an overview of some known results related to this problem and a summary of our contributions.

1.1 Collapsing monoids

A monoid M on A is called **collapsing** if the interval $\mathrm{Int}(M)$ has only one element, namely the essentially unary clone generated by M.

The first result exhibiting a large family of collapsing monoids is due to P. P. Pálfy [Pal84]; soon after its discovery the result became influential in development of the structure theory of finite algebras called 'tame congruence theory'. Pálfy's theorem states that if M is a transformation monoid on a finite set A with more than two elements such that M contains all constant transformations and each nonconstant member of M is a permutation, then M is collapsing unless M is the monoid of unary polynomial operations of a vector space.

In Theorem 5.1 we generalize this theorem as follows: we determine all collapsing monoids M on a finite set with at least two elements that consists of at least one unary constant operation and some permutations (cf. M. Dormán [Dor08], Theorem 3.1.). This theorem generalizes the main result of A. Fearnley and I. G. Rosenberg in [FR03], as well.

Despite the fact that 'for most M' the monoidal interval $\mathrm{Int}(M)$ is expected to contain uncountably many clones, there are large intervals in the submonoid lattice of the full transformation monoid such that all members of these intervals are collapsing. In Theorem 3.4 we proved that for a finite set A with at least four elements in the submonoid lattice of the full transformation monoid $T(A)$ there are intervals that contain only collapsing monoids and whose cardinalities are greater than $2^{2^{c|A|}}$ for some positive constant c (cf. M. Dormán [Dor02], Proposition 2.4.).

For permutation groups the results known so far indicate that 'large' permutation groups, e.g. all primitive permutation groups, are collapsing (cf. P. P. Pálfy and Á. Szendrei [PSz82] and K. A. Kearnes and Á. Szendrei [KSz01]). This motivated us in extending the investigation of collapsing monoids to 'large' inverse monoids. In Chapter 4 we investigate the monoidal intervals $\mathrm{Int}(M)$ where M belongs to a class of inverse transformation monoids constructed from finite lattices. These inverse monoids arise from finite lattices by applying the construction introduced by T. Saito and M. Katsura in [SK92] to describe maximal inverse transformation monoids. We describe a necessary and sufficient condition for an inverse monoid constructed from a finite lattice to be collapsing (Theorem 4.3).

1.2 Finite monoidal intervals with more than one element

The earliest result concerning monoidal intervals was the description of the monoidal interval that corresponds to the full transformation monoid $T(A)$: this interval is an $(|A|+1)$-element chain (G. A. Burle [Bur67]).

In Pálfy's theorem, if M coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field then $\mathrm{Int}(M)$ is a 2-element chain (cf. P. P. Pálfy [Pal84] and Theorem 2.2).

Our result in Chapter 5 is closely related to the latter statement. Let M be a non-collapsing monoid on a finite set A with $|A| \geqslant 3$ such that M consists of at least two unary constant operations and some permutations and satisfies the conditions (i)–(iii) of Theorem 5.3. If M contains more than two unary constant operations then $\mathrm{Int}(M)$ is a 2-element chain, otherwise, either $\mathrm{Int}(M)$ is isomorphic to the direct square of the 2-element chain or $\mathrm{Int}(M)$ is a 3-element chain.

1.3 Infinite monoidal intervals

We discussed earlier that $\aleph_0$ and $2^{\aleph_0}$ are the only possibilities for the cardinality of an infinite monoidal interval if the base set A is finite. As for $\aleph_0$, the first, and so far the only construction of a transformation monoid M with $|\mathrm{Int}(M)| = \aleph_0$ is due to A. A. Krokhin in [Kro97b].

In contrast, we know a lot of examples of transformation monoids M for which $|\mathrm{Int}(M)| = 2^{\aleph_0}$ holds. For example, the monoidal interval corresponding to the one-element monoid (consisting of the identity operation only) satisfies this condition.

In Chapter 4 we present a family of inverse transformation monoids constructed from finite lattices using the Saito–Katsura construction in [SK92], for which the corresponding monoidal intervals have cardinality $2^{\aleph_0}$ (Theorem 4.17 and Theorem 4.22).

Chapter 2

Preliminaries

This section is devoted to a survey of the basic concepts and techniques that will be used in this dissertation. In Section 2.1 we discuss clones on finite sets and their properties, while in Section 2.2 we discuss monoidal intervals.

As usual, for a set X the set of all subsets of X will be denoted by $P(X)$. Let X, Y, Y', and Z be sets for which $Y \subseteq Y'$ holds. By the composition of the maps $\alpha\colon X \to Y$ and $\beta\colon Y' \to Z$ we will mean the map $X \to Z,\ x \mapsto (x\alpha)\beta$, denoted by $\alpha \circ \beta$. For arbitrary subset W of X the restriction of the map α to the set W is the map $\alpha|_W\colon W \to \alpha(W),\ x \mapsto \alpha(x)$.

For a finite set A we will denote the full transformation semigroup, the symmetric group, and the set of unary constant operations on A by $T(A)$, $S(A)$, and $C(A)$, respectively. For an arbitrary element a of A we will use the notation c_a for the unary constant operation on A with value a.

For the set of positive integers we will use the notation $\mathbb{N}$, and we will refer to them as natural numbers.

2.1 Clones

Let A be a set and n be a positive integer. An **n-ary operation on A** is a function $f\colon A^n \to A$. An operation is called **finitary** if it is n-ary for a positive integer n. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. The **superposition** of an n-ary operation $f \in \mathcal{O}_A$ by a k-ary operations $g_1, \dots, g_n \in \mathcal{O}_A$ is the k-ary operation $f(g_1, \dots, g_n) \in \mathcal{O}_A$ defined by the rule

$$f(g_1, \dots, g_n)(x_1, \dots, x_k) = f\big(g_1(x_1, \dots, x_k), \dots, g_n(x_1, \dots, x_k)\big),$$

and for positive integers n and $i \leqslant n$ the **i-th n-ary projection** is the operation

$$e_i^{(n)}\colon A^n \to A,\ e_i^{(n)}(x_1, \dots, x_n) \mapsto x_i.$$

A set $\mathcal{C}$ of finitary operations on a set A is said to be a **clone** if it contains all the projections and is closed under superposition of operations. It is obvious that $\mathcal{O}_A$ and the set $\mathcal{P}_A$ of all projections on A are clones. Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. This lattice will be denoted by $\mathbb{CL}_A$. The greatest and the least elements of $\mathbb{CL}_A$ are $\mathcal{O}_A$ and $\mathcal{P}_A$, respectively. Furthermore, we can define the **clone generated by a subset** F of $\mathcal{O}_A$ as the intersection of all clones that contain F. This clone will be denoted by $\langle F\rangle$. It is easy to see that $\langle F\rangle$ is the least clone containing F. If F is a finite subset of $\mathcal{O}_A$, say $F = \{f_1, \dots, f_s\}$, then we write $\langle f_1, \dots, f_s\rangle$ instead of $\langle\{f_1, \dots, f_s\}\rangle$. For a positive integer n, the set of all n-ary operations of in a clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(n)}$.

For an algebra $\mathbb{A} = (A; F)$ there is a clone that can be naturally attached to it, the **clone of term operations** of $\mathbb{A}$, which is the clone generated by its fundamental operations F. This clone will be denoted by $\mathrm{Clo}\,(\mathbb{A})$. The algebras $\mathbb{A}$ and $\mathbb{B}$ with the same universe are said to be **term equivalent** if their clones of term operations coincide, i.e., $\mathrm{Clo}\,(\mathbb{A}) = \mathrm{Clo}\,(\mathbb{B})$. It is worth mentioning that term equivalent algebras have the same subalgebras and congruences. It is easy to see that every clone on A can be obtained as a clone of term operations of a suitable algebra with universe A.

An n-ary operation $f \in \mathcal{O}_A$ is said to **depend on its i-th variable** ($1 \leqslant i \leqslant n$) if there are elements $a_1, \dots, a_{i-1}, a_i, a_i', a_{i+1}, \dots, a_n$ of A such that

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_n).$$

Otherwise the i-th variable of f is called **fictitious**. We call the operation f **essentially k-ary** if it depends on exactly k variables of its all variables.

For a natural number k a **k-ary relation on** A is a subset of A^k. A relation is finitary if it is k-ary for a positive integer k. We will denote by $\mathcal{R}_A$ the set of all finitary relations on A. Let m and n be positive integers, and let $\varrho \in \mathcal{R}_A$ be an m-ary relation and $f \in \mathcal{O}_A$ be an n-ary operation. We call an $n \times m$ matrix $X = (x_{i,j})$ over A a **ϱ-matrix** if the rows of X belong to ϱ, i.e., $(x_{i,1}, \dots, x_{i,m}) \in \varrho$ for all i ($1 \leqslant i \leqslant n$). The operation f is said to **preserve** the relation ϱ if for every ϱ-matrix $(x_{i,j}) \in A^{n\times m}$ the m-tuple

$$\big(f(x_{1,1}, \dots, x_{n,1}), \dots, f(x_{1,m}, \dots, x_{n,m})\big)$$

also belongs to ϱ. It is obvious that the operation f preserves the relation ϱ if and only if ϱ is a subalgebra of the algebra $(A; f)^m$.

For a subset R of $\mathcal{R}_A$ the set of all finitary operations on A that preserve each member of R will be denoted by $\mathrm{Pol}(R)$. If R is finite, say $R = \{\varrho_1, \dots, \varrho_s\}$, then we simply write $\mathrm{Pol}(\varrho_1, \dots, \varrho_s)$. On the other hand, for a subset F of $\mathcal{O}_A$ the set of all finitary relations on A that are preserved by each member of F will be denoted by $\mathrm{Inv}(F)$. If F is finite, say $F = \{f_1, \dots, f_s\}$, then we simply write $\mathrm{Inv}(f_1, \dots, f_s)$.

For every finite set A the maps

$$\begin{aligned}\mathrm{Inv}\colon P(\mathcal{O}_A) \to P(\mathcal{R}_A),\ F \mapsto \mathrm{Inv}(F),\\ \mathrm{Pol}\colon P(\mathcal{R}_A) \to P(\mathcal{O}_A),\ R \mapsto \mathrm{Pol}(R)\end{aligned}$$

define a Galois connection between sets of operations and sets of relations. This is summarized in Theorem 2.1 below. For more details about Galois connections in general consult with the books [DP02], [PK79], and [BSCs88].

Theorem 2.1. *Let A be a finite set. Then for arbitrary subsets F and F' of $\mathcal{O}_A$ and for arbitrary subsets R and R' of $\mathcal{R}_A$ we have that*

(i) *if* $F \subseteq F'$ *then* $\mathrm{Inv}(F) \supseteq \mathrm{Inv}(F')$, *and if* $R \subseteq R'$ *then* $\mathrm{Pol}(R) \supseteq \mathrm{Pol}(R')$;

(ii) $F \subseteq \mathrm{Pol}(\mathrm{Inv}(F))$ *and* $R \subseteq \mathrm{Inv}(\mathrm{Pol}(R))$;

(iii) *the clones on A are exactly the sets of the form* $\mathrm{Pol}(R)$ $(R \subseteq \mathcal{O}_A)$;

(iv) *the fixed points of the map* $\mathrm{Pol} \circ \mathrm{Inv}$ *are exactly the sets of the form* $\mathrm{Inv}(F)$ $(F \subseteq \mathcal{O}_A)$, *which constitute a complete lattice* $\mathbb{RL}_A$;

(v) *the lattices $\mathbb{CL}_A$ and $\mathbb{RL}_A$ are dually isomorphic, moreover, the maps* $\mathrm{Inv}|_{\mathbb{CL}_A}$ *and* $\mathrm{Pol}|_{\mathbb{RL}_A}$ *are mutually inverse dual isomorphisms.*

2.2 Monoidal intervals

Throughout this dissertation we will assume that the base set A is a finite set with more than one element. To give a more detailed introduction into the concept of a monoidal interval let M be a transformation monoid on A, and let $\mathrm{Int}(M)$ denote the collection of all clones $\mathcal{C}$ on A such that the set of unary operations of $\mathcal{C}$ is M. The clone $\langle M\rangle$ of essentially unary operations generated by M is a member of $\mathrm{Int}(M)$, in fact, it is the least member of $\mathrm{Int}(M)$, so $\mathrm{Int}(M)$ is non-empty. Furthermore, it is clear that every clone $\mathcal{C}$ in $\mathrm{Int}(M)$ is contained in the set

$$\mathrm{Sta}(M) = \bigl\{f(x_1,\dots,x_n) \in \mathcal{O}_A \mid n \in \mathbb{N},\ \text{and}\\ f(m_1(x),\dots,m_n(x)) \in M \text{ for all } m_1,\dots,m_n \in M\bigr\},$$

which is called the **stabilizer** of the monoid M. It is easy to verify that $\mathrm{Sta}(M)$ is a clone on A, therefore $\mathrm{Sta}(M)$ is the largest member of $\mathrm{Int}(M)$. Moreover, we see that a clone $\mathcal{C} \in \mathbb{CL}_A$ belongs to $\mathrm{Int}(M)$ if and only if $\langle M\rangle \subseteq \mathcal{C} \subseteq \mathrm{Sta}(M)$. Thus $\mathrm{Int}(M)$ is the interval $[\langle M\rangle, \mathrm{Sta}(M)]$ in the clone lattice $\mathbb{CL}_A$. Such an interval is called a **monoidal interval**.

With the help of Post's theorem one can easily describe the monoidal intervals in $\mathbb{CL}_A$ for $A = \{0,1\}$. There are six monoidal intervals, and exactly the monoidal intervals corresponding to the monoids $\{\mathrm{id}_A, c_0, c_1, r\}$, $\{\mathrm{id}_A, c_0, c_1\}$, and $\{\mathrm{id}_A, r\}$, respectively, are finite. These finite intervals can be seen in Figure 1, where $+$ denotes the addition modulo 2, $\wedge$ and $\vee$ denote the meet and join operations with respect to the partial order $\leqslant = \{(0,0),(1,1),(0,1)\}$ on A, while r is the unary operation $r\colon A \to A,\ x \mapsto x+1$, and $\mathcal{D}_3$ is the clone of all self-dual operations on A.

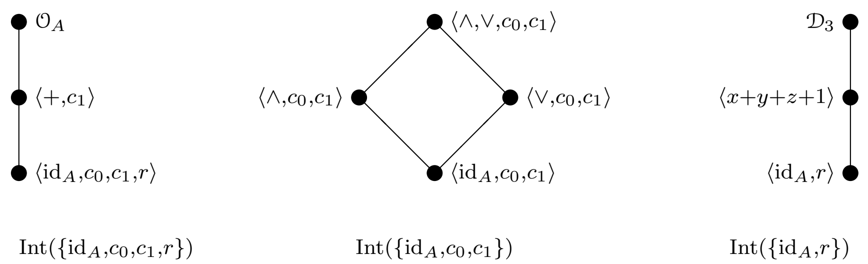

Figure 1: Finite monoidal intervals in $\mathbb{CL}_{\{0,1\}}$.

The remaining three monoidal intervals, which correspond to the monoids $\{\mathrm{id}_A\}$, $\{\mathrm{id}_A, c_1\}$, and $\{\mathrm{id}_A, c_0\}$, have cardinality $\aleph_0$. The corresponding monoidal intervals can be seen in Figure 2 and Figure 3, where we use the following notation:

$$\mathcal{F}_4^k = \mathrm{Pol}(\{0,1\}^k \setminus \{(0,0,\dots,0)\}) \quad (k \in \mathbb{N},\ k \geqslant 2), \quad \mathcal{F}_4^\infty = \textstyle\bigcap_{k=2}^\infty \mathcal{F}_4^k,$$
$$\mathcal{F}_8^k = \mathrm{Pol}(\{0,1\}^k \setminus \{(1,1,\dots,1)\}) \quad (k \in \mathbb{N},\ k \geqslant 2), \quad \mathcal{F}_8^\infty = \textstyle\bigcap_{k=2}^\infty \mathcal{F}_8^k,$$

for $k \in \mathbb{N} \cup \{\infty\},\ k \geqslant 2$

$$\mathcal{F}_1^k = \mathcal{F}_4^k \cap \mathrm{Pol}(\{0\},\{1\}), \qquad \mathcal{F}_5^k = \mathcal{F}_8^k \cap \mathrm{Pol}(\{0\},\{1\}),$$
$$\mathcal{F}_2^k = \mathcal{F}_1^k \cap \mathrm{Pol}(\{0\},\{1\},\leqslant), \quad \mathcal{F}_6^k = \mathcal{F}_5^k \cap \mathrm{Pol}(\{0\},\{1\},\leqslant),$$
$$\mathcal{F}_3^k = \mathcal{F}_4^k \cap \mathrm{Pol}(\{1\},\leqslant), \qquad \mathcal{F}_7^k = \mathcal{F}_8^k \cap \mathrm{Pol}(\{0\},\leqslant),$$
$$\mathcal{A}_4 = \mathrm{Pol}(\{0\},\{1\},\leqslant),$$
$$\mathcal{D}_1 = \mathcal{D}_3 \cap \mathrm{Pol}(\{0\},\{1\}), \qquad \mathcal{D}_2 = \mathcal{F}_2^2 \cap \mathcal{F}_6^2,$$
$$\mathcal{L}_2 = \mathrm{Pol}(\{1\}) \cap \langle +, c_1\rangle, \qquad \mathcal{L}_3 = \mathrm{Pol}(\{0\}) \cap \langle +, c_1\rangle$$
$$\mathcal{S}_3 = \mathcal{F}_4^\infty \cap \langle \vee, c_0, c_1\rangle, \qquad \mathcal{P}_3 = \mathcal{F}_8^\infty \cap \langle \wedge, c_0, c_1\rangle,$$
$$\mathcal{S}_5 = \mathrm{Pol}(\{0\},\leqslant) \cap \langle \vee, c_0, c_1\rangle, \quad \mathcal{P}_5 = \mathrm{Pol}(\{1\},\leqslant) \cap \langle \wedge, c_0, c_1\rangle.$$

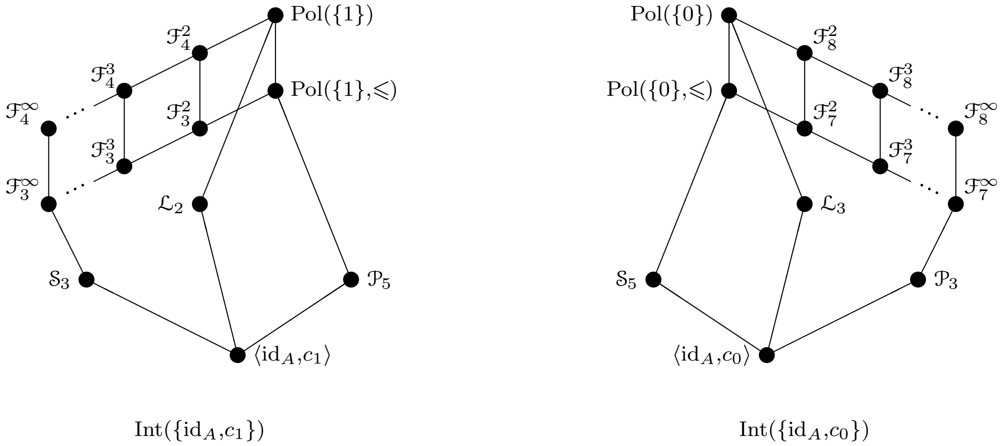

Figure 2: The monoidal intervals $\mathrm{Int}(\{\mathrm{id}_A, c_1\})$ and $\mathrm{Int}(\{\mathrm{id}_A, c_0\})$.

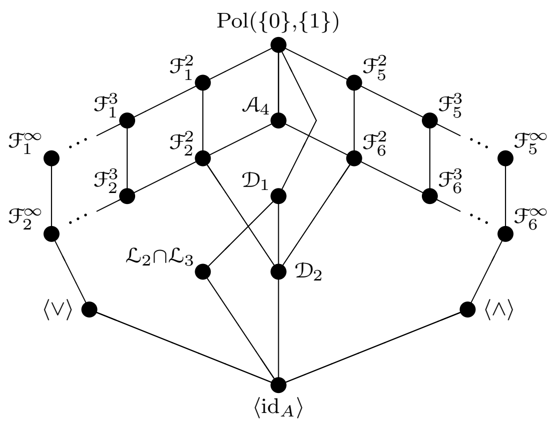

Figure 3: The monoidal interval $\mathrm{Int}(\{\mathrm{id}_A\})$.

Recall from the introduction that if a monoidal interval $\mathrm{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**. In this case the only element of $\mathrm{Int}(M)$ is $\langle M \rangle$. By a result of J.-U. Grabowski [Gra97], M is collapsing if and only if $\mathrm{Sta}(M)$ contains no essentially binary operations. Hence it is decidable for a monoid M whether M is collapsing. However, since there are at least $|A|^{|A|^2} - 2 \cdot |A|^{|A|} > 0.99 \cdot |A|^{|A|^2}$ essentially binary operations on A if $|A| \geqslant 3$, therefore in practice it is rather difficult to decide whether or not a monoid is collapsing.

There is a simple but useful necessary condition for a transformation monoid M on A to be collapsing: if M is collapsing, then M is **weakly transitive**, that is, there is an element $a \in A$ such that $\{m(a) : m \in M\}$ coincides with A (T. Ihringer and R. Pöschel [IP93]).

For further reference we state Pálfy's theorem that was mentioned in the introduction.

Theorem 2.2 (cf. P. P. Pálfy [Pal84]). *Let A be a finite set with $|A| \geqslant 3$, and let M be a transformation monoid on A that contains all the unary constant operations and whose nonconstant operations are permutations. Then $|\mathrm{Int}(M)| \leqslant 2$; moreover, $|\mathrm{Int}(M)| = 1$ unless M coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field.*

To prove that for a transformation monoid M the monoidal interval $\mathrm{Int}(M)$ has cardinality $2^{\aleph_0}$ the following method of J. Demetrovics and L. Hannák in [DH97] will be useful.

Let I be a set and $\mathfrak{C} = \{\mathcal{C}_i : i \in I\}$ is a set of clones on A, furthermore, let $\mathfrak{R} = \{\varrho_i : i \in I\}$ is a set of finitary relations on A. The set $\mathfrak{C}$ is said to be **independent** if for all $i \in I$ we have that

$$\mathcal{C}_i \not\subseteq \left\langle \bigcup \{\mathcal{C}_j : j \in I \setminus \{i\}\} \right\rangle .$$

An easy consequence of the independence of clones is the following: if $\mathfrak{C}$ is a complete join-subsemilattice of $\mathbb{CL}_A$ that contains an infinite independent subset then $\mathfrak{C}$ has cardinality $2^{\aleph_0}$ (cf. [DH97], Proposition 1). We remark that a monoidal interval is an example for a complete join-subsemilattice of $\mathbb{CL}_A$.

The set $\mathfrak{C}$ is **separated by** $\mathfrak{R}$ if for all $m, n \in I$ we have that $\mathcal{C}_m \subseteq \mathrm{Pol}(\varrho_n)$ if and only if $m \neq n$. The significance of separation is that independence is a consequence of it, that is, if $\mathfrak{C}$ is separated by $\mathfrak{R}$ then $\mathfrak{C}$ is independent.

