On Two Problems Concerning Partially Ordered Sets

Ph.D. Thesis

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Contents

Acknowledgements

First of all, I thank my two supervisors, Miklós Maróti and László Zádori. Miklós launched me into research giving me my first topic. The first chapter of this thesis is what I have squeezed out of this very topic since. He was supporting me even in times when we were parted by the Atlantic Ocean. Professor Zádori introduced me to the topic investigated in the second chapter. When Miklós was overseas, he was the one guiding me, not letting me feel unattended.

During the past few years, I had the privilege to work with great coauthors. Beyond the two mentioned already, I must name the other two, Gábor Czédli and Gergő Gyenizse. They motivated me in many ways that I could encapsulate in the phrase "setting the bar high".

Through my adolescence, I always got lucky to have exceptional math teachers, which must have contributed to me seeing absolutely no other option but to become a mathematician.

Last but not least, many of my thanks go to my family. When, abruptly, I fell in love with mathematics, my mother started supporting me right away by both showing no signs of skepticism and buying as many math books as I dared to ask. Later, at the university, I learned from the borrowed notes of a beautiful classmate called Fanni. Without her (notes) I might still be an undergraduate. She has been putting up with me since, from a certain point as my wife, which is not easy, especially when I submerge deeply into my own thoughts.

Foreword

This dissertation is about two topics, both concerning partially ordered sets, in short *posets*. Accordingly, it is split into two chapters. It is built around four papers of the author [7–10].

The first chapter deals with the author's oldest research topic and is based on the papers [7–9]. In 2009–2010 Jaroslav Ježek and Ralph McKenzie published a series of papers [3–6] in which they examined (among other things) the first-order definability in substructure orderings (some particular posets) of finite mathematical structures with a given type, and determined the automorphism group of these orderings. They considered finite semilattices [3], ordered sets [6], distributive lattices [4] and lattices [5]. Similar investigations [12, 15–17] have emerged since. Substructureness induces a natural order on the set of (isomorphism types of) structures of a fixed type, e. g. finite semilattices. By definability in substructure orderings, they meant first-order definability in the language of partially ordered sets in these particular substructure-orders. In such languages, the internal structure of the structures that are the vertices of the poset cannot be invoked (formally). Only their relations to each other, i. e. the substructure relation between them can be referenced. By examining such languages, we mean investigating their expressive power: what can and what can't they tell? Those limits are intimately related to the automorphism groups of the orderings and that is one of the reasons why Ježek and McKenzie dealt with them in their papers.

The author, as a BSc student at the time, set out to continue this research for finite directed graphs. He was advised to do so by his first supervisor, Miklós Maróti, who, telling it by heart, told the basic concept not quite in line with the Ježek-McKenzie papers. Neither of them had any idea how outstandingly lucky this mistake was going to turn out eventually. Given the problem, the author immediately started working on it. Only when already having results did he and his first supervisor realize the difference in the basic concepts. At that point, they decided to carry on, not to let go of the research that already had been done. Instead of the substructure concept used by Ježek and McKenzie, the author started to work on the embeddability concept for directed graphs which is related the constraint satisfaction problem. As their names suggest, both speak of directed graphs being parts of each other somehow, but there is a huge difference in the how. Substructureness lets one delete only vertices to get its smaller parts, while to get embeddable directed graphs, we can delete both edges and vertices. To get a feeling of the difference, see Figures 1 and 2, showing the lower segments of the embeddability and substructure orderings, respectively.

In this thesis, we will deal with both concepts for directed graphs, dividing Chapter 1 into two parts correspondingly. In the first years, the author was considering exclusively

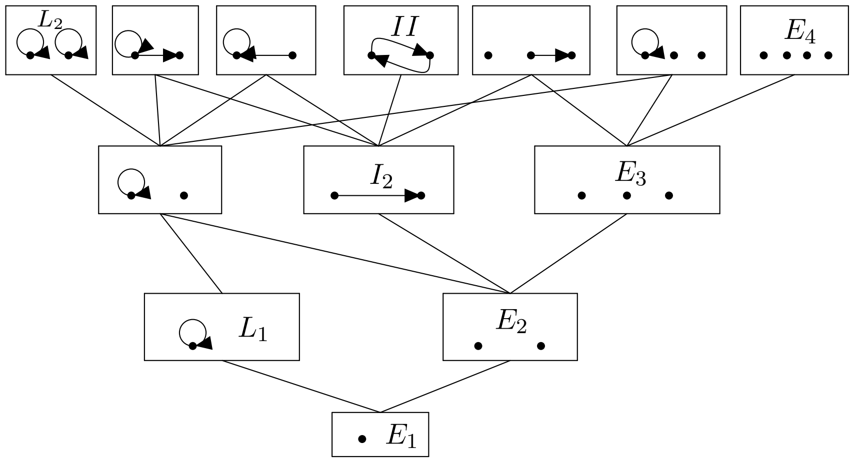


Figure 1: The initial segment of the Hasse diagram of the embeddability ordering of digraphs, that is $(\mathcal{D}; \leq)$.

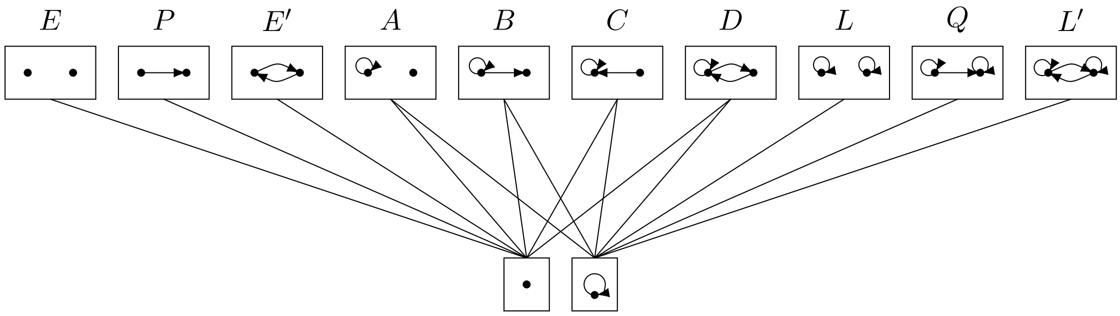


Figure 2: The initial segment of the Hasse diagram of the substructure ordering of digraphs, that is $(\mathcal{D}; \sqsubseteq)$. (The labels are only used for a specific purpose in the thesis. Please disregard them unless they are being specifically referred to, for they are not the general notations of the digraphs under them.)

the embeddability concept. Both his BSc and MSc theses were written around this topic and he had two papers published [8, 9], almost literal adaptations of these. The Ježek-McKenzie papers, in spite of dealing with four different structure types, had a particular joint taste. For one, the automorphism groups of the partial orders all turned out to be either trivial or the two element group. The first two studies of the author [8, 9] shared this taste despite working with embeddability instead of substructureness. The author did have a foggy feeling how the substructure case might play out, but not until having a sense of closure for the embeddability research did he start to think seriously about the substructure concept, the original one studied by Ježek and McKenzie. Upon having an earnest look at the problem, automorphisms were beginning to turn up. Gradually, more and more automorphisms were being found. This, being unprecedented in the line of this particular research topic, startled the author. He finally found 768 automorphisms, forming a group far from trivial. As already mentioned, the automorphism groups are closely related to the power of the languages being probed here. The bigger the automorphism group, the weaker the language. Hence it is not surprising that this vast automorphism group makes the research veer off of its usual trajectory. Unfortunately, the author cannot prove there are no automorphisms beyond the ones he has found. Yet, we will prove that there are finitely many automorphisms. As for the expressive power of the first-order language in question, that is where the enormous luck of getting the main concept wrong at the beginning comes in. What we will prove is roughly the following. Modulo the orbits of the (finite) automorphism group, the first-order language of the substructure ordering can express that of the embeddability ordering. This is a big deal only because the first-order language of the embeddability ordering has already been examined, and it has turned out to be very strong. Now we see how the author (and his first supervisor) got lucky. If faced with the substructure-problem first, the author probably would have gotten deterred at the beginning, stumbling into the hardness of the problem, having no tools to tackle it.

The second chapter investigates a completely different problem, still having posets as main players in it though. Ivo G. Rosenberg, in a classical result [13], classified the maximal clones into six classes. For five of the six classes it has been shown that the clones of these classes are finitely generated. The unsettled class is the class of clones containing monotone operations of bounded partial orders, that is posets having both least and largest elements. Some partial results have already been obtained. Monotone clones of at most seven element posets are proven to be finitely generated and so are posets with an at least ternary monotone near unanimity operation. In a brilliant paper from 1986, Gábor Tardos [14] shows that the clone of a particular eight element poset is not finitely generated. This was the first proof showing a maximal clone to be not finitely generated. In a 1993 paper [18], László Zádori, the second supervisor of the author, generalized Tardos's result by describing all series parallel posets having not finitely generated clones. Since Zádori, up until recently no one found non-finitely generated maximal clones, though one may conjecture there are a lot of them. In the second chapter, we present the recent paper [10] finding new such clones, written by the author of this Thesis and his two supervisors. We come up with a new family of finite bounded posets whose clones of monotone operations are not finitely generated and suggest some directions where we think this research might evolve in the future.

In the first part of the chapter, as already mentioned, we present a new family of finite bounded posets whose clones of monotone operations are not finitely generated. The proofs of these results are analogues of those of Tardos. Another interesting family of finite posets from the finite generability point of view is the family of locked crowns. To decide whether the clone of a locked crown, where the crown is of at least six elements, is finitely generated or not, one needs to go beyond the scope of Tardos's proof. Although our investigations are not complete in this direction, they led to the results in the second part of the chapter.

We call a monotone operation ascending if it is greater than or equal to some projection. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of ascending idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. We provide an example of a half bounded finite poset whose clone of ascending idempotent operations is finitely generated but whose clone is not finitely generated. Another interesting consequence of our result is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations.

Chapter 1

Definability in the Embeddability and Substructure Orderings of Finite Directed Graphs

1.1 General Introduction

This chapter synthesizes the papers [7–9]. Though only two of them, namely [9] and [7], are included as they are. This is because there is an intricate connection between the papers [8] and [9]. Oversimplified, we could say the main result of [9] yields that of [8]. But there is more to it than that, which is to be explained later. Nevertheless, this saves us from including [8] as is.

Before introducing the most basic concepts, a few words on the notations in general. There is a table of notations at the end of the chapter, starting on page 48, to save the readers from getting lost among the many notations used throughout. It would be nice to keep the notations of the original papers, but it is impossible as they have conflicting notations. This is not because the author did not try to stay consistent. The papers share the following characteristic. Their main theorems are proven with a main construction that requires a lot of preparation involving a lot of special digraphs to be introduced, all of them needing notations, naturally. The author tried to make the notations intuitive, e. g. the circle graph having n vertices is denoted by O_n. The main constructions do resemble at first sight but differ in the details, so do the technicalities leading up to them. There were cases when, in separate papers, slightly different notions cried for the same notations, or at least it was natural to use the same notation for a different, though similar, concept. This, happening in separate papers, did not cause conflict. However, obviously, such ambiguities must not happen inside such a thesis, therefore change of some notations is unavoidable. We will tackle this the following way. As we go forward, we keep the original notations as long as it causes no conflict. If, to avoid a conflict, a change of notation is implemented, the reader is always notified.

It is time to introduce the most basic mathematical concepts of this chapter precisely. Let us consider a nonempty set V and a binary relation $E \subseteq V^2$. We call the pair $G = (V, E)$ a *directed graph* or just *digraph*. Let $\mathcal{D}$ denote the set of isomorphism types of finite digraphs.

The elements of $V(= V(G))$ and $E(= E(G))$ are called the *vertices* and *edges* of G, respectively. A digraph G is said to be *embeddable* into G', and we write $G \leq G'$, if there exists an injective homomorphism $\varphi : G \to G'$, i.e. an injective map for which $(v_1, v_2) \in E(G)$ implies $(\varphi(v_1), \varphi(v_2)) \in E(G')$. A digraph G is a *substructure* of G', and we write $G \sqsubseteq G'$, if it is isomorphic to an induced substructure (on some subset of the vertices) of G' . In graph theory, the term *subgraph* is used rather for the embeddability concept, so will it be used here. Every substructure is embeddable but the converse is not true. The names of these two concepts often mix both orally and on paper when it is clear from the context which notion we are using the whole time. In this thesis, however, we must be very cautious as both concepts are used alternately throughout. It is easy to see that both $\leq$ and $\sqsubseteq$ are partial orders on $\mathcal{D}$. Both partially ordered sets are naturally graded. The digraph G is on the nth level of $(\mathcal{D}; \leq)$ or $(\mathcal{D}; \sqsubseteq)$ if $|V(G)|+|E(G)| = n$ or $|V(G)| = n$, respectively. See Figures 1 and 2 for the initial segments of the Hasse diagrams of the two partial orders. For digraphs $G, G' \in \mathcal{D}$, let $G \mathbin{\dot\cup} G'$ denote their disjoint union, as usual. We use the term *weakly connected* in the usual sense, i. e. just disregarding the direction of the edges. Our digraphs split into *weakly connected components*, naturally. Let us use the abbreviation *wcc*="weakly connected component" and *wccs* for the plural because there are proofs mentioning this particular expression many many times.

Let $(\mathcal{A}; \leq)$ be an arbitrary poset. An n-ary relation R is said to be (first-order) definable in $(\mathcal{A}; \leq)$ if there exists a first-order formula $\Psi(x_1, x_2, \ldots, x_n)$ with free variables $x_1, x_2, \ldots, x_n$ in the language of partially ordered sets such that for any $a_1, a_2, \ldots, a_n \in \mathcal{A}$, $\Psi(a_1, a_2, \ldots, a_n)$ holds in $(\mathcal{A}; \leq)$ if and only if $(a_1, a_2, \ldots, a_n) \in R$. A subset of $\mathcal{A}$ is definable if it is definable as a unary relation. An element $a \in \mathcal{A}$ is said to be definable if the set $\{a\}$ is definable.

