On Two Problems Concerning Partially Ordered Sets

Outline of Ph.D. Thesis by

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The thesis is about two problems both concerning partially ordered sets, shortly, *posets*. Though being connected by the type of their main objects of interest, i.e. posets, the two problems are unrelated in their depths.

1 Definability in the Embeddability and Substructure Orderings of Finite Directed Graphs

The first problem, which fills up Chapter 1, is *first-order definability in substructure and embeddability orderings*. Chapter 1 is based on three papers of the author [5–7]. To put it on the huge palette of mathematics merely, one could say the following. The questions seem like logic: try to grasp the expressive power of a certain first-order language in a given structure. To get answers, we use basic, finite, combinatorial thinking, no more. What looms behind the problems though, is the symmetries of some particular, complicated, infinite posets. This research is, in fact, a continuation of a series of papers by Jaroslav Ježek and Ralph McKenzie [1–4], published in 2009-2010. Beyond the author of this thesis, others have picked up on this topic [9, 12–14].

Let us go into detail a little. Let $\mathcal{D}$ be the set of (the isomorphism types of) finite directed graphs, shortly, digraphs. For two digraphs $G, G' \in \mathcal{D}$, let $G \leq G'$ denote that G is *embeddable* into G', that is we can get G from G' by leaving out some vertices and edges. Equivalently, there exists an injective map from G to G' preserving the edges. An ostensibly similar notion follows. Let $G \sqsubseteq G'$ denote that G is a *substructure* of G', that is we can get G from G' by leaving out vertices only. Equivalently, there exists an injective map from G to G' preserving both edges and non-edges (i. e. the absence of edges). What we have so far is two partially ordered sets: $(\mathcal{D}; \leq)$ and $(\mathcal{D}; \sqsubseteq)$ (see Figs. 1 and 2). In the first chapter of the thesis, we investigate the expressive power of the first-order language of partially ordered sets for these two particular posets.

Probably, the most natural question is elementwise definability. Can you identify every single element in either $(\mathcal{D}; \leq)$ or $(\mathcal{D}; \sqsubseteq)$ by

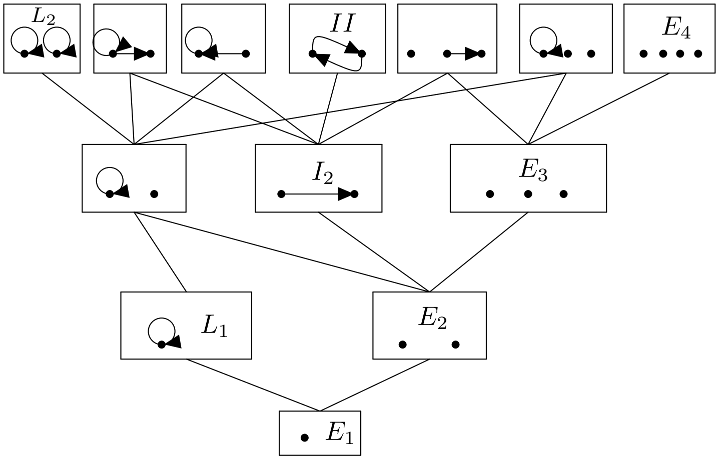

Figure 1: The initial segment of the Hasse diagram of the embeddability ordering, $(\mathcal{D}; \leq)$.

a first-order formula in the language of posets? This is where symmetries, i. e. automorphisms, come into play. Say, in a poset P, the element p is taken by an automorphism to a different element p'. Then, naturally, first-order formulas cannot distinguish p from p' as they share the exact same structural properties in P.

With regard to both the automorphisms and definability, $(\mathcal{D}; \leq)$ is a much easier nut to crack. Therefore, we start Chapter 1 with the embeddability ordering. The automorphism that sends G to its transpose G^T, that is just reversing all edges, is easy to discover. Consequently, the strongest we can hope, in terms of elementwise definability, is that the set $\{G, G^T\}$ is first-order definable for every digraph $G \in \mathcal{D}$. Indeed, this is proven in the thesis.

Theorem 1. *In the poset $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is first-order definable for all finite digraph $G \in \mathcal{D}$.*

Using this theorem, we can show that there is no other nontrivial automorphism, pointing to the strong, back-and-forth connection between the definability we investigate and the automorphisms.