Theorem 2.3 (cf. [DH97], Proposition 3.). *Let $\mathfrak{C} = \{\mathcal{C}_i : i \in \mathbb{N}\}$ be a set of clones separated by a set of relations $\mathfrak{R} = \{\varrho_i : i \in \mathbb{N}\}$ on A. Let $\mathcal{K}_1 \subseteq \mathcal{K}_2$ be clones on A such that $\mathcal{C}_i \subseteq \mathcal{K}_2$ and $\mathcal{K}_1 \subseteq \mathrm{Pol}(\varrho_i)$ hold for all $i \in \mathbb{N}$. Then the interval $[\mathcal{K}_1, \mathcal{K}_2] = \{\mathcal{C} \in \mathbb{CL}_A : \mathcal{K}_1 \subseteq \mathcal{C} \subseteq \mathcal{K}_2\}$ has cardinality $2^{\aleph_0}$.*

The case when each memeber of $\mathfrak{C}$ is generated by a single element of $\mathcal{O}_A$, say $\mathcal{C}_i = \langle f_i \rangle$ for all $i \in I$, is especially important for the construction of monoidal intervals of cardinality $2^{\aleph_0}$. The following corollary of Theorem 2.3 will handle this case.

Corollary 2.4. *Let M be a transformation monoid on A, let $\mathfrak{C} = \{\langle f_i \rangle : i \in \mathbb{N}\}$ be a set of subclones of* $\mathrm{Sta}(M)$ *and let $\mathfrak{R} = \{\varrho_i : i \in \mathbb{N}\}$ be a set of* *relations on* A*. If* $\mathfrak{C}$ *is separated by* $\mathfrak{R}$ *and* $M \subseteq \mathrm{Pol}(\varrho_i)$ *hold for all* $i \in \mathbb{N}$ *then the monoidal interval* $\mathrm{Int}(M)$ *has cardinality* $2^{\aleph_0}$.

Proof. Set $\mathcal{K}_1 = \langle M \rangle$, $\mathcal{K}_2 = \mathrm{Sta}(M)$ and $\mathfrak{C}' = \{\langle M \cup \{f_i\}\rangle : i \in \mathbb{N}\}$. Then $\mathfrak{C}'$ is also separated by $\mathfrak{R}$, and by Theorem 2.3 the interval $\mathrm{Int}(M) = [\mathcal{K}_1, \mathcal{K}_2]$ contains $2^{\aleph_0}$ clones. □

Chapter 3

Large intervals of collapsing monoids

In this chapter we present some new families of collapsing monoids (Theorem 3.1). These monoids form 'large' intervals in the submonoid lattices of the full transformation semigroups in the sense that their cardinalities are $2^{2^{cn}}$ for some positive c, where n is the size of the base set. Unfortunately, the construction that was used in the proof of Theorem 3.1 does not work for 3-element sets. However, a similar construction has already worked for sets with $|A| = 3$ (Theorem 3.6). With the help of it and many earlier results we managed to describe all collapsing monoids on $\{0, 1, 2\}$.

3.1 Intervals of collapsing monoids

Let A be a finite set with at least 4 elements. Let P, Q, and R be pairwise disjoint nonempty subsets of A such that $|R| \geqslant 2$. Let $T(P, Q, R)$ be the set of all transformations $t \in T(A)$, such that for all $p \in P, q \in Q$ and $r, r' \in R$ if $t(r) = t(r')$ then $t(p) \in \{t(q), t(r)\}$. Let M be an arbitrary transformation monoid on A. The monoid M is said to be **rich** with respect to P, Q, R if for some $s \in A$, and for all $a, b \in A$ such that $a \neq b$ and $s \in \{a, b\}$, M contains transformations m and n such that $m(P) = m(Q) = \{a\}$, $m(R) = \{b\}$ and $n(P) = n(R) = \{a\}$, $n(Q) = \{b\}$. If P, Q, R are clear from the context, then we will simply say that M is rich.

The main result of this section is the following.

Theorem 3.1 ([Dor02]). *Let A be a finite set with at least four elements, and let P, Q, R be disjoint nonempty subsets of A such that $|R| \geqslant 2$. Then every rich monoid $M \subseteq T(P, Q, R)$ is collapsing.*

Example 3.2. *Let B be the set $\{0, 1, 2, 3\}$, and M be the monoid generated by all transformations $m \in T(B)$ such that $m(2) = m(3)$, $m(0) \in \{m(1), m(2)\}$ and $0 \in m(\{0, 1, 2\})$. If we choose the sets $P = \{0\}$, $Q = \{1\}$ and $R = \{2, 3\}$, then it is obvious that the monoid M is rich, and it is contained in $T(P, Q, R)$.*

The proof of Theorem 3.1 is based on the lemma below, which states that in the operation table of an essentially binary operation a particular configuration always occurs.

Lemma 3.3. *Let $f \in \mathcal{O}_A$ be an essentially binary operation on a finite set A with at least two elements. Then, for every element $s \in A$ there exist $a, b, c, d \in A$ such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$.*

Proof. Let s be a fixed element of A. Suppose first that the unary operation $f(s, x)$ is constant. Since f depends on its second variable, there are elements $u, v \in A$ such that $f(u, s) \neq f(u, v)$. Thus, if $f(s, v) = f(u, v)$, then $f(s, s) = f(s, v) = f(u, v) \neq f(u, s)$, so the elements $a = s, b = u, c = v, d = s$ satisfy the requirements. Otherwise we have $f(s, v) \neq f(u, v) \neq f(u, s)$, hence the choice $a = s, b = u, c = s, d = v$ is appropriate. The same argument yields suitable elements if the unary operation $f(x, s)$ is constant. Finally, if none of the unary operations $f(s, x)$ and $f(x, s)$ are constant, then there are elements $a, c \in A$ such that $f(a, s) \neq f(s, s) \neq f(s, c)$, hence the elements $a, b = s, c$, and $d = s$ satisfy the requirements of the lemma. □

Proof of Theorem 3.1. Let M be a monoid, which is rich with respect to $P,\ Q,\ R$, and let $s \in A$ be an element witnessing the richness of M. Choose elements $p, q, r, r' \in A$ such that $p \in P,\ q \in Q$, and $r, r' \in R$. We will show that the stabilizer of M contains no essentially binary operation. Let $f \in \mathcal{O}_A$ be an essentially binary operation. By Lemma 3.3, there are elements a, b, c, d such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$. Since M is a rich monoid, there are transformations $t', t'' \in M$ such that

$$t'(p) = t'(q) = b,\ t'(r) = t'(r') = a,$$

and

$$t''(p) = t''(r) = t''(r') = d,\ t''(q) = c.$$

Let $t(x) = f(t'(x), t''(x))$ $(x \in A)$. Then

$$t(p) = f(b, d),\ t(q) = f(b, c),$$

and

$$t(r) = t(r') = f(a, d).$$

If $f(a, d) \neq f(b, c)$ then $t(p), t(q)$ and $t(r)$ are pairwise distinct, while, if $f(a, d) = f(b, c)$ then $t(p) \neq t(q) = t(r)$. Hence, in both cases, the transformation t is not in $T(P, Q, R)$, thus, t is not in M. That is, the operation f does not belong to the stabilizer of M. □

Further on we show that for a finite set A with $|A| \geqslant 6$ in the submonoid lattice of $T(A)$ there are large intervals, which contain only collapsing monoids. In the sequel, the monoid generated by a set H of transformations will be denoted by $\langle H \rangle$. If $H = \{t_1, \dots, t_r\}$ then we will write $\langle t_1, \dots, t_r \rangle$ instead of $\langle \{t_1, \dots, t_r\} \rangle$.

Let A be a finite set with $|A| \geqslant 6$. Let the elements $p, q, r, r' \in A$ be pairwise distinct, and let $P = \{p\}$, $Q = \{q\}$, $R = \{r, r'\}$, $A' = A \backslash (P \cup Q \cup R)$. We define the monoid N on A to be the monoid generated by the set of all transformations $t \in T(P, Q, R)$ for which $t(r) = t(r')$ and the restriction of t onto A' is the identity operation on A'. It is easy to see that N is contained in $T(P, Q, R)$. For an arbitrary monoid $K \subseteq T(A')$ we will denote by $\hat{K}$ the monoid which consists of all transformations from $T(A)$ whose restriction onto A' is a member of K, and whose restriction onto the set $P \cup Q \cup R$ is the identity operation. Since $t \in \langle N \cup \hat{K} \rangle$ implies that $t|_{A'} \in K$, we get that if K_1, K_2 are submonoids of $T(A')$ and $K_1 \neq K_2$ then $\langle N \cup \hat{K}_1 \rangle \neq \langle N \cup \hat{K}_2 \rangle$. Furthermore, $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P, Q, R)$, and N is rich.

Theorem 3.4 ([Dor02]). *Let A be a finite set with $|A| = n \geqslant 6$. Then all members of the interval $[N, \langle N \cup \widehat{T(A')} \rangle]$ is collapsing, and this interval has cardinality greater than $2^{2^{c'n}}$ for some positive constant c'.*

To prove Theorem 3.4, we need a simple estimate of the cardinality of the subsemigroup lattice of $T(A)$.

Lemma 3.5. *Let A be a finite set with $|A| = n \geqslant 3$. Then the full transformation semigroup $T(A)$ has at least $2^{2^{cn}}$ subsemigroups for some positive constant c.*

Proof. Let $P'(A)$ be the set of all subsets of A which have cardinality $[n/2]$. Let U be an arbitrary element of $P'(A)$, and let M_U be the semigroup of all transformations whose ranges are contained in U. It is easy to see that if H is a subset of $P'(A)$ then $T_H = \bigcup_{U \in H} M_U$ is a subsemigroup of $T(A)$. Furthermore, if $H_1, H_2 \subseteq P'(A)$ and $H_1 \neq H_2$ then $T_{H_1} \neq T_{H_2}$. Thus, we have that the subsemigroups T_H $(H \subseteq P'(A))$ are pairwise distinct, and

$$|\{T_H \mid H \subseteq P'(A)\}| = 2^{|P'(A)|} = 2^{\binom{n}{[n/2]}} \geqslant 2^{4^{[n/2]}/(2[n/2])} \geqslant 2^{2^{cn}}$$

for some positive constant c. □

Proof of Proposition 3.4. Since the monoid N is rich and $\langle N \cup \widehat{T(A')}\rangle$ is a subset of $T(P,Q,R)$, we see that from Theorem 3.1 that every monoid in the interval $[N, \langle N \cup \widehat{T(A')}\rangle]$ is collapsing. Furthermore by Lemma 3.5, this interval has cardinality greater than the number of subsemigroups of $T(A')$, that is $|[N, \langle N \cup \widehat{T(A')}\rangle]| \geqslant 2^{2^{c'n}}$ for some positive constant c'. □

3.2 Collapsing monoids on a three-element set

In this section we will describe all collapsing monoids on a 3-element set.

Let A be a 3-element set. We will define two sets of transformations on A. Let $p, s \in A$ be arbitrary elements of A. Let T_p denote the set of all transformations $t \in T(A)$ such that either t is a permutation fixing p or t is not a permutation, and $t(p) \in \{t(q), t(r)\}$, where $\{p, q, r\} = A$. Furthermore, let $M_{p,s}$ be the set of all transformations $t \in T_p$ such that $t(A) \subseteq \{s, a\}$ for some $a \in A \setminus \{s\}$ or t is the identity operation. It is easy to see that both T_p and $M_{p,s}$ are transformation monoids on A, and with the aid of these monoids we get a similar description as in Theorem 3.1

Theorem 3.6 ([Dor02]). *Let A be a 3-element set. Then each monoid M for which there are elements $p, s \in A$ such that $M_{p,s} \subseteq M \subseteq T_p$ is collapsing.*

Proof. Let M be a monoid on the set A such that $M_{p,s} \subseteq M \subseteq T_p$ for some $p, s \in A$. Let $f \in \mathcal{O}_A$ be an essentially binary operation. Suppose that f is in the stabilizer of M. By Lemma 3.3, there are elements $a, b, c, d \in A$ such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$. Since $M_{p,s} \subseteq M$, there are transformations $t_1, t_2 \in M$ such that

$$t_1(p) = t_1(q) = b,\ t_1(r) = a \text{ and } t_2(p) = t_2(r) = d,\ t_2(q) = c.$$

Let $t(x) = f(t_1(x), t_2(x))$ $(x \in A)$. Then $t \in M$, and

$$t(p) = f(b, d),\ t(q) = f(b, c),\ \text{and } t(r) = f(a, d).$$

By the assumptions on M, $t(q) = t(r) \neq t(p)$ is impossible, and we get that t is a permutation which fixes the element p. We will show that this leads to a contradiction. Since the elements $f(b, d), f(b, c)$, and $f(a, d)$ are pairwise distinct, we have that $f(a, c) \in \{f(b, d), f(b, c), f(a, d)\}$.

Case 1: $f(a, c) = f(a, d)$ or $f(a, c) = f(b, c)$. Assume the first equality holds and let $t_3, t_4 \in M_{p,s}$ be transformations such that

$$t_3(p) = t_3(r) = b,\ t_3(q) = a,\ \text{and } t_4(p) = t_2(q) = c,\ t_4(r) = d.$$

Let $t'(x) = f(t_3(x), t_4(x))$ $(x \in A)$. Then t' is a permutation and $t'(p) = f(b, c) \neq p$, which is a contradiction. Mutatis mutandis for the case $f(a, c) = f(b, c)$.

Case 2: $f(a, c) = f(b, d)$. Let $t_5, t_6 \in M_{p,s}$ be transformations such that

$$t_5(p) = t_5(q) = b,\ t_5(r) = a, \text{ and } t_6(p) = t_6(r) = c,\ t_6(q) = d.$$

Let $t''(x) = f(t_5(x), t_6(x))$ $(x \in A)$. Then $t'' \in M$, and $t''(r) = t''(q) \neq t''(p)$, which is again a contradiction. □

Now we are in a position to give a complete list of collapsing monoids on a 3-element set.

Let A be a 3-element set. Without loss of generality, we may assume that $A = \{0, 1, 2\}$. Let M_1 and M_2 be submonoids of $T(A)$. We say that M_1 is **equivalent** to M_2, and we write $M_1 \bowtie M_2$, iff there is a permutation α on A such that $M_2 = \{\alpha^{-1} m \alpha \mid m \in M_1\}$, that is M_2 is the conjugate of M_1 by α. It is straightforward to check that $\bowtie$ is an equivalence relation on the set of submonoids of $T(A)$. Furthermore, if $M_1 \bowtie M_2$ then $\operatorname{Int}(M_1) \cong \operatorname{Int}(M_2)$. On a 3-element set the subsemigroups of $T(A)$ were described by D. Lau (cf. [Lau84], [Lau95]). Using this description, we get 699 submonoids on A in 160 $\bowtie$-classes. We will use several earlier results to obtain the complete list of collapsing monoids on A. These results are due to Demetrovics–Hannák [DH79], [DH97], Fearnley–Rosenberg [FR03], Krokhin [Kro95], [Kro97a], Pálfy [Pal84], Pálfy–Szendrei [PSz82]. We will use the following notation for the constants and the permutations in $T(A)$.

x	$c_0(x)$	$c_1(x)$	$c_2(x)$	$\mathrm{id}(x)$	$\tau_0(x)$	$\tau_1(x)$	$\tau_2(x)$	$\sigma(x)$	$\sigma^2(x)$
0	0	1	2	0	0	2	1	1	2
1	0	1	2	1	2	1	0	2	0
2	0	1	2	2	1	0	2	0	1

Theorem 3.7 ([Dor02]). *On the* 3*-element set* $A = \{0, 1, 2\}$ *there are* 30 *collapsing monoids in* 11 $\bowtie$*-classes. If* M *is a collapsing monoid on* A*, then* M *is equivalent to exactly one of the following monoids:*

(1) $\langle c_0, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, \tau_2\}$,

(2) $\langle c_0, c_1, c_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2\}$,

(3) $\langle c_0, c_2, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \tau_0\}$,

(4) $\langle c_0, \sigma \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \sigma, \sigma^2\}$,

(5) S_3,

(6) $M_{2,0}$,

(7) $M_{2,2}$,

(8) $\langle M_{2,2} \cup \{\tau_2\} \rangle = M_{2,2} \cup \{\tau_2\}$,

(9) $T_2 \setminus \{\tau_2\}$,

(10) T_2,

where T_2 *is the monoid of all transformations* $t \in T(A)$ *such that either* $t = \mathrm{id}_A$ *or* $t(2) \in \{t(0), t(1)\}$*, while* $M_{2,r}$ $(r \in \{0, 2\})$ *is the monoid of all transformations* $t \in T_0$ *for which either* $|t(A)| \leqslant 2$ *and* $r \in t(A)$ *or* $t = \mathrm{id}_A$ *or* t *is constant.*

Proof. From earlier results in [FR03], [Pal84], and [PSz82], it follows that the monoids (1), (2)–(4), and (5) are collapsing, while for the monoids (6)–(11) this property is the consequence of Theorem 3.6. To check that transformation monoids which are equivalent to neither of (1)–(10) are not collapsing we used the results in [Bur67], [DH79], [DH97], [IP93], [Kro95], [Kro97a], [Mar82], [Pal84], [PSz82], and a simple computer program (written in PASCAL) based on the result of [Gra97]. □

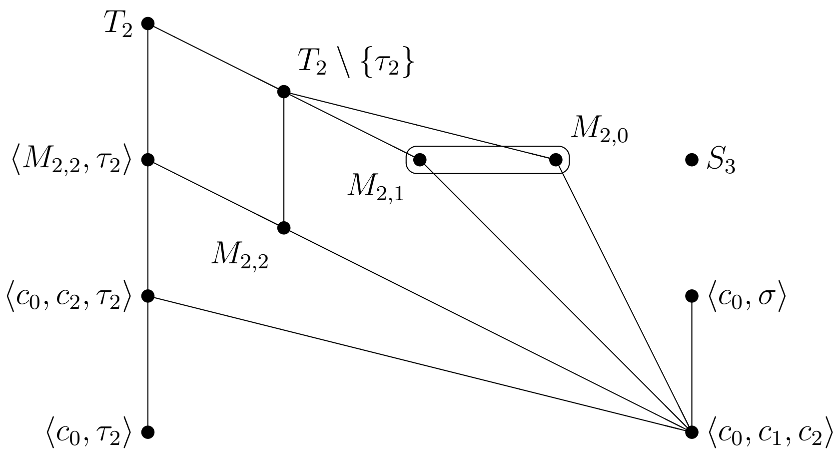

Figure 4.

In Figure 4 a fragment of the poset of collapsing monoids on $\{0, 1, 2\}$ can be seen. The whole poset can be obtained by rotating this fragment about 'axis' S_3, $\langle c_0, \sigma \rangle$, $\langle c_0, c_1, c_2 \rangle$ through $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$. Rotating the whole poset through $\frac{2}{3}\pi$ corresponds to conjugating the monoids by σ. Hence the three monoids on the 'axis' form singleton $\bowtie$-classes, the $\bowtie$-class of $M_{2,0}$ has six elements, while all the other $\bowtie$-classes contain exactly three elements.

Chapter 4

Collapsing inverse monoids

In this chapter we investigate the monoidal intervals corresponding to inverse transformation monoids constructed from finite lattices. These inverse monoids arise from finite lattices by applying the construction introduced by T. Saito and M. Katsura in [SK92] to describe maximal inverse transformation monoids. We describe a necessary and sufficient condition for an inverse monoid constructed from a finite lattice to be collapsing (Theorem 4.3). This work was inspired by earlier results on permutation groups, which indicated that 'large' permutation groups, e.g. all primitive permutation groups, are collapsing (cf. Pálfy–Szendrei [PSz82] and Kearnes–Szendrei [KSz01]).

In the last section of this chapter we present some examples of maximal inverse monoids for which the corresponding monoidal intervals are large. Using the result of Demetrovics–Hannák [DH97] for the construction of large sets of clones (cf. Theorem 2.4), we prove in Theorem 4.17 that for a 3-element chain the corresponding monoidal interval has cardinality $2^{\aleph_0}$. With the help of this result and Theorem 4.22 we can present a large set of lattices for which the associated monoidal intervals also have cardinality $2^{\aleph_0}$.

4.1 The inverse monoid $\mathrm{IS}(\mathbb{L})$

Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite lattice. The least and greatest elements of $\mathbb{L}$ will be denoted by $0_{\mathbb{L}}$ and $1_{\mathbb{L}}$, respectively. If the lattice is clear from the context then we omit the subscript, and simply write 0 and 1, respectively. The set of atoms and the set of join-irreducible elements of $\mathbb{L}$ will be denoted by $\mathcal{A}(\mathbb{L})$ and $\mathcal{J}(\mathbb{L})$, respectively, and we put $\mathcal{A}_0(\mathbb{L}) = \mathcal{A}(\mathbb{L}) \cup \{0\}$. If there is no danger of confusion, we simply write $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{J}$, respectively. Two elements a and b of $\mathbb{L}$ will be called **similar** iff the principal ideals $(a]$ and $(b]$ are isomorphic. We write $a \sim b$ to denote that a is similar to b. The relation $\sim$ is an equivalence relation on L. If the $\sim$-class containing a has only one element then a will be called **isolated**. For every element $a \in L$ we define a unary operation φ_a by the rule $\varphi_a(x) = x \wedge a$ $(x \in L)$. In particular, φ_0 is constant with range $\{0\}$. For similar elements $a, b \in L$ the symbol $\beta_{a,b}$ will denote an isomorphism between the principal ideals $(a]$ and $(b]$. Define a set

IS($\mathbb{L}$) of transformations on L in the following way:

$$\text{IS}(\mathbb{L}) = \{\beta_{v,w} \circ \varphi_v \mid v, w \in L,\ v \sim w,\ \text{and}\ \beta_{v,w} \colon (v] \to (w] \text{ is an isomorphism}\}.$$

Then IS($\mathbb{L}$) is an inverse submonoid of the full transformation semigroup on L (cf. Saito–Katsura [SK92], Lemma 3.1). We note that, with the help of Proposition 4.1 (b), one can easily verify that the set IS($\mathbb{L}$) is closed under composition.

Let $M = \text{IS}(\mathbb{L})$ be the inverse monoid determined by the lattice $\mathbb{L}$.

Proposition 4.1. *Let m be an arbitrary transformation from M. Then*

(a) *m is monotone;*

(b) *there is a unique element $v \sim m(1)$ of L such that $m = \beta_{v,m(1)} \circ \varphi_v$ for some isomorphism $\beta_{v,m(1)} \colon (v] \to (m(1)]$; furthermore, for any $u \in L$, $m(u) = m(1)$ if and only if $v \leqslant u$;*

(c) *$m(\mathcal{A}_0) \subseteq \mathcal{A}_0$, and $m(0) = 0$, moreover for arbitrary atom d we have that $0 < m(d)$ if and only if $d \leqslant v$;*

(d) *if $m(d) = 0$ for every atom d of $\mathbb{L}$ then $m = \varphi_0$.*

Proof. (a) As the transformation m is in M, there are elements $v, w \in L$ and an isomorphism $\beta_{v,w} \colon (v] \to (w]$ such that $m = \beta_{v,w} \circ \varphi_v$. Since both φ_v and $\beta_{v,w}$ are monotone, the operation m is monotone, as well.

(b) For the element w in part (a) we get that $w = \beta_{v,w}(v) = \beta_{v,w}(1 \wedge v) = m(1)$. Since $\beta_{v,w}$ is an isomorphism, the elements v and w are similar, and so $v \sim m(1)$. Furthermore, for arbitrary element u of L we have that

$$\begin{aligned} m(u) = w \quad &\iff \quad \beta_{v,w}(u \wedge v) = w \\ &\iff \quad u \wedge v = v \\ &\iff \quad v \leqslant u. \end{aligned}$$

This proves that the element v must be the least element of the set

$$\{u \in L \mid m(u) = m(1)\},$$

and hence it is uniquely determined by m.

(c) It is straightforward to check that $m(0) = 0$. Let $d \in L$ be an arbitrary atom. If $d \not\leqslant v$ then $d \wedge v = 0$ implies that $m(d) = \beta_{v,w}(d \wedge v) = \beta_{v,w}(0) = 0$.