1.2 Embeddability

1.2.1 Introduction

The directed graph $G^T := (V, E^{-1})$ is called the *transpose* of G, where E^{-1} denotes the inverse relation of E. In the poset $(\mathcal{D}, \leq)$ let $G \prec G'$ denote that G' covers G. Obviously $\prec$ is a definable relation in $(\mathcal{D}, \leq)$. In [8], the main result is

Theorem 1.1 (Theorem 2.38 [8])**.** *In the poset $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is first-order definable for all finite digraph $G \in \mathcal{D}$.*

This theorem is the best possible in the following sense. Observe, that $G \mapsto G^T$ is an automorphism of $(\mathcal{D}; \leq)$. This implies that the digraphs G and G^T cannot be distinguished with first-order formulas of $(\mathcal{D}; \leq)$. What does Theorem 1.1 tell about first-order definability in $(\mathcal{D}; \leq)$? It tells the following

Corollary 1.2. *A finite set H of digraphs is definable if and only if*

$$\forall G \in \mathcal{D} : \quad G \in H \Rightarrow G^T \in H.$$

So the first-order definability of finite subsets in $(\mathcal{D}; \leq)$ is settled. What about infinite subsets? One might ask if the set of weakly connected digraphs is first-order definable in

$(\mathcal{D}; \leq)$ as a standard model-theoretic argument shows that it is not definable in the first-order language of digraphs. The answer to this question appears to be out of reach with the result of [8]. In [9] though, some apparatus is built to handle a number of such questions. In doing so, a path laid by Ježek and McKenzie in [6] was followed. In particular, the set of weakly connected digraphs turned out to be definable.

This section's main theorem coincides with that of [9], so we build from this paper mostly. The method (of [9]) is the following. We add a constant—a particular digraph that is not isomorphic to its transpose—A to the structure $(\mathcal{D}; \leq)$ to get $(\mathcal{D}; \leq, A)$. We define an enriched small category $\mathcal{CD}'$ and show that its first-order language is quite strong: it contains the full second-order language of digraphs. Finally, we show that first-order definability in $\mathcal{CD}'$ (after factoring by isomorphism) is equivalent to first-order definability in $(\mathcal{D}; \leq, A)$. This result gives Theorem 1.1 as an easy corollary and a lot more.

We offer two approaches for the proof (of the main theorem). We either use the result of [8], Theorem 1.1, and do not get it as a corollary but have a more elegant proof for our main result. Or we do not use it, instead we get it as a corollary but we have a little more tiresome proof for the main result.

1.2.2 Precise Formulation of the Section's Main Theorem and Some Display of its Power

Once more, we emphasize that the approach we present in this section is from Ježek and McKenzie [6].
Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for all $n \in \mathbb{N}$. Let us define the small category $\mathcal{CD}$ of finite digraphs the following way. The set $\mathrm{ob}(\mathcal{CD})$ of objects consists of digraphs on $[n]$ for some $n \in \mathbb{N}$. For all $A, B \in \mathrm{ob}(\mathcal{CD})$ let $\hom(A, B)$ consist of triples $f = (A, \alpha, B)$ where $\alpha : A \to B$ is a homomorphism, meaning $(x, y) \in E(A)$ implies $(\alpha(x), \alpha(y)) \in E(B)$. Composition of morphisms are made the following way. For arbitrary objects $A, B, C \in \mathrm{ob}(\mathcal{CD})$ if $f = (A, \alpha, B)$ and $g = (B, \beta, C)$, then

$$fg = (A, \beta \circ \alpha, C).$$

It is easy to see that $f \in \hom(A, B)$ is injective if and only if for all $X \in \mathrm{ob}(\mathcal{CD})$

$$\forall g, h \in \hom(X, A) : \quad gf = hf \Leftrightarrow g = h. \tag{1.1}$$

Similarly $f \in \hom(A, B)$ is surjective is and only if for all $X \in \mathrm{ob}(\mathcal{CD})$

$$\forall g, h \in \hom(B, X) : \quad fg = fh \Leftrightarrow g = h.$$

These are first-order definitions in the (first-order) language of categories, hence in $\mathcal{CD}$, isomorphism and embeddability are first-order definable. This implies that all first-order definable relations in $(\mathcal{D}, \leq)$ are definable in $\mathcal{CD}$ too. To put it more precisely, if $\rho \subseteq \mathcal{D}^n$ is an n-ary relation definable in $(\mathcal{D}; \leq)$ then

$$\{(A_1, \dots, A_n) : A_i \in \mathrm{ob}(\mathcal{CD}), (\bar{A}_1, \dots, \bar{A}_n) \in \rho\}$$

is definable in $\mathcal{CD}$, where $\bar{A}_i$ denotes the isomorphism type of A_i.

Definition 1.3. Let us introduce some objects and morphisms:

$$\begin{aligned}
\mathbf{E}_1 \in \mathrm{ob}(\mathcal{CD}): &\quad V(\mathbf{E}_1) = [1],\ E(\mathbf{E}_1) = \emptyset,\\
\mathbf{I}_2 \in \mathrm{ob}(\mathcal{CD}): &\quad V(\mathbf{I}_2) = [2],\ E(\mathbf{E}_1) = \{(1,2)\},\\
\mathbf{f}_1 \in \mathrm{hom}(\mathbf{E}_1, \mathbf{I}_2): &\quad \mathbf{f}_1 = (\mathbf{E}_1, \{(1,1)\}, \mathbf{I}_2),\\
\mathbf{f}_2 \in \mathrm{hom}(\mathbf{E}_1, \mathbf{I}_2): &\quad \mathbf{f}_2 = (\mathbf{E}_1, \{(1,2)\}, \mathbf{I}_2).
\end{aligned}$$

Adding these four constants to $\mathcal{CD}$ we get $\mathcal{CD}'$.

In the first-order language of $(\mathcal{D}, \leq)$, formulas can only operate with the facts whether digraphs as a whole are embeddable into each other or not, the inner structure of digraphs is (officially) unavailable. In the first-order language of $\mathcal{CD}'$ though, we can capture embeddability (as we have seen above) but it is possible to capture the first-order language of digraphs too. The latter is far from trivial, but the following argument explains it. For any $X \in \mathrm{ob}(\mathcal{CD})$ the set of morphisms $\mathrm{hom}(\mathbf{E}_1, X)$ is naturally bijective with the elements of X. Observe that if $f, g \in \mathrm{hom}(\mathbf{E}_1, X)$ are

$$f = (\mathbf{E}_1, \{(1,x)\}, X),\quad g = (\mathbf{E}_1, \{(1,y)\}, X)\quad (x, y \in V(X)),$$

then $(x,y) \in E(X)$ holds if and only if

$$\exists h \in \mathrm{hom}(\mathbf{I}_2, X): \quad \mathbf{f}_1 h = f,\ \mathbf{f}_2 h = g. \tag{1.2}$$

To put it briefly, $X \cong CD_X$, where

$$V(CD_X) = \mathrm{hom}(\mathbf{E}_1, X),\quad E(CD_X) = \{(f,g) : f, g \in \mathrm{hom}(\mathbf{E}_1, X),\ (1.2)\text{ holds}\}.$$

This shows how we can reach the inner structure of digraphs with the first-order language of $\mathcal{CD}'$. So the first-order language of $\mathcal{CD}'$ is much richer than that of $(\mathcal{D}, \leq)$. We can go even further. One can show that the first-order language of $\mathcal{CD}'$ can express the full second-order language of digraphs. To formulate this more precisely, the first-order language of $\mathcal{CD}'$ can express a language containing not only variables ranging over objects and morphisms of $\mathcal{CD}'$ but also

(I) quantifiable variables ranging over

  (a) elements of any object,
  (b) arbitrary subsets of objects,
  (c) arbitrary functions between two objects,
  (d) arbitrary subsets of products of finitely many objects (heterogeneous relations),

(II) dependent variables giving the universe and the edge relation of an object,

(III) the apparatus to denote

  (a) edge relation between elements,
  (b) application of a function to an element,
  (c) membership of a tuple of elements in a relation.

For example, let us see how (Ib), (Id) and (IIIc) can be "modeled" in $\mathcal{CD}'$.
Let us start with (Ib). Let $E_n \in \mathrm{ob}(\mathcal{CD}')$ denote the empty digraph on $[n]$. The set

$$E = \{E_n \in \mathrm{ob}(\mathcal{CD}') : n \in \mathbb{N}\}$$

is easily definable in $\mathcal{CD}'$. Let $A \in \mathrm{ob}(\mathcal{CD}')$ be an arbitrary object and $S \subseteq A$ a subset of it. Let γ be a bijection $V(E_{|S|}) \to S$. Let us define the morphism

$$p : E_{|S|} \to A, \;\; p(x) = \gamma(x) \;\; (x \in V(E_{|S|})).$$

It is easy to see that we represented the subset S with the pair $(E_{|S|}, p)$. A universal quantification over the subsets of A would look like

$$(\forall E_{|S|} \in E)(\forall p \in \hom(E_{|S|}, A)),$$

with the addition of p being injective, which is expressible, see (1.1). Next, let us consider (Id). Let $A_1, \dots, A_n \in \mathrm{ob}(\mathcal{CD}')$ be arbitrary objects and let $R \subseteq A_1 \times \cdots \times A_n$ be nonempty. Let $\pi_i(r)$ be the ith projection of $r \in R$. The functions $\pi_1, \dots, \pi_n$ "determine" the relation R in the following sense:

$$(a_1, \dots, a_n) \in R \;\; \Leftrightarrow \;\; \exists r \in R : \pi_i(r) = a_i \;\; (i = 1, \dots, n).$$

We will represent the functions π_i the following way. Let $\gamma : V(E_{|R|}) \to R$ be a bijection. Let us define the morphisms p_i:

$$p_i : E_{|R|} \to A_i, \;\; p_i(x) = \pi_i(\gamma(x)) \;\; (x \in V(E_{|R|}))$$

It is easy to see that we represented the relation R uniquely with $(E_{|R|}, p_1, \dots, p_n)$. So an example of an existential quantification of type (Id) is

$$(\exists E_{|R|} \in E)(\exists p_1 \in \hom(E_{|R|}, A_1)) \dots (\exists p_n \in \hom(E_{|R|}, A_n)).$$

For (IIIc), an element of $A_1 \times \cdots \times A_n$ is represented with an element of

$$\hom(E_1, A_1) \times \cdots \times \hom(E_1, A_n) \tag{1.3}$$

and if $(E_{|R|}, p_1, \dots, p_n)$ belongs to $R \subseteq A_1 \times \cdots \times A_n$ and $(f_1, \dots, f_n)$, an element of (1.3), belongs to $x \in A_1 \times \cdots \times A_n$, then $x \in R$ can be expressed in the way

$$(\exists f \in \hom(E_1, E_{|R|}))(fp_1 = f_1 \wedge \dots \wedge fp_1 = f_1).$$

Let $\mathbf{A} \in \mathrm{ob}(\mathcal{CD})$ denote the digraph $V(\mathbf{A}) = [3]$, $E(\mathbf{A}) = \{(1,3), (2,3)\}$. Now from the fact that in $\mathcal{CD}'$ isomorphism and embeddability are definable and from Theorem 1.1, the set

$$\{X \in \mathrm{ob}(\mathcal{CD}) : X \cong \mathbf{A} \text{ or } X \cong \mathbf{A}^T\}$$

is definable in $\mathcal{CD}'$. From this set, the formula (using the first order language of digraphs turned out to be expressible above)

$$(\exists x \in X)(\forall y \in X)(y \neq x \;\Rightarrow\; (y, x) \in E(X))$$

chooses the set

$$\{X \in \mathrm{ob}(\mathcal{CD}) : X \cong \mathbf{A}\}.$$

This shows that the first order language of $\mathcal{CD}'$ is stronger then the first-order language of $(\mathcal{D}, \leq)$ because in the latter, the isomorphism type of $\mathbf{A}$, denoted by A is not definable as it is not isomorphic to its transpose.

Definition 1.4. By adding the isomorphism type of **A** as a constant to $(\mathcal{D}, \leq)$ we get $(\mathcal{D}; \leq, A)$. Let us denote this structure by $\mathcal{D}'$.

We say that the relation $\rho \subseteq (\mathrm{ob}(\mathcal{CD}))^n$ is *isomorphism invariant* if when for $A_i, B_i \in \mathrm{ob}(\mathcal{CD})$, $A_i \cong B_i$ $(1 \leq i \leq n)$ hold, then

$$(A_1, \ldots, A_n) \in \rho \iff (B_1, \ldots, B_n) \in \rho.$$

The set of isomorphism invariant relations of $\mathrm{ob}(\mathcal{CD})$ is naturally bijective with the relations of $\mathcal{D}$. The main result of the section is the following

Theorem 1.5. *A relation is first-order definable in $\mathcal{D}'$ if and only if the corresponding isomorphism invariant relation of $\mathcal{CD}'$ is first-order definable in $\mathcal{CD}'$.*

We have already seen the proof of the easy(=only if) direction of this theorem. We prove the difficult direction in Subsection 1.2.3 by creating a model of $\mathcal{CD}'$ in $\mathcal{D}'$.

Definition 1.6. A relation $R \subseteq \mathrm{ob}(\mathcal{CD})^n$ is called *transposition invariant* if it is isomorphism invariant and $(G_1, \ldots, G_n) \in R$ implies $(G_1^T, \ldots, G_n^T) \in R$.