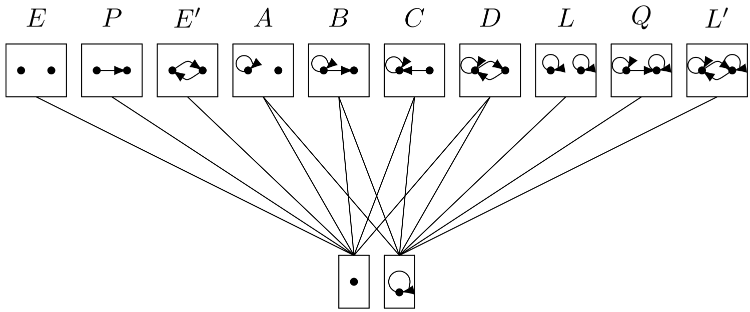

Figure 2: The initial segment of the Hasse diagram of the substructure ordering, $(\mathcal{D}; \sqsubseteq)$.

Theorem 2. *The poset $(\mathcal{D}; \leq)$ has exactly two automorphisms, namely the trivial and the one that maps every digraph to its transpose. Consequently, the automorphism group of $(\mathcal{D}; \leq)$ is isomorphic to $\mathbb{Z}_2$.*

So far, what we have settled is the definability of finite subsets of $(\mathcal{D}; \leq)$: a finite subset $S \subset \mathcal{D}$ is first-order definable if and only if $G \in S$ implies $G^T \in S$. Hence, to move forward, we must ask about infinite subsets.

As a famous statement in model theory reveals, there is no first-order formula defining the set of weakly connected digraphs in their own first-order language. Surprisingly, we do have such a formula in our language. We even show that with the addition of just a single constant, a digraph that is not isomorphic to its transpose, the whole second-order language of directed graphs is expressible in our language. Technically, what we do is go on the path laid by Ježek and McKenzie in [4]. We define a new language. This language is seemingly much stronger than the first-order language in question. Nonetheless we show that, in fact, it possesses the same expressive power.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for all $n \in \mathbb{N}$. Let us define the small category $\mathcal{CD}$ of finite digraphs the following way. The set $\text{ob}(\mathcal{CD})$ of objects consists of digraphs on $[n]$ for some $n \in \mathbb{N}$. For all $A, B \in \text{ob}(\mathcal{CD})$ let $\hom(A, B)$ consist of triples $f = (A, \alpha, B)$ where $\alpha : A \to B$ is edge-preserving. Composition of morphisms is made the following way. For arbitrary objects $A, B, C \in \mathrm{ob}(\mathcal{CD})$ if $f = (A, \alpha, B)$ and $g = (B, \beta, C)$, then

$$fg = (A, \beta \circ \alpha, C).$$

It is easy to see that $f \in \hom(A, B)$ is injective if and only if for all $X \in \mathrm{ob}(\mathcal{CD})$

$$\forall g, h \in \hom(X, A) : \quad gf = hf \Leftrightarrow g = h. \tag{1}$$

Similarly $f \in \hom(A, B)$ is surjective is and only if for all $X \in \mathrm{ob}(\mathcal{CD})$

$$\forall g, h \in \hom(B, X) : \quad fg = fh \Leftrightarrow g = h.$$

These are first-order definitions in the (first-order) language of categories, hence in $\mathcal{CD}$, isomorphism and embeddability are first-order definable. This implies that all first-order definable relations in $(\mathcal{D}, \leq)$ are definable in $\mathcal{CD}$ too.

Let us introduce some objects and morphisms:

$$\begin{aligned} \mathbf{E}_1 \in \mathrm{ob}(\mathcal{CD}) : &\quad V(\mathbf{E}_1) = [1],\ E(\mathbf{E}_1) = \emptyset, \\ \mathbf{I}_2 \in \mathrm{ob}(\mathcal{CD}) : &\quad V(\mathbf{I}_2) = [2],\ E(\mathbf{E}_1) = \{(1,2)\}, \\ \mathbf{f}_1 \in \hom(\mathbf{E}_1, \mathbf{I}_2) : &\quad \mathbf{f}_1 = (\mathbf{E}_1, \{(1,1)\}, \mathbf{I}_2), \\ \mathbf{f}_2 \in \hom(\mathbf{E}_1, \mathbf{I}_2) : &\quad \mathbf{f}_2 = (\mathbf{E}_1, \{(1,2)\}, \mathbf{I}_2). \end{aligned}$$

Adding these four constants to $\mathcal{CD}$ we get $\mathcal{CD}'$.