If $d \leqslant v$ then $m(d) = \beta_{v,w}(d \wedge v) = \beta_{v,w}(d)$. Since $0 \prec d$ and $\beta_{v,w}$ is an isomorphism, we get that $0 \prec \beta_{v,w}(d)$. Hence, $m(d) = \beta_{v,w}(d)$ is an atom, as well. Thus, the inclusion $m(\mathcal{A}_0) \subseteq \mathcal{A}_0$ and all other claims in (c) are proved.

(d) Assume that $m(d) = 0$ holds for every atom d of $\mathbb{L}$. If the inequality $0 < v$ were true then there would be an atom $d \leqslant v$. Then by part (c), $0 < m(d)$ would hold, which contradicts the assumption. Hence, $v = 0$. Then for arbitrary element $x \in L$ we get that $m(x) = \beta_{0,w}(x \wedge 0) = \beta_{0,w}(0) = 0$. Therefore, $m = \varphi_0$. □

Lemma 4.2. *Suppose $\mathbb{L}$ has at least two atoms. If f is a binary operation in the stabilizer of M then*

(a) *$f(\mathcal{A}_0 \times \mathcal{A}_0) \subseteq \mathcal{A}_0$ and $f(0,0) = 0$;*

(b) *$f|_{\mathcal{A}_0}$ is an essentially unary operation;*

(c) *if $f|_{\mathcal{A}_0}$ does not depend on its first [second] variable then $f(x,0) = 0$ [$f(0,x) = 0$] for all $x \in L$.*

Proof. Throughout this proof we will repeatedly use the fact that for any two atoms k, l of $\mathbb{L}$ there is a unique isomorphism $\beta_{k,l} \colon (k] \to (l]$, and hence the transformations φ_k and $\beta_{k,l} \circ \varphi_k$ belong to M. Now choose and fix two distinct atoms d_0 and d_1 of $\mathbb{L}$, and let d and d' be arbitrary atoms of $\mathbb{L}$.

(a) Since $f \in \mathrm{Sta}(M)$, the transformations $t = f(\varphi_d, \beta_{d,d'} \circ \varphi_d)$, $r = f(\varphi_d, \varphi_0)$, and $l = f(\varphi_0, \varphi_d)$ belong to M. Thus we get from Proposition 4.1 (c) that $t(d) = f(d, d')$, $r(d) = f(d, 0)$, and $l(d) = f(0, d)$ belong to $\mathcal{A}_0$, and $0 = t(0) = f(0,0)$. This proves the first statement.

(b) To prove the second statement, define two unary transformations m and n as follows:

$$m = f(\beta_{d_0,d} \circ \varphi_{d_0}, \beta_{d_1,d'} \circ \varphi_{d_1}), \qquad n = f(\beta_{d_0,d} \circ \varphi_{d_0}, \beta_{d_0,d'} \circ \varphi_{d_0}).$$

Again, $f \in \mathrm{Sta}(M)$ implies that $m, n \in M$. Furthermore, we have

$$\begin{aligned} m(x) &= f(\beta_{d_0,d}(x \wedge d_0), \beta_{d_1,d'}(x \wedge d_1)) \\ &= \begin{cases} f(d,0) & \text{if } x = d_0, \\ f(0,d') & \text{if } x = d_1, \\ f(0,0) = 0 & \text{if } x \in \mathcal{A}_0 \setminus \{d_0, d_1\}, \end{cases} \end{aligned} \tag{1}$$

and

$$\begin{aligned} n(x) &= f(\beta_{d_0,d}(x \wedge d_0), \beta_{d_0,d'}(x \wedge d_0)) \\ &= \begin{cases} f(d,d') & \text{if } x = d_0, \\ f(0,0) = 0 & \text{if } x \in \mathcal{A}_0 \setminus \{d_0\}. \end{cases} \end{aligned} \tag{2}$$

First we will prove that at least one of the elements $f(d,0)$ and $f(0,d')$ is 0. We proceed by contradiction. Suppose that $f(d,0), f(0,d') > 0$. By part (a), the element $f(d,d') = n(d_0)$ is in $\mathcal{A}_0$, and $m(d_0) = f(d,0) > 0$, $m(d_1) = f(0,d') > 0$. Since $m \in M$, there are similar elements $v, w \in L$ and an isomorphism $\beta_{v,w}\colon (v] \to (w]$ such that $m = \beta_{v,w} \circ \varphi_v$. Hence, by Proposition 4.1 (c), this implies that $d_0, d_1 \leqslant v$, and

$$m(d_0) = \beta_{v,w}(d_0 \wedge v) = \beta_{v,w}(d_0) \neq \beta_{v,w}(d_1) = \beta_{v,w}(d_1 \wedge v) = m(d_1),$$

since d_0 and d_1 are distinct atoms. Thus $f(d,d') = m(1) \geqslant m(d_0) \vee m(d_1) \notin \mathcal{A}_0$. This contradicts part (a), and therefore proves that $f(d,0)$ or $f(0,d')$ is 0.

If $f(d,0) = f(0,d') = 0$ then by formula (1) the value of m is 0 for all atoms of $\mathbb{L}$, therefore $m = \varphi_0$ by Proposition 4.1 (d). Thus, $f(d,d') = m(1) = 0 = f(d,0) = f(0,d')$.

Suppose that $f(d,0) > 0$ or $f(0,d') > 0$ holds. Without loss of generality, we may suppose that $f(d,0) > 0$. Then $f(0,d') = 0$, so by formula (1) we have that $m(d_0) = f(d,0) > 0$, and $m(c) = 0$ for all atoms c distinct from d_0. Furthermore, $m(1) = f(d,d') = n(d_0) \in \mathcal{A}_0$, hence the monotonicity of m implies that $f(d,d') = f(d,0)$.

Thus, we get that if for all atoms d, d' of $\mathbb{L}$ the equalities $f(d,0) = 0$ and $f(0,d') = 0$ hold then $f|_{\mathcal{A}_0}$ is constantly 0. Otherwise, there is an atom d of $\mathbb{L}$ for which either $f(d,0) > 0$ or $f(0,d) > 0$. If $f(d,0) > 0$ then $f(0,d') = 0$ for all atoms d' of $\mathbb{L}$. Hence, by the preceeding argument $f(k,l) = f(k,0)$ for all atoms $k, l \in \mathcal{A}$. Therefore, the operation $f|_{\mathcal{A}_0}$ is essentially unary. A similar argument shows that if $f(0,d) > 0$ then $f|_{\mathcal{A}_0}$ is also an essentially unary unary operation.

(c) Without loss of generality, we may assume that $f|_{\mathcal{A}_0}$ does not depend on its second variable. Then $f(0,d) = f(0,0) = 0$ for all atoms d of $\mathbb{L}$, by part (a). For the operation $t = f(\varphi_0, \varphi_1) \in M$, this means that t is 0 for every atom of $\mathbb{L}$. Then by Proposition 4.1 (d), $t = \varphi_0$. Hence, $0 = t(x) = f(\varphi_0(x), \varphi_1(x)) = f(0,x)$ for all $x \in L$. □

4.2 When the monoid IS($\mathbb{L}$) is collapsing

This section is devoted to the proof of the following theorem which characterizes the collapsing monoids among the inverse monoids of the form IS($\mathbb{L}$) where $\mathbb{L}$ is a finite lattice.

First, we need the following definition. Let a and b be arbitrary elements of L. We will say that the element b is **dwarfed** by a if for all elements $b' \in L$ such that $b' \sim b$ we have that $b' \leqslant a$. We will use the notation $b \ll a$ to denote that a dwarfs b. Now we are in a position to state the central result of this chapter.

Theorem 4.3 ([Dor07])**.** *Let $\mathbb{L}$ be a finite lattice such that $|L| \geqslant 3$. Then the inverse monoid $M = \mathrm{IS}(\mathbb{L})$ is collapsing if and only if no element of $\mathcal{J} \setminus \mathcal{A}$ dwarfs a nonzero element of L.*

Proof. Suppose that there are elements $a \in \mathcal{J} \setminus \mathcal{A}$ and $b \in L \setminus \{0\}$ such that $b \ll a$. Then $b \leqslant a$ since we have that

$$b \leqslant \bigvee \{b' \in L \mid b' \sim b\} \leqslant a.$$

We will construct an essentially binary operation f that belongs to the stabilizer of M. Let $\bar{a}$ be the unique lower cover of a, and define f in the following way:

$$f(x, y) = \begin{cases} x \wedge a = x \wedge \bar{a} & \text{if } a \nleqslant x, \\ \bar{a} = x \wedge \bar{a} & \text{if } a \leqslant x,\ b \nleqslant y, \\ a & \text{if } a \leqslant x, b \leqslant y. \end{cases}$$

Since we have $f(0, 1) = 0$, $f(1, 1) = a$ and $f(1, 0) = \bar{a}$, therefore f is an essentially binary operation. To check that f belongs to the stabilizer of M, consider arbitrary elements $m_1 = \beta_{u_1,v_1} \circ \varphi_{u_1}$ and $m_2 = \beta_{u_2,v_2} \circ \varphi_{u_2}$ of M, and set $t = f(m_1, m_2)$.

If $a \nleqslant v_1$ or $b \nleqslant v_2$ then $a \nleqslant m_1(x)$ for every $x \in L$ or $b \nleqslant m_2(x)$ for every $x \in L$. Thus

$$t(x) = f(m_1(x), m_2(x)) = m_1(x) \wedge \bar{a} = \varphi_{\bar{a}}(m_1(x)) \text{ for all } x \in L.$$

Hence $t = \varphi_{\bar{a}} \circ m_1 \in M$.

Now assume that $a \leqslant v_1$ and $b \leqslant v_2$. Then $t(1) = f(v_1, v_2) = a$, and there exist elements $a' \leqslant u_1$, $b' \leqslant u_2$ such that $\beta_{u_1,v_1}(a') = a$ and $\beta_{u_2,v_2}(b') = b$. Next we prove that $b' \leqslant a'$.

Claim 4.4. *For any elements $c, d \in L$ the following statements are equivalent:*

(i) *The element d is dwarfed by c.*

(ii) *The inequality $d' \leqslant c'$ holds for all elements $c', d' \in \mathbb{L}$ for which $c' \sim c$ and $d' \sim d$.*

The implication (ii) $\Longrightarrow$ (i) is an easy consequence of the definition.

To prove that (i) implies (ii) choose an arbitrary element $c' \in L$ such that $c' \sim c$, and let $\beta \colon [c) \to [c')$ be an isomorphism. Furthermore, let H_d denote the set $\{d' \in L \mid d' \sim d\}$. By (i), for arbitrary element $d' \in H_d$ we have that $d' \leqslant c$. Since β is an isomorphism, we get that

$$d \sim \beta(d') \leqslant \beta(c) = c',$$

and so $\beta(d') \in H_d$. Therefore the isomorphism β induces a permutation of H_d. Hence, $d' \leqslant c'$ holds for arbitrary elements $c', d' \in L$ for which $c' \sim c$ and $d' \sim d$, that is, $d \ll c$. This completes the proof of Claim 4.4.

Since $b \ll a$ and $a' \sim a$, $b' \sim b$, by Claim 4.4, we get that $b' \leqslant a'$. Let x be an arbitrary element of L. If $a' \nleqslant x$ then $a' \nleqslant x \wedge u_1$. Hence, $a \nleqslant \beta_{u_1,v_1}(x \wedge u_1) = m_1(x)$; therefore

$$t(x) = f(m_1(x), m_2(x)) = m_1(x) \wedge a.$$

If $a' \leqslant x$ then $a' \leqslant x \wedge u_1$ and because of $b' \leqslant a' \leqslant x$ we have $b' \leqslant x \wedge u_2$. This implies that $a \leqslant \beta_{u_1,v_1}(x \wedge u_1) = m_1(x)$ and $b \leqslant \beta_{u_2,v_2}(x \wedge u_2) = m_2(x)$, therefore

$$t(x) = f(m_1(x), m_2(x)) = a = m_1(x) \wedge a.$$

Thus $t(x) = m_1(x) \wedge a$ for all $x \in L$, showing that $t = \varphi_a \circ m_1 \in M$. This proves that the binary operation f is in the stabilizer of M. Hence, M is not collapsing.

Now suppose that the monoid M is not collapsing. We will show that there is an element of $\mathcal{J} \setminus \mathcal{A}$ that dwarfs a nonzero element of L.

If the lattice $\mathbb{L}$ has only one atom then let the element b be the unique atom of $\mathbb{L}$ and let a be an upper cover of b. Then a is not an atom, but join-irreducible and $b \ll a$ holds.

From now on, we will suppose that the lattice $\mathbb{L}$ has at least two (distinct) atoms. For an arbitrary element u of L define a set F_u as follows:

$$F_u = \big\{ f \in \mathrm{Sta}(M)^{(2)} \,\big|\, f|_{\mathcal{A}_0} \text{ does not depend on its second variable,} \\ \text{and there are elements } y_1, y_2 \in L \text{ such that } f(u, y_1) \neq f(u, y_2) \big\}.$$

Furthermore, let W be the set $\{u \in L \mid F_u \neq \emptyset\}$. By the result of Jens-Uwe Grabowski [Gra97], the stabilizer of M contains an essentially binary operation, which ensures the set W to be non-empty. By Lemma 4.2 (c) $F_0 = \emptyset$, therefore $0 \notin W$. Notice that every operation $f \in F_u$ ($u \in W$) is essentially binary. Indeed, f depends on its second variable because there are elements $y_1, y_2 \in L$ such that $f(u, y_1) \neq f(u, y_2)$. In view of Lemma 4.2 (a) and (c) we have $f(0, y_1) = f(0, y_2) = 0$. Since $f(u, y_1) \neq f(u, y_2)$, at least one of the sets $\{f(0, y_1), f(u, y_1)\}$ and $\{f(0, y_2), f(u, y_2)\}$ has more than one element, which proves that f depends on its first variable.

Choose a minimal element a from W, and let p be a minimal element of the set $\{h(a,0) \mid h \in F_a\}$, which is not empty, since $F_a \neq \emptyset$. This means that the set $\{g \in F_a \mid g(a,0) = p\}$ is a non-empty finite set, so let f be an element of this set that is minimal with respect to the pointwise order of operations in F_a. Finally, let b be a minimal element of the set $\{d \in L \mid f(a,0) \neq f(a,d)\}$. The elements a, b and the operation f selected this way will be fixed for the rest of the proof. Some of their basic properties are summarized in the next claim.

Claim 4.5. *We have*

(a) $0 < a$ *and* $0 < b$*;*

(b) $f(x,y) = f(x,0)$ *whenever* $x, y \in L$ *and* $x < a$*;*

(c) $h(a,0) < f(a,0)$ *for no* $h \in F_a$*;*

(d) *if* $g \in F_a$ *is such that* $g(a,0) = f(a,0)$ *and* $g(x,y) \leqslant f(x,y)$ *for all* $x, y \in L$*, then* $g = f$*;*

(e) *if* $c < b$ *then* $f(a,c) = f(a,0)$*.*

In (a) $0 < b$ follows from the choice of b, and $0 < a$ from the fact that $0 \notin W$. The minimality of a implies that $F_x = \emptyset$ for every elements $x < a$ ($x \in L$). In particular, for each such x we have $f \notin F_x$ although $f \in \mathrm{Sta}(M)^{(2)}$ and $f|_{\mathcal{A}_0}$ does not depend on its second variable. Thus, for such an x, $f(x,y)$ cannot depend on its variable y. This proves (b). Properties (c), (d) and (e) are immediate consequences of the minimality of $f(a,0) = p$, the minimality of f, and the minimality of b, respectively.

Claim 4.6. *If c is an element of L such that $c \nleqslant f(a,0)$ and $c \leqslant f(a,b)$ then the operation $\varphi_c \circ f$ is in F_a.*

Let $\bar{f}$ be the binary operation $\varphi_c \circ f \in \mathrm{Sta}(M)$. Then $\bar{f}(x,y) = f(x,y) \wedge c$ for all $x, y \in L$. Therefore $\bar{f}|_{\mathcal{A}_0}$ does not depend on its second variable, because $f|_{\mathcal{A}_0}$ has this property. Furthermore,

$$\bar{f}(a,0) = f(a,0) \wedge c \leqslant f(a,0) \wedge f(a,b) < c = \bar{f}(a,b).$$

Thus $\bar{f} \in F_a$, completing the proof of Claim 4.6.

Claim 4.7. *The elements $f(a,0)$ and $f(a,b)$ are comparable.*

Suppose that $f(a,0) \parallel f(a,b)$, and let $\bar{f}$ be the binary operation $\varphi_{f(a,b)} \circ f \in \mathrm{Sta}(M)$. By Claim 4.6, $\bar{f} \in F_a$. However, $\bar{f}(a,0) = f(a,0) \wedge f(a,b) < f(a,0)$, which contradicts Claim 4.5 (c). This completes the proof of the claim.

Let m and n be the unary operations $f(\varphi_a, \varphi_0)$ and $f(\varphi_a, \varphi_b)$, respectively; that is, $m(x) = f(a \wedge x, 0)$ and $n(x) = f(a \wedge x, b \wedge x)$ for all $x \in L$. For the operation n we get that

$$f(a, a \wedge b) = n(a) \leqslant n(1) = f(a, b). \tag{3}$$

Claim 4.8. *For the operation $g(x,y) = f(\varphi_a(x), \varphi_b(y))$ we have that*

$$g(x,y) = f(x \wedge a, y \wedge b) = \begin{cases} f(x \wedge a, 0) \leqslant f(a, 0) & \text{if } a \nleqslant x, \\ f(a, 0) & \text{if } a \leqslant x,\ b \nleqslant y, \\ f(a, b) & \text{if } a \leqslant x,\ b \leqslant y. \end{cases}$$

Indeed, if $a \nleqslant x$ then $x \wedge a < a$. Hence by Claim 4.5 (b), $f(x \wedge a, y \wedge b) = f(x \wedge a, 0) = m(x) \leqslant m(1) = f(a, 0)$. If $a \leqslant x$, $b \nleqslant y$ then $x \wedge a = a$, $y \wedge b < b$. Then by Claim 4.5 (e), $f(x \wedge a, y \wedge b) = f(a, y \wedge b) = f(a, 0)$. Finally, if $a \leqslant x$, $b \leqslant y$ then $f(x \wedge a, y \wedge b) = f(a, b)$. This proves Claim 4.8.

From now on the argument splits according to whether $<$ or $=$ holds in (3).

Case 1: $n(a) < n(1)$.

In this case, we have $f(a, a \wedge b) < f(a, b)$ by (3), which implies that $a \wedge b < b$, in particular, $a \neq b$. It follows from Claim 4.5 (e) that $f(a, 0) = f(a, a \wedge b)$, and therefore $f(a, 0) < f(a, b)$.

Claim 4.9. $0 < f(a, 0)$.

Assume that $f(a, 0) = 0$. This assumption implies that $m(1) = f(a, 0) = 0$, and by Proposition 4.1 (d), $m = \varphi_0$. Since $n(x) = g(x, x)$ and $f(x \wedge a, 0) = m(x) = 0$ for all $x \in L$, we get from Claim 4.8 that

$$n(x) = \begin{cases} f(a, b) & \text{if } a \leqslant x,\ b \leqslant x, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of M, the range $\{0, f(a, b)\}$ of n is the ideal $(f(a, b)]$ and by Proposition 4.1 (b), there is an isomorphism $\beta_{a \vee b, f(a,b)} \colon (a \vee b] \to (f(a, b)]$ such that $n = \beta_{a \vee b, f(a,b)} \circ \varphi_{a \vee b}$. The equality $(f(a, b)] = \{0, f(a, b)\}$ implies that $f(a, b)$ is an atom, and so $a \vee b$ is an atom, as well. Therefore by Claim 4.5 (a), the elements a and b are atoms, furthermore $a \neq b$. Hence $a \vee b$ cannot be an atom. This contradiction proves Claim 4.9.

Claim 4.10. *$f(a, 0) \prec f(a, b)$ and $f(a, b)$ is join-irreducible.*

Let $c \in L$ be an element such that $f(a,0) \prec c \leqslant f(a,b)$, and let $\bar{f}$ be the operation $\varphi_c \circ f \in \mathrm{Sta}(M)$. By Claim 4.6, the operation $\bar{f}$ is in F_a. Since $\bar{f}(a,0) = f(a,0)$, and $\bar{f}(x,y) = f(x,y) \wedge c \leqslant f(x,y)$ for all $x, y \in L$, we get from Claim 4.5 (d) that $\bar{f} = f$. Hence, $f(a,0) \prec c = \bar{f}(a,b) = f(a,b)$.

The element $f(a,b)$ is the join of all the join-irreducible elements u for which $u \leqslant f(a,b)$. Since $f(a,0) < f(a,b)$, there is an element $u_0 \in \mathcal{J} \cap (f(a,b)]$ such that $u_0 \nleqslant f(a,0)$, that is $f(a,0) < u_0$ or $u_0 \parallel f(a,0)$. In the latter case, by Claim 4.6, the operation $\tilde{f} = \varphi_{u_0} \circ f$ is in F_a. Moreover, $\tilde{f}(a,0) < f(a,0)$ which contradicts Claim 4.5 (c). Thus we must have $f(a,0) < u_0 \leqslant f(a,b)$ whence $f(a,b) = u_0 \in \mathcal{J}$, since $u_0 \leqslant f(a,b)$ and $f(a,0) \prec f(a,b)$. This completes the proof of Claim 4.10.

Claim 4.11. *The element b is similar to $f(a,b)$, hence it is join-irreducible, and $a < b$.*

By Claim 4.8, for the unary operation n we have that

$$n(x) = g(x,x) = \begin{cases} f(a \wedge x, 0) & \text{if } a \nleqslant x, \\ f(a,0) & \text{if } a \leqslant x,\ b \nleqslant x, \\ f(a,b) & \text{if } a \leqslant x,\ b \leqslant x. \end{cases}$$

Since $f(a,0) < f(a,b) = n(1)$, we get from Proposition 4.1 (b) that $n = \beta_{a \vee b, f(a,b)} \circ \varphi_{a \vee b}$ and $a \vee b \sim f(a,b) \in \mathcal{J}$. Thus $a \vee b \in \mathcal{J}$, therefore $\{a,b\} \cap \mathcal{J} \neq \emptyset$ and a, b are comparable. Since $a \wedge b < b$, we get that $a < b \in \mathcal{J}$, completing the proof of Claim 4.11.

Thus, $n = \beta_{b,f(a,b)} \circ \varphi_b$. For arbitrary element c of the interval $[a,b)$ we get from Claim 4.5 (e) that $f(a,c) = f(a,0) = f(a,a)$ since $a, c < b$. Therefore

$$n(c) = f(c \wedge a, c \wedge b) = f(a,c) = f(a,a) = n(a).$$

Thus $c = a$ since $a \leqslant c < b$ and $\beta_{b,f(a,b)}$ is an isomorphism. This proves that a is the unique lower cover of b. Hence $\beta_{b,f(a,b)}$ maps b to $f(a,b)$ and a to $f(a,0)$.