Corollary 1.7. *A relation is first-order definable in $\mathcal{D}$ if and only if the corresponding isomorphism invariant relation of $\mathcal{CD}'$ is both transposition invariant and first-order definable in $\mathcal{CD}'$.*

Proof. The "only if" direction is obvious. For the "if" direction, let $R \subseteq \mathcal{D}^n$ be a relation that corresponds to a transposition invariant and first-order definable relation of $\mathcal{CD}'$. We need to show that R is first-order definable in $\mathcal{D}$. We know, by Theorem 1.5, that it is first-order definable in $\mathcal{D}'$. Let $\Phi(x_1, \ldots, x_n)$ be a formula that defines it. Let $\Phi'(y, x_1, \ldots, x_n)$ denote the formula that we get from $\Phi(x_1, \ldots, x_n)$ by replacing the constant A with y at all of its occurrences. The set $\{A, A^T\}$ is easily defined (even without the usage of Theorem 1.1) in $\mathcal{D}$. Let us define

$$\Phi''(x_1, \ldots, x_n) := \exists y (y \in \{A, A^T\} \wedge \Phi'(y, x_1, \ldots, x_n)). \tag{1.4}$$

We claim that for $S := \{(x_1, \ldots, x_n) : \Phi''(x_1, \ldots, x_n)\}$, $S = R$ holds. $R \subseteq S$ is clear as $\Phi'(A, x_1, \ldots, x_n)$ defines R. Let $s \in S$. If this particular tuple s is defined with $y = A$ in Φ'' then $s \in R$ is obvious. If s is defined with $y = A^T$ then s^T can be defined with $y = A$ in Φ'' and this yields $s^T \in R$, where the transpose is taken componentwise. Finally, the transposition invariance of R implies $s \in R$. □

We have already seen that in the first-order language of $\mathcal{CD}'$ we have access to the first-order language of digraphs. Let $G = (V, E)$ be an arbitrary fixed digraph with $V = \{v_1, \ldots, v_n\}$. Then the formula

$$\exists x_1 \ldots \exists x_n \forall y \Big(\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \ \wedge \bigvee_{i=1}^{n} y = x_i \ \wedge \bigwedge_{(v_i, v_j) \in E} (x_i, x_j) \in E \ \wedge \bigwedge_{(v_i, v_j) \notin E} (x_i, x_j) \notin E \Big)$$

defines G in the first-order language of digraphs. This leads to the following corollary of Theorem 1.5.

Corollary 1.8. *In $\mathcal{D}'$, all elements are first-order definable.*

Corollary 1.9 (=Theorem 1.1)**.** *For all $G \in \mathcal{D}$, the set $\{G, G^T\}$ is first-order definable in $(\mathcal{D}, \leq)$.*

Proof. By Corollary 1.8, we have a formula $\phi_G(x, y)$ for which $\phi_G(x, A)$ defines G. We can now conclude with the argument seen at (1.4). □

The previous two statements will only earn the "title" corollary truly, if we prove Theorem 1.5 without using them, which will be one way to approach the proof of Theorem 1.5.

In the second-order language of digraphs—which has turned out to be available in the first-order language of $\mathcal{CD}'$—the formula

$$\exists H \subseteq G(\exists v, w \in G(v \in H \wedge w \notin H) \ \wedge \ \forall x, y \in G(x \rightarrow y \Rightarrow (x, y \in H \ \vee \ x, y \notin H)))$$

defines the set of not weakly connected digraphs. This means that the set of weakly connected digraphs is first-order definable in $\mathcal{D}$, by Corollary 1.7. That fact seems quite nontrivial to prove without Theorem 1.5. This definability is surprising as the set of weakly connected digraphs is not definable in the first-order language of digraphs (by a standard model-theoretic argument).

1.2.3 The Proof of the Section's Main Theorem (Theorem 1.5)

In this subsection, we prove the "if" direction of Theorem 1.5. Throughout the proof, the elementwise definability of some particular digraphs is being used. To make it precise, see the following lemma. Note that some of the lemmas digraphs are not introduced yet.

Lemma 1.10. *The following digraphs (of at most 9 elements) are first-order definable in $\mathcal{D}'$: I_2, L_1, E_2, A, A^T, and the digraphs under (1.28), (1.30), and (1.36).*

Proof. We offer two separate approaches. The first is based on the observation that we only need to consider some (finite) levels at the "bottom" of the poset $\mathcal{D}$. This means it is only a matter of time for someone to create this proof. The detailed proof would be technical and it would bring nothing new to the table, so we skip it. If contemplating having a go at it, some computer calculation could be evoked to help.

The second approach is using the statement of Corollary 1.8. Our lemma at hand is just a special case of that. We have to be very cautious with this though. The statement of Corollary 1.8 was a consequence of the very theorem whose proof we are dealing with at the moment (Thm. 1.5). Hence, the statement of Corollary 1.8 must be achieved alternatively in this approach. What gives us this as a viable option is Theorem 1.1 being proven in [8] from the ground, using nothing. Thus we can use this and conclude with showing that the statement of Corollary 1.8 is a consequence of Theorem 1.1. This is not trivial, but manageable nonetheless. The reader finds this argument at point of the Thesis where it fits better: see the second proof of Theorem 1.49 located on page 29. □

Some basic definitions and concepts follow.

Definition 1.11. Let E_n $(n = 1, 2, \dots)$ denote the "empty" digraph with n vertices and F_n $(n = 1, 2, \dots)$ denote the "full" digraph with n vertices:

$$V(E_n) = \{v_1, v_2, \dots, v_n\}, \quad E(E_n) = \emptyset,$$

$$V(F_n) = \{v_1, v_2, \dots, v_n\}, \quad E(F_n) = V(F_n)^2.$$

The following definition, numbered 1.12, shows up in each of the papers [7–9] introducing the families of digraphs I_n, O_n, and L_n. Unfortunately these occurrences do not agree on where they run the ns of the families from. These alterations were for technical reasons at the time but, clearly, now we have to come up with one universal version for the whole thesis. We have decided to use the variant of the paper [7]. Consequently, regarding this particular issue, modifications are required in the present section, but the following section (*Substructure*) remains unaffected.

Definition 1.12. Let I_n for $n \in \{1, 2, \dots\}$, O_n for $n \in \{3, 4 \dots\}$, and L_n for $n \in \{1, 2, \dots\}$ be the following (fig. 1.1.) digraphs:

$$V(I_n) = V(O_n) = V(L_n) = \{v_1, v_2, \dots, v_n\},$$

$$E(I_n) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)\},$$
$$E(O_n) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\},$$
$$E(L_n) = \{(v_1, v_1), (v_2, v_2), \dots, (v_n, v_n)\}.$$

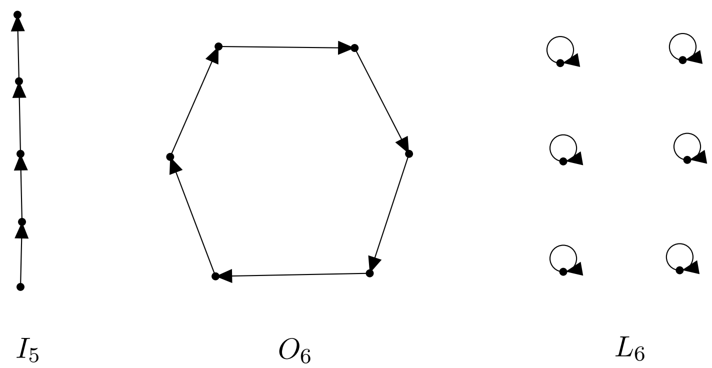


Figure 1.1: I_5, O_6, I_6

Definition 1.13. For $G \in \mathcal{D}$, let $L(G)$ denote the digraph that we get from G by adding all loops possible. For $\mathcal{G} \subseteq \mathcal{D}$, let us define $\mathcal{L}(\mathcal{G}) = \{L(G) : G \in \mathcal{G}\}$.

We remark that this previous definition was a little different in [8]. We then assumed that G has no loops which we do not do here.

Definition 1.14. For $G \in \mathcal{D}$, let $M(G)$ the digraph that we get from G be by leaving all the loops out. For $\mathcal{G} \subseteq \mathcal{D}$, let us define $\mathcal{M}(\mathcal{G}) := \{M(G) : G \in \mathcal{G}\}$.

Here, by "definability", we will always mean first-order definability in $\mathcal{D}'$.

Lemma 1.15. *The sets* $\mathcal{E} := \{E_n : n \in \mathbb{N}\}$, $\mathcal{L} := \{L_n : n \in \mathbb{N}\}$ *and the relation* $\{(L_n, E_n) : n \in \mathbb{N}\}$ *are definable.*

Proof. $\mathcal{E}$ is the set of $X \in \mathcal{D}$ for which $I_2 \not\leq X$ and $L_1 \not\leq X$. $\mathcal{L}$ is the set of those digraphs $X \in \mathcal{D}$ for which there exists $E_i \in \mathcal{E}$ such that X is maximal with the properties $E_i \leq X$, $E_{i+1} \not\leq X$ and $I_2 \not\leq X$. (E_{i+1} is easily defined using E_i as it is the only cover of E_i in the set $\mathcal{E}$.)
The relation consists of those pairs $(X, Y) \in \mathcal{D}^2$ for which $X \in \mathcal{L}$, and Y is maximal element of $\mathcal{E}$ that is embeddable into X. □

The relations

$$\{(G, E_n) : E_n \leq G,\ E_{n+1} \not\leq G\}, \quad \text{and} \tag{1.5}$$

$$\{(G, L_n) : L_n \leq G,\ L_{n+1} \not\leq G\} \tag{1.6}$$

are obviously definable, from which the following relations are definable too:

Definition 1.16.

$$\mathfrak{E} := \{(G, K) : \exists\, E_n \in \mathcal{E}, \text{ for which } (G, E_n), (K, E_n) \in (1.5)\},$$

$$\mathfrak{L} := \{(G, K) : \exists\, L_n \in \mathcal{L}, \text{ for which } (G, L_n), (K, L_n) \in (1.6)\}.$$

Definition 1.17. Let $\mathcal{O}$ denote the set of those digraphs that are disjoint unions of circles (O_n for $n \geq 3$) of not necessarily different sizes.

Lemma 1.18. $\mathcal{O}$ *is definable.*

Proof. Let us define the digraph II of Fig. 1 first. This is definable as this is the only digraph on the fourth level, above I_2, that is a cover of only one. Let $\mathcal{H}$ be the set consisting of those $X \in \mathcal{D}$ for which there exists $E_n \in \mathcal{E}$ such that X is maximal with the properties

$$E_2 \leq X, \quad A, A^T, L_1, II \not\leq X, \text{ and } (X, E_n) \in \mathfrak{E}. \tag{1.7}$$

We state that

$$\mathcal{H} = \mathcal{O} \cup \{G \,\dot\cup\, E_1 : G \in \mathcal{O}\}. \tag{1.8}$$

Let $G \in \mathcal{H}$. It is easy to see that there can be at most 1 weakly connected component of G that has only 1 vertex (and hence is isomorphic to E_1) as the opposite would conflict the maximality of G. The conditions $A \not\leq X$ and $A^T \not\leq X$ mean there is no vertex in G that is either an ending or a starting point of two separate edges, respectively. Therefore every weakly connected component of G is either the digraph II, a circle, or only one element. II is excluded by (1.7). Finally, $\mathcal{O}$ is the set of $X \in \mathcal{D}$ for which $X \in \mathcal{H}$ but there is no such $Y \in \mathcal{H}$ that $Y \prec X$. □

Lemma 1.19. *The following sets and relations are definable:*

$$\mathcal{O}_\cup := \{O_n : n \geq 3\}, \quad \{(O_n, E_n) : n \geq 3\},$$

$$\{F_n : n \in \mathbb{N}\}, \quad \{(F_n, E_n) : n \in \mathbb{N}\}, \tag{1.9}$$

$$\{(G, M(G)) : G \in \mathcal{D}\}, \tag{1.10}$$

$$\mathfrak{M} := \{(X, Y) : \exists Z((X, Z), (Y, Z) \in (1.10))\},$$

$$\{(G, L(G)) : G \in \mathcal{D}\}. \tag{1.11}$$

Proof. $\mathcal{O}_\cup$ is the set of digraphs $X \in \mathcal{D}$ for which $X \in \mathcal{O}$ but there is no $Y \in \mathcal{O}$ such that $Y < X$. The corresponding relation $\{(O_n, E_n) : n \geq 3\}$ is definable with (1.5).
The set under (1.9) consists of those $X \in \mathcal{D}$ for which $X < Y$ implies $(X, Y) \notin \mathfrak{E}$. The corresponding relation is defined as above.
(1.10) is the set of pairs $(X, Y) \in \mathcal{D}^2$ for which Y is maximal with the conditions $Y \leq X$ and $L_1 \not\leq Y$.
$\mathfrak{M}$ is already given by a first-order definition.
(1.11) is the set of pairs $(X, Y) \in \mathcal{D}^2$ for which Y is maximal with the property that $(X, Y) \in \mathfrak{M}$. □

Lemma 1.20. *The following relation is definable:*

$$\mathfrak{E}_+ := \{(E_n, E_m, E_{n+m}) : n, m \in \mathbb{N}\}.$$

Proof. The relation $\mathfrak{E}_+$ consists of the triples $(X, Y, Z) \in \mathcal{D}^3$ that satisfy the following conditions. $X, Y \in \mathcal{E}$, meaning $X = E_i$ and $Y = E_j$ for some $i, j \in \mathbb{N}$. With Lemma 1.19, $M(F_j)$ can be defined (with E_j). Let F_j^* denote the digraph the we get from $M(F_j)$ by adding one loop. This is the only digraph $W \in \mathcal{D}$ for which $M(F_j) \prec W$ and $L_1 \leq W$. Now the digraph $L_i \,\dot\cup\, M(F_j)$ is definable as the digraph $Q \in \mathcal{D}$ which is minimal with the conditions $L_i \leq Q$, $M(F_j) \leq Q$ and $F_j^* \not\leq Q$. Finally, $Z \in \mathcal{E}$ such that $(Z, L_i \,\dot\cup\, M(F_j)) \in \mathfrak{E}$. □

Lemma 1.21. *The following relation is definable:*

$$\{(E_n, E_m) : 3 \leq n + 2 \leq m \leq 2n + 1\}. \tag{1.12}$$

Proof. The relation is the set of those pairs $(X, Y) \in \mathcal{D}^2$ which satisfy the following conditions. For $X \in \mathcal{E}$, meaning $X = E_n$, we can define E_{2n} to be the element from the set $\mathcal{E}$ for which $(E_n, E_n, E_{2n}) \in \mathfrak{E}_+$. E_{2n+1} and E_{n+1} are defined similarly, with

$$(E_{2n}, E_1, E_{2n+1}) \in \mathfrak{E}_+, \text{ and } (E_n, E_1, E_{n+1}) \in \mathfrak{E}_+$$

Finally, $Y \in \mathcal{E}$ and $E_{n+1} < Y \leq E_{2n+1}$. □

Definition 1.22. Let $O_n^* := O_{n+2} \,\dot\cup\, O_{n+3} \,\dot\cup\, \dots \,\dot\cup\, O_{2n+1}$ for $n \in \{1, 2, \dots\}$.