In the first-order language of $(\mathcal{D}, \leq)$, formulas can only operate with facts telling whether digraphs as a whole are embeddable into each other or not, the inner structure of digraphs is (officially) unavailable. In the first-order language of $\mathcal{CD}'$ though, we can capture embeddability (as we have seen above) but it is possible to capture the first-order language of digraphs too. The latter is not trivial, but the following argument explains it. For any $X \in \mathrm{ob}(\mathcal{CD})$ the set of morphisms $\hom(\mathbf{E}_1, X)$ is naturally bijective with the elements of X. Observe that if $f, g \in \hom(\mathbf{E}_1, X)$ are

$$f = (\mathbf{E}_1, \{(1,x)\}, X), \quad g = (\mathbf{E}_1, \{(1,y)\}, X) \quad (x, y \in V(X)),$$

then (x, y) is an edge of X if and only if

$$\exists h \in \hom(\mathbf{I}_2, X) : \quad \mathbf{f}_1 h = f,\ \mathbf{f}_2 h = g. \tag{2}$$

This shows how we can reach the inner structure of digraphs with the first-order language of $\mathcal{CD}'$. So the first-order language of $\mathcal{CD}'$ is much richer than that of $(\mathcal{D}, \leq)$. We can go even further. One can show that the first-order language of $\mathcal{CD}'$ can express the full second-order language of digraphs. To formulate this more precisely, we will show that the first-order language of $\mathcal{CD}'$ can express a language containing not only variables ranging over objects and morphisms of $\mathcal{CD}'$ but also

(I) quantifiable variables ranging over

(a) elements of any object,

(b) arbitrary subsets of objects,

(c) arbitrary functions between two objects,

(d) arbitrary subsets of products of finitely many objects (heterogeneous relations),

(II) dependent variables giving the universe and the edge relation of an object,

(III) the apparatus to denote

(a) edge relation between elements,

(b) application of a function to an element,

(c) membership of a tuple of elements in a relation.

We say that the relation $\rho \subseteq (\mathrm{ob}(\mathcal{CD}))^n$ is *isomorphism invariant* if whenever $A_i, B_i \in \mathrm{ob}(\mathcal{CD})$, and $A_i \cong B_i$ $(1 \leq i \leq n)$, then

$$(A_1, \ldots, A_n) \in \rho \Leftrightarrow (B_1, \ldots, B_n) \in \rho.$$

The set of isomorphism invariant relations of $\mathrm{ob}(\mathcal{CD})$ is naturally bijective with the relations of $\mathcal{D}$.

Let A denote the (isomorphism type of the) digraph with edges (a, c) and (b, c), on three vertices: a, b and c. Then what we prove is the following.

Theorem 3. *A relation is definable using the first-order language of $(\mathcal{D}; \leq, A)$ if and only if the corresponding isomorphism invariant relation of $\mathcal{CD}'$ is first-order definable in $\mathcal{CD}'$.*

To prove this, we somehow "model" the workings of this category using the first-or language of $(\mathcal{D}; \leq, A)$. This is a long and technical proof.

The second part of Chapter 1 examines the *substructure* ordering, $(\mathcal{D}; \sqsubseteq)$. Here, we are faced with something new right away. Unprecedented in the line of this topic, we find nontrivial automorphisms. Though we present a conjecture for the automorphism group, it is unproven at the moment.

Conjecture 4. *The automorphism group of* $(\mathcal{D}; \sqsubseteq)$ *is isomorphic to the 768-element group,* $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$*, with a given* α *in the semidirect product.*

We try to offer some sense of this vast automorphism group. To define an automorphism φ, we need to tell how to get $\varphi(G)$ from G. All the automorphisms, that we know of at the moment, share a particular characteristic. They are all, say, *local* in the following sense. Roughly speaking, to get $\varphi(G)$ from G, one only needs to consider and modify G's at most two element substructures according to some given rule. To make this clearer, we give a nontrivial example. Let $\varphi(G)$ be the digraph that we get from G such that we change the direction of the edges on those two element substructures of G that have loops on both vertices. It is easy to see that this defines an automorphism, indeed. (Perhaps, one would quickly discover the automorphism that gets $\varphi(G)$ by reversing all edges of G, but this is different.) Observe that, in this example, the modification of G happens locally, namely on 2-element substructures. All the automorphisms, that we know of, share this property. Now, we define some of our automorphisms φ_i. To do so, we just tell how to get $\varphi_i(G)$ from G. One of the most trivial automorphisms is

- φ_1: where there is a loop, clear it, and vice versa, to the vertices with no loop, insert one.