Now, we will prove that the element a is isolated, that is, $a' \sim a$ implies $a' = a$. Suppose that the element $a' \in L$ is similar to a, and set $s = g(\beta_{a',a} \circ \varphi_{a'}, \varphi_b)$ where g is the operation defined in Claim 4.8. Then $s \in M$ and $s(1) = g(a,b) = f(a,b)$. Hence $s = \beta_{b',f(a,b)} \circ \varphi_{b'}$ for some element $b' \in L$, which is similar to $f(a,b)$. Then

$$f(a,b) = s(b') = g(\beta_{a',a}(b' \wedge a'), b' \wedge b).$$

Hence Claim 4.8 implies that $a \leqslant \beta_{a',a}(b' \wedge a')$ and $b \leqslant b' \wedge b$. Since $\beta_{a',a}(b' \wedge a') \leqslant a$ and $b' \wedge b \leqslant b$, we have that $\beta_{a',a}(b' \wedge a') = a$ and $b' \wedge b = b$. The second equality shows that $b \leqslant b'$, but since $b' \sim f(a,b) \sim b$ we get that $b = b'$. The first equality implies that $b' \wedge a' = a'$. Thus $a' \leqslant b'$, and equality cannot hold because that would imply that $a \sim a' = b' = b$, which is impossible. Hence, $a' < b' = b$. Since a is the unique lower cover of b and $a \sim a'$, we get that $a' = a$. This proves that the element a is isolated. Since $a \sim f(a,0)$, as witnessed by $\beta_{b,f(a,b)}$, we conclude that $f(a,0) = a$.

Since a is isolated, the join of elements similar to a is a. Since $0 < a$, the element b cannot be an atom. Therefore, for the elements $b \in \mathcal{J} \setminus \mathcal{A}$, $a \in L \setminus \{0\}$ we have that $a \ll b$.

Case 2: $n(a) = n(1)$.

Claim 4.12. *$b \leqslant a$.*

In this case, (3) shows that $f(a, a \wedge b) = f(a,b)$, so by Claim 4.5 (e), we have $a \wedge b = b$, i.e., $b \leqslant a$.

Since the transformations m and n are in M, there are elements u, v, u', v' in L and isomorphisms $\beta_{u,v} \colon (u] \to (v]$, $\beta_{u',v'} \colon (u'] \to (v']$ such that

$$m = f(\varphi_a, \varphi_0) = \beta_{u,v} \circ \varphi_u, \qquad \text{and} \qquad n = f(\varphi_a, \varphi_b) = \beta_{u',v'} \circ \varphi_{u'}.$$

We will denote the isomorphisms $\beta_{u,v}$ and $\beta_{u',v'}$ by β and β', respectively. Thus

$$f(x \wedge a, 0) = m(x) = \beta(x \wedge u),$$

and

$$f(x \wedge a, x \wedge b) = n(x) = \beta'(x \wedge u')$$

for all $x \in L$.

Claim 4.13. *If $x \in L$ is such that $a \nleqslant x$ then $m(x) = n(x)$.*

If $a \nleqslant x$ then $x \wedge a < a$ and by Claim 4.5 (b),

$$n(x) = f(x \wedge a, x \wedge b) = f(x \wedge a, 0) = m(x)$$

holds. This completes the proof of Claim 4.13.

Since $m(a) = f(a,0) = m(1)$ and $n(a) = f(a, a \wedge b) = f(a,b) = n(1)$ we have that $u \leqslant a$ and $u' \leqslant a$, by Proposition 4.1 (b). Now we distinguish cases according to whether $=$ or $<$ holds. The followings are easy consequences of the definition of m and n:

$$\begin{aligned} f(a,0) &= m(1) = v = m(u) = \beta(u) \sim u, \\ f(a,b) &= n(1) = v' = n(u') = \beta'(u') \sim u'. \end{aligned} \tag{4}$$

Case 2.1: $u, u' < a$.

Then $a \nleqslant u, u'$, and by Claim 4.13 and (4),

$$f(a,0) = m(u) = n(u) = \beta'(u \wedge u') \leqslant v' = f(a,b),$$
$$f(a,b) = n(u') = m(u') = \beta(u' \wedge u) \leqslant v = f(a,0).$$

Hence $f(a,0) = f(a,b)$, which contradicts the choice of the elements a and b.

Case 2.2: $u = a$ and $u' < a$.
Then $a \nleqslant u'$, and by Claim 4.13 and (4),

$$f(a,b) = n(u') = m(u') = \beta(u' \wedge u) = \beta(u') < \beta(u) = v = f(a,0).$$

Next we want to show that $a \in \mathcal{J}$. Suppose that there are distinct elements $r_1, r_2 \in L$ such that $r_1, r_2 \prec a = u$. Then $m(r_1) = \beta(r_1) \neq \beta(r_2) = m(r_2)$ since β is an isomorphism, and $n(r_j) \leqslant n(1) = f(a,b)$ $(j = 1,2)$ by the monotonicity of n and by (4). Furthermore, $m(r_j) = n(r_j)$ $(j = 1,2)$ by Claim 4.13. Thus, combining these with (4) and the fact that β is an isomorphism we get that

$$\begin{aligned} f(a,0) &= \beta(u) = \beta(r_1 \vee r_2) = \beta(r_1) \vee \beta(r_2) \\ &= m(r_1) \vee m(r_2) = n(r_1) \vee n(r_2) \leqslant f(a,b). \end{aligned}$$

Since $f(a,b) < f(a,0)$, this is impossible. Hence $a \in \mathcal{J}$, and for the the unique lower cover $\bar{a}$ of a we have $u' \leqslant \bar{a} \prec a = u$, therefore by (4) and Claim 4.13

$$f(a,b) = \beta'(u') = \beta'(\bar{a} \wedge u') = n(\bar{a}) = m(\bar{a}) = \beta(\bar{a}) \prec \beta(a) = f(a,0),$$

where $\beta(a) \sim a$. Hence, $f(a,b) \prec f(a,0) \sim a \in \mathcal{J}$.

From Claim 4.12 we know that $b \leqslant a$. First we will argue the case $a = b$. Suppose $a = b$. Then $f(a,a) \prec f(a,0) \sim a \in \mathcal{J}$. We want to show that the element a is isolated. Let z be an arbitrary element that is similar to a, and set $t = f(\varphi_a, \beta_{z,a} \circ \varphi_z) \in M$. Then $t(1) = f(a,a)$ and by Claim 4.5 (e), $t(a) = f(a, \beta_{z,a}(a \wedge z)) \in \{f(a,0), f(a,a)\}$. The monotonicity of t implies that $t(a)$ must be equal to $f(a,a)$, and so $\beta_{z,a}(a \wedge z) = a$. Since $\beta_{z,a}$ is an isomorphism, we get that $a \wedge z = z$, i.e., $z \leqslant a$. As $z \sim a$, we have that $z = a$. Therefore, a is isolated, and it cannot be an atom, because our assumption that $\mathbb{L}$ has at least two atoms ensures that atoms are not isolated in $\mathbb{L}$. Hence $b = a \in \mathcal{J} \setminus \mathcal{A}$, and $b \ll a$.

Now we suppose that $b < a$. Let b' be an arbitrary elements of L such that $b' \sim b$, and let t be the unary operation $f(\varphi_a, \beta_{b',b} \circ \varphi_{b'}) \in M$. Suppose that $b' \nleqslant a$. Then $a \wedge b' < b'$ implies that $\beta_{b',b}(a \wedge b') < b$ and so $t(a) =$

$f(a, \beta_{b',b}(a \wedge b')) = f(a, 0)$ by Claim 4.5 (e). Furthermore, $t(1) = f(a, b)$. The monotonicity of t implies that $f(a, 0) = t(a) \leqslant t(1) = f(a, b)$, however, this is impossible. Hence, $b' \leqslant a$, proving that $\bigvee\{b' \in L \mid b' \sim b\} \leqslant a$. Since $0 < b < a \in \mathcal{J}$, this proves that a is not an atom and $b \ll a$.

Case 2.3: $u < a$ and $u' = a$.

Then $a \nleqslant u$, and by Claim 4.13 and (4),

$$f(a, 0) = m(u) = n(u) = \beta'(u \wedge u') = \beta'(u) \leqslant \beta'(u') = f(a, b).$$

We want to show that $a \in \mathcal{J}$. Suppose that there are distinct elements $r_1, r_2 \in L$ such that $r_1, r_2 \prec a = u'$. Then $n(r_1) = \beta'(r_1) \neq \beta'(r_2) = n(r_2)$ since β' is an isomorphism, and $m(r_j) \leqslant m(1) = f(a, 0)$ $(j = 1, 2)$ by the monotonicity of m and by (4). Moreover, $m(r_j) = n(r_j)$ $(j = 1, 2)$ by Claim 4.13. Thus, in the same way as in Case 2.2, we get that

$$\begin{aligned} f(a, b) = \beta'(u') = \beta'(r_1 \vee r_2) &= \beta'(r_1) \vee \beta'(r_2) \\ = n(r_1) \vee n(r_2) &= m(r_1) \vee m(r_2) \leqslant f(a, 0). \end{aligned}$$

Since $f(a, 0) < f(a, b)$, this is impossible. Hence $a \in \mathcal{J}$, and for the unique lower cover $\bar{a}$ of a we have $u \leqslant \bar{a} \prec a = u'$, therefore by (4) and Claim 4.13

$$f(a, 0) = \beta(u) = \beta(\bar{a} \wedge u) = m(\bar{a}) = n(\bar{a}) = \beta'(\bar{a}) \prec \beta'(a) = f(a, b),$$

where $\beta'(a) \sim a$. Hence, $f(a, 0) \prec f(a, b) \sim a \in \mathcal{J}$. Since $b \leqslant a$ by Claim 4.12, we get that either $a = b$ or $b < a$.

If $a = b$ then $f(a, 0) \prec f(a, a) \sim a \in \mathcal{J}$. Let z be an arbitrary element that is similar to a, and set $t = f(\varphi_a, \beta_{z,a} \circ \varphi_z) \in M$. Let x be an arbitrary element of L. If $a \nleqslant x$ then $x \wedge a < a$ and by Claim 4.5 (b), $t(x) = f(x \wedge a, \beta z, a(x \wedge z)) = f(x \wedge a, 0) = m(x) \leqslant m(1) = f(a, 0)$. If $a \leqslant x$ and $z \nleqslant x$ then $x \wedge a = a$ and $x \wedge z < z$. The latter implies that $\beta_{z,a}(x \wedge z) < a$ since $\beta_{z,a}$ is an isomorphism. Hence by Claim 4.5 (e), $t(x) = f(x \wedge a, \beta_{z,a}(x \wedge z)) = f(a, \beta_{z,a}(x \wedge z)) = f(a, 0)$. Finally, if $a, z \leqslant x$ then $t(x) = f(a, a)$. Since $t(1) = f(a, a)$, we have by Proposition 4.1 (b) that $t = \beta_{a \vee z, f(a,a)} \circ \varphi_{a \vee z}$ with $a \vee z \sim f(a, a)$. Thus $a \vee z \sim f(a, a) \sim a$. Since a is join-irreducible and $a \sim z$, the element z must be equal to a. Hence, a is a join-irreducible isolated element of $\mathbb{L}$, and $b \ll a$. The element a is not an atom, because our assumption that $\mathbb{L}$ has at least two atoms ensures that atoms are not isolated in $\mathbb{L}$.

Now assume that $b < a$. Let b' be arbitrary element of L such that $b' \sim b$, and set $t = f(\varphi_a, \beta_{b',b} \circ \varphi_{b'})$. Then $t(1) = f(a, b)$, and by Proposition 4.1 (b), there is an element $a' \sim t(1)$ such that $t(a') = t(1)$. Thus $t(a') = f(a, b)$. Hence, by the definition of t we have that

$$f(a, b) = t(a') = f(a' \wedge a, \beta_{b',b}(a' \wedge b')). \tag{5}$$

Since $\sim$ is an equivalence relation and $a' \sim t(1) = f(a, b) \sim a$, we have that $a' \sim a$. Therefore either $a' = a$ or $a' \parallel a$. Assume that $a' \parallel a$. Then $a' \wedge a < a$. Applying first (5), then Claim 4.5 (b), and finally the definition and the monotonicity of m, we get that

$$\begin{aligned} f(a, b) &= f(a' \wedge a, \beta_{b',b}(a' \wedge b')) = f(a' \wedge a), 0) \\ &= m(a' \wedge a) \leqslant m(a) = f(a, 0). \end{aligned}$$

This is a contradiction since $f(a, 0) < f(a, b)$. Therefore, $a' = a$, and so

$$f(a, b) = t(a') = t(a) = f(a, \beta_{b',b}(a \wedge b')).$$

Since $\beta_{b',b}(a \wedge b') \leqslant b$, we get from Claim 4.5 (e) that $\beta_{b',b}(a \wedge b')$ must equal b', which implies that $b' \leqslant a \wedge b'$, i.e., $b' \leqslant a$. Hence, $b \ll a$. Here a is not an atom, because $0 < b < a$.

Case 2.4: $u = u' = a$.

Then by (4), $f(a, 0) \sim u \sim u' \sim f(a, b)$. Since $f(a, 0) \neq f(a, b)$, this implies that $f(a, 0) \parallel f(a, b)$, which contradicts Claim 4.7.

This completes the proof of Theorem 4.3. □

As a corollary of Theorem 4.3, we show that for all atomistic lattices $\mathbb{L}$ with at least three elements the inverse monoids IS($\mathbb{L}$) are collapsing. Furthermore, we describe all lattices $\mathbb{L}$ with at most 6 elements for which the inverse monoids IS($\mathbb{L}$) are collapsing.

From now on, we will assume $\mathbb{L}$ to be a finite lattice with at least 3 elements. The lattice $\mathbb{L}$ will be called **atomistic** if every element of $L \setminus \{0\}$ is a join of atoms.

Corollary 4.14. *If* $\mathbb{L}$ *is an atomistic lattice then* IS($\mathbb{L}$) *is collapsing.*

Proof. If $\mathbb{L}$ is an atomistic lattice then the set of join-irreducible elements coincides with the set of atoms. Therefore by Theorem 4.3, the monoid IS($\mathbb{L}$) is collapsing. □

As another application of Theorem 4.3 we determine all lattices $\mathbb{L}$ with at most six elements for which the inverse monoids IS($\mathbb{L}$) are collapsing.

Corollary 4.15. *For a lattice* $\mathbb{L}$ *such that* $3 \leqslant |L| \leqslant 6$, IS($\mathbb{L}$) *is collapsing if and only if* $\mathbb{L}$ *is isomorphic to one of the lattices in Figure 5.*

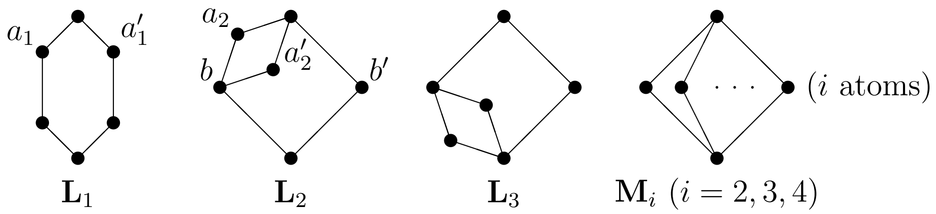

Figure 5.

Proof. Since the lattices $\mathbf{L}_3$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$, are atomistic, we get from Corollary 4.14, that the inverse monoids $\mathrm{IS}(\mathbb{L}_3)$, $\mathrm{IS}(\mathbf{M}_2)$, $\mathrm{IS}(\mathbf{M}_3)$, and $\mathrm{IS}(\mathbf{M}_4)$ are collapsing.

In the lattice $\mathbb{L}_1$ there are exactly two join-irreducible elements that are not atoms: a_1 and a_1'. Furthermore, these elements are similar, and $a_1 \wedge a_1' = 0_{\mathbb{L}_1}$. Hence there is no element other then 0 which is dwarfed by a. In the lattice $\mathbb{L}_2$ the join-irreducible elements that are not atoms are a_2 and a_2'. These elements are similar, and $a_2 \wedge a_2' = b \succ 0$. Since $b \sim b'$ and $b \vee b' = 1_{\mathbb{L}_2}$ we get that there is no element other than 0 which is dwarfed by a. Thus by Theorem 4.3, the inverse monoids $\mathrm{IS}(\mathbb{L}_1)$ and $\mathrm{IS}(\mathbb{L}_2)$ are collapsing.

It is straightforward to check that every lattice L $(3 \leqslant |L| \leqslant 6)$ that is not isomorphic to either of the lattices in Figure 5 is isomorphic to one of the four lattices at the top of Figure 6 or has exactly one atom, or has exactly one coatom, that is, it has the form shown at the bottom of Figure 6.

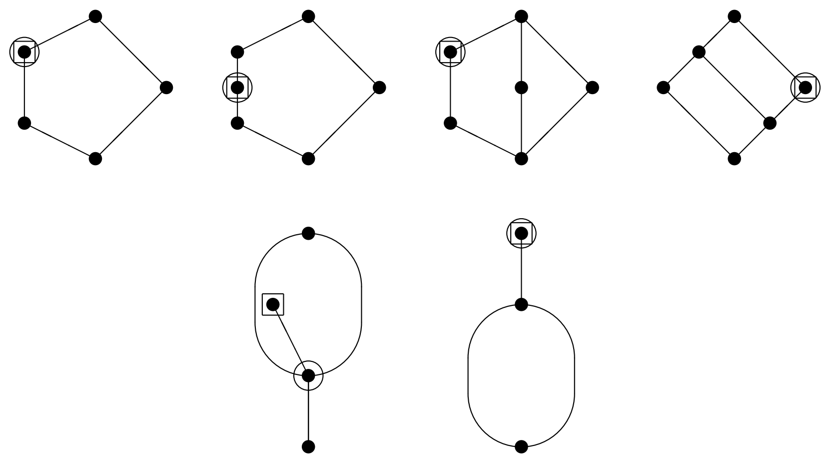

Figure 6.

To prove that for these lattices $\mathbb{L}$ the inverse monoids $\mathrm{IS}(\mathbb{L})$ is not collapsing, we provide elements $a \in \mathcal{J}(\mathbb{L}) \setminus \mathcal{A}(\mathbb{L})$ and $b \in L \setminus \{0\}$ such that $b \ll a$. In Figure 6 the boxed element is a, and the encircled element is b. This completes the proof of Corollary 4.15. $\square$

4.3 Examples when Int(IS($\mathbb{L}$)) is large

In this section we present some examples of lattices $\mathbb{L}$ for which the interval Int(IS($\mathbb{L}$)) are infinite.

On the 2-element set $A = \{0, 1\}$, there is only one lattice, up to isomorphism, namely the 2-element chain $\mathbf{C}_2$ with the partial order $0 < 1$. Then $\mathrm{IS}(\mathbf{C}_2)$ consists of the unary operations $\varphi_1 = \mathrm{id}_A$ and φ_0. Using Post's results (cf. Post [Pos41]), we get the following.

Theorem 4.16. *The monoidal interval corresponding to* $\mathrm{IS}(\mathbf{C}_2)$ *contains countably infinite clones.*

From now on, we will assume $\mathbb{L}$ to be a finite lattice with at least 3 elements.

Theorem 4.17 ([Dor07])**.** *For a 3-element chain* $\mathbb{L}$ *we have* $|\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| = 2^{\aleph_0}$.

Proof. Let $\mathbb{L}$ be the chain $0 < 1 < 2$ on $\{0, 1, 2\}$. Then $M = \mathrm{IS}(\mathbb{L}) = \{\varphi_0, \varphi_1, \varphi_2\}$. Now we describe the operations in $\mathrm{Sta}(M)$.

Claim 4.18. *An n-ary operation $f \in \mathcal{O}_L$ belongs to the stabilizer of M if and only if $f(0, \dots, 0) = 0$ and $f(1 \wedge s_1, \dots, 1 \wedge s_n) = 1 \wedge f(s_1, \dots, s_n)$ holds for all elements $s_1, \dots, s_n \in L$.*

Let $f \in \mathcal{O}_L$ be an n-ary operation satisfying the requirements of the claim, and let $s_1, \dots, s_n \in L$ be arbitrary elements of L. Set $t = f(\varphi_{s_1}, \dots, \varphi_{s_n})$. Then $t(0) = f(0, \dots, 0) = 0 = 0 \wedge f(s_1, \dots, s_n)$, and $t(2) = f(2 \wedge s_1, \dots, 2 \wedge s_n) = f(s_1, \dots, s_n) = 2 \wedge f(s_1, \dots, s_n)$. The assumption on f and these equalities imply that $t = \varphi_{f(s_1,\dots,s_n)}$, whence $t \in M$. This proves that $f \in \mathrm{Sta}(M)$.

Conversely, if $f \in \mathrm{Sta}(M)$ is an n-ary operation then for all $s_1, \dots, s_n \in L$ we have that $t = f(\varphi_{s_1}, \dots, \varphi_{s_n}) \in M$. Since $t(2) = f(2 \wedge s_1, \dots, 2 \wedge s_n) = f(s_1, \dots, s_n)$, we get that $t = \varphi_{f(s_1,\dots,s_n)}$. Hence, $f(0, \dots, 0) = t(0) = 0$, and $f(1 \wedge s_1, \dots, 1 \wedge s_n) = t(1) = 1 \wedge f(s_1, \dots, s_n)$. This completes the proof of Claim 4.18.

Let $U_k, V_k, W_k \subseteq L^k$ ($k \in \mathbb{N},\ k \geqslant 3$) denote the following sets

$$\begin{aligned}
U_k &= \big\{(0,2,2,\dots,2), (2,0,2,\dots,2), \dots, (2,2,2,\dots,2,0)\big\},\\
V_k &= \big\{(0,1,1,\dots,1), (1,0,1,\dots,1), \dots, (1,1,1,\dots,1,0)\big\},\\
W_k &= \{1,2\}^k \setminus \big\{(2,2,2,\dots,2)\big\}.
\end{aligned}$$

Define an n-ary operation f_n and an m-ary relation ϱ_m ($m, n \in \mathbb{N},\ m, n \geqslant 3$) on L as follows:

$$f_n(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_1 = x_2 = x_3 = \cdots = x_n = 0, \\ 2 & \text{if } (x_1, x_2, x_3, \dots, x_n) \in U_n, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\varrho_m = \big\{(0, 0, 0, \dots, 0)\big\} \cup U_m \cup V_m \cup W_m \subseteq L^m.$$

Next, we summarize some easy observations on the relation ϱ_m for later reference. All these facts are simple consequences of the definition of ϱ_m.

Claim 4.19. *Let $(a_1, a_2, a_3, \dots, a_m)$ be an arbitrary element of ϱ_m. Then*

(a) *if $0 \in \{a_1, a_2, a_3, \dots, a_m\}$ then $(a_1, \dots, a_m) \in \big\{(0, \dots, 0)\big\} \cup U_m \cup V_m$;*

(b) *if there exist indices $1 \leqslant i < i' \leqslant m$ such that $a_i = a_{i'} = 0$ then $a_j = 0$ for all j ($1 \leqslant j \leqslant m$).*

From the definition of f_n we get that

$$1 \wedge f(s_1, \dots, s_n) = f(1 \wedge s_1, \dots, 1 \wedge s_n) = \begin{cases} 0 & \text{if } s_1 = \cdots = s_n = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for all elements $s_1, \dots, s_n \in L$. Hence by Claim 4.18, the operation f_n belongs to $\mathrm{Sta}(M)$ ($n \in \mathbb{N},\ n \geqslant 3$).

Claim 4.20. $M \subseteq \mathrm{Pol}(\varrho_m)$ ($m \in \mathbb{N},\ m \geqslant 3$).

Let $(a_1, \dots, a_m) \in L^m$ be an arbitrary element of ϱ_m, and let $t = \varphi_s$ be an arbitrary element of M. If $a_1 = \cdots = a_m = 0$ or $s = 0$ then $(t(a_1), \dots, t(a_m)) = (0, \dots, 0) \in \varrho_m$. If $t = \varphi_2$ then $(t(a_1), \dots, t(a_m)) = (2 \wedge a_1, \dots, 2 \wedge a_m) = (a_1, \dots, a_m) \in \varrho_m$. Finally, if $t = \varphi_1$ and $(a_1, \dots, a_m) \neq (0, \dots, 0)$ then $(a_1, \dots, a_n) \in U_m \cup V_m \cup W_m$. If $(a_1, \dots, a_n) \in U_m \cup V_m$ then $(t(a_1), \dots, t(a_m)) \in V_m \subseteq \varrho_m$, while if $(a_1, \dots, a_m) \in W_m$ then

$$(t(a_1), \dots, t(a_m)) = (1, \dots, 1) \in \varrho_m.$$

This proves Claim 4.20.