Lemma 1.23. *The relation*

$$\{(O_n^*, E_n) : n \in \mathbb{N}\} \tag{1.13}$$

and the set $\{O_n^* : n \in \mathbb{N}\}$ *are definable.*

Proof. The relation (1.13) can be defined as the set of pairs $(X, Y) \in \mathcal{D}^2$ satisfying the following conditions. $Y \in \mathcal{E}$, meaning $Y = E_n$. X satisfies $X \in \mathcal{O}$ and is minimal with the following property: for all $O_i \in \mathcal{O}_\cup$ for which $(E_n, E_i) \in (1.12)$ holds, $O_i \leq X$.
With the relation (1.13), the set is easily defined the usual way. □

Definition 1.24. Let $\mathcal{O}_n^{\rightarrow}$ denote the set of digraphs X which we get by adding an edge that is not a loop to O_n.

Note that $X \succ O_n$ for all $X \in \mathcal{O}_n^{\rightarrow}$.

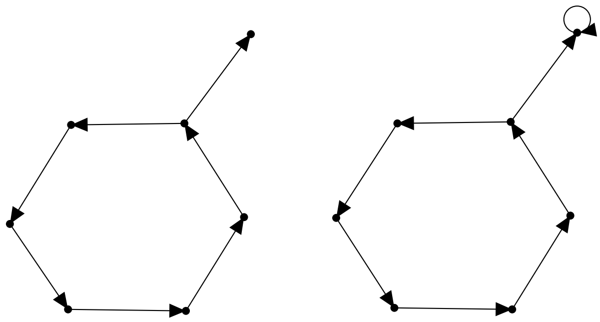
Lemma 1.25. *The following relation is definable:*

$$\{(X, E_n) : \ 2 < n, \ \ X \in \mathcal{O}_n^{\rightarrow}\}.$$

Proof. The relation consists of those pairs $(X, Y) \in \mathcal{D}^2$ that satisfy the following conditions. $Y \in \mathcal{E}$ and $E_2 < Y$, meaning $Y = E_n$, where $2 < n$. $O_n \prec X$, $L_1 \nleq X$, and $(X, O_n) \in \mathfrak{C}$. □

Definition 1.26. Let $♂_n$ (see Fig. 1.2) be the digraph with $n+1$ vertices $V(♂_n) = \{v_1, \dots, v_{n+1}\}$ for which v_1, v_2, …, v_n constitute a circle O_n and the only additional edge in $♂_n$ is (v_n, v_{n+1}). Let $♂_n^L$ be the previous digraph plus one loop:

$$E(♂_n^L) = E(♂_n) \cup \{(v_{n+1}, v_{n+1})\}.$$



$♂_6$ $\qquad$ $♂_6^L$

Figure 1.2: $♂_6$ and $♂_6^L$

Definition 1.27. Let $2 < i, j$ be integers and let us consider the circles O_i, O_j and E_1 with

$$V(O_i) = \{v_1, \dots, v_i\}, \ \ V(O_j) = \{v^1, \dots, v^j\}, \ \ V(E_1) = \{u\}.$$

Let $♂_{i,j}^L$ denote the following digraph:

$$V(♂_{i,j}^L) := V(O_i) \cup V(O_j) \cup V(E_1), \ \ E(♂_{i,j}^L) := E(O_i) \cup E(O_j) \cup \{(v_1, u), (v^1, u), (u, u)\}.$$

Definition 1.28. Let $O_{n,L}$ (see Fig. 1.3) be the following digraph: $V(O_{n,L}) = \{v_1, v_2, \dots, v_n\}$, $E(O_{n,L}) = E(O_n) \cup \{(v_1, v_1)\}$, meaning

$$E(O_{n,L}) = \{(v_1, v_1), (v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}.$$

Lemma 1.29. *Let*

$$O_{n,L}^* := O_{n+1,L} \mathbin{\dot\cup} O_{n+2,L} \mathbin{\dot\cup} \dots \mathbin{\dot\cup} O_{2n,L}.$$

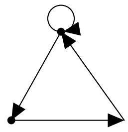


Figure 1.3: $O_{3,L}$

The following sets and relations are definable:

$$\{(♂_n, E_n) : n > 2\}, \quad \{♂_n : n > 2\}, \tag{1.14}$$

$$\{(♂^L_n, E_n) : n > 2\}, \ \{♂^L_n : n > 2\}, \tag{1.15}$$

$$\{(♂^L_{i,j}, E_i, E_j) : 2 < i, j, \ i \neq j\}, \ \{♂^L_{i,j} : 2 < i, j, \ i \neq j\}, \tag{1.16}$$

$$\{O_{n,L} : n > 2\}, \quad \{(O_{n,L}, E_n) : n > 2\}, \tag{1.17}$$

$$\{O^*_{n,L} : n \in \mathbb{N}\}, \quad \{(O^*_{n,L}, E_n) : n \in \mathbb{N}\}. \tag{1.18}$$

Proof. The relation (1.14) consists of those pairs $(X, Y) \in \mathcal{D}^2$ that satisfy the following. $Y \in \mathcal{E}$, meaning $Y = E_n$. There exists $Z \in \mathcal{D}$ for which $O_n \prec Z \prec X$, $(E_{n+1}, X) \in \mathfrak{E}$, and $L_1 \nleq X$. There exists no $Z \in \mathcal{O}^{\rightarrow}_n$ for which $Z \leq X$. Finally, $A^T \leq X$. The corresponding set is easily defined using the relation we just defined.
The set under (1.17) consists of those digraphs $X \in \mathcal{D}$ for which there exists $O_n \in \mathcal{O}_{\cup}$ such that $O_n \prec X$ and $L_1 \leq X$. The corresponding relation is easily defined.
The relation (1.15) consists of those pairs $(X, Y) \in \mathcal{D}^2$ that satisfy the following. $Y \in \mathcal{E}$, meaning $Y = E_n$. With the relation (1.14), $♂_n$ is definable. Now X is determined by the following properties: $♂_n \prec X$, $L_1 \leq X$ and $O_{n,L} \nleq X$. The corresponding set is easily defined using the relation we just defined.
The relation (1.16) consists of those triples $(X, Y, Z) \in \mathcal{D}^3$ that satisfy the following. $Y, Z \in \mathcal{E}$ such that $E_2 < Y, Z$ and $Y \neq Z$, meaning $Y = E_i$, $Z = E_j$ for some $2 < i, j$, $i \neq j$. Now $O_i \dot\cup O_j$ is the digraph $W \in \mathcal{D}$ determined by $W \in \mathcal{O}$, $(W, E_{i+j}) \in \mathfrak{E}$ and $O_i, O_j \leq W$. $O_i \dot\cup O_j \dot\cup E_1$ is the digraph W determined by $O_i \dot\cup O_j \prec W$ and $(O_i \dot\cup O_j, W) \notin \mathfrak{E}$. Finally, X is defined by:

$$\exists W_1, W_2 : O_i \dot\cup O_j \dot\cup E_1 \prec W_1 \prec W_2 \prec X, \ \ (X, O_i \dot\cup O_j \dot\cup E_1) \in \mathfrak{E},$$

$$L_1 \leq X, \ \ O_{i,L} \nleq X, \ \ O_{j,L} \nleq X, \ \ ♂^L_i \leq X, \ \ ♂^L_j \leq X.$$

The corresponding set is easily defined using the relation we just defined.
The relation $\{(O^*_{n,L}, E_n) : n \in \mathbb{N}\}$ consists of those pairs $(X, Y) \in \mathcal{D}^2$ that satisfy the following conditions. $Y \in \mathcal{E}$, meaning $Y = E_n$. For X, the following properties hold:

- $O^*_n \leq X$ and $(X, O^*_n) \in \mathfrak{E}$,
- $O_i \leq O^*_n \Rightarrow (Y \in \mathcal{O}^{\rightarrow}_i \Rightarrow Y \nleq X)$,
- $O_i \leq O^*_n \Rightarrow ♂_i \nleq X$,
- $O_i \leq O^*_n \Rightarrow O_{i,L} \leq X$,

- $L_{n+1} \not\leq X$.

With the relation we just defined the corresponding set is easily defined. $\square$

Definition 1.30. Let us denote the vertices of O_i and O_j with

$$V(O_i) = \{v_1, \dots, v_i\}, \;\; V(O_j) = \{v^1, \dots, v^j\}.$$

Let $O_{i \to j}$ denote the digraph

$$V(O_{i\to j}) = V(O_i) \cup V(O_j), \quad E(O_{i\to j}) = E(O_i) \cup E(O_j) \cup \{(v_1, v^1)\}.$$

Lemma 1.31. *The following relation and set are definable:*

$$\{(O_{i\to j}, E_i, E_j) : i, j > 2\}, \quad \{O_{i\to j} : i, j > 2\} \tag{1.19}$$

Proof. The relation (1.19) consists of those triples $(X, Y, Z) \in \mathcal{D}^3$ for which the following conditions hold. $Y, Z \in \mathcal{E}$ satisfy $E_2 < Y, Z$, meaning $Y = E_i$ and $Z = E_j$, where $i, j > 2$. $(X, E_{i+j}) \in \mathfrak{E}$ and $W \prec X$, where $W \in \mathcal{O}$ is such that $(W, X) \in \mathfrak{E}$ and precisely O_i and O_j are embeddable into W from the set $\mathcal{O}_\cup$ (here $i = j$ is possible). Finally, $♂_i \leq X$. The set is easily defined using the relation. $\square$

The proof of the crucial Lemma 1.36 requires a lot of nontrivial preparation which we begin here.

Definition 1.32. Let $\mathcal{W}(G)$ denote the set of weakly connected components of G.

Now we have to announce a change to the original notations of the paper [9]. Unfortunately, the set of symbols $\{\sqsubseteq, \sqsubset, \sqsupseteq, \sqsupset\}$, which we use here to express substructureness, got used for a small part of the paper for a completely different concept. We replace with $\{\trianglelefteq, \triangleleft, \trianglerighteq, \triangleright\}$, see the following definition.

Definition 1.33. Let

$$G \trianglelefteq G' \Leftrightarrow M(G) \leq M(G'), \;\text{ and }\; G \triangleleft G' \Leftrightarrow M(G) < M(G'),$$

$$G \equiv G' \;\Leftrightarrow\; M(G) = M(G') \; (\Leftrightarrow\; (G, G') \in \mathfrak{M}), \text{ that is } \equiv \;=\; \trianglelefteq \cap \trianglelefteq^{-1},$$

$$\equiv_G^C \;:= \{H \in \mathcal{W}(G) : H \equiv C\}, \text{ and similarly}$$

$$=_G^C \;:= \{H \in \mathcal{W}(G) : H = C\}.$$

Recall that we use the abbreviation *wcc*="weakly connected component" and *wccs* for the plural. $\trianglelefteq$ is obviously a quasiorder and $\equiv_G^C$ is the set of the wccs of G that are equivalent to C with respect to the equivalence $\equiv$.

We say that a wcc W of G is *raised* by the embedding $\varphi : G \to G'$ if for the wcc W' of G' that it embeds into, i. e. $\varphi(W) \subseteq W'$, $W \triangleleft W'$ holds. In this case, we say that W *is raised into* W'. A wcc W of G is either raised or embeds into $\equiv_{G'}^W$ (considered now as a subgraph of G').

Lemma 1.34. *Let G and G' be digraphs having n vertices such that $G \equiv G'$. Let φ be an embedding $G \to G' \,\dot\cup\, O_n^*$. Let us suppose that W and W' are wccs of G and G' respectively, such that W is raised into W'. Then $W' \equiv I_m$ for some m, and consequently $W \equiv I_{m'}$ for some $m' < m$.*

Proof. It suffices to show that $M(W')$ can be embedded into O_n^*, that is what we are going to do. For an arbitrary wcc V of G, it is clear that $\equiv_G^V$ and $\equiv_{G'}^V$ are either bijective under φ (considered as subgraphs of G and G') or a wcc of $\equiv_G^V$ is raised. The fact that W is raised into W' excludes $\equiv_G^{W'}$ and $\equiv_{G'}^{W'}$ being bijective as these two subgraphs are $\equiv$–equivalent, so a bijection would only be possible if only $\equiv_G^{W'}$ was mapped into $\equiv_{G'}^{W'}$. This means that a wcc W_1 of $\equiv_G^{W'}$ is raised into some wcc W_1'. If W_1' is a wcc of O_n^*, then we are done as clearly

$$W \lhd W_1 \lhd W_1'.$$

If this is not the case, then we repeat the same argument to get wccs $W_2 \in \equiv_G^{W_1'}$, and W_2' such that W_2 is raised into W_2'. Again, if W_2' is in O_n^*, then we are done as

$$W \lhd W_1 \lhd W_2 \lhd W_2'.$$

If not, we repeat the argument. Since an infinite chain of wccs with strictly increasing size is impossible, we will get to our claim eventually. □

We are in the middle of the preparation for Lemma 1.36. The following Lemma 1.35 is the key, the most difficult part of the paper. Before the lemma, we give an example to aid the understanding of its statement. We consider the digraphs G and $G' \dot\cup O_n^*$ and we are interested if the assumptions

- $G \leq G' \dot\cup O_n^*$,
- $G \equiv G'$, and
- G and G' have the same number of loops

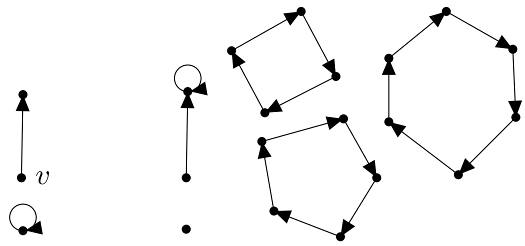
force $G = G'$? The answer is negative and a counterexample is shown in Figure 1.4. To prove Lemma 1.36, we will need to ensure that $G = G'$ with a first-order definition. Observe the following. Let $\overline{G}$ denote the digraph we get from G by adding a loop to the vertex labeled with v. Now it is impossible to add one loop to G' such that we get a $\overline{G'}$ for which $\overline{G} \leq \overline{G'} \dot\cup O_3^*$ holds. We just showed the following property: we can add some loops to G, getting $\overline{G}$, such that it is impossible to add the same number of loops to G', getting $\overline{G'}$, such that $\overline{G} \leq \overline{G'} \dot\cup O_3^*$ holds. If we have $G = G'$ this property does not hold, obviously. Have we found a property that, together with the three above, ensures $G = G'$? The following lemma answers this question affirmatively.