Observe that this automorphism operates with the 1-element substructures. Now we start to make use of the labels of Fig. 2.

- φ_2: change the substructures (isomorphic to) E to E' and vice versa.

- φ_3: change the substructures (isomorphic to) L to L' and vice versa.
- φ_4: reverse the edges in the substructures (isomorphic to) P.
- φ_5: reverse the edges in the substructures (isomorphic to) Q.

Let S_4 denote the symmetric group over the four-element set $\{A, B, C, D\}$, and $\pi \in S_4$. We define

- φ_π: We change the substructures (isomorphic to) $X \in \{A, B, C, D\}$ to $\pi(X)$ (such that the loops remain in place).

Observe that, with the exception of φ_1, the automorphisms defined above do not touch loops (when getting $\varphi_i(G)$ from G). We conjecture that these automorphisms generate the whole automorphism group. After seeing these generators, the 768-element group, $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$, may feel more natural to the reader.

We have already seen that there is a strong connection between the expressive power of the first-order language of posets and their automorphism groups. Does this mean that the uncertain automorphism group blocks us from getting any definability result? Though this could very well be the case, fortunately, it is not. What we show is the following.

Theorem 5. *With the addition of finitely many constants, the first-order language of* $(\mathcal{D}; \sqsubseteq)$ *can express that of* $(\mathcal{D}; \leq)$.

Note that this theorem carries weight only because, at this point, we've already established that the first-order language of $(\mathcal{D}; \leq)$ is very strong.

How do we prove such an expressibility statement though? Despite the fact that the proof is long and technical, it is based on a simple idea, which we outline here. Remember that we get substructures of a directed graph by leaving out vertices, while, to get embeddable digraphs, we can leave out both vertices and edges. We want to define the latter, so we should be able to 'simulate' leaving out edges somehow. Our approach is the following. In a digraph G, if there is an edge (u, v), then we add a vertex and two edges to 'support' the edge (u, v). Namely, we add w to the set of vertices, and the edges (u, w) and (w, v) to the set of edges. After the addition, we say that the edge (u, v) is 'supported'. The idea is that the supportedness of an edge can be terminated by leaving out a vertex, i.e. w in the previous example, what we can do by taking substructures. Roughly, what we should do is: support all edges, take a substructure, and in one more step, leave only the supported edges in. Of course, there seem to be many problems with this. Firstly, how can we distinguish between the supporting vertices and the original ones? This appears to be an essential part of the plan. Secondly, the plan ended with "leave only the supported edges in" which just looks like running into the original problem again: We cannot leave edges out. Even though the plan seems flawed for these reasons, it is manageable. That is what we do in the thesis.

Finding a minimal list of the constants of the theorem above is almost equivalent to determining the automorphism group. Hence such a minimal list is not provided. A possible, far-from-minimal list consists of the digraphs of at most 12 elements. It might seem odd that we do not "know" what constant digraphs we use through our proof. This is because some of our arguments go the following way. Some properties of digraphs can be told by saying something about them *locally*. For example, one can judge if a digraph has a non-loop edge by the set of its (at most) 2-element substructures. Far more complicated properties can be told in this way. It would get overwhelmingly tedious to list all the digraphs that are used in this manner. And even if we did so, though we would get a much more concrete list, it would still be quite far from minimal. Therefore, analyzing this particular proof to get a minimal list seems hopeless (at least to the author).

As a corollary to the theorem above, we get, nevertheless, that the automorphism group is finite—the best we can prove as for now.

Theorem 6. *The automorphism group of the poset* $(\mathcal{D}; \sqsubseteq)$ *is finite.*

2 On Finite Generability of Clones of Finite Posets

This chapter investigates a completely different problem, still having posets as main players in it. A set of finitary operations is called a clone if it contains all projections and is closed under superposition (composition). In this thesis, we always assume the base set of our operations to be finite. Clearly, the set of all operations (on a finite base set) is a clone. The largest (with respect to inclusion) clones that are smaller than this one are called maximal clones. Ivo G. Rosenberg, in a classical result [10], classified the maximal clones into six classes. For five of the six classes it has been shown that the clones of these classes are finitely generated. The unsettled class is the class of clones consisting of the monotone operations of bounded partial orders, that is posets having both least and largest elements. Some partial results have already been obtained. Monotone clones of at most seven element posets are proven to be finitely generated and so are posets with a monotone near unanimity operation. In a brilliant paper [11] from 1986, Gábor Tardos shows that the clone of a particular eight element poset is not finitely generated. This was the first proof showing a maximal clone to be not finitely generated. In a 1993 paper [15], László Zádori generalized Tardos's result by describing all series parallel posets having not finitely generated clones. Since Zádori, up until recently no one found non-finitely generated maximal clones, though one may conjecture that there are a lot of them. We present the recent paper [8] finding new such clones in Chapter 2. The author submerged in this topic as a PhD student guided by his second supervisor, the professor Zádori just mentioned. Miklós Maróti, the first supervisor of the author, also joined. The three of them wrote the paper [8] that comes up with a new family of finite bounded posets whose clones of monotone operations are not finitely generated and suggests some directions where, the authors think, this research might evolve in the future.