Claim 4.21. *For $m, n \geqslant 3$, the operation f_n preserves the relation ϱ_m if and only if $m \neq n$.*

Clearly, the operation f_n preserves the relation ϱ_m if and only if for every $n \times m$ matrix whose rows belong to ϱ_m, the m-tuple of column values of f_n belongs to ϱ_m, as well.

The rows of the $m \times m$ matrix

$$\begin{pmatrix} 0 & 2 & 2 & \cdots & 2 \\ 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 0 \end{pmatrix}$$

belong to ϱ_m, however, the m-tuple of the column values of f_m is

$$(2, 2, 2, \dots, 2),$$

which is not in ϱ_m. Therefore the operation f_m does not preserve the relation ϱ_m.

From now on, we will suppose that $m \neq n$. Let $H = (h_{i\,j})_{n \times m}$ be an arbitrary $n \times m$ matrix whose rows belong to ϱ_m. The rows and the transposed columns of H will be denoted by r_i and c_j ($1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$), and we set $h = (f_n(c_1), \dots, f_n(c_m))$. Our aim is to prove that $h \in \varrho_m$.

If there exist indices $1 \leqslant j < j' \leqslant m$ such that $c_j = c_{j'} = (0, 0, 0, \dots, 0)$ then by Claim 4.19 (b), $r_i = (0, 0, 0, \dots, 0)$ for all i ($1 \leqslant i \leqslant n$). Then $h = (0, 0, 0, \dots, 0) \in \varrho_m$.

If there is exactly one $j_0 \in \{1, \dots, m\}$ for which $c_{j_0} = (0, 0, 0, \dots, 0)$ then by Claim 4.19 (a), the rows of H belong to $\big\{(0, \dots, 0)\big\} \cup U_m \cup V_m$. If every row of H belongs to $U_m \cup V_m$ then $c_j \in \{1, 2\}^n$ for all j ($1 \leqslant j \leqslant m,\ j \neq j_0$). Therefore $f_n(c_j) = 1$ for all j ($1 \leqslant j \leqslant m,\ j \neq j_0$). Since $f_n(c_{j_0}) = 0$ we get that $h \in V_m \subseteq \varrho_m$. If there exist indices $1 \leqslant i < i' \leqslant n$ such that $r_i = r_{i'} = (0, 0, 0, \dots, 0)$ then for all j ($1 \leqslant j \leqslant m,\ j \neq j_0$) we get that $f_n(c_j) = 1$. Hence, $h \in V_m \subseteq \varrho_m$ since $f_n(c_{j_0}) = 0$. If there is exactly one $i_0 \in \{1, \dots, n\}$ such that $r_{i_0} = (0, 0, 0, \dots, 0)$ then either there is an $i_1 \in \{1, \dots, n\}$, $i_1 \neq i_0$, such that $r_{i_1} \in V_m$ or $r_1 = \dots = r_{i_0 - 1} = r_{i_0 + 1} = \dots = r_n \in U_m$. In the first case H is of the form

$$\begin{array}{c} \\ i_0 \\ \\ i_1 \\ \\ \end{array} \begin{pmatrix} \vdots & & \vdots & \overset{j_0}{\vdots} & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$

and so, $h = (1, \ldots, 1, 0, 1, \ldots, 1) \in V_m \subseteq \varrho_m$. In the second case we get that H is of the form

$$\begin{array}{c} \\ \\ \\ i_0 \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & j_0 & & & \end{array} \\ \begin{pmatrix} 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \end{pmatrix} \end{array},$$

and so, $h = (2, \ldots, 2, 0, 2, \ldots, 2) \in U_m \subseteq \varrho_m$.

If $c_j \neq (0, 0, 0, \ldots, 0)$ for all j $(1 \leqslant j \leqslant m)$ then $f_n(c_j) \in \{1, 2\}$ for all j $(1 \leqslant j \leqslant m)$, and so $h \in \{1, 2\}^m$. Our aim is to show that $h \neq (2, 2, 2, \ldots, 2)$. By the definition of f_n, the equality $h = (2, 2, 2, \ldots, 2)$ holds if and only if $c_1, \ldots, c_m \in U_n$. Since $U_n \subseteq \{0, 2\}^n$ we get that if H has a 1 entry then $h \neq (2, 2, 2, \ldots, 2)$. Further on we may suppose that all entries of H are 0 or 2. If there is an $i_0 \in \{1, \ldots, n\}$ such that $r_{i_0} = (0, 0, 0, \ldots, 0)$ then there must be a column c_{j_0} of H such that $c_{j_0} \notin U_n$. Then $f_n(c_{j_0}) = 1$ implies that $h \in W_m \subseteq \varrho_m$. Otherwise, $r_i \neq (0, 0, 0, \ldots, 0)$ for all i $(1 \leqslant i \leqslant n)$. Therefore, every row of H contains at most one 0 entry. Since $m \neq n$, there is a column c_{j_0} of H which contains either no 0 entries or more than two 0 entries. Then $f_n(c_{j_0}) = 1$, and so $h \in W_m \subseteq \varrho_m$. This completes the proof of Claim 4.21.

Let I be an arbitrary subset of $\{k \in \mathbb{N} \mid k \geqslant 3\}$, and set

$$F_I = \langle \{f_i \mid i \in I\} \cup M \rangle .$$

Then $F_I \subseteq \mathrm{Sta}(M)$. If $I_1, I_2 \subseteq \{k \in \mathbb{N} \mid k \geqslant 3\}$, $I_1 \neq I_2$, then we may suppose, without restricting generality, that there is an element $i \in \mathbb{N}$ such that $i \in I_1$ and $i \notin I_2$. Therefore $F_{I_2} \neq F_{I_1}$, since by Claims 4.20 and 4.21, $F_{I_2} \subseteq \mathrm{Pol}(\varrho_i)$ but $F_{I_1} \nsubseteq \mathrm{Pol}(\varrho_i)$. Hence,

$$2^{\aleph_0} = \left|\left\{F_I \,\middle|\, I \subseteq \{k \in \mathbb{N} \mid k \geqslant 3\}\right\}\right| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| \leqslant 2^{\aleph_0},$$

which proves that $|\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| = 2^{\aleph_0}$. □

We conclude the paper with a discussion of lattices $\mathbb{L}$ for which the monoidal interval $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ has cardinality $2^{\aleph_0}$. For elements $u \leqslant v$ of $\mathbb{L}$, we will use the notation $[u, v]$ for the interval $\{x \in L \mid u \leqslant x \leqslant v\}$. We will call a lattice $\mathbb{L}$ **pinched** if L contains an element $b \in L \setminus \{0, 1\}$ such that $L = [0, b] \cup [b, 1]$.

Theorem 4.22 ([Dor07]). *Let $\mathbb{L}$ be a pinched lattice, and let $b \in L \setminus \{0,1\}$ be an element such that $L = [0,b] \cup [b,1]$. Then $|\mathrm{Int}(\mathrm{IS}([0,b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$.*

Proof. It is easy to see that we have $c \leqslant b$ or $b \leqslant c$ for every element $c \in L$. Hence, the element b is isolated. Let M and $\widetilde{M}$ be the inverse monoids $\mathrm{IS}(\mathbb{L})$ and $\mathrm{IS}([0,b])$, respectively.

Claim 4.23. *$[0,b]$ is closed under each operation $f \in \mathrm{Sta}(M)$.*

Let $c_1, \dots, c_n$ be arbitrary elements of $[0,b]$, and set $t = f(\varphi_{c_1}, \dots, \varphi_{c_n})$. Then by Proposition 4.1 (b), there are similar elements $u, v \in L$ and an isomorphism $\beta\colon (u] \to (v]$ such that $t = \beta_{u,v} \circ \varphi_u$. Since $c_1, \dots, c_n \leqslant b$, we get that

$$t(1) = f(c_1, \dots, c_n) = f(b \wedge c_1, \dots, b \wedge c_n) = t(b).$$

Then by Proposition 4.1 (b), $f(c_1, \dots, c_n) = v \leqslant b$. Hence, $f(c_1, \dots, c_n) \in [0,b]$. This proves Claim 4.23.

Claim 4.24. *For all transformations $m \in M$ and for all elements $c \in [b,1]$ we have that $m(c) \wedge b = m(b) \wedge b$.*

By Proposition 4.1 (b), there are similar elements $u, v \in L$ and isomorphism $\beta_{u,v}\colon (u] \to (v]$ such that $m = \beta_{u,v} \circ \varphi_u$. If $u \leqslant b$ then again by Proposition 4.1 (b), we get that $m(c) = m(b)$ for all $c \in [b,1]$. Hence, $m(c) \wedge b = m(b) \wedge b$ for all $c \in [b,1]$. On the other hand, if $u > b$ then by Proposition 4.1 (a) and by the fact that b is isolated, we get that $b = m(b) < m(u) = v$. Hence, $b = m(b) \wedge b \leqslant m(c) \wedge b \leqslant b$ implies that $m(c) \wedge b = m(b) \wedge b$ for all $c \in [b,1]$. This proves Claim 4.24.

Let u be an arbitrary element of the interval $[0,b]$. By Claim 4.23, the unary operation φ_u can be restricted to $[0,b]$, the restriction $\varphi_u|_{[0,b]}$ will be denoted by $\widetilde{\varphi}_u$.

Claim 4.25. *$M|_{[0,b]} = \widetilde{M}$.*

First we prove that $M|_{[0,b]} \subseteq \widetilde{M}$. Let t be an arbitrary transformation from M. Then by Proposition 4.1 (b), there are similar elements $u, v \in L$ and an isomorphism $\beta_{u,v}\colon (u] \to (v]$ such that $t = \beta_{u,v} \circ \varphi_u$. If $u \leqslant b$ then $v \leqslant b$ also holds, since $u \sim v$. Therefore the elements u, v are similar in the interval $[0,b]$. Hence, $t|_{[0,b]} = \beta_{u,v} \circ \widetilde{\varphi}_u \in \widetilde{M}$. If $u > b$ then $v > b$, since $u \sim v$. Furthermore, $\beta_{u,v}|_{[0,b]}$ is the isomorphism $\beta_{b,b}\colon [0,b] \to [0,b]$, $c \mapsto \beta_{u,v}(c)$, since b is isolated. Hence, for all $c \in [0,b]$ we get that $t(c) = \beta_{u,v}(c \wedge u) = \beta_{u,v}(c) = \beta_{b,b}(c) = \beta_{b,b}(c \wedge b)$. Thus, $t|_{[0,b]} = \beta_{b,b} \circ \widetilde{\varphi}_b \in \widetilde{M}$.

To prove the reverse inclusion, choose an arbitrary transformation $s \in \widetilde{M}$. Define the unary transformation f_s on L as follows:

$$f_s\colon L \to L,\ f_s(x) = s(x \wedge b).$$

By Proposition 4.1 (b), there are similar elements $u, v \in [0, b]$ such that $s = \beta_{u,v} \circ \widetilde{\varphi}_u$. If $x \leqslant b$ then $x \wedge b = x$ and $f_s(x) = s(x \wedge b) = s(x) = \beta_{u,v}(x \wedge u)$. If $x \geqslant b$ then $x \wedge b \geqslant u$ and $f_s(x) = s(x \wedge b) = \beta_{u,v}((x \wedge b) \wedge u) = \beta_{u,v}(x \wedge u)$. Hence, $f_s = \beta_{u,v} \circ \varphi_u \in M$. This concludes the proof of Claim 4.25.

Let g be an arbitrary n-ary operation from $\mathrm{Sta}(\widetilde{M})$, and define the n-ary operation f_g on L as follows:

$$f_g\colon L^n \to L,\ (a_1, \ldots, a_n) \mapsto g(a_1 \wedge b, \ldots, a_n \wedge b).$$

Claim 4.26. *For all $g \in \mathrm{Sta}(\widetilde{M})$ we have that $f_g|_{[0,b]} = g$ and $f_g \in \mathrm{Sta}(M)$.*

It is straightforward to check that the first statement is true. To prove the second statement choose arbitrary transformations $m_1, \ldots, m_n \in M$, and set $t = f_g(m_1, \ldots, m_n)$. By Proposition 4.1 (b), there are similar elements $u_i, v_i \in L$ and isomorphisms $\beta_i\colon (u_i] \to (v_i]$ for every i $(1 \leqslant i \leqslant n)$ such that $m_i = \beta_{u_i,v_i} \circ \varphi_{u_i}$. As $m_1|_{[0,b]}, \ldots, m_n|_{[0,b]} \in \widetilde{M}$ by Claim 4.25, we get that

$$t|_{[0,b]} = f_g|_{[0,b]}(m_1|_{[0,b]}, \ldots, m_n|_{[0,b]}) = g(m_1|_{[0,b]}, \ldots, m_n|_{[0,b]}) \in \widetilde{M}.$$

Then by Proposition 4.1 (b), there are similar elements $u, v \in [0, b]$ and an isomorphisms $\beta\colon (u] \to (v]$ such that $t|_{[0,b]} = \beta_{u,v} \circ \widetilde{\varphi}_u$. By Claim 4.24, for all $c \geqslant b$ we get that

$$\begin{aligned} t(c) &= f_g(m_1(c), \ldots, m_n(c)) \\ &= g(m_1(c) \wedge b, \ldots, m_n(c) \wedge b) \\ &= g(m_1(b) \wedge b, \ldots, m_n(b) \wedge b) \\ &= t(b). \end{aligned}$$

Thus, $t = \beta_{u,v} \circ \varphi_u \in M$. This proves Claim 4.26.

For an arbitrary clone $\mathcal{D} \in \mathrm{Int}(\widetilde{M})$ define the clone $\mathcal{C}_{\mathcal{D}}$ in the following way:

$$\mathcal{C}_{\mathcal{D}} = \langle \{ f_g \mid g \in \mathcal{D} \} \rangle .$$

Since $M \subseteq \{ f_g \mid g \in \mathcal{D} \} \subseteq \mathrm{Sta}(M)$, we get that $\mathcal{C}_{\mathcal{D}}$ is in $\mathrm{Int}(M)$. Furthermore, by Claims 4.25 and 4.26, we get that

$$\mathcal{C}_{\mathcal{D}}|_{[0,b]} = \left\langle \left\{ f_g|_{[0,b]} \mid g \in \mathcal{D} \right\} \right\rangle = \langle \{ g \mid g \in \mathcal{D} \} \rangle = \mathcal{D}.$$

Hence, the map

$$\varphi\colon \mathrm{Int}(\widetilde{M}) \to \mathrm{Int}(M),\ \mathcal{D} \mapsto \mathcal{C}_{\mathcal{D}}$$

is an injection, which proves that $|\mathrm{Int}(\mathrm{IS}([0, b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$. □

Corollary 4.27. *If $\mathbb{L}$ is a finite lattice which has a unique atom then the monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ *is infinite.*

Proof. Let $0 \prec b$ be the unique atom of $\mathbb{L}$. Then $\mathbb{L}$ is pinched: $L = [0, b] \cup [b, 1]$. Therefore by Theorem 4.22, $\aleph_0 = |\mathrm{Int}(\mathrm{IS}([0, b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$. □

Corollary 4.28. *If $\mathbb{L}$ is a finite chain with at least 3 elements then the monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ *has cardinality* $2^{\aleph_0}$.

Proof. We may assume that the finite chain $\mathbb{L}$ is the n-element chain on the set $\{0, 1, 2, \dots, n-1\}$ ($n \geqslant 3$) with the order $0 < 1 < 2 < \cdots < n-1$. Then the lattice $\mathbb{L}$ is pinched: $L = [0, 2] \cup [2, n-1]$. Therefore by Theorems 4.22 and 4.17, $2^{\aleph_0} = |\mathrm{Int}(\mathrm{IS}([0, 2]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$. Hence, $|\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| = 2^{\aleph_0}$. □

Chapter 5

Monoids with constants and permutations

In this chapter we study monoidal intervals that correspond to monoids, which consist of at least one unary constant operation and whose nonconstant operations are permutations.

In Section 4.2, using E. L. Post's result in [Pos41], we prove that if such a monoid contains exactly one unary constants operation then the corresponding monoidal interval is infinite.

Sections 4.3 is mainly devoted to determining all collapsing transformation monoids among those mentioned in the first paragraph (Theorem 5.1). This result generalizes the well-known theorem of P. P. Pálfy in [Pal84], which is concerned with monoids that contain all unary constant operations and whose nonconstant operations are permutations. Furthermore, we describe a family of transformation monoids that consist of at least three unary constant operations and some permutations for which the corresponding monoidal intervals are 2-element chains.

5.1 Main results of this chapter

The set of all unary constant operations and the set of all permutations on A will be denoted by $C(A)$ and $S(A)$, respectively. For arbitrary element $v \in A$ we will use the notation c_v for the unary constant operation with value v. Throughout this chapter, the monoid M is supposed to be contained in $C(A) \cup S(A)$, moreover we will assume that M contains at least one but not all unary constant operations. We note that for collapsing monoids that contain all the unary constant operations and whose non-constant operations are permutations a complete description is provided by P. P. Pálfy [Pal84], as we mentioned in the introduction. Hence we will also assume that M does not contain all unary constant operations. Let V be the set of all elements $v \in A$ such that $c_v \in M$, and set $W = A \setminus V$. By the assumptions on the monoid M, we have that $\emptyset \subsetneq V, W \subsetneq A$. Define P to be the set of all permutations contained in M. The facts that A is finite and M is closed under composition ensure that P is a permutation group on A and

$$\alpha(V) = V, \qquad \alpha(W) = W \tag{6}$$

hold for all $\alpha \in P$. These equalities allow us to restrict P to V and W, and obtain the permutation groups

$$P_V = \{\alpha|_V \mid \alpha \in P\} \subseteq S(V),$$
$$P_W = \{\alpha|_W \mid \alpha \in P\} \subseteq S(W).$$

Furthermore, let $\mathfrak{i}_V$ be the restriction map $\mathfrak{i}_V\colon P \to P|_V,\ \alpha \mapsto \alpha|_V$. If the map $\mathfrak{i}_V$ is injective, then for every transformation $m \in M$ the unique extension of the map $m|_V$ to A is m. Hence, if the map $\mathfrak{i}_V$ is injective, the map

$$\mathfrak{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W.$$

is well-defined.

Our first theorem characterizes all collapsing monoids that consist of permutations and at least one unary constant operation. This extends the results obtained by A. Fearnley and I. Rosenberg in [FR03].

Theorem 5.1 ([Dor08])**.** *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least one unary constant operation and some permutations. Then M is collapsing if and only if*

(i) $|V| \geqslant 2$,

(ii) *P_W is transitive,*

(iii) *$\mathfrak{i}_V$ is injective, and*

(iv) *one of the following conditions holds:*

  (a) *the monoid $M|_V$ is collapsing,*

  (b) *the map $\mathfrak{j}$ is not injective,*

  (c) *the permutation group P_W is not regular.*

Thus, a transformation monoid M that consists of at least one unary constant operation and some permutations is not collapsing, i.e., staisfies $|\mathrm{Int}(M)| \geqslant 2$, iff one of (i)–(iv) fails for M. In some of these cases, namely if (i) fails or if (i)–(iii) hold but (iv) fails, we know more about the monoidal interval $\mathrm{Int}(M)$. Theorem 5.2 below treats the case when (i) fails, and Theorem 5.3 the case when (i)–(iii) hold but (iv) fails.

Theorem 5.2 ([Dor08])**.** *Let $M \subseteq C(A) \cup S(A)$ be a monoid such that it contains only one unary constant operation. Then the monoidal interval* $\mathrm{Int}(M)$ *is infinite.*

Theorem 5.3 ([Dor08]). *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least two unary constant operations and some permutations. If conditions* (i)–(iii) *of Theorem 5.1 hold but condition* (iv) *of Theorem 5.1 fails for M then* $\mathrm{Int}(M)$ *is isomorphic to* $\mathrm{Int}(M|_V)$*. Hence,*

- *if $|V| = 2$ and $M|_V$ is the monoid $\{\mathrm{id}_V, c_0|_V, c_1|_V\}$, then $|W| = 1$, $M = \{\mathrm{id}_A, c_0, c_1\}$, and* $\mathrm{Int}(M)$ *is isomorphic to the direct square of the 2-element chain;*
- *if $|V| = 2$ and $M|_V$ is the full transformation semigroup on V, then $|W| = 2$, $M = \{\mathrm{id}_A, c_0, c_1, (0\,1)(2\,3)\}$, and* $\mathrm{Int}(M)$ *is a 3-element chain;*
- *if $|V| \geqslant 3$, then* $\mathrm{Int}(M)$ *is a 2-element chain.*

5.2 Monoids with infinite monoidal intervals

In this section we prove Theorem 5.2.

Proof of Theorem 5.2. We may assume without loss of generality that $0 \in A$ and $c_0 \in M$. For every natural number $n \geqslant 4$ define the n-ary operation f_n as follows:

$$f_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } |\{i \mid x_i = 0\}| \geqslant 2, \\ x_{\min\{j \mid x_j \neq 0\}} & \text{otherwise.} \end{cases}$$

We will prove that the operations f_n $(n \geqslant 4)$ are in $\mathrm{Sta}(M)$ and the clones $\mathcal{C}_n = \langle \{f_n\} \cup M \rangle$ $(n \geqslant 4)$ are pairwise distinct.

Consider arbitrary transformations $m_1, \ldots, m_n \in M$, and let m be the unary operation $f_n(m_1, \ldots, m_n)$. To prove that $m \in M$ suppose first that $|\{i \mid m_i = c_0\}| \geqslant 2$. Then there are indices $1 \leqslant j < k \leqslant n$ such that $m_j = m_k = c_0$. Let a be an arbitrary element of A, and set $\mathbf{a} = (m_1(a), \ldots, m_n(a))$. Then $f_n(\mathbf{a}) = 0$ since the equalities $m_j(a) = 0$ and $m_k(a) = 0$ ensure that more than one component of $\mathbf{a}$ is 0. Hence $m(a) = f_n(\mathbf{a}) = 0$ for every element $a \in A$, proving that $m = c_0 \in M$. It remains to consider the case when $|\{i \mid m_i = c_0\}| \leqslant 1$. In this case either $m_i \neq c_0$ for all i $(1 \leqslant i \leqslant n)$ or there is exactly one $i \in \{1, \ldots, n\}$ such that $m_i = c_0$. Hence by (6), $|\{i \mid m_i(a) = 0\}| = n \geqslant 2$ if $a = 0$, and $|\{i \mid m_i(a) = 0\}| \leqslant 1$ if $a \neq 0$. Then by the definition of f_n we get that for arbitrary element $a \in A$

$$m(a) = f_n(\mathbf{a}) = \begin{cases} 0 & \text{if } a = 0, \\ m_2(a) & \text{if } a \neq 0 \text{ and } m_1 = c_0, \\ m_1(a) & \text{otherwise.} \end{cases}$$

Hence the unary operation

$$f_n(m_1, \dots, m_n) = m = \begin{cases} m_2 & \text{if } m_1 = c_0, \\ m_1 & \text{otherwise} \end{cases}$$

is in M, proving that the operations f_n $(n \geqslant 4)$ are in $\mathrm{Sta}(M)$.