Lemma 1.35. *Let G, G' be digraphs with n vertices and with the same number of loops. Let us suppose $G \equiv G'$ and $G \leq G' \dot\cup O_n^*$. Then $G \neq G'$ holds if and only if we can add some loops to G so that we get the digraph $\overline{G}$ such that it is impossible to add the same number of loops to G', getting the digraph $\overline{G'}$, such that $\overline{G} \leq \overline{G'} \dot\cup O_n^*$. In formulas this is: there exists a digraph $\overline{G}$ for which*

$$G \leq \overline{G}, \ \ G \equiv \overline{G}$$

such that there exists no digraph X for which

$$G' \dot\cup O_n^* \leq X, \ \ X \equiv G' \dot\cup O_n^*, \ \ X \leq L(G) \dot\cup O_n^*, \ \ (\overline{G}, X) \in \mathfrak{L}.$$



G $\qquad$ $G' \,\dot\cup\, O_3^*$

Figure 1.4: A G and a corresponding $G' \,\dot\cup\, O_3^*$ forming a counterexample

Proof. The direction $\Leftarrow$ (or rather its contrapositive) is obvious. Accordingly, let us suppose $G \neq G'$.

Let C denote the largest joint subgraph consisting of whole wccs of both G and G'. Let us introduce the so-called *reduced subgraphs*:

$$G = C \,\dot\cup\, G_R, \text{ and } G' = C \,\dot\cup\, G'_R. \tag{1.20}$$

Observe that the digraphs G_R and G'_R are not empty and $G_R \equiv G'_R$.

Let W denote a $\trianglelefteq$-maximal wcc of G_R. We claim $W \equiv I_k$ for some $k > 1$, and

$$|=_G^{I_k}| - |=_{G'}^{I_k}| = |\equiv_{G_R}^{I_k}|, \tag{1.21}$$

or equivalently, all wccs of $\equiv_{G_R}^{I_k}$ are loop-free. Let φ be an embedding $G \to G' \,\dot\cup\, O_n^*$. Observe that φ raises a wcc isomorphic to W as G' has less wccs isomorphic to W by the definitions of the reduced subgraphs. Hence, by Lemma 1.34, we have $W \equiv I_k$ for some $k \geq 1$. This is less then what we claimed, the exclusion of the case $k = 1$ remains to be seen yet. First, we prove (1.21) for $k \geq 1$, then by using that, we prove $k \neq 1$. It is easy to see from the definitions that (1.21) is equivalent to the fact that all wccs of $\equiv_{G_R}^{I_k}$ are loop-free. Let us suppose, for contradiction, that a wcc V of $\equiv_{G_R}^{I_k}$ has a loop in it. Observe that the loops of G and G' are bijective under φ. Moreover, from the maximality of W, it is easy to see that for a wcc $U \rhd I_k$ of G, the loops of $\equiv_G^U$ are bijective with the loops of $\equiv_{G'}^U$ under φ. Consequently, none of the wccs of $=_G^V$ is raised as, by our previous argument, there is no component to be raised into. Hence $|=_G^V| \leq |=_{G'}^V|$, which clearly contradicts the fact that V is an element of $\equiv_{G_R}^{I_k}$. We have proven (1.21), only the exclusion of $k = 1$ remains from our claim above. Let us suppose $k = 1$ for contradiction. An arbitrary wcc K of G is either $K \equiv I_1$ or $K \rhd I_1$. In the latter case, as we have seen above, the loops of $\equiv_G^K$ are bijective with $\equiv_{G'}^K$. If $K \equiv I_1$, then from (1.21) the nonempty set $\equiv_{G_R}^{I_1}$ is loop-free. Consequently, $\equiv_{G'_R}^{I_1}$, that has the same number of elements, consists of L_1s. This means G has more loops then G' does, a contradiction. We have entirely proven our claim.

Observe that, from our claim above, the nonempty set $\equiv_{G'_R}^{I_k}$ contains no loop-free elements. Take $W' \in \equiv_{G'_R}^{I_k}$. We make the digraph $\overline{W}$ from I_k by adding 1 loop so that $\overline{W} \neq W'$. This is possible because either W' has loops on all of its vertices, then (using $k > 1$) adding the loop arbitrarily suffices; or there is a vertex that has no loop on it, then adding the loop to this vertex in I_k does.

Now we create the digraph $\overline{G}$ of the theorem by adding 1 loop to each loop-free wcc of G. To the wccs of $=^{I_k}_{G}$ we add 1 loop each such that they all become $\overline{W}$. To all other loop-free wccs of G, we add 1 loop each arbitrarily.

To prove that $\overline{G}$ is sufficient, we suppose, for contradiction, that, by adding the same number of loops to G', we can get some $\overline{G'}$ for which $\overline{G} \leq \overline{G'} \,\dot\cup\, O^*_n$. Let ϕ be an embedding $\overline{G} \to \overline{G'} \,\dot\cup\, O^*_n$. For each wcc has a loop in $\overline{G}$, ϕ is technically an isomorphism $\phi : \overline{G} \to \overline{G'}$. Our final claim is,

$$| =^{\overline{W}}_{\overline{G}} | > | =^{\overline{W}}_{\overline{G'}} |, \tag{1.22}$$

which contradicts the existence of the isomorphism $\phi : \overline{G} \to \overline{G'}$. If (1.22) gets proven, we are done. Using the decomposition (1.20) and the knowledge on how $\overline{G}$ was created, the left side of (1.22) is

$$| =^{\overline{W}}_{\overline{G}} | \;\; = \;\; | =^{I_k}_{G} | + | =^{\overline{W}}_{G_R} | + | =^{\overline{W}}_{C} | \;\; = \;\; | =^{I_k}_{G} | + | =^{\overline{W}}_{C} |, \tag{1.23}$$

since $\equiv^{\overline{W}}_{G_R} \;=\; \equiv^{I_k}_{G_R}$ was shown to be loop-free above. Observe that even though we do not know exactly how $\overline{G'}$ was created, a component isomorphic to $\overline{W}$ can only appear in it if either it was already in G' and no loop was added to that specific component, or the component was isomorphic to I_k in G', but a loop was added to the right place. This implies

$$| =^{\overline{W}}_{\overline{G'}} | \;\; \leq \;\; | =^{I_k}_{G'} | + | =^{\overline{W}}_{G'_R} | + | =^{\overline{W}}_{C} |. \tag{1.24}$$

Using (1.23) and (1.24), it is enough to show that

$$| =^{I_k}_{G} | + | =^{\overline{W}}_{C} | \;\; > \;\; | =^{I_k}_{G'} | + | =^{\overline{W}}_{G'_R} | + | =^{\overline{W}}_{C} |,$$

or equivalently,

$$| =^{I_k}_{G} | - | =^{I_k}_{G'} | \;\; > \;\; | =^{\overline{W}}_{G'_R} |.$$

Using (1.21), this turns into $| \equiv^{I_k}_{G_R} | > | =^{\overline{W}}_{G'_R} |$, which is obvious considering how $\overline{W}$ was created. We have proven (1.22), we are done. □

Lemma 1.36. *The following relation is definable:*

$$\{(G, G \,\dot\cup\, O^*_n) : G \in \mathcal{D},\ |V(G)| = n\}. \tag{1.25}$$

Proof. The relation in question is the set of pairs $(X, Y) \in \mathcal{D}^2$ that satisfy the following conditions. Let $(X, E_n) \in \mathfrak{E}$. Now $L(X) \,\dot\cup\, O^*_n$ is the minimal digraph $W \in \mathcal{D}$ with the following conditions: $L(X) \leq W$, $O^*_n \leq W$, there is no $O^*_n \prec Z$ for which $L_1 \leq Z$ and $Z \leq W$. (Here we used the fact that O^*_n has so big circles that cannot fit into X.) Now Lemma 1.35 tells us that the set of the following first-order conditions suffice:

- $Y \equiv L(X) \,\dot\cup\, O^*_n$,
- $X \leq Y$,
- $(X, Y) \in \mathfrak{L}$, and
- (taken from the end of the statement of Lemma 1.35:) there exists NO digraph $\overline{X}$ for which:

– $X \leq \overline{X}$, $X \equiv \overline{X}$, and

– there exists no digraph Z for which $Y \leq Z$, $Z \equiv Y$, $Z \leq L(X) \dot{\cup} O_n^*$, and $(\overline{X}, Z) \in \mathfrak{L}$.

□

Definition 1.37. Let $G \in \mathcal{D}$ be a digraph having n vertices. Let us denote the vertices of O_n^* with

$$V(O_n^*) := \{v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq n+1+i\}$$

such that $V(O_{n+i}) = \{v_{i,j} : 1 \leq j \leq n+1+i\}$. Let $\underline{v} := (v^1, \ldots, v^n)$ be a tuple of the vertices of G. Let us define the digraph $G \overset{\underline{v}}{\leftarrow} O_n^*$ the following way:

$$V(G \overset{\underline{v}}{\leftarrow} O_n^*) := V(G \dot{\cup} O_n^*), \quad E(G \overset{\underline{v}}{\leftarrow} O_n^*) := E(G \dot{\cup} O_n^*) \cup \{(v_{i,1}, v^i) : 1 \leq i \leq n\}.$$

Lemma 1.38. *The following relation is definable:*

$$\{(G, G \overset{\underline{v}}{\leftarrow} O_n^*) : G \in \mathcal{D},\ |V(G)| = n \text{ and } \underline{v} \text{ is a tuple of the vertices of } G\}. \tag{1.26}$$

Proof. First, we define the relation

$$\{(G, L(G) \overset{\underline{v}}{\leftarrow} O_n^*) : G \in \mathcal{D},\ |V(G)| = n \text{ and } \underline{v} \text{ is a tuple of the vertices of } L(G)\}. \tag{1.27}$$

This relation consists of those pairs $(X, Y) \in \mathcal{D}^2$ for which the following holds. Let $(X, E_n) \in \mathfrak{E}$. From X, $L(X)$ is definable. Hence, with the relation (1.25), $L(X) \dot{\cup} O_n^*$ is definable. Now Y is minimal with the following properties:

- $L(X) \dot{\cup} O_n^* \leq Y$ and $(Y, L(X) \dot{\cup} O_n^*) \in \mathfrak{E}$.
- There is no $L(X) \prec Z$ for which $(L(X), Z) \in \mathfrak{E}$ and $Z \leq Y$.
- There is no $O_n^* \prec Z$ for which $(O_n^*, Z) \in \mathfrak{E}$ and $Z \leq Y$.
- For all $O_i \in \mathcal{O}_\cup$, $O_i \leq O_n^*$ implies $♂_i^L \leq Y$.
- There are no $O_i, O_j \in \mathcal{O}_\cup$ for which $O_i \neq O_j$, $O_i, O_j \leq O_n^*$ and $♂_{i,j}^L \leq Y$.

Finally, the relation (1.26) consists of those pairs $(X, Y) \in \mathcal{D}^2$ which satisfy the following conditions. Let $(X, E_n) \in \mathfrak{E}$ again. Then Y satisfies: there exists $L(X) \overset{\underline{v}}{\leftarrow} O_n^*$ for which

$$(L(X) \overset{\underline{v}}{\leftarrow} O_n^*, Y) \in \mathfrak{M}, \quad X \dot{\cup} O_n^* \leq Y \leq L(X) \overset{\underline{v}}{\leftarrow} O_n^*, \quad (X, Y) \in \mathfrak{L}.$$

□

Definition 1.39. Let v_1 and v^1 denote the vertices of $♂_i$ and $♂_j$ with degree 1. Let us define $♂_i \to ♂_j$ the following way:

$$V(♂_i \to ♂_j) := V(♂_i \dot{\cup} ♂_j), \quad E(♂_i \to ♂_j) := E(♂_i \dot{\cup} ♂_j) \cup \{(v_1, v^1)\}.$$

Lemma 1.40. *The following relation is definable:*

$$\{(♂_i \to ♂_j, E_i, E_j) : 2 < i, j,\ \ i \neq j\}.$$

Proof. The relation above consists of those pairs $(X, Y, Z) \in \mathcal{D}^3$ which satisfy the following. $Y, Z \in \mathcal{E}$, $E_2 < Y, Z$ and $Y \neq Z$, meaning $Y = E_i$, $Z = E_j$, where $2 < i, j$ and $i \neq j$. Now $O_i \dot\cup O_j \dot\cup E_1$ can be similarly defined as in Lemma 1.29. From this, $O_i \dot\cup O_j \dot\cup E_2$ is the only digraph $W \in \mathcal{D}$ for which $O_i \dot\cup O_j \dot\cup E_1 \prec W$ and $(W, O_i \dot\cup O_j \dot\cup E_1) \notin \mathfrak{E}$. Now $O_i \dot\cup O_j \dot\cup L_2$ is the only digraph $W \in \mathcal{D}$ for which there exists $V \in \mathcal{D}$ such that

$$O_i \dot\cup O_j \dot\cup E_2 \prec V \prec W, \quad L_2 \leq W, \quad O_{i,L} \nleq W \text{ and } O_{j,L} \nleq W.$$

$\male_i^L \dot\cup \male_j^L$ is the only digraph $W \in \mathcal{D}$ for which there exists $V \in \mathcal{D}$ such that

$$O_i \dot\cup O_j \dot\cup L_2 \prec V \prec W, \quad \male_i^L \leq W, \quad \male_j^L \leq W, \text{ but } \male_{i,j}^L \nleq W.$$

Let I denote the digraph

$$V(I) = \{u, v\}, \quad E(I) := \{(u, v), (u, u), (v, v)\}. \tag{1.28}$$

The set

$$\{\male_i^L \to \male_j^L, \male_j^L \to \male_i^L\} \tag{1.29}$$

consists of those $W \in \mathcal{D}$ for which $\male_i^L \dot\cup \male_j^L \prec W$, $I \leq W$. The digraph $\male_i^L \dot\cup E_1$ is defined as usual. From this, the digraph $\male_i^L \dot\cup L_1$ is definable as the only $W \in \mathcal{D}$ for which $\male_i^L \dot\cup E_1 \prec W$, $L_2 \leq W$ and there is no $V \in \mathcal{D}$ such that $\male_i^L \prec V$, $L_2 \leq V$ and $V \leq W$. Let v denote the vertex of $\male_i^L$ that has a loop on it and let x be the only vertex of L_1. Let $\male_i^{L\to}$ and I^* be the following digraphs:

$$V(\male_i^{L\to}) := V(\male_i^L \dot\cup L_1), \quad E(\male_i^{L\to}) := E(\male_i^L \dot\cup L_1) \cup \{(v, x)\}$$
$$V(I^*) := \{u, v, w\}, \quad E(I^*) := \{(v, v), (w, w), (u, v), (v, w)\}. \tag{1.30}$$

Now $\male_i^{L\to}$ is the only digraph $W \in \mathcal{D}$ for which $\male_i^L \dot\cup L_1 \prec W$ and $I^* \leq W$. From the set (1.29) we can choose $\male_i^L \to \male_j^L$ with the fact

$$\male_i^{L\to} \leq \male_i^L \to \male_j^L, \quad \male_i^{L\to} \nleq \male_j^L \to \male_i^L.$$

Finally, $X = M(\male_i^L \to \male_j^L)$. □

Lemma 1.41. *The following relation and set are definable:*

$$\{(O_{i,i}, E_i) : 2 < i\}, \quad \{O_{i,i} : 2 < i\}, \tag{1.31}$$

where $O_{i,i} := O_i \dot\cup O_i$.