In the first part of the chapter, we present this new family of finite bounded posets whose clones of monotone operations are not finitely generated. Let $\mathbf{k}$ denote the k-element antichain. Let A_n be the poset obtained from the Boolean lattice with n atoms by removing its greatest element, and B_n the dual of A_n. Let $C_{m,n} = A_m + \mathbf{2} + B_n$ (see Fig. 3). We prove the following.

Theorem 7. *If $m, n \geq 2$, then the clones and idempotent clones of $C_{m,n}$ are non-finitely generated.*

Here, by the idempotent clone, as usual, we mean the clone of those monotone operations that satisfy the identity $f(x, \ldots, x) = x$. The proof of this statement is an analogue of the one in the famous paper of Tardos.

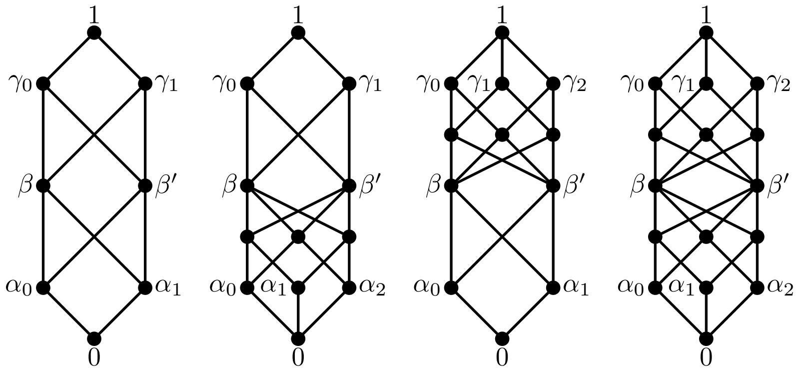

Figure 3: The posets $C_{2,2}$, $C_{3,2}$, $C_{2,3}$, and $C_{3,3}$

Sketchily, the Tardos-proof goes the following way. It presents relations which the small-arity operations preserve, while the large-arity ones do not. Say, you come up with a relation r that is preserved by operations of arity at most n, but there is a larger-arity operation that does not preserve r. This means that the (finite) set of the monotone operations of arity at most n does not generate the clone.

How do we produce a particular "larger-arity" operation that does not preserve our relation? Tardos defines it partially, only on some carefully chosen elements of the domain so that it shows that, indeed, it does not preserve his relation. Then he states that his partial operation is extendible fully. To show the extendability, he leans heavily on the description of the so-called *obstructions* of his poset which he provides earlier. (Actually, he did not call them obstructions at the time, he called them zigzags for their shape in his particular case.) Obstructions are the minimal causes that block partial operations from being extendible. Using them, the proof of the extendability of a partial function is just checking it does not contain any of them. The toughest part of carrying over the Tardos-proof to other posets turns out to be the description of obstructions. Fortunately, we managed to describe them for $C_{m,n}$.

Another interesting family of finite posets, from the finite generability point of view, is the family of locked crowns. To decide whether the clone of a locked crown, where the crown is of at least six elements, is finitely generated or not, one needs to go beyond the scope of Tardos's proof for the description of obstructions in this case seems hopeless. Although our investigations are not complete in this direction, they led to the results in the second part of the chapter.

We call a monotone operation ascending if it is greater than or equal to some projection. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of ascending idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. We provide an example of a half bounded finite poset whose clone of ascending idempotent operations is finitely generated but whose clone is not finitely generated.

Theorem 8. *The clone of ascending idempotent operations of the poset* $\mathbf{2} + \mathbf{2} + \mathbf{1}$ *is finitely generated. Meanwhile, its clone is nonfinitely generated.*

Another interesting consequence of our results is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations.

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