Now we prove that the clones $\mathcal{C}_n$ $(n \geqslant 4)$ are pairwise distinct. The binary relation $\varrho = \{(0,0)\} \cup (W \times W)$ is a congruence relation of the algebras $\mathbb{A}_n = (A; \mathcal{C}_n)$ $(n \geqslant 4)$. Identify the sets $\{0\}$ and W with 0 and 1, respectively. The meet and join operations with respect to the partial order $0 < 1$ will be denoted by $\wedge$ and $\vee$, respectively. Then the clone of term operations of the quotient algebra $\mathrm{Clo}\,(\mathbb{A}_n/\varrho)$ is

$$\mathrm{Clo}\,(\mathbb{A}_n/\varrho) = \{f/\varrho \mid f \in \mathcal{C}_n\} = \langle \{f_n/\varrho\} \cup \{m/\varrho \mid m \in M\} \rangle = \langle f_n/\varrho, c_0/\varrho \rangle\,,$$

since the clone $\mathcal{C}_n$ is generated by the set $\{f_n\} \cup \{m \mid m \in M\}$ and we have that $\{m/\varrho \mid m \in M\} = \{\mathrm{id}_A/\varrho, c_0/\varrho\}$. The operation c_0/ϱ is the unary constant operation on $\{0, 1\}$ with value 0, and the operation f_n/ϱ is the following:

$$(f_n/\varrho)(x_1, \dots, x_n) = \bigvee_{k=1}^{n} (x_1 \wedge \dots \wedge x_{k-1} \wedge x_{k+1} \wedge \dots \wedge x_n).$$

Using Post's results in [Pos41], we get that the clones

$$\mathrm{Clo}\,(\mathbb{A}_n/\varrho) = \langle f_n/\varrho, c_0/\varrho \rangle \quad (n \geqslant 4)$$

are pairwise distinct. Hence the clones $\mathcal{C}_n$ $(n \geqslant 4)$ are pairwise distinct, as well. This completes the proof of the theorem. □

5.3 Monoids with finite monoidal intervals

Lemma 5.4. *If the permutation group P_W is intransitive, then the monoid M is not collapsing.*

Proof. Assume that the permutation group P_W is intransitive. Then for all elements $w \in W$ we have that $\{\alpha(w) \mid \alpha \in P\} \subsetneq W$. Consider an arbitrary element $a \in A$. Then

$$\{m(a) \mid m \in M\} = V \cup \{\alpha(a) \mid \alpha \in P\} \subsetneq V \cup W = A.$$

Hence, M is not weakly transitive, and by a result of Ihringer–Pöschel [IP93], the monoid M is not collapsing. □

Lemma 5.5. *Every operation in* $\mathrm{Sta}(M)$ *can be restricted to* V.

Proof. Let f be an arbitrary n-ary operation in $\mathrm{Sta}(M)$, and choose arbitrary elements $v_1, \ldots, v_n$ in V. The unary operation $m = f(c_{v_1}, \ldots, c_{v_n})$ is constant with value $f(v_1, \ldots, v_n)$, and m belongs to M, since $c_{v_1}, \ldots, c_{v_n} \in M$ and $f \in \mathrm{Sta}(M)$. Thus it follows from the definition of V that $f(v_1, \ldots, v_n) \in V$. □

We will denote the set $\{f|_V \mid f \in \mathrm{Sta}(M)\}$ of restrictions of operations in $\mathrm{Sta}(M)$ by $\mathrm{Sta}(M)|_V$.

Lemma 5.6. *Suppose that*

(i) $|V| \geqslant 2$, *and*

(ii) *the permutation group* P_W *is transitive.*

If the map $\mathfrak{i}_V$ *is not injective then* M *is not collapsing.*

Proof. Suppose that the map $\mathfrak{i}_V$ is not injective. We will prove that the monoid M is not collapsing by exhibiting an essentially binary operation in the stabilizer of M. Since $\mathfrak{i}_V$ is not injective, there are permutations $\alpha, \beta \in P$ such that $\alpha \neq \beta$ but $\alpha|_V = \beta|_V$. Choose distinct elements v and v' from V, and define the binary operation f as follows:

$$f(x,y) = \begin{cases} \alpha(x) & \text{if } x \in V, \text{ or } x \in W \text{ and } y \neq v', \\ \beta(x) & \text{if } x \in W \text{ and } y = v'. \end{cases}$$

It follows from this definition that $f(x, v) = \alpha(x)$ for all $x \in A$. Since α is a permutation, there are distinct elements $a_1,\ a_2 \in A$ such that $f(a_1, v) \neq f(a_2, v)$. Furthermore, there is an element $w \in W$ such that $\alpha(w) \neq \beta(w)$, and so $f(w, v) = \alpha(w) \neq \beta(w) = f(w, v')$. These equalities prove that the operation f is essentially binary.

To prove that f is in the stabilizer of M, choose arbitrary transformations $m_1, m_2 \in M$. Assume first that m_1 is a permutation. Then by (6), $m_1(a) \in V$ for every element $a \in V$, and $m_1(b) \in W$ for every element $b \in W$. Since $\alpha|_V = \beta|_V$, we have that $\alpha(m_1(a)) = \beta(m_1(a))$, hence

$$f(m_1(a), m_2(a)) = \alpha(m_1(a)) = \beta(m_1(a)). \tag{7}$$

If m_2 is the unary constant operation with value v', then for every element $b \in W$ we have that $(m_1(b), m_2(b)) \in W \times \{v'\}$, and so, $f(m_1(b), m_2(b)) = \beta(m_1(b))$ holds by the definition of f. Therefore $f(m_1, m_2)$ is the unary operation $\beta \circ m_1 \in M$ by (7). If $m_2 \neq c_{v'}$, then $m_2(b) \neq v'$ holds for every element $b \in W$. Hence, by the definition of f, for all elements $b \in W$ we have that $f(m_1(b), m_2(b)) = \alpha(m_1(b))$. Therefore $f(m_1, m_2) = \alpha \circ m_1 \in M$ holds by (7).

If m_1 is not a permutation, then $m_1 = c_a$ for some element $a \in V$. Then $m_1(x) = a \in V$ for every element $x \in A$. Hence $f(m_1(x), m_2(x)) = \alpha(x)$ for every element $x \in A$, that is, $f(m_1, m_2) = c_{\alpha(a)}$ is in M.

Hence f is an essentially binary operation in the stabilizer of M, which proves that the monoid M is not collapsing. □

The next three lemmas are concerned with the case when (i)–(iii) hold for M.

Lemma 5.7. *Suppose that*

(i) $|V| \geqslant 2$,

(ii) *the permutation group* P_W *is transitive, and*

(iii) *the map* $\mathfrak{i}_V$ *is injective.*

If $\mathrm{Sta}(M)|_V$ *contains only essentially unary operations, then* M *is collapsing.*

Proof. Let f be an arbitrary n-ary operation in $\mathrm{Sta}(M)$. By Lemma 5.5, the operation f can be restricted to V, moreover by the assumption, the restriction $f|_V$ of f to V is an essentially unary operation. Hence there is an index $i \in \{1, \dots, n\}$ and there is a unary operation $m \in M$ for which $f(v_1, \dots, v_n) = m(v_i)$ holds for every n-tuple $(v_1, \dots, v_n) \in V^n$. Our aim is to prove that f is essentially unary. To prove this, fix an element $w_0 \in W$, and let $(a_1, \dots, a_n)$ be an arbitrary n-tuple in A^n. Then there are transformations $t_1, \dots, t_n \in M$ such that $t_i(w_0) = a_i$ $(1 \leqslant i \leqslant n)$. Set $t = f(t_1, \dots, t_n)$. Since f is in $\mathrm{Sta}(M)$, the unary operation t is in M. Furthermore,

$$t(w_0) = f(t_1, \dots, t_n)(w_0) = f(t_1(w_0), \dots, t_n(w_0)) = f(a_1, \dots, a_n), \qquad (8)$$

and for every element $a \in V$ we have that

$$t(a) = f(t_1, \dots, t_n)(a) = f(t_1(a), \dots, t_n(a)) = m(t_i(a)). \qquad (9)$$

The unary operation $m \in M$ is either a permutation or a unary constant operation. If m is a unary constant operation, then there is an elements $v \in V$ such that $m = c_v$. Then (9) implies that $t(a) = v$ for all elements $a \in V$. However, since $|V| \geqslant 2$ this latter fact shows that $t = c_v$, and so by (8), $f(a_1, \dots, a_n) = t(w_0) = v = m(a_i)$. Otherwise, if m is a permutation, then $t|_V = (mt_i)|_V$ by (9), and the injectivity of $\mathfrak{i}_V$ implies that $t = mt_i$.

Hence $f(a_1, \dots, a_n) = t(w_0) = (mt_i)(w_0) = m(t_i(w_0)) = m(a_i)$. Therefore, in both cases we get that for arbitrary n-tuple $(a_1, \dots, a_n) \in A^n$ the equality

$$f(a_1, \dots, a_n) = m(a_i)$$

holds, proving that f is essentially unary. Hence M is collapsing, since the stabilizer of M contains only essentially unary operations. This completes the proof. □

If the monoid $M|_V$ is collapsing, then the monoid M is also collapsing, by Lemma 5.7. Henceforth we will investigate the monoids M for which $M|_V$ is not collapsing.

Lemma 5.8. *Suppose that*

(i) $|V| \geqslant 2$,

(ii) *the permutation group* P_W *is transitive,*

(iii) *the map* $\mathfrak{i}_V$ *is injective, and*

(iv) *the monoid* $M|_V$ *is not collapsing.*

If the map $\mathfrak{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W$ *is not injective or the permutation group* P_W *is not regular, then* M *is collapsing.*

Proof. To prove the statement, suppose that either the map $\mathfrak{j}$ is not injective or the permutation group P_W is not regular. Then there are permutations $\alpha, \beta \in P$ such that $\alpha|_V \neq \beta|_V$ and $\alpha(w^*) = \beta(w^*)$ for some element $w^* \in W$. By Proposition 5.7, it is enough to prove that $\mathrm{Sta}(M)|_V$ contains only essentially unary operations.

Suppose that $\mathrm{Sta}(M)|_V$ contains an operation that is not essentially unary. Thus $\mathrm{Sta}(M)|_V$ is a clone on V with unary part $M|_V$ that is different from the essentially unary clone $\langle M|_V \rangle$. If $|V| \geqslant 3$, this implies by the result of Pálfy [Pal84]that the clone $\mathrm{Sta}(M)|_V$ is the set of all polynomial operations of some finite vector space $(V;\, +\,, \lambda\cdot\ \ (\lambda \in K))$ over a finite field K. If $|V| = 2$, say $V = \{0, 1\}$, then the assumtion that $\mathfrak{j}$ is not injective or P_W is not regular implies that $|P_V| \neq 1$. Therefore $M|_V$ is the full transformation semigroup on V, and the monoidal interval $\mathrm{Int}(M|_V)$ is the 3-element chain $\langle M|_V \rangle \subsetneq \langle +, c_1 \rangle \subsetneq \mathcal{O}_V$, where $+$ is addition modulo 2. Hence, either $\mathrm{Sta}(M)|_V = \langle +, c_1 \rangle$ or $\mathrm{Sta}(M)|_V = \mathcal{O}_V$. Thus we get that, in all cases, the monoid $M|_V$ is the set of all unary polynomial operations of some finite vector space $(V;\, +\,, \lambda\cdot\ \ (\lambda \in K))$ over a finite field K, moreover, the binary operation $x - y$ is in $\mathrm{Sta}(M)|_V$. Then there are elements $\lambda_1, \lambda_2 \in K$ and $v_1, v_2 \in V$ such that $\alpha|_V = \lambda_1 \cdot x + v_1$ and $\beta|_V = \lambda_2 \cdot x + v_2$, and there is a binary operation $f \in \mathrm{Sta}(M)$ for which $f|_V(x, y) = x - y$. Define the unary transformations t_1 and t_2 to be the transformations $f(\alpha, \alpha)$ and $f(\alpha, \beta)$, respectively. Then for all $v \in V$ we have that

$$\begin{aligned} t_1|_V(v) &= f|_V(\alpha|_V, \beta|_V)(v) \\ &= \alpha(v) - \alpha(v) \\ &= 0, \\ t_2|_V(v) &= f|_V(\alpha|_V, \beta|_V)(v) \\ &= \alpha|_V(v) - \beta|_V(v) \\ &= (\lambda_1 - \lambda_2) \cdot v + (v_1 - v_2). \end{aligned}$$

Hence, $t_1 = c_0$. Suppose that $\lambda_1 \neq \lambda_2$. Then $t_2|_V$ is a permutation, hence t_2 must be a permutation. Since $\alpha(w^*) = \beta(w^*)$ and $w^* \in W$, we get that

$$\begin{aligned} 0 = t_1(w^*) &= f(\alpha, \alpha)(w^*) = f(\alpha(w^*), \alpha(w^*)) \\ &= f(\alpha(w^*), \beta(w^*)) = t_2(w^*) \in W. \end{aligned}$$

This is a contradiction, since $0 \in V$. Thus $\lambda_1 = \lambda_2$. This implies that $v_1 \neq v_2$, since $\alpha|_V \neq \beta|_V$, and that $t_2|_V$ is constant with value $v_1 - v_2$. Hence t_2 is the unary constant operation $c_{v_1 - v_2}$, and we have

$$\begin{aligned} 0 = t_1(w^*) &= f(\alpha, \alpha)(w^*) = f(\alpha(w^*), \alpha(w^*)) \\ &= f(\alpha(w^*), \beta(w^*)) = f(\alpha, \beta)(w^*) = t_2(w^*) = v_1 - v_2. \end{aligned}$$

This contradiction proves that $\mathrm{Sta}(M)|_V$ contains only essentially unary operations. It follows from Lemma 5.7 that the monoid M is collapsing, and this completes the proof of Lemma 5.8. □

Lemma 5.9. *Suppose that*

(i) $|V| \geqslant 2$,

(ii) *the permutation group P_W is transitive,*

(iii) *the map $\mathfrak{i}_V$ is injective, and*

If the map $\mathfrak{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W$ is injective and the permutation group P_W is regular, then the intervals $\mathrm{Int}(M)$ *and* $\mathrm{Int}(M|_V)$ *are isomorphic.*

Proof. We note that the map $\mathfrak{j}$ is well-defined, since $\mathfrak{i}_V$ is injective by the assumption.

Claim 5.10. *The restriction map $\sigma\colon M \to M|_V,\ m \mapsto m|_V$ is an isomorphism between the monoids M and $M|_V$. Hence, every transformation in $M|_V$ has a unique extension in M.*

Since equality $(mm')|_V = m|_V m'|_V$ holds for arbitrary transformations m and m' in M, the map σ is a homomorphism. Let t be an arbitrary element of $M|_V$. Then there is a transformation $m \in M$ such that $m|_V = t$. This proves that σ is surjective. The injectivity of σ follows from (i) and (iii). The proof of Claim 5.10 is completed.

Our aim is to prove that the restriction map

$$\Sigma\colon \mathrm{Sta}(M) \to \mathrm{Sta}(M|_V),\ f \mapsto f|_V$$

is an isomorphism between tha stabilizers $\mathrm{Sta}(M)$ and $\mathrm{Sta}(M|_V)$.

By Lemma 5.5, the map Σ is well-defined since $\mathrm{Sta}(M)|_V \subseteq \mathrm{Sta}(M|_V)$. As it is obvious that Σ is a homomorphism, it remains to show that Σ is bijective.

Claim 5.11. *For arbitrary elements $a \in A$ and $w \in W$ there is a unique transformation $m_{w,a}$ in M such that $m_{w,a}(w) = a$. Moreover, if $a \in V$, then $m_{w,a}$ is the unary constant operation c_a, and if $a \in W$, then $m_{w,a}$ is a permutation.*

If $a \in V$, then $m_{w,a}$ must be the unary operation c_a, since for all permutations $m \in P$ we have that $m(w) \in W$ by (6). If $a \in W$ then (6) shows that $m_{w,a}$ must be a permutation, and the regularity of P_W ensures the existence and uniqueness of such permutation. This completes the proof of Claim 5.11.

For an arbitrary n-ary operation g in $\mathrm{Sta}(M|_V)$ we will define an n-ary operation $\widehat{g}$ on A^n in the following way. Choose and fix an element w_0 in W, and consider arbitrary elements $a_1, \dots, a_n$ of A. Let $m \in M$ be the unique extension of $g(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V) \in M|_V$, and let the value of $\widehat{g}$ on the n-tuple $(a_1, \dots, a_n)$ be $m(w_0)$.

Claim 5.12. *The value of $\widehat{g}$ on the n-tuple $(a_1, \dots, a_n)$ does not depend on the choice of w_0.*

Let w_0' be an arbitrary element of W, and let $m' \in M$ be the unary operation for which

$$m'|_V = g(m_{w_0',a_1}|_V, \dots, m_{w_0',a_n}|_V).$$

Our goal is to prove that $m'(w_0') = m(w_0)$. By Claim 5.11 we get that

$$m_{w_0,a_i} = m_{w_0',a_i} m_{w_0,w_0'} \quad (1 \leqslant i \leqslant n),$$

and so, for every element $v \in V$ we get that

$$\begin{aligned}
m|_V(v) &= g(m_{w_0,a_1}|_V, \ldots, m_{w_0,a_n}|_V)(v) \\
&= g(m_{w_0,a_1}|_V(v), \ldots, m_{w_0,a_n}|_V(v)) \\
&= g(m_{w_0,a_1}(v), \ldots, m_{w_0,a_1}(v)) \\
&= g((m_{w_0',a_1} m_{w_0,w_0'})(v), \ldots, (m_{w_0',a_n} m_{w_0,w_0'})(v)) \\
&= g(m_{w_0',a_1}(m_{w_0,w_0'}(v)), \ldots, m_{w_0',a_n}(m_{w_0,w_0'}(v))) \\
&= g(m_{w_0',a_1}|_V(m_{w_0,w_0'}(v)), \ldots, m_{w_0',a_n}|_V(m_{w_0,w_0'}(v))) \\
&= g(m_{w_0',a_1}|_V, \ldots, m_{w_0',a_n}|_V)(m_{w_0,w_0'}(v)) \\
&= m'(m_{w_0,w_0'}(v)) \\
&= (m' m_{w_0,w_0'})(v) \\
&= (m' m_{w_0,w_0'})|_V(v).
\end{aligned}$$

Hence $m|_V = (m' m_{w_0,w_0'})|_V$, and Claim 5.10 implies that $m = m' m_{w_0,w_0'}$. Therefore

$$m'(w_0') = m'(m_{w_0,w_0'}(w_0)) = (m' m_{w_0,w_0'})(w_0) = m(w_0).$$

This completes the proof of Claim 5.12.

Claim 5.13. *For arbitrary unary operations $t_1, \ldots, t_n \in M$ and for arbitrary element $w \in W$ we have that*

$$\widehat{g}(t_1(w), \ldots, t_n(w)) = t(w),$$

where $t \in M$ is the unique extension of the unary operation $g(t_1|_V, \ldots, t_n|_V)$.

By Claim 5.11 we get that

$$t|_V = g(t_1|_V, \ldots, t_n|_V) = g(m_{w,t_1(w)}|_V, \ldots, m_{w,t_n(w)}|_V),$$

and so, $\widehat{g}(t_1(w), \ldots, t_n(w)) = t(w)$ by Claim 5.12 and by the definition of $\widehat{g}$. This proves Claim 5.13.

Claim 5.14. *The operation $\widehat{g}$ is the unique extension of g in the stabilizer of M.*

First we show that $\widehat{g}$ is an extension of g. Consider arbitrary elements $v_1, \ldots, v_n$ of V. By definition, $\widehat{g}(v_1, \ldots, v_n) = m(w_0)$ where m is the unique extension of $g(m_{w_0,v_1}|_V, \ldots, m_{w_0,v_n}|_V)$. By Claim 5.11,

$$g(m_{w_0,v_1}|_V, \ldots, m_{w_0,v_n}|_V) = g(c_{v_1}|_V, \ldots, c_{v_n}|_V) = c_{g(v_1,\ldots,v_n)}|_V,$$

and so, (i) and Claim 5.10 imply that $m = c_{g(v_1,\dots,v_n)}$. Hence

$$\widehat{g}(v_1,\dots,v_n) = m(w_0) = g(v_1,\dots,v_n).$$

This proves that $\widehat{g}$ is an extension of g.

Next we prove that $\widehat{g}$ is in the stabilizer of M. Consider arbitrary elements $t_1,\dots,t_n$ of M, and set $t = \widehat{g}(t_1,\dots,t_n)$. Our aim is to prove that t belongs to M. By the preceding paragraph, the restriction of t to V is the unary operation

$$t|_V = \widehat{g}(t_1,\dots,t_n)|_V = \widehat{g}|_V(t_1|_V,\dots,t_n|_V) = g(t_1|_V,\dots,t_n|_V) \in M|_V.$$

Let $\widehat{t} \in M$ be the unique extension of $g(t_1|_V,\dots,t_n|_V) \in M|_V$ to A. Then $\widehat{t}|_V = t|_V$, and for arbitrary element $w \in W$ we have that

$$t(w) = \widehat{g}(t_1,\dots,t_n)(w) = \widehat{g}(t_1(w),\dots,t_n(w)) = \widehat{t}(w),$$

where the first equality follows from the definition of $\widehat{g}$ and the last equality from Claim 5.13. Since $t|_V = \widehat{t}|_V$, this proves that $t = \widehat{t} \in M$. Hence the operation $\widehat{g}$ is in $\mathrm{Sta}(M)$.

Finally, we show that there are no other extensions of g in $\mathrm{Sta}(M)$. Assume that $\widetilde{g}$ is an extension of g in the stabilizer of M. Then for every n-tuple $(v_1,\dots,v_n) \in V^n$ we have that $\widehat{g}(v_1,\dots,v_n) = \widetilde{g}(v_1,\dots,v_n) = g(v_1,\dots,v_n)$. Consider arbitrary n-tuple $(a_1,\dots,a_n) \in A^n$, and let $m \in M$ be the unique extension of $g(m_{w_0,a_1}|_V,\dots,m_{w_0,a_n}|_V)$. Then

$$\begin{aligned} m|_V &= g(m_{w_0,a_1}|_V,\dots,m_{w_0,a_n}|_V) \\ &= \widetilde{g}|_V(m_{w_0,a_1}|_V,\dots,m_{w_0,a_n}|_V) \\ &= \widetilde{g}(m_{w_0,a_1},\dots,m_{w_0,a_n})|_V. \end{aligned}$$

Since $\widetilde{g} \in \mathrm{Sta}(M)$, the unary operation $\widetilde{g}(m_{w_0,a_1},\dots,m_{w_0,a_n})$ is in M, and Claim 5.10 implies that $\widetilde{g}(m_{w_0,a_1},\dots,m_{w_0,a_n}) = m$. Furthermore, we get that

$$\begin{aligned} \widetilde{g}(a_1,\dots,a_n) &= \widetilde{g}(m_{w_0,a_1}(w_0),\dots,m_{w_0,a_n}(w_0)) \\ &= \widetilde{g}(m_{w_0,a_1},\dots,m_{w_0,a_n})(w_0) \\ &= m(w_0) \\ &= \widehat{g}(a_1,\dots,a_n). \end{aligned}$$

Thus $\widetilde{g}(a_1,\dots,a_n) = \widehat{g}(a_1,\dots,a_n)$ holds for arbitrary n-tuples $(a_1,\dots,a_n) \in A^n$, and so $\widetilde{g} = \widehat{g}$. This proves Claim 5.14.