Proof. The relation (1.31) consists of those pairs $(X, Y) \in \mathcal{D}^2$ for which the following holds. $Y \in \mathcal{E}$, meaning $Y = E_i$. $X \in \mathcal{O}$, $(X, E_{2i}) \in \mathfrak{E}$, and from the set $\mathcal{O}_\cup$, O_i is the only element that is embeddable into X. The corresponding set can now be easily defined. □

Lemma 1.42. *The following relation is definable:*

$$\{(O_i^* \dot\cup O_{j,L}^*, E_i, E_j) : 1 \leq i, j\}.$$

Proof. The relation above is the set of triples $(X, Y, Z) \in \mathcal{D}^3$ which satisfy the following. $Y, Z \in \mathcal{E}$, $E_1 \leq Y, Z$, meaning $Y = E_i$, $Z = E_j$, where $1 \leq i, j$. Now $O_i^* \dot\cup O_j^*$ is the digraph W satisfying the following:

- $W \in \mathcal{O}$
- If $E_x, E_y \in \mathcal{E}$ satisfy $(O_i^*, E_x) \in \mathfrak{E}$ and $(O_j^*, E_y) \in \mathfrak{E}$, then $(W, E_{x+y}) \in \mathfrak{E}$.
- For all $O_n \in \mathcal{O}_\cup$ that satisfy $O_n \leq O_i^*$ or $O_n \leq O_j^*$, $O_n \leq W$ holds.
- For all $O_n \in \mathcal{O}_\cup$ which satisfy $O_n \leq O_i^*$ and $O_n \leq O_j^*$, $O_{n,n} \leq W$ holds.

Finally, X is the minimal digraph with $O_i^* \dot\cup O_j^* \leq X \leq L(O_i^* \dot\cup O_j^*)$ and $O_{j,L}^* \leq X$. □

Definition 1.43. Let us denote the vertices of O_n^* by

$$V(O_n^*) := \{v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq n+1+i\}$$

such that the circle O_{n+1+i} consists of $\{v_{i,j} : 1 \leq j \leq n+1+i\}$. Similarly, let us denote the vertices of $O_{m,L}^*$ by

$$V(O_{m,L}^*) := \{v^{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq m+1+i\}$$

such that the circle $O_{m+1+i,L}$ consists of $\{v^{i,j} : 1 \leq j \leq m+1+i\}$ and the loops are on the vertices $\{v^{i,1} : 1 \leq i \leq m\}$. For a map $\alpha : [n] \to [m]$, we define the digraph $F_\alpha(n,m)$ as

$$V(F_\alpha(n,m)) := V(O_n^* \dot\cup O_{m,L}^*),$$

$$E(F_\alpha(n,m)) := E(O_n^* \dot\cup O_{m,L}^*) \cup \{(v_{i,1}, v^{\alpha(i),1}) : 1 \leq i \leq n\}.$$

Let

$$\mathcal{F}(n,m) := \{F_\alpha(n,m) :\ \alpha : [n] \to [m]\}.$$

Lemma 1.44. *The following relation is definable:*

$$\{(F_\alpha(n,m), E_n, E_m) : 1 \leq n, m,\ \ \alpha : [n] \to [m]\}. \tag{1.32}$$

Proof. The relation above consists of those triples $(X, Y, Z) \in \mathcal{D}^3$ that satisfy the following. $Y, Z \in \mathcal{E}$, meaning $Y = E_n$, $Z = E_m$, where $1 \leq n, m$. Now X is a minimal digraph with the following conditions:

- $O_n^* \dot\cup O_{m,L}^* \leq X$ and $(O_n^* \dot\cup O_{m,L}^*, X) \in \mathfrak{E} \cap \mathfrak{L}$.
- $O_i \leq O_n^*$ implies $♂_i^L \leq X$.
- $O_i \leq O_n^* \dot\cup O_{m,L}^*$ implies there is no $W \in \mathcal{O}_i^{\to}$, for which $W \leq X$.
- There is no V for which $V \leq X$ and $♂_i \prec V$, such that $♂_i \leq X$ and $♂_i^L \not\leq V$.
- There is no $♂_i^L \prec V$ for which $V \leq X$ and $L_2 \leq V$.

□

Lemma 1.45. *The following relation is definable:*

$$\{(F_{\mathrm{id}_{[n]}}(n,n), E_n, E_n) : 1 \leq n\}. \tag{1.33}$$

Proof. The relation in question consists of those triples $(X, Y, Z) \in (1.32)$ for which $Y = Z \in \mathcal{E}$ and for $i, j \geq 2$ we have

$$O_{i\to j} \leq X \Rightarrow E_i = E_j.$$

□

Lemma 1.46. *The following relation is definable:*

$$\{(F_\alpha(n,m), F_\beta(m,l), F_{\beta\circ\alpha}(n,l), E_n, E_m, E_l) : 1 \leq n, m, l, \quad \alpha : [n] \to [m], \quad \beta : [m] \to [l]\}. \tag{1.34}$$

Proof. The relation in question is the set of those 6-tuples $(X_1, \ldots, X_6) \in \mathcal{D}^6$ which satisfy the following. $X_4, X_5, X_6 \in \mathcal{E}$, meaning $X_4 = E_n$, $X_5 = E_m$ and $X_6 = E_l$ where $1 \leq n, m, l$. $X_1 \in \mathcal{F}(n, m)$, $X_2 \in \mathcal{F}(m, l)$ and $X_3 \in \mathcal{F}(n, l)$. Finally:

$$(O_{i\to j} \leq X_1 \text{ and } O_{j\to k} \leq X_2) \;\Rightarrow\; O_{i\to k} \leq X_3.$$

□

Definition 1.47. There is a bijection between the digraphs $G \overset{v}{\leftarrow} O_n^*$ and the elements of $\mathrm{ob}(\mathcal{CD})$. Let us observe that the vertices of G are labeled with the circles

$$O_{(n+1)+1}, O_{(n+1)+2}, \ldots, O_{(n+1)+n}$$

in $G \overset{v}{\leftarrow} O_n^*$. On the other hand, in $\mathrm{ob}(\mathcal{CD})$, they are labeled with $1, \ldots, n$. The element of $\mathrm{ob}(\mathcal{CD})$ that corresponds to $G \overset{v}{\leftarrow} O_n^*$ will be denoted by $(G \overset{v}{\leftarrow} O_n^*)_{\mathcal{CD}}$ from now on.

Lemma 1.48. *The following relation is definable:*

$$\begin{gathered}\{(X, F_\alpha(n,m), Y) \in \mathcal{D}^3 : X = G \overset{v}{\leftarrow} O_n^*,\ Y = H \overset{w}{\leftarrow} O_m^* \textit{ for some } \underline{v} \textit{ and } \underline{w}, \textit{ and} \\ ((X)_{\mathcal{CD}}, \alpha, (Y)_{\mathcal{CD}}) \in \hom((X)_{\mathcal{CD}}, (Y)_{\mathcal{CD}})\}\end{gathered} \tag{1.35}$$

Proof. The relation in question is the set of those pairs $(X, F, Y) \in \mathcal{D}^3$ which satisfy the following. There exist G and H such that $(G, X) \in (1.26)$ and $(H, Y) \in (1.26)$. Finally, F satisfies

- $(F, E_n, E_m) \in (1.32)$,
- $(\male_i \to \male_j \leq G \overset{v}{\leftarrow} O_n^*(= X),\ \ O_i, O_j \leq O_n^* \text{ and } O_{i\to k}, O_{j\to l} \leq F) \Longrightarrow$ $((O_k \neq O_l \text{ and } \male_k \to \male_l \leq H \overset{w}{\leftarrow} O_m^*) \ \vee\ (O_k = O_l \text{ and } \male_k^L \leq H \overset{w}{\leftarrow} O_m^*))$,
- $(\male_i^L \leq G \overset{v}{\leftarrow} O_n^*,\ \ O_i \leq O_n^* \text{ and } O_{i\to k} \leq F) \Rightarrow \male_k^L \leq H \overset{w}{\leftarrow} O_m^*$.

□

The proof of Theorem 1.5 is now properly prepared for, we only need to put the pieces together.

Proof of the main theorem: Theorem 1.5. We have already seen in Section 1.2.2 that all relations first-order definable in $\mathcal{D}'$ are definable in $\mathcal{CD}'$ as well. So we only need to deal with the converse. We wish to build a copy of $\mathcal{CD}'$ inside $\mathcal{D}'$ so that all things we can formulate in the first-order language of $\mathcal{CD}'$ becomes accessible in its model in $\mathcal{D}'$. Let the set of objects be

$$\{G \overset{\underline{v}}{\leftarrow} O_n^* : G \in \mathcal{D},\ |V(G)| = n \text{ and } \underline{v} \text{ is a vector of the vertices of } G\},$$

and the set of morphisms be (1.35). We can define both as Lemma 1.48 shows. Identity morphisms can be defined with Lemma 1.45. For the triples

$$(X_1, Z_1, Y_1), (X_2, Z_2, Y_2), (X_3, Z_3, Y_3) \in \mathcal{D}^3$$

the condition $(X_i, Z_i, Y_i) \in (1.35)$ ensures that there exist α_i such that

$$((X_i)_{\mathcal{CD}}, \alpha_i, (Y_i)_{\mathcal{CD}}) \in \hom((X_i)_{\mathcal{CD}}, (Y_i)_{\mathcal{CD}}).$$

Moreover, if we suppose $Y_1 = X_2$, $X_3 = X_1$, $Y_3 = Y_2$ and that there exists a 6-tuple in (1.34) of the form $(Z_1, Z_2, Z_3, *, *, *)$, we have forced

$$((X_1)_{\mathcal{CD}}, \alpha_1, (Y_1)_{\mathcal{CD}})((X_2)_{\mathcal{CD}}, \alpha_2, (Y_2)_{\mathcal{CD}}) = ((X_3)_{\mathcal{CD}}, \alpha_3, (Y_3)_{\mathcal{CD}}).$$

The four constants in $\mathcal{CD}'$ require 4 digraphs, say,

$$C_1, C_2, C_3, \text{ and } C_4 \tag{1.36}$$

of $\mathcal{D}'$ to be defined such that

$$(C_1)_{\mathcal{CD}} = \mathbf{E}_1,\ (C_2)_{\mathcal{CD}} = \mathbf{I}_2$$

and C_3, and C_4 are the elements of the set $\mathcal{F}(1,2)$. Now we have all the "tools" accessible in $\mathcal{CD}'$. Finally, the relation (1.26) lets us "convert" the elements of $\mathcal{D}'$ and $\mathcal{CD}'$ back and forth. We are done. □

1.2.4 The Automorphism Group

So far we know two automorphisms of $(\mathcal{D}, \leq)$, namely the trivial one and $G \mapsto G^T$. In this subsection we prove that there is no other, meaning that the automorphism group of $(\mathcal{D}, \leq)$ is isomorphic to $\mathbb{Z}_2$.

We offer two approaches. First, we build on the strong Theorem 1.5, and we reach our goal easier, naturally. Second, we just use the much weaker Theorem 1.1, making it trickier to succeed. On the second path, there is an argument quite useful when going for the second approach of the proof of Lemma 1.10.

Determining the automorphism group using Theorem 1.5

Theorem 1.49. *The poset $(\mathcal{D}, \leq)$ has exactly two automorphisms, namely the trivial and the one that maps every digraph to its transpose. Consequently, the automorphism group of $(\mathcal{D}, \leq)$ is isomorphic to $\mathbb{Z}_2$.*

Proof. Here, we are allowed to take adventage of Corollary 1.8. Regarding the automorphisms, it says that the only automorphism fixing the digraph A is the identity. To prove this, let φ be an automorphism fixing A, and let G be an arbitrary digraph. What we need to show is $\varphi(G) = G$. Let $\phi(x)$ be the formula defining G using the constant A. We apply the automorphism φ to the formula ϕ, that is we change all appearances of A to $\varphi(A)$. This way, we get a formula ϕ', defining $\varphi(G)$, naturally. To conclude, $\varphi(A) = A$ gives us $\phi = \phi'$, and, as a consequence, $\varphi(G) = G$.