The bijectitvity of Σ follows from Claim 5.14. Hence, Σ is an isomorphism between the clones $\mathrm{Sta}(M)$ and $\mathrm{Sta}(M|_V)$. Then restriction to V induces an isomorphism between the subclone lattices of $\mathrm{Sta}(M)$ and $\mathrm{Sta}(M|_V)$. By this induced isomorphism the clones $\langle M\rangle$ and $\langle M|_V\rangle$ correspond to each other, which implies that the monoidal intervals $\mathrm{Int}(M)$ and $\mathrm{Int}(M|_V)$ are isomorphic.

This completes the proof of Lemma 5.9. □

Proof of Theorem 5.1. Suppose that the monoid M is collapsing. Then the number of unary constant operations in M is greater than 1 by Therorem 5.2. Hence $|V| \geqslant 2$, and (i) holds. Lemma 5.4 shows that the permutation group P_W must be transitive, while the injectivity of the map $\mathfrak{i}_V$ follows from Lemma 5.6. These prove that M has the properties (ii) and (iii). If neither one of the conditions (b) and (c) of (iv) holds for M, then the intervals $\mathrm{Int}(M|_V)$ and $\mathrm{Int}(M)$ are isomorphic by Lemma 5.9, and so, the monoid $M|_V$ is collapsing, since M is. Then condition (a) of (iv) holds for M. This shows that (iv) also holds for the monoid M.

Suppose that conditions (i)–(iv) hold for the monoid M. The assumption that (iv) holds for M means that either $M|_V$ is collapsing or $\mathfrak{j}$ is not injective or P_W is not regular. If $M|_V$ is collapsing, then $\mathrm{Sta}(M)|_V \subseteq \mathrm{Sta}(M|_V)$ contains only essentially unary operations, and therefore M is collapsing by Lemma 5.7. If $M|_V$ is not collapsing then (iv) implies that either $\mathfrak{j}$ is not injective or P_W is not regular. Therefore, M is collapsing by Lemma 5.8.

This completes the proof of Theorem 5.1. □

Proof of Theorem 5.3. If conditions (i)–(iii) of Theorem 5.1 hold but condition (iv) of Theorem 5.1 fails for M then $\mathrm{Int}(M)$ is isomorphic to $\mathrm{Int}(M|_V)$ by Lemma 5.9. If $|V| \geqslant 3$ then by Theorem 2.2 (cf. P. P. Pálfy [Pal84]), the monoidal interval $\mathrm{Int}(M|_V)$ is a 2-element chain, and so, $\mathrm{Int}(M)$ is a 2-element chain, as well. To finish the proof we note that if $|V| = 2$, say $V = \{0,1\}$, then $M|_V$ is either the monoid $\{c_0|_V, c_1|_V, \mathrm{id}_V\}$ and $\mathrm{Int}(M|_V)$ is isomorphic to he direct square of the 2-element chain or $M|_V$ is the full transformation semigroup and $\mathrm{Int}(M|_V)$ is isomorphic to the 3-element chain (cf. E. L. Post [Pos41] and Á. Szendrei [Sze86]). Therefore, in the former case $\mathrm{Int}(M)$ is isomorphic to he direct square of the 2-element chain, and in the latter case $\mathrm{Int}(M)$ is isomorphic to the 3-element chain. The proof of Theorem 5.3 is complete. □

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Summary

The dissertation is based on the articles [Dor02], [Dor07], and [Dor08], which are contained in Chapters 3, 4, and 5, respectively. The topic of the dissertation is the 'theory' of monoidal intervals on finite sets. The main results are concerned with collapsing monoids. In Chapter 3 we present large intervals in the submonoid lattice of the full transformation semigroup that consist of collapsing monoids, and describe the collapsing monoids on three-element sets. In Chapter 4 we determine all collapsing (inverse) monoids arise from finite lattices by applying the construction introduced by T. Saito and M. Katsura in [SK92], and describe a family of inverse monoids for which the corresponding monoidal intervals have cardinality $2^{\aleph_0}$. Finally, in Chapter 5 extending the well-known theorem of P. P. Pálfy in [Pal84], we give a complete description for collapsing monoids that contain at least one unary operations and whose nonconstant operations are permutations. Moreover, we present a family of monoids with two-element monoidal intervals.

Clones and monoidal intervals

A set of finitary operations on a set is said to be a **clone** if it contains all the projections and is closed under composition of operations. The set of all clones on a set A constitutes a complete lattice $\mathbb{CL}_A$ with respect to the set-theoretic inclusion. Post's result in [Pos41] is a complete description of all members of the clone lattice $\mathbb{CL}_{\{0,1\}}$. It turns out that if A has two elements, then there are $\aleph_0$ clones on A. The situation changes dramatically when A has more than two elements. In [JM59] Ju. I. Janov and A. A. Mučnik proved that on a finite set A with more than two elements there are $2^{\aleph_0}$ clones. Moreover, as the results of A. A. Bulatov in [Bul92] and [Bul94] show, the structure of the clone lattice is rather complicated.

Let A be a set. For an arbitrary clone $\mathcal{C}$ on A the set of unary operations in $\mathcal{C}$ constitutes a transformation monoid. Furthermore, it is not hard to show (see Á. Szendrei [Sze86]) that for an arbitrary transformation monoid M on A the clones whose subset of unary operations is M form an interval $\mathrm{Int}(M)$ in the clone lattice $\mathbb{CL}_A$. Such an interval is called a **monoidal interval**.

The least element of $\mathrm{Int}(M)$ is the clone $\langle M \rangle$ of essentially unary operations generated by M, and the greatest member of $\mathrm{Int}(M)$ is

$$\mathrm{Sta}(M) = \big\{ f(x_1, \ldots, x_n) \in \mathcal{O}_A \mid n \in \mathbb{N}, \text{ and } f(m_1(x), \ldots, m_n(x)) \in M \text{ for all } m_1, \ldots, m_n \in M \big\},$$

the **stabilizer** of the monoid M. If the interval $\mathrm{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**.

If A is finite, then there are only finitely many transformation monoids on A. Hence the monoidal intervals partition the clone lattice $\mathbb{CL}_A$ into finitely many blocks. Since $\mathbb{CL}_A$ has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one might expect that 'for most M' the monoidal interval $\mathrm{Int}(M)$ contains uncountably many clones. This expectation is justified by the fact that if $|A| = 3$, then more than half of the monoidal intervals have cardinality $2^{\aleph_0}$. Nevertheless, it turns out that for many interesting transformation monoids M the intervals $\mathrm{Int}(M)$ are countable. So, studying these intervals may lead to a better understanding of some parts of the clone lattice $\mathbb{CL}_A$.

The monoidal intervals are also related to the following unsolved problem on the congruences of the clone lattice: If A is a finite set with more than two elements, does $\mathbb{CL}_A$ have a nontrivial congruence? The relationship is revealed by a result of A. A. Krokhin [Kro01b] proving that any proper congruence of $\mathbb{CL}_A$ is a subrelation of the equivalence relation whose equivalence classes are the monoidal intervals. We note that for the case when A has only two elements, the congruences of $\mathbb{CL}_A$ have been determined by Krokhin–Semigrodskikh [KS01], using Post's description of $\mathbb{CL}_A$.

The problem of classifying all monoids on a finite set A according to the cardinalities of the corresponding monoidal intervals was first raised by Á. Szendrei [Sze86]. For the case when A is a two-element set Post's description of the clone lattice provides a complete solution to this problem: there are three finite and three infinite intervals. For the case when A is a finite set with more than two elements, and hence the clone lattice has cardinality $2^{\aleph_0}$, I. G. Rosenberg and N. Sauer in [RS] observed that each monoidal interval in $\mathbb{CL}_A$ either has cardinality $2^{\aleph_0}$ or is countable (see also M. Pinsker [Pin08]). Thus, Szendrei's problem can be refined as follows (see A. A. Krokhin [Kro97b]): for which transformation monoids does the corresponding monoidal interval have cardinality

- 1,
- finite but greater than 1,
- $\aleph_0$,
- $2^{\aleph_0}$?

In the remaining part we present our results. For easier reference, we kept the original numbering of theorems of the dissertation.

Large intervals of collapsing monoids

As the title indicates, we will prove that in the submonoid lattice of the full transformation semigroup on a finite set with at least 6 elements there are 'large' intervals such that all of their members are collapsing. Now, we describe the construction that leads to these monoids, which works for an on at least four-element set.

Let A be a finite set with at least 4 elements. Let P, Q, and R be pairwise disjoint nonempty subsets of A such that $|R| \geqslant 2$. Let $T(P,Q,R)$ be the set of all transformations $t \in T(A)$, such that for all $p \in P, q \in Q$ and $r, r' \in R$ if $t(r) = t(r')$ then $t(p) \in \{t(q), t(r)\}$. Let M be an arbitrary transformation monoid on A. The monoid M is said to be **rich** with respect to P, Q, R if for some $s \in A$, and for all $a, b \in A$ such that $a \neq b$ and $s \in \{a, b\}$, M contains transformations m and n such that $m(P) = m(Q) = \{a\},\ m(R) = \{b\}$ and $n(P) = n(R) = \{a\},\ n(Q) = \{b\}$.

The following theorem shows the importance of rich monoids.

Theorem 3.1 ([Dor02]). *Let A be a finite set with at least four elements, and let P, Q, R be disjoint nonempty subsets of A such that $|R| \geqslant 2$. Then every rich monoid $M \subseteq T(P,Q,R)$ is collapsing.*

This theorem allows us to construct 'large' intervals consisting of collapsing monoids.

Let A be a finite set with $|A| \geqslant 6$. Let the elements $p, q, r, r' \in A$ be pairwise distinct, and let $P = \{p\},\ Q = \{q\},\ R = \{r, r'\}, A' = A \backslash (P \cup Q \cup R)$. We define the monoid N on A to be the monoid generated by the set of all transformations $t \in T(P,Q,R)$ for which $t(r) = t(r')$ and the restriction of t onto A' is the identity operation on A'. It is easy to see that N is contained in $T(P,Q,R)$. For an arbitrary monoid $K \in T(A')$ we will denote by $\hat{K}$ the monoid which consists of all transformations from $T(A)$ whose restriction onto A' is a member of K, and whose restriction onto the set $P \cup Q \cup R$ is the identity operation. Since $t \in \langle N \cup \hat{K} \rangle$ implies that $t|_{A'} \in K$, we get that if K_1, K_2 are submonoids of $T(A')$ and $K_1 \neq K_2$ then $\langle N \cup \hat{K}_1 \rangle \neq \langle N \cup \hat{K}_2 \rangle$. Furthermore, $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P,Q,R)$, and N is rich.

Using the fact that the cardinality of the submonoid lattice of the full transformation semigroup on A is greater than $2^{2^{c|A|}}$ for some positive constant c and Theorem 3.1 we get the following theorem.

Theorem 3.4 ([Dor02]). *Let A be a finite set with $|A| = n \geqslant 6$. Then all members of the interval $[N, \langle N \cup \widehat{T(A')} \rangle]$ is collapsing, and this interval has cardinality greater than $2^{2^{c'n}}$ for some positive constant c'.*

On a 3-element set we can not use the previous construction, however, a similar method will work in this case.

Let A be a 3-element set. We will define two sets of transformations on A. Let $p, s \in A$ be arbitrary elements of A. Let T_p denote the set of all transformations $t \in T(A)$ such that either t is a permutation fixing p or t is not a permutation, and $t(p) \in \{t(q), t(r)\}$, where $\{p, q, r\} = A$. Furthermore, let $M_{p,s}$ be the set of all transformations $t \in T_p$ such that $t(A) \subseteq \{s, a\}$ for some $a \in A \setminus \{s\}$ or t is the identity operation. It is easy to see that both T_p and $M_{p,s}$ are transformation monoids on A.

With the aid of these monoids we get a similar description as in Theorem 3.1.

Theorem 3.6 ([Dor02]). *Let A be a 3-element set. Then each monoid M for which there are elements $p, s \in A$ such that $M_{p,s} \subseteq M \subseteq T_p$ is collapsing.*

Let $\bowtie$ be the relation on $T(A)$ defined in the following way: transformation monoids M_1 and M_2 on A are $\bowtie$-related if there is permutation $\pi \in S(A)$ for which $M_2 = \{\pi^{-1} m \pi : m \in M_1\}$ hold. It is easy to see that $\bowtie$ is an equivalence relation, and the monoidal intervals that correspond to $\bowtie$-related monoids are isomorphic. On a 3-element set there are 699 monoids in 160 $\bowtie$-classes. The last theorem in this section characterizes the collapsing monoids among them.

Theorem 3.7 ([Dor02]). *On the 3-element set $A = \{0, 1, 2\}$ there are 30 collapsing monoids in 11 $\bowtie$-classes. If M is a collapsing monoid on A, then M is equivalent to exactly one of the following monoids:*

(1) $\langle c_0, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, \tau_2\}$,

(2) $\langle c_0, c_1, c_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2\}$,

(3) $\langle c_0, c_2, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \tau_0\}$,

(4) $\langle c_0, \sigma \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \sigma, \sigma^2\}$,

(5) S_3,

(6) $M_{2,0}$,

(7) $M_{2,2}$,

(8) $\langle M_{2,2} \cup \{\tau_2\} \rangle = M_{2,2} \cup \{\tau_2\}$,

(9) $T_2 \setminus \{\tau_2\}$,

(10) T_2,

where T_2 is the monoid of all transformations $t \in T(A)$ such that either $t = \mathrm{id}_A$ or $t(2) \in \{t(0), t(1)\}$, while $M_{2,r}$ ($r \in \{0, 2\}$) is the monoid of all transformations $t \in T_0$ for which either $|t(A)| \leqslant 2$ and $r \in t(A)$ or $t = \mathrm{id}_A$ or t is constant.

Collapsing inverse monoids

Earlier results on permutation groups indicate that 'large' permutation groups, e.g. all primitive permutation groups, are collapsing (cf. Pálfy–Szendrei [PSz82] and Kearnes–Szendrei [KSz01]). This motivated us in extending the investigation of collapsing monoids to 'large' inverse monoids.

To formulate our results we need some definitions and concepts.

Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite lattice. The least and greatest elements of $\mathbb{L}$ will be denoted by 0 and 1, respectively. The set of atoms and the set of join-irreducible elements of $\mathbb{L}$ will be denoted by $\mathcal{A}(\mathbb{L})$ and $\mathcal{J}(\mathbb{L})$, respectively, and we put $\mathcal{A}_0(\mathbb{L}) = \mathcal{A}(\mathbb{L}) \cup \{0\}$. If there is no danger of confusion, we simply write $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{J}$, respectively. Two elements a and b of $\mathbb{L}$ will be called **similar** if and only if the principal ideals $(a]$ and $(b]$ are isomorphic. We write $a \sim b$ to denote that a is similar to b. The relation $\sim$ is an equivalence relation on L. If the $\sim$-class containing a has only one element then a will be called **isolated**. For every element $a \in L$ we define a unary operation φ_a by the rule $\varphi_a(x) = x \wedge a$ ($x \in L$). In particular, φ_0 is constant with range $\{0\}$. For similar elements $a, b \in L$ the symbol $\beta_{a,b}$ will denote an isomorphism between the principal ideals $(a]$ and $(b]$. Define a set $\mathrm{IS}(\mathbb{L})$ of transformations on L in the following way:

$$\mathrm{IS}(\mathbb{L}) = \{\beta_{v,w} \circ \varphi_v \mid v, w \in L,\ v \sim w,\ \text{and} \\ \beta_{v,w} \colon (v] \to (w] \text{ is an isomorphism}\}.$$

Then $\mathrm{IS}(\mathbb{L})$ is an inverse submonoid of the full transformation semigroup on L (cf. Saito–Katsura [SK92], Lemma 3.1).

First, we need the following definition. Let a and b be arbitrary elements of L. We will say that the element b is **dwarfed** by a if for all elements $b' \in L$ such that $b' \sim b$ we have that $b' \leqslant a$. We will use the notation $b \ll a$ to denote that a dwarfs b. Now we are in a position to state the central result of this section.

Theorem 4.3 ([Dor07]). *Let $\mathbb{L}$ be a finite lattice such that $|L| \geqslant 3$. Then the inverse monoid $M = \mathrm{IS}(\mathbb{L})$ is collapsing if and only if no element of $\mathcal{J} \setminus \mathcal{A}$ dwarfs a nonzero element of L.*

If $\mathbb{L}$ is an atomistic lattice then $\mathcal{J} = \mathcal{A}$, and so, the conditions of the previous theorem are satisfied.

Corollary 4.14. *If* $\mathbb{L}$ *is an atomistic lattice then* $\mathrm{IS}(\mathbb{L})$ *is collapsing.*

Describing lattices $\mathbb{L}$ for which $\mathrm{IS}(\mathbb{L})$ is collapsing we turn our attention to large monoidal intervals. We conclude this section with a discussion of lattices $\mathbb{L}$ for which the monoidal interval $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ has cardinality $2^{\aleph_0}$. For elements $u \leqslant v$ of $\mathbb{L}$, we will use the notation $[u, v]$ for the interval $\{x \in L \mid u \leqslant x \leqslant v\}$. We will call a lattice $\mathbb{L}$ **pinched** if L contains an element $b \in L \setminus \{0, 1\}$ such that $L = [0, b] \cup [b, 1]$.

Next theorem services as a basis for further constructions.

Theorem 4.22 ([Dor07])**.** *Let* $\mathbb{L}$ *be a pinched lattice, and let* $b \in L \setminus \{0, 1\}$ *be an element such that* $L = [0, b] \cup [b, 1]$. *Then* $|\mathrm{Int}(\mathrm{IS}([0, b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$.

The most natural examples for pinched lattices are finite chains with at least 3 elements.

Theorem 4.17 ([Dor07])**.** *For a* 3*-element chain* $\mathbb{L}$ *we have* $|\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| = 2^{\aleph_0}$.

Corollary 4.28. *If* $\mathbb{L}$ *is a finite chain with at least* 3 *elements then the monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ *has cardinality* $2^{\aleph_0}$.

Finally, combining Post's results for description of $\mathbb{CL}_{\{0,1\}}$ and Theorem 4.22, we can state the following.

Corollary 4.27. *If* $\mathbb{L}$ *is a finite lattice which has a unique atom then the monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ *is infinite.*

Collapsing monoids consisting of constants and permutations

The well-known result of P. P. Pálfy, inspired the investigation of monoids consisting of constants and permutations. Pálfy's theorem in [Pal84] for finite sets states the following.

Theorem (cf. P. P. Pálfy [Pal84])**.** *Let* A *be a finite set with* $|A| \geqslant 3$, *and let* M *be a transformation monoid on* A *that contains all the unary constant operations and whose nonconstant operations are permutations. Then* $|\mathrm{Int}(M)| \leqslant 2$*; moreover,* $|\mathrm{Int}(M)| = 1$ *unless* M *coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field.*

We restrict our efforts to monoids that contain at least ont but not all unary constant operations and whose nonconstant operations are permutations. We need further definitions to form our results.

Let V be the set of all elements $v \in A$ such that $c_v \in M$, and set $W = A \setminus V$. Assume that $\emptyset \subsetneq V, W \subsetneq A$. Define P to be the set of all permutations contained in M. The facts that A is finite and M is closed under composition ensure that P is a permutation group on A and

$$\alpha(V) = V, \qquad \alpha(W) = W \tag{10}$$

hold for all $\alpha \in P$. These equalities allow us to restrict P to V and W, and obtain the permutation groups

$$\begin{aligned} P_V &= \{\alpha|_V \mid \alpha \in P\} \subseteq S(V), \\ P_W &= \{\alpha|_W \mid \alpha \in P\} \subseteq S(W). \end{aligned}$$

Furthermore, let $\mathfrak{i}_V$ be the restriction map $\mathfrak{i}_V \colon P \to P|_V,\ \alpha \mapsto \alpha|_V$. If the map $\mathfrak{i}_V$ is injective, then for every transformation $m \in M$ the unique extension of the map $m|_V$ to A is m. Hence, if the map $\mathfrak{i}_V$ is injective, the map

$$\mathfrak{j} \colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W.$$

is well-defined.

Our first theorem in this section characterizes all collapsing monoids $M \subseteq C(A) \cup S(A)$ that contain at least one unary constant operation. This extends the results obtained by A. Fearnley and I. Rosenberg in [FR03].

Theorem 5.1 ([Dor08])**.** *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least one unary constant operation and some permutations. Then M is collapsing if and only if*

(i) $|V| \geqslant 2$,

(ii) P_W *is transitive,*

(iii) $\mathfrak{i}_V$ *is injective, and*

(iv) *one of the following conditions holds:*

  (a) *the monoid $M|_V$ is collapsing,*

  (b) *the map $\mathfrak{j}$ is not injective,*

  (c) *the permutation group P_W is not regular.*

As a consequence of the previous theorem we get that a monoid $M \subseteq C(A) \cup S(A)$ containing exactly one unary constant operation cannot be collapsing. The following theorem states a bit more.

Theorem 5.2 ([Dor08]). *Let $M \subseteq C(A) \cup S(A)$ be a monoid such that it contains only one unary constant operation. Then the monoidal interval* $\mathrm{Int}(M)$ *is infinite.*

Theorem 5.3 ([Dor08]). *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least two unary constant operations and some permutations. If conditions* (i)–(iii) *of Theorem 5.1 hold but condition* (iv) *of Theorem 5.1 fails for M then* $\mathrm{Int}(M)$ *is isomorphic to* $\mathrm{Int}(M|_V)$. *Hence,*

- *if $|V| = 2$ and $M|_V$ is the monoid $\{\mathrm{id}_V, c_0|_V, c_1|_V\}$, then $|W| = 1$, $M = \{\mathrm{id}_A, c_0, c_1\}$, and* $\mathrm{Int}(M)$ *is isomorphic to the direct square of the* 2*-element chain;*
- *if $|V| = 2$ and $M|_V$ is the full transformation semigroup on V, then $|W| = 2$, $M = \{\mathrm{id}_A, c_0, c_1, (0\,1)(2\,3)\}$, and* $\mathrm{Int}(M)$ *is a* 3*-element chain;*
- *if $|V| \geqslant 3$, then* $\mathrm{Int}(M)$ *is a* 2*-element chain.*

Összefoglaló

A disszertáció a [Dor02], [Dor07] és [Dor08] publikációk felhasználásával készült, a publikációkban foglalt eredményeket, rendre a 3., 4. és 5. Fejezetek tartalmazzák. A dolgozat célja a monadikus intervallumok vizsgálata véges halmazokon. Legfontosabb eredményeink összeejtő monoidokkal kapcsolatosak. A 3. Fejezetben megmutatjuk, hogy a teljes transzformáció-félcsoport részmonoid hálójában vannak olyan „nagy” elemszámú intervallumok, amelyek csak összeejtő monoidokat tartalmaznak, valamint tetszőleges 3-elemű halmazon meghatározzuk az összes összeejtő monoidot. A 4. Fejezetben azokat az összeejtő (inverz) monoidokat írjuk le, amelyek a T. Saito és M. Katsura által [SK92]-ben leírt konstrukcióval kaphatók meg valamely véges hálóból. Továbbá, monoidok egy olyan családját is leírjuk, amelyekhez $2^{\aleph_0}$ számosságú monadikus intervallum tartozik. Végül, az 5. Fejezetben általánosítjuk Pálfy P. P. [Pal84]-ben található tételét: teljes leírást adunk mindazon összeejtő monoidokról, amelyek legalább egy darab egyváltozós konstans műveletet tartalmaznak és minden konstanstól különböző műveletük permutáció. Valamint, olyan monoidokat is mutatunk, amelyekhez kételemű monadikus intervallum tartozik.

Klónok és monadikus intervallumok

Egy adott halmazon értelmezett véges változós műveletek valamely halmazát **klónnak** nevezzük, ha tartalmazza a projekciókat és zárt a műveletek kompozíciójára. Az összes klónok halmaza az A halmazon teljes hálót alkot a (halmazelméleti) tartalmazásra vonatkozóan, amelyet $\mathbb{CL}_A$-val jelölünk.

Ha az A halmaz kételemű, akkor a $\mathbb{CL}_A$ klónháló megszámlálhatóan végtelen sok elemet tartalmaz, és szerkezete E. L. Post eredményeinek nyomán teljesen ismert (vö. [Pos41]). Míg kételemű halmazon $\aleph_0$ klón van, legalább háromelemű halmaz esetében a helyzet lényegesen bonyolultabb. Ju. I. Janov és A. A. Mučnik [JM59]-ban megmutatta, hogy legalább háromelemű véges halmazon $2^{\aleph_0}$ klón van. A. A. Bulatov eredményei a [Bul92] és [Bul94] cikkekben pedig azt mutatják, hogy $\mathbb{CL}_A$ szerkezete rendkívül bonyolult.

Legyen A tetszőleges halmaz. Ekkor bármely $\mathcal{C} \in \mathbb{CL}_A$ klón esetén a klónbeli egyváltozós műveletek halmaza transzformáció-monoidot alkot az A halmazon. Valamint, az is egyszerűen látható, hogy adott M transzformáció-monoidra azon klónok összessége, amelyek egyváltozós függvényeinek halmaza éppen M a $\mathbb{CL}_A$ klónháló egy $\mathrm{Int}(M)$ intervallumát alkotják (ld. Á. Szendrei [Sze86]). A klónháló azon intervallumát, amely a fenti módon keletkezik **monadikus intervallumnak** nevezzük. Az $\mathrm{Int}(M)$ intervallum legkisebb eleme $\langle M\rangle$ (az M által generált lényegében egyváltozós műveletek halmaza), a legnagyobb eleme pedig

$$\mathrm{Sta}(M) = \big\{ f(x_1,\dots,x_n) \in \mathcal{O}_A \mid n \in \mathbb{N}, \text{ and} \\ f(m_1(x),\dots,m_n(x)) \in M \text{ for all } m_1,\dots,m_n \in M \big\},$$

az M monoid **stabilizátora**. Az M monoidot **összeejtő monoidnak** nevezzük, ha az $\mathrm{Int}(M)$ intervallum egyetlen elemet tartalmaz.

Mivel véges A halmazon véges sok transzformáció-monoid van, így a hozzájuk tartozó monadikus intervallumok egy olyan partícióját adják a $\mathbb{CL}_A$ klónhálónak, amelynek véges sok blokkja van. Mivel $|A| \geqslant 3$ esetén a $\mathbb{CL}_A$ klónháló számossága $2^{\aleph_0}$, ezért azt várhatjuk, hogy monadikus intervallumaink „nagy részének" számossága szintén ennyi lesz. Mindenesetre, az eddig elért eredmények azt mutatják, hogy sok olyan transzformáció-monoid van, amelyhez megszámlálható monadikus intervallum tartozik. A monadikus intervallumok vizsgálata által a klónháló szerkezetének alaposabb feltárását remélhetjük.

A monadikus intervallumokkal kapcsolatban érdemes megemlíteni a következő, a klónháló kongruenciáihoz kötődő problémát: ha A legalább 3-elemű véges halmaz, akkor van-e a $\mathbb{CL}_A$ klónhálónak nemtriviális kongruenciája? A kapcsolatot A. A. Krokhin [Kro01b] azon eredménye teremti meg, amely szerint $\mathbb{CL}_A$ minden nemtriviális kongruenciája részrelációja annak az ekvivalenciarelációnak, amelynek blokkjai a monadikus intervallumok. Kételemű A halmaz esetén a $\mathbb{CL}_A$ háló kongruenciáit A. A. Krokhin és A. P. Semigrodskikh határozták meg [KS01]-ben, a klónháló Post-féle leírásának felhasználásával.

Szendrei Á. vetette fel [Sze86]-ban a monadikus intervallumok számosságuk szerinti osztályozásának problémáját. Post leírásának segítségével egyszerűen meghatározhatjuk $\mathbb{CL}_{\{0,1\}}$ monadikus intervallumainak számosságát: hat darab monadikus intervallum van, amelyek közül pontosan három darab véges. Ha az A halmaznak legalább három eleme van, akkor egyrészt a $\mathbb{CL}_A$ klónháló számossága $2^{\aleph_0}$, másrészt a Rosenberg–Sauer [RS]-beli eredménye szerint bármely monadikus intervallum számossága $2^{\aleph_0}$ vagy megszámlálható (ld. még M. Pinsker [Pin08]). Ezért a Szendrei-probléma a következőképpen finomítható: melyek azok a transzformációmonoidok, amelyekhez tartozó monadikus intervallum számossága

- 1,
- véges, de 1-nél nagyobb,
- $\aleph_0$,
- $2^{\aleph_0}$?

A fennmaradó részben, a tételek disszertációbeli eredeti számozását megtartva, az egyes fejezetekben foglalt eredményeinket ismertetjük.

Összeejtő monoidok nagy intervallumai

A címben foglaltaknak megfelelően, megmutatjuk, hogy legalább hatelemű alaphalmaz esetén a teljes transzformáció-félcsoport részmonoid hálójában olyan „nagy” elemszámú intervallumok is vannak, amelyek elemei mind összeejtő monoidok. Az alábbiakban azt a konstrukciót ismertetjük, amelynek segítségével, legalább négyelemű alaphalmazon, ezeket az összeejtő monoidokat megkaphatjuk.

Legyen A legalább négyelemű véges halmaz, és legyenek P, Q és R olyan páronként diszjunkt részhalmazai A-nak, amelyekre $|R| \geqslant 2$ teljesül. Legyen $T(P, Q, R)$ mindazon $t \in T(A)$ transzformációk halmaza az A halmazon, amelyekre igaz, hogy tetszőleges $p \in P, q \in Q$ és $r, r' \in R$ estén, ha $t(r) = t(r')$, akkor $t(p) \in \{t(q), t(r)\}$. Azt mondjuk, hogy az $M \subseteq T(A)$ transzformáció-monoid **gazdag** a P, Q, R halmazokra vonatkozóan, ha van olyan $s \in A$ elem, amelyre minden az $a \neq b$ és $s \in \{a, b\}$ összefüggéseknek eleget tevő $a, b \in A$ esetén M tartalmaz olyan m és n transzformációkat, amelyekre $m(P) = m(Q) = \{a\}$, $m(R) = \{b\}$ és $n(P) = n(R) = \{a\}$, $n(Q) = \{b\}$ teljesül.

A következő tétel mutatja, hogy miért is fontosak a gazdag monoidok.

3.1. Tétel ([Dor02]). *Legyen A legalább négyelemű véges halmaz, továbbá legyenek P, Q, R páronként diszjunkt részhalmazai A-nak úgy, hogy $|R| \geqslant 2$ teljesüljön. Ekkor minden olyan M gazdag monoid, amely részhalmaza $T(P, Q, R)$-nek összeejtő.*

Ezen tétel képezi alapját annak a konstrukciónak, amely lehetővé teszi, hogy „nagy” csak összeejtő monoidokat tartalmazó intervallumokat találjunk a teljes transzformáció-félcsoport részmonoid hálójában.

Legyen A legalább hatelemű véges halmaz, legyenek $p, q, r, r' \in A$ páronként különböző elemek, valamint legyen $P = \{p\}$, $Q = \{q\}$, $R = \{r, r'\}$ és $A' = A \setminus (P \cup Q \cup R)$. Az N monoidot az A halmazon definiáljuk a következőképpen. Legyen N az a monoid, amelyet mindazon $T(P,Q,R)$-beli t transzformációk generálnak, amelyekre teljesül, hogy $t(r) = t(r')$ és t-nek az A' halmazra való megszorítása éppen A' identikus transzformációja. Egyszerűen igazolható, hogy N része $T(P,Q,R)$-nek. Tetszőleges $K \subseteq T(A')$ transzformáció-monoidra $\hat{K}$ jelölje azt a monoidot, amely mindazon $T(A)$-beli transzformációkból áll, amelyek A'-re való megszorítása eleme K-nak, és amelyek $(P \cup Q \cup R)$-re való megszorítása a megfelelő identikus transzformáció. Mivel $t \in \langle N \cup \hat{K} \rangle$ esetén $t|_{A'} \in K$, ezért a $T(A')$ transzformáció-félcsoport tetszőleges K_1, K_2 részmonoidjára, ha $K_1 \neq K_2$, akkor $\langle N \cup \hat{K}_1 \rangle \neq \langle N \cup \hat{K}_2 \rangle$. Továbbá, az is teljesül, hogy $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P,Q,R)$, és az N monoid gazdag.

Felhasználva, hogy a $T(A)$ transzformáció-félcsoport részmonoid hálójának számossága nagyobb, mint $2^{2^{c|A|}}$, ahol c valamely alkalmas pozitív konstans, és a 3.1. Tételt, az alábbi állítást kapjuk.

3.4. Tétel ([Dor02]). *Legyen A legalább hatelemű véges halmaz. Ekkor valamennyi az $[N, \langle N \cup \widehat{T(A')} \rangle]$ intervallumba eső monoid összeejtő, és ezen intervallum elemszáma nagyobb, mint $2^{2^{c'n}}$, ahol c' alkalmas pozitív konstans.*

Háromelemű halmazon a fentiekben vázolt konstrukció nem alkalmazható, de egy hozzá hasonló módszer már ebben az esetben is működik.

Legyen A háromelemű halmaz, $p, s \in A$. Jelölje T_p mindazon $t \in T(A)$ transzformációk halmazát, amelyekre teljesülnek a következők: t permutáció, amelynek p fixpontja vagy $t(p) \in \{t(q), t(r)\}$, ahol $\{p,q,r\} = A$. Továbbá, álljon $M_{p,s}$ azokból a T_p-beli transzformációkból, amelyekre $t(A) \subseteq \{s, a\}$ valamely $a \in A \setminus \{s\}$ vagy t az A halmaz identikus művelete. Könnyen igazolható, hogy teljes transzformáció-félcsoport T_p és $M_{p,s}$ részhalmazai monoidok az A halmazon.

Ezen monoidok segítségével a 3.1. Tételben foglaltakhoz hasonló leírást adhatunk bizonyos összeejtő monoidokra.

3.6. Tétel ([Dor02]). *Legyen A háromelemű halmaz. Ekkor minden olyan $M \subseteq T(A)$ transformáció-monoid összeejtő, amelyre vannak olyan $p, s \in A$ elemek, hogy M-re $M_{p,s} \subseteq M \subseteq T_p$ teljesül.*

Definiáljuk a $\bowtie$ relációt a $T(A)$ halmazon a következőképpen: az M_1 és M_2 transzformáció-monoidok az A halmazon pontosan akkor legyenek $\bowtie$ relációban, ha van olyan $\pi \in S(A)$ permutáció, amelyre $M_2 = \{\pi^{-1} m \pi : m \in M_1\}$ teljesül. Ekkor a $\bowtie$ reláció ekvivalenciareláció, és azok a monadikus intervallumok, amelyek $\bowtie$ relációban álló monoidokhoz tartoznak izomorfak. Tetszőleges háromelemű halmazon 699 darab transzformáció-monoid van 160

$\bowtie$-osztályban. Ezen rész utolsó tétele az összeejtő monoidok leírását adja meg.

3.7. Tétel ([Dor02]). *Az $A = \{0, 1, 2\}$ halmazon 30 darab összeejtő monoid van 11 $\bowtie$-osztályban. Ha az $M \subseteq T(A)$ monoid összeejtő, akkor $M \bowtie$ relációban áll az alábbi monoidok közül pontosan eggyel:*

(1) $\langle c_0, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, \tau_2\}$,

(2) $\langle c_0, c_1, c_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2\}$,

(3) $\langle c_0, c_2, \tau_2 \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \tau_0\}$,

(4) $\langle c_0, \sigma \rangle = \{\mathrm{id}_A, c_0, c_1, c_2, \sigma, \sigma^2\}$,

(5) S_3,

(6) $M_{2,0}$,

(7) $M_{2,2}$,

(8) $\langle M_{2,2} \cup \{\tau_2\} \rangle = M_{2,2} \cup \{\tau_2\}$,

(9) $T_2 \setminus \{\tau_2\}$,

(10) T_2.

Összeejtő inverz monoidok

Az ezen részben foglaltakat azon permutációcsoportokra vonatkozó eredmények készítették elő, amelyek szerint a „nagy" permutációcsoportok (pl. a primitív permutációcsoportok) összeejtőek. (vö. Pálfy–Szendrei [PSz82] és Kearnes–Szendrei [KSz01]). Így permutációcsoportok után természetesnek tűnt a vizsgálatokat „nagy" (= maximális) inverz monoidokra is kiterjeszteni.

Eredményeink ismertetéséhez szükségünk lesz az alábbi jelölésekre, definíciókra és fogalmakra.

Legyen $\mathbb{L} = (L; \vee, \wedge)$ véges háló, amelynek legkisebb, illetve legnagyobb elemét jelölje 0, illetve 1. Az $\mathbb{L}$ háló atomjainak, illetve egyesítés irreducibilis elemeinek halmazát jelölje $\mathcal{A}(\mathbb{L})$, illetve $\mathcal{J}(\mathbb{L})$, és legyen $\mathcal{A}_0(\mathbb{L}) = \mathcal{A}(\mathbb{L}) \cup \{0\}$. Ha nem áll fenn a félreértésnek a veszélye, akkor a fentiek helyett egyszerűen csak az $\mathcal{A}$, $\mathcal{A}_0$ és $\mathcal{J}$ jelöléseket használjuk. Az $\mathbb{L}$ háló a és b elemeit **hasonlónak** nevezzük, ha az általuk generált $(a]$ és $(b]$ főideálok izomorfak. Az a és b elemek hasonlóságát $a \sim b$-vel jelöljük. A hasonlóság ekvivalenciareláció az L halmazon. Ha az a elemet tartalmazó $\sim$-osztály egyetlen elemet tartalmaz, akkor azt mondjuk, hogy az a elem **izolált**. Tetszőleges $a \in L$ esetén a φ_a egyváltozós műveletet a $\varphi_a(x) = x \wedge a$ $(x \in L)$ szabállyal definiáljuk. Például, φ_0 a konstans 0 művelet az L halmazon. Ha az a és b

elemek ($a, b \in L$) hasonlóak, akkor $\beta_{a,b}$ egy $(a]$ és $(b]$ közötti izomorfizmust fog jelölni. Legyen $\mathrm{IS}(\mathbb{L})$ az alábbi részhalmaza $T(L)$-nek:

$$\mathrm{IS}(\mathbb{L}) = \{\beta_{v,w} \circ \varphi_v \mid v, w \in L,\ v \sim w \text{ és } \beta_{v,w} \colon (v] \to (w] \text{ izomorfizmus}\}.$$

Az $\mathrm{IS}(\mathbb{L})$ részhalmaz a teljes transzformáció-félcsoport egy inverz részmonoidja az L halmazon (vö. Saito–Katsura [SK92], Lemma 3.1).

Tekintsük az L halmaz a és b elemeit. Azt mondjuk, hogy az a elem **erősen nagyobb**, mint b, ha valahányszor a $b' \in L$ elemre $b' \sim b$ teljesül, mindannyiszor $b' \leqslant a$. Azt, hogy az a elem erősen nagyobb, mint b az $b \ll a$ szimbólummal jelöljük. Most már meg tudjuk fogalmazni ezen rész legfontosabb eredményét.

4.3. Tétel ([Dor07]). *Legyen $\mathbb{L}$ véges háló, amelynek legalább három eleme van. Ekkor az $M = \mathrm{IS}(\mathbb{L})$ inverz monoid pontosan akkor összeejtő, ha nincs olyan $\mathcal{J} \setminus \mathcal{A}$-beli elem, amely erősen nagyobb lenne, mint $L \setminus \{0\}$ valamely eleme.*

Ha az $\mathbb{L}$ háló atomisztikus, akkor $\mathcal{J} = \mathcal{A}$, és így az előző tétel feltételei nyilván teljesülnek.

4.14. Következmény. *Ha az $\mathbb{L}$ háló atomisztikus, akkor $\mathrm{IS}(\mathbb{L})$ összeejtő.*

Miután leírtuk azokat a hálókat, amelyekre $\mathrm{IS}(\mathbb{L})$ összeejtő, figyelmünket a „nagy” monadikus intervallumok felé fordítjuk. Ezen részt azzal zárjuk, hogy $\mathbb{L}$ hálók egy olyan családját adjuk meg, amelyekre az $\mathrm{IS}(\mathbb{L})$ monoidhoz $2^{\aleph_0}$ számosságú monadikus intervallum tartozik. Az L-beli $u \leqslant v$ elemek esetén az $\{x \in L \mid u \leqslant x \leqslant v\}$ intervallumot $[u, v]$-vel jelöljük. Azt mondjuk, hogy az $\mathbb{L}$ háló **megosztott**, ha van olyan $b \in L \setminus \{0, 1\}$ elem, amelyre $L = [0, b] \cup [b, 1]$ teljesül.

4.22. Tétel ([Dor07]). *Legyen $\mathbb{L}$ megosztott háló, és legyen $b \in L \setminus \{0, 1\}$ olyan elem, amelyre $L = [0, b] \cup [b, 1]$ teljesül. Ekkor $|\mathrm{Int}(\mathrm{IS}([0, b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbb{L}))|$.*

A legegyszerűbb megosztott hálók a legalább háromelemű láncok.

4.17. Tétel ([Dor07]). *Ha $\mathbb{L}$ háromelemű lánc, akkor $|\mathrm{Int}(\mathrm{IS}(\mathbb{L}))| = 2^{\aleph_0}$.*

4.28. Következmény. *Ha az $\mathbb{L}$ legalább háromelemű véges lánc, akkor az $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ monadikus intervallum számossága $2^{\aleph_0}$.*

Végül, E. L. Postnak a $\mathbb{CL}_{\{0,1\}}$ háló leírására vonatkozó eredményeiből a 4.22. Tételnek az alkalmazásával a következőt kapjuk.

4.27. Következmény. *Ha az $\mathbb{L}$ véges háló egyetlen atomot tartalmaz, akkor az $\mathrm{Int}(\mathrm{IS}(\mathbb{L}))$ monadikus intervallum végtelen.*

Konstansokból és permutációkból álló összeejtő monoidok

Az alábbi eredmények kiindulópontjául Pálfy P. P. jól ismert tétele szolgál. Pálfy tétele [Pal84]-ben véges halmazokra a következőt állítja.

Tétel (cf. P. P. Pálfy [Pal84]). *Legyen A véges halmaz, amelynek legalább három elem van, és legyen M olyan transzformáció-monoid az A halmazon, amely az összes egyváltozós konstans műveletet tartalmazza és minden nemkonstans művelete permutáció. Ekkor* $|\mathrm{Int}(M)| \leqslant 2$*; pontosabban,* $|\mathrm{Int}(M)| = 2$ *pontosan akkor teljesül, ha M megegyezik valamely véges vektortér egyváltozós polinom függvényeinek monoidjával.*

Az elkövetkezőkben csak olyan $M \subseteq C(A) \cup S(A)$ monoidokra szorítkozunk, amelyek legalább egy, de nem az összes egyváltozós konstans műveletet tartalmazzák. Az eredmények formába öntéséhez azonban további jelölésekre lesz szükségünk.

Legyen V mindazon $v \in A$ elemek halmaza, amelyekre $c_v \in M$, és legyen $W = A \setminus V$. A továbbiakban feltesszük, hogy $\emptyset \subsetneq V, W \subsetneq A$. Legyen P az M-beli permutációk halmaza. Az a tény, hogy A véges M zárt a műveletek kompozíciójára biztosítja, hogy P permutációcsoport az A halmazon, és

$$\alpha(V) = V, \qquad \alpha(W) = W$$

teljesül tetszőleges $\alpha \in P$-re. Ezen egyenlőségek lehetővé teszik, hogy a P-beli permutációkat megszorítsuk a V és W halmazokra, így a következő permutációcsoportokat kapjuk:

$$P_V = \{\alpha|_V \mid \alpha \in P\} \subseteq S(V),$$
$$P_W = \{\alpha|_W \mid \alpha \in P\} \subseteq S(W).$$

Továbbá, jelölje $\mathfrak{i}_V$ a $P \to P|_V,\ \alpha \mapsto \alpha|_V$ leképezést. Ha az $\mathfrak{i}_V$ leképezés injektív, akkor minden $m \in M$ transzformáció esetén az $m|_V$ leképezés egyetlen módon terjeszthető ki A-ra úgy, hogy M-beli leképezést kapjunk. Ezért, ha az $\mathfrak{i}_V$ leképezés injektív, akkor a

$$\mathsf{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W.$$

leképezés jól definiált.

Első tételünk az összeejtő monoidokat karakterizálja mindazon transzformáció-monoidok között, amelyek permutációkból és legalább egy egyváltozós kontans műveletből állnak. Ezen tétel egyben az A. Fearnley és I. Rosenberg [FR03]-ban kapott egyik eredményének is a kiterjesztése.

5.1. Tétel ([Dor08]). *Legyen A legalább kételemű véges halmaz, és legyen M olyan transzformáció-monoid az A halmazon, amely legalább egy egyváltozós konstans műveletből és permutációkból áll. Ekkor M pontosan akkor összeejtő, ha*

(i) $|V| \geqslant 2$,

(ii) P_W *transitív,*

(iii) $\mathfrak{i}_V$ *injektív, és*

(iv) *az alábbi feltételek közül legalább egy teljesül:*

  (a) *az* $M|_V = \{m|_V : m \in M\}$ *monoid összeejtő,*

  (b) *a* $\mathfrak{j}$ *leképezés nem injektív,*

  (c) *a* P_W *permutációcsoport nem reguláris.*

Az 5.1. Tétel következményeként azonnal adódik, hogy olyan $M \subseteq C(A) \cup S(A)$ monoid, amely pontosan egy egyváltozós konstans műveletet tartalmaz nem lehet összeejtő. A következő tétel ennél egy kicsit többet állít.

5.1. Tétel ([Dor08]). *Legyen $M \subseteq C(A) \cup S(A)$ olyan monoid, amely pontosan egy egyváltozós kontans műveletet tartalmaz. Ekkor az* $\mathrm{Int}(M)$ *monadikus intervallum végtelen.*

5.3. Tétel ([Dor08]). *Legyen A olyan véges halmaz, amely legalább két elemet, és legyen $M \subseteq C(A) \cup S(A)$ olyan transzformáció-monoid, amely legalább két egyváltozós konstans műveletet tartalmaz. Ha az 5.1. Tétel* (i)–(iii) *feltételei teljesülnek, de a* (iv) *feltétel nem, akkor* $\mathrm{Int}(M)$ *izomorf* $\mathrm{Int}(M|_V)$-*vel. Így,*

- *ha* $|V| = 2$ *és* $M|_V$ *megegyezik az* $\{\mathrm{id}_V, c_0|_V, c_1|_V\}$ *monoiddal, akkor* $|W| = 1$, $M = \{\mathrm{id}_A, c_0, c_1\}$, *and* $\mathrm{Int}(M)$ *izomorf a kételemű lánc direkt négyzetével;*
- *ha* $|V| = 2$ *és* $M|_V$ *a teljes transzformáció-félcsoport* V*-n, akkor* $|W| = 2$, $M = \{\mathrm{id}_A, c_0, c_1, (0\,1)(2\,3)\}$, *és* $\mathrm{Int}(M)$ *egy háromelemű lánc;*
- *ha* $|V| \geqslant 3$, *akkor* $\mathrm{Int}(M)$ *egy kételemű lánc.*