Finally we use Corollary 1.9. As an automorphism cannot take an element outside from a definable set, for an arbitrary automorphism α, $\alpha(A) \in \{A, A^T\}$ holds. If $\alpha(A) = A$, then α is the identity, as proven above. If the other possibility is realized, then with α_T denoting the transposition automorphism, we have $\alpha_T^{-1} \circ \alpha(A) = A$, resulting in $\alpha_T^{-1} \circ \alpha(A)$ having to be the identity. Hence $\alpha = \alpha_T$, indeed. □

Determining the automorphism group using only Theorem 1.1

Lemma 1.50. *$G^T \leq G \,\dot\cup\, O_n$ implies $G = G^T$ for every finite digraph G and integer $2 < n$.*

Proof. Our first easy observation is that $X \leq O_n$ implies $X = X^T$. Let us denote the weakly connected components of G by $\{G_a\}_{a \in A}$. Let $A = B \,\dot\cup\, C$ such that $b \in B$ if and only if G_b is embeddable into O_n. Now let us suppose that $G^T \leq G \,\dot\cup\, O_n$. With the notation just introduced

$$\left(\dot{\bigcup_{a\in A}} G_a\right)^T \leq \left(\dot{\bigcup_{a\in A}} G_a\right) \dot\cup\, O_n$$

$$\dot{\bigcup_{a\in A}} G_a^T \leq \left(\dot{\bigcup_{a\in A}} G_a\right) \dot\cup\, O_n$$

$$\left(\dot{\bigcup_{b\in B}} G_b^T\right) \dot\cup \left(\dot{\bigcup_{c\in C}} G_c^T\right) \leq \left(\dot{\bigcup_{b\in B}} G_b\right) \dot\cup \left(\dot{\bigcup_{c\in C}} G_c\right) \dot\cup\, O_n$$

which obviously implies

$$\dot{\bigcup_{c\in C}} G_c^T \leq \left(\dot{\bigcup_{c\in C}} G_c\right) \dot\cup \underbrace{\left(\dot{\bigcup_{b\in B}} G_b\right) \dot\cup\, O_n}_{=:X}. \tag{1.37}$$

If there exists a $c \in C$ for which $G_c^T \leq X$, then $G_c^T \leq O_n$, for X consists of weakly connected components embeddable into O_n. Then, according to our first observation, $G_c^T = G_c$ which means $G_c \leq O_n$, a contradiction. This means there is no $c \in C$ for which $G_c^T \leq X$ so from (1.37) we deduce

$$\dot{\bigcup_{c\in C}} G_c^T \leq \dot{\bigcup_{c\in C}} G_c, \text{ and}$$

$$\left(\dot{\bigcup_{c\in C}} G_c\right)^T \leq \dot{\bigcup_{c\in C}} G_c.$$

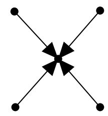


Figure 1.5: The digraph X_4.

By transposing both sides the direction of the embeddability stays the same obviously, but we get the converse, implying

$$\left(\dot{\bigcup_{c\in C}} G_c\right)^T = \dot{\bigcup_{c\in C}} G_c.$$

Using our first observation once more, we obtain

$$\left(\dot{\bigcup_{b\in B}} G_b\right)^T = \dot{\bigcup_{b\in B}} G_b^T = \dot{\bigcup_{b\in B}} G_b.$$

Finally, putting together what we have we get

$$G = \dot{\bigcup_{a\in A}} G_a = \left(\dot{\bigcup_{b\in B}} G_b\right) \dot\cup \left(\dot{\bigcup_{c\in C}} G_c\right) = \left(\dot{\bigcup_{b\in B}} G_b\right)^T \dot\cup \left(\dot{\bigcup_{c\in C}} G_c\right)^T = G^T.$$

□

Definition 1.51. Let us call X_n (see Fig. 1.5) the digraph with

$$V(X_n) = \{v, v_1, v_2, \ldots, v_n\}, \quad E(X_n) = \{(v_1, v), (v_2, v), \ldots, (v_n, v)\}.$$

Proof of Theorem 1.49 using just Theorem 1.1. It is easily seen that an automorphism can only move the elements of $\mathcal{D}$ inside definable sets, therefore from Theorem 1.1 it follows that it either does not move an element or maps it to its transpose. Let us consider an automorphism $\varphi : \mathcal{D} \mapsto \mathcal{D}$ for which there exists $G \in \mathcal{D}$ such that $G \neq G^T$ and $\varphi(G) = G$. We must show that φ is the identity function. This can be done by showing that adding G to the language of partially ordered sets as a constant results in every element of $\mathcal{D}$ becoming definable. So let us add G to the language of partially ordered sets as a constant and pick an arbitrary $F \in \mathcal{D}$ that is not isomorphic to its transpose (those digraphs that are isomorphic to their transposes are definable by Theorem 1.1). Our goal will be to show that F is definable.
Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Lemma 1.50 lets us define $G \dot\cup O_{n+1}$ as the unique element from the definable set

$$\{G \dot\cup O_{n+1}, (G \dot\cup O_{n+1})^T = G^T \dot\cup O_{n+1}\}$$

that G is embeddable into. Let us use the notation

$$V(G \dot\cup O_{n+1}) = V(G) \cup \{v'_1, v'_2, \ldots, v'_{n+1}\}.$$

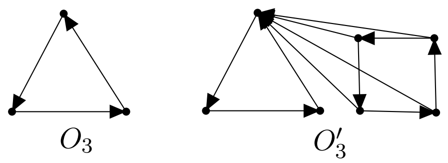


Figure 1.6: The digraph O_3 and a corresponding O'_3.

We create a digraph G' (see Fig. 1.6) by adding edges to $G \dot\cup O_{n+1}$ as follows:

$$E(G') = E(G \dot\cup O_{n+1}) \cup \{(v'_1, v_1), (v'_2, v_1), (v'_3, v_1), \ldots, (v'_{n+1}, v_1)\}. \tag{1.38}$$

Now G' is definable as the unique element of the set $\{G', (G')^T\}$ into which $G \dot\cup O_{n+1}$ is embeddable. X_{n+1} is the unique element from the set $\{X_{n+1}, (X_{n+1})^T\}$ that is embeddable into G' so it is definable too. A is the unique element from the set $\{A, A^T\}$ that is embeddable into X_{n+1}. So far we have proven that A is definable.
Now we do the same as above, but backwards. Let m be the number of vertices of F. X_{m+1} is the unique element in the set $\{X_{m+1}, (X_{m+1})^T\}$ that A is embeddable into. Let F' be created from F analogously to how G' was created from G in (1.38) (see Fig. 1.6). Now F' is the only element from the set $\{F', (F')^T\}$ that X_{m+1} is embeddable into. Next, $F \dot\cup O_{m+1}$ is the only element from the definable set

$$\{F \dot\cup O_{m+1}, (F \dot\cup O_{m+1})^T = F^T \dot\cup O_{m+1}\}$$

that is embeddable into F'. Finally, by Lemma 1.50, F is definable as the only element from the set $\{F, F^T\}$ that is embeddable into $F \dot\cup O_{m+1}$. □

1.3 Substructure

1.3.1 Introduction

To recall the definition of substructureness and the most basic concepts about it, see Section 1.1, General Introduction.

The main result of the section is the following.

Theorem 1.52. *There exists a finite set of finite directed graphs $\{C_1, \ldots, C_k\}$ such that the binary embeddability relation,*

$$\{(G, G') : G \leq G'\},$$

is definable in the first-order language of $(\mathcal{D}; \sqsubseteq, C_1, \ldots, C_k)$. Consequently, every relation definable in the first-order language of $(\mathcal{D}; \leq)$ is definable in that of $(\mathcal{D}; \sqsubseteq, C_1, \ldots, C_k)$.

In itself, this theorem is quite weightless, what fills it with content is that we already know that the first-order language of $(\mathcal{D}; \leq)$ is surprisingly strong, recall Subsection 1.2.2. With Theorem 1.52, everything we proved in the previous section for the first-order language of $(\mathcal{D}; \leq)$ transforms automatically into statements for the first-order language of $(\mathcal{D}; \sqsubseteq, C_1, \ldots, C_k)$. For example:

Corollary 1.53. *There exists a finite set of finite directed graphs* $\{C_1, \dots, C_k\}$ *such that in the first-order language of* $(\mathcal{D}; \sqsubseteq, C_1, \dots, C_k)$

- *every single digraph* G *is definable,*
- *the set of weakly connected digraphs is definable, moreover,*
- *the full second-order language of digraphs becomes available.*

We remark that the notations of Theorem 1.52 and Corollary 1.53 may suggest that the set $\{C_1, \dots, C_k\}$ in the two statements can be the same. This is not necessarily true, even though there is a strong connection between the two sets. Depending on the set of Theorem 1.52, an additional digraph might have to be added the get the corresponding set of Corollary 1.53. This is due to the fact that the first-order language of $(\mathcal{D}; \leq)$ does not yield the listed statements of Corollary 1.53 in itself. As we saw in the previous section, a constant (a particular digraph), e.g. the digraph A, has to be added to the first-order language of $(\mathcal{D}; \leq)$ to make these true. If this constant is not already there in the set of $\{C_1, \dots, C_k\}$ of Theorem 1.52 then its addition might be required to get that of Corollary 1.53. As the equality of the sets is not stated anywhere, this technically is not a problem.

We wish to make another remark on the lists $\{C_1, \dots, C_k\}$ to avoid false expectations. Naturally, as we proceed with our proof the lists $\{C_1, \dots, C_k\}$ will be continuously growing. The final list is revealed gradually throughout the section, and that is why we now outline it in advance. To do so, we describe a family of our arguments used in the last, technical section of the paper. Some properties of digraphs can be told by saying something about the list of their, say, at most 4-element substructures (without multiplicity, naturally). For example one can tell if a digraph has loops based on the list of its 1-element substructures. Similarly, one can judge if it has a non-loop edge by the list of its (at most) 2-element substructures. Far more complicated properties can be told in this way, say, *locally.* We adopt this thinking in the last section of the paper. This will force our lists $\{C_1, \dots, C_k\}$ to be {at most 12-element digraphs}. This list is long but finite nevertheless.

The main reason for a minimal list $\{C_1, \dots, C_k\}$ being out of our grasp lies in the complexity of the automorphism group of $(\mathcal{D}; \sqsubseteq)$. Unfortunately, we are not able to determine it, we can only prove it is finite.

Theorem 1.54. *The automorphism group of* $(\mathcal{D}, \sqsubseteq)$ *is finite.*

The proof is in the next subsection.

Though unable to prove it, we will formulate a conjecture for the automorphism group in the next subsection.

In the next subsection, numbered 1.3.2, we prove what we know about the automorphisms group, namely Theorem 1.54, and tell our conjecture in detail. The subsection after that, Subsection 1.3.3, contains the proof of the main theorem, Theorem 1.52, without some technicalities. In the following one, Subsection 1.3.4, the reader finds the technicalities skipped before.

1.3.2 On the Automorphism Group

First, we prove Theorem 1.54 using Theorem 1.52.

Proof of Theorem 1.54. It is clear that the orbits of the automorphism group are finite as an automorphism can only move a digraph inside its level in $(\mathcal{D}, \sqsubseteq)$. Let $o(G)$ denote the size of the orbit of the digraph G (which is therefore a positive integer).

We state that it suffices to present a finite set of digraphs such that the only automorphism fixing them all is the identity. To prove that, let $\{C_1, \ldots, C_k\}$ be such a set and φ be an arbitrary automorphism. Observe that the images of C_i under φ determine φ completely, or in other words, the only automorphism agreeing with φ on $\{C_1, \ldots, C_k\}$ is φ. Indeed, with the notations

$$S = \{\alpha \in \mathrm{Aut}(\mathcal{D}; \sqsubseteq) : \alpha(C_i) = \varphi(C_i), \ \ i = 1, \ldots, k\},$$

and $S' = \{\alpha\varphi^{-1} : \alpha \in S\}$, $|S| = |S'|$ holds, and $|S'| = 1$ for all elements of S' fix all of $\{C_1, \ldots, C_k\}$. The fact that an automorphism is completely determined by its action on $\{C_1, \ldots, C_k\}$ means that the automorphism group has at most $o(C_1) \cdot \ldots \cdot o(C_k)$ elements. That proves our statement.

Finally, we claim that $\{C_1, \ldots, C_k\}$ of Theorem 1.52 suffices for the purpose above, namely the only automorphism fixing them all is the identity. Let φ be an automorphism that fixes all C_i. Let $G \in \mathcal{D}$ be arbitrary. We need to show that $\varphi(G) = G$. We know from Corollary 1.53 that there exists a formula $\phi_G(x)$ with one free variable, that defines G in the first-order language of $(\mathcal{D}, \sqsubseteq, C_1, \ldots, C_k)$. If we change all occurrences of C_i to $\varphi(C_i)$ in $\phi_G(x)$, then we get a formula $\phi_{\varphi(G)}(x)$ defining $\varphi(G)$. For φ fixes all C_is, $\phi_G(x) = \phi_{\varphi(G)}(x)$, implying $G = \varphi(G)$. □

In the remaining part of the subsection, we present the automorphisms that we know of. Here, no claim is proven rigorously, they are all rather conjectures. Our intention is just to offer some insight on how the author sees the automorphism group at the moment.

Before the (semi-)precise definition of our automorphisms, we feel it is useful to give a nontechnical glimpse at them. Automorphisms map digraphs to digraphs in $\mathcal{D}$. To define an automorphism φ, we need to tell how to get $\varphi(G)$ from G. All the automorphisms, that we know of at the moment, share a particular characteristic. They are all, say, *local* in the following sense. Roughly speaking, to get $\varphi(G)$ from G, one only needs to consider and modify G's at most two element substructures according to some given rule.

To make this clearer, we give an example. Let $\varphi(G)$ be the digraph that we get from G such that we change the direction of the edges on those two element substructures of G that have loops on both vertices. It is easy to see that this defines an automorphism, indeed. Perhaps, one would quickly discover the automorphism that gets $\varphi(G)$ by reversing all edges of G, but this is different. In this example, the modification of G happens only locally, namely on 2-element substructures. All the automorphisms, that we know of, share this property.

Now, we define some of our automorphisms, φ_i, (semi-)precisely. We tell how to get $\varphi_i(G)$ from G. One of the most trivial automorphisms is

- φ_1: where there is a loop, clear it, and vice versa, to the vertices with no loop, insert one.

Observe that this automorphism operates with the 1-element substructures. Now we start to make use of the labels of Fig. 2.

- φ_2: change the substructures (isomorphic to) E to E' and vice versa.
- φ_3: change the substructures (isomorphic to) L to L' and vice versa.
- φ_4: reverse the edges in the substructures (isomorphic to) P.
- φ_5: reverse the edges in the substructures (isomorphic to) Q.

Let S_4 denote the symmetric group over the four-element set $\{A, B, C, D\}$, and $\pi \in S_4$. We define

- φ_π: We change the substructures (isomorphic to) $X \in \{A, B, C, D\}$ to $\pi(X)$ (such that the loops remain in place).

Observe that, with the exception of φ_1, the automorphisms defined above do not touch loops (when getting $\varphi_i(G)$ from G). We conjecture that these automorphisms generate the whole automorphism group.

Finally, we investigate the structure of the group of our conjecture. Let I denote the set of possible indexes of our φs, namely

$$I = \{1, \dots, 5\} \cup \{\pi \in S_4\}.$$

Let $\langle\rangle$ stand for subgroup generation. Let $S = \langle \varphi_i : i \in I \rangle$ denote the group of our conjecture. It seems that S splits into the internal semidirect product

$$S = \langle \varphi_i : i \in I \setminus \{1\} \rangle \rtimes \langle \varphi_1 \rangle.$$

Furthermore, the first factor appears to be a(n internal) direct product

$$\langle \varphi_2 \rangle \times \langle \varphi_3 \rangle \times \langle \varphi_4 \rangle \times \langle \varphi_5 \rangle \times \langle \varphi_\pi : \pi \in S_4 \rangle.$$

Here, at the last factor, the subgroup generation is just a technicality as, clearly, the φ_πs constitute a subgroup themselves. These observations all need a proper checking, but they give rise to the conjecture that S is isomorphic to

$$(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2,$$

where S_4, again, denotes the symmetric group over the set $\{A, B, C, D\}$, and α is the following. Obviously, $\alpha(0) = \mathrm{id} \in \mathrm{Aut}(\mathbb{Z}_2^4 \times S_4)$. To define $\alpha(1)$, let $p, q, r, s \in \{0, 1\}$ and $\pi \in S_4$. Then

$$\alpha(1) : (p, q, r, s, \pi) \mapsto (q, p, s, r, (BC)\pi(BC)), \tag{1.39}$$

where (BC) is just the usual cycle notation of the permutation of S_4 that takes B to C and vice versa. Note that the group of our conjecture has 768 elements. Even though we cannot prove that there are no more automorphisms beyond the ones in S, we conjecture so.

Conjecture 1.55. *The automorphism group of the partial order $(\mathcal{D}; \sqsubseteq)$ is isomorphic to $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$, with the α defined above (around (1.39)).*

1.3.3 The Proof of the Section's Main Theorem (Theorem 1.52) Without Some Technicalities

As long and technical as it may seem, the whole proof of Theorem 1.52 is based on a simple idea, which we outline here. We get substructures of a directed graph by leaving out vertices, while, to get embeddable digraphs, we can leave out vertices and edges both. We want to define the latter, so we should be able to 'simulate' leaving out edges somehow. Our approach is the following. In a digraph G, if there is an edge $(u, v) \in E(G)$, then we add a vertex and two edges to "support" the edge (u, v). Namely, we add w to the set of vertices, and the edges (u, w) and (w, v) to the set of edges. After the addition, we say that the edge (u, v) is "supported". The idea is that the supportedness of an edge can be terminated by leaving out a vertex, in the previous example w, what we can do by taking substructures. Roughly, what we should do is: support all edges, take a substructure, and in one more step, leave only the supported edges in. Of course, there seem to be many problems with this (if told in such a simplified way). Firstly, how can we distinguish between the supporting vertices and the original ones? This appears to be an essential part of the plan. Secondly, the plan ended with "leave only the supported edges in" which just looks running into the original problem again: We cannot leave edges out. Even though the plan seems flawed for these reasons, it is manageable. The whole section is no more than building the apparatus and carrying it out.

Definition 1.56. In this section, we use two particular automorphisms:

- the *loop-exchange automorphism*, denoted by l, which is φ_1 (of the previous section),
- the *complement automorphism*, denoted by c, which replaces $E(G)$ with its complement, $V(G)^2 \setminus E(G)$.

Definition 1.57. A directed graph is called an *IO-graph* if it satisfies the following conditions. The only one-element substructure of it is E_1. If X is a two-element substructure then it is either E_2 or I_2. If X is a three-element substructure then X is E_3, or $I_2 \dot\cup E_1$, or I_3, or O_3. Let the set of IO-graphs be denoted by IO.

Lemma 1.58. *The set IO is definable.*

Proof. Observe that the set IO is already given by a first-order definition, using the one, two, and three element digraphs as constants. □

Observe that the set IO is closed under taking substructures. The following lemma motivates our notation IO.

Lemma 1.59. *A directed graph is an IO-graph if and only if it is a disjoint union of lines and/or circles.*

Proof. Straightforward induction on the number of vertices suffices, using the closedness mentioned prior to the lemma. □

Lemma 1.60. *The set $\{O_n : n \geq 3\}$ is definable.*

Proof. It is clear that all elements of the set are IO-graphs, we just need to choose which. It is easy to see that, in IO, those that have a unique lower-cover (within IO) are:

$$\underbrace{G \dot\cup \cdots \dot\cup G}_{k \text{ copies}}, \text{ where } G \in \{E_1, I_2\} \cup \{O_n : n \geq 3\},$$

for $k \geq 1$ except when $X = E_1$, then $k > 1$. In this set, the desired digraphs are exactly those that are minimal (in this particular set) and have I_3 or O_3 as a substructure. □

Definition 1.61. A digraph is called *loop-full* if all vertices have loops on them, and *loop-free* if none. The *loop-full part* of a digraph is the maximal loop-full substructure of it, and the *loop-free part* is the maximal loop-free substructure.

Lemma 1.62. *The relation*

$$\{(G, F, G \dot\cup F) : G, F \in \mathcal{D},\ G \text{ is loop-full and } F \text{ is loop-free}\}$$

is definable.

Proof. The relation consists of those triples (X, Y, Z) for which

- X is the loop-full part of Z,
- Y is the loop-free part of Z, and
- there is no two element substructure of Z that consists exactly one loop and has a non-loop edge in it.

□

Definition 1.63. Let $L_{\to}$ denote the digraph with

$$V(L_{\to}) = \{v_1, v_2\}, \text{ and } E(G) = \{(v_1, v_1), (v_1, v_2)\}.$$

Definition 1.64. Let G be a loop-full digraph with $V(G) = \{v_1, \ldots, v_n\}$. Then $l(G)$ is loop-free. Let the set of its vertices be $l(G) = \{v'_1, \ldots, v'_n\}$ (to recall l, see Def. 1.56) with

$$\text{for } i \neq j:\ (v'_i, v'_j) \in E(l(G)) \Leftrightarrow (v_i, v_j) \in E(G).$$

Let $G \to l(G)$ denote the digraph for which

$$V(G \to l(G)) = V(G) \cup V(l(G)), \text{ and}$$

$$E(G \to l(G)) = E(G) \cup E(l(G)) \cup \{(v_i, v'_i) : 1 \leq i \leq n\}.$$

Lemma 1.65. *The relation*

$$\{(G, l(G), G \to l(G)) : G \in \mathcal{D},\ G \text{ is loop-full}\}$$

is definable.

Proof. Let us consider the triples (X, Y, Z) for which

- X is the loop-full part of Z, and Y is the loop-free part of Z,

- $X \dot\cup E_1 \not\sqsubseteq Z$, and $Y \dot\cup L_1 \not\sqsubseteq Z$ (both are definable by Lemma 1.62),
- on two points, the only substructure having exactly one loop and at least one non-loop edge is $L_\to$, and
- no digraph of the first two pictures of Fig. 1.7 is a substructure. We consider the dashed edges possibilities, either we draw them (individually) or not. In this way, there are 6 (isomorphism types) encoded into the first two pictures of Fig. 1.7. We exclude them all.

Now we have ensured that the edges $L_\to$ constitute a bijection between the vertices of X and Y in Z. It only remains to force this bijection to be edge and non-edge preserving as well. This can be done by requiring the additional the property

- Consider the third picture of Figure 1.7 as before, the dashed edges are possibilities. We forbid those from being substructures in which the dashed edges are not symmetrically drawn on the two (loop-full and loop-free) sides.

□

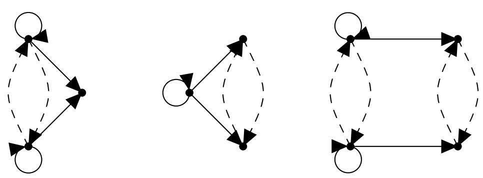


Figure 1.7:

We are going to need some basic arithmetic later. We define addition in the following lemma.

Lemma 1.66. *The following relation is definable:*

$$\{(E_n, E_m, E_{n+m}) : n, m \geq 1\}.$$

Proof. The set $\{E_n\}$ is definable as it consists of E_1 plus those digraphs which have only E_2 as a two-element substructure. $E_n \dot\cup (L_m \to E_m)$ is the digraph X for which

- $E_n \dot\cup L_m \sqsubseteq X$ (using Lemmas 1.62 and 1.65),
- $L_m \to E_m \sqsubseteq X$ (using Lemma 1.65),
- the second digraph of Fig. 1.7, without the dashed edges, is not a substructure,
- $E_{n+1} \dot\cup L_m \not\sqsubseteq X$ (E_{n+1} is just the cover of E_n in $\{E_n\}$),
- on two vertices, the only substructure having a non-loop edge is $L_\to$,
- the maximal loop-full substructure of X is L_m, and
- the maximal loop-free substructure of X is of the form E_i.

The E_i of the last condition is E_{n+m}. ☐

Lemma 1.67. *The following relation is definable:*

$$\{(G, F) : G \text{ and } F \text{ have the same number of vertices}\}. \tag{1.40}$$

Proof. We "determine" the number of vertices for the loop-full and the loop-free parts of the graphs separately and add them using Lemma 1.66. Let G_1 denote the loop-full part of G, and G_2 denote the loop-free part. Let X denote the digraph with the following properties:

- The loop-full part of X is G_1, and the loop-free part is E_i for some i.
- On two points, the only substructure having exactly one loop and at least one non-loop edge is $L_\to$.
- $G_1 \dot\cup E_1 \not\leq X$, and $E_i \dot\cup L_1 \not\leq X$.
- Just as in the proof of Lemma 1.65, no digraph of the 6 digraphs of the first two pictures of Fig. 1.7 is a substructure. (No matter, we wouldn't even need all 6 in this case.)

Observe that in X, the edges $L_\to$ constitute a bijection between G_1 and E_i, consequently i in the first condition is $|V(G_1)|$.

Now we proceed analogously for the loop-free part, G_2. We do not write all the conditions down again, as they are just the ones above converted with the automorphism l. This way, we get L_j with $j = |V(G_2)|$. We already have E_i and L_j defined, such that $i + j = |V(G)|$. To conclude, we use the relation of Lemma 1.65 to get E_j and Lemma 1.66 to obtain the desired E_{i+j}, marking the number of vertices of G.

Finally, $(G, F) \in (1.40)$ holds if and only if, by doing the same, we get the same $E_{i'+j'}$ marking the number of vertices. ☐

We define some more arithmetic in the following lemma, namely multiplication.

Lemma 1.68. *The following relation is definable:*

$$\{(E_n, E_m, E_{nm}) : n, m \geq 1\}.$$

Proof. The relation $\{(E_i, F_i) : i = 1, 2, \dots\}$ is definable as, beyond (E_1, F_1), for $i > 1$, F_i is the only digraph having the same vertices as E_i that has only F_2 as a two element substructure. Let X be a digraph that is maximal with the following properties:

1. $E_1 \not\leq X$ to ensure that the relation $E(X)$ is reflexive.
2. $l(I_2) \not\leq X$ to ensure that the relation $E(X)$ is symmetric.
3. The digraph of Fig. 1.8 is not a substructure of X to ensure that the relation $E(X)$ is transitive.
4. L_n is the maximal L_i substructure.
5. F_m is the maximal F_i substructure.

The conditions 1-3 force $E(X)$ to be an equivalence. Condition 4 tells the equivalence has at most n classes and condition 5 requires the classes to have at most m elements. It is easy to see that such an equivalence relation has a base set of at most nm elements, hence $|V(X)| = nm$. Thus, using Lemma 1.67, we are done. □

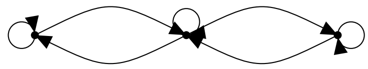


Figure 1.8:

Lemma 1.69. *Disjoint union of IO graphs is definable, i.e. the following relation is definable:*

$$\{(G_1, G_2, G_1 \dot\cup G_2) : G_1, G_2 \in IO\}.$$

Proof. Though the notion $l(G) \to G$ was not defined as is in Definition 1.64, it is such an analogue that the reader surely can decipher without effort. Using G_1 and G_2, we want to define

$$G_1 \dot\cup (l(G_2) \to G_2), \tag{1.41}$$

whose loop-free part is the sought $G_1 \dot\cup G_2$. For this, let X satisfy the following conditions.

- $|V(X)| = |V(G_1)| + 2|V(G_2)|$ (using Lemmas 1.67 and 1.66),
- $G_1 \dot\cup l(G_2) \sqsubseteq X$ (using Lemma 1.62),
- $l(G_2) \to G_2 \sqsubseteq X$ (using Lemma 1.65), and
- $l(F_2) \not\sqsubseteq X$.

It easy to see that these three conditions ensure that (1.41) is embeddable (not substructure!) into X: there can be edges between the substructures G_1 and G_2 which we need to exclude. If there is an edge from G_2 to G_1 (in this particular direction), then the first graph of Fig. 1.9 is a substructure, without the dashed edges. Analogously, if an edge goes from G_1 to G_2, then the second digraph of Fig. 1.9 is a substructure, without the dashed edges. Edges going both directions is forbidden by the last condition above. Thus we need to exclude these two substructures. Let Y satisfy the following conditions.

- $|V(Y)| = |V(G_2)| + 2$, and $Y \sqsupseteq l(G_2)$.
- I_2 and $L_\to$ are substructures of Y.
- The digraph of Fig. 1.10 is not a substructure of Y.

These three conditions do not define the two digraphs of Fig. 1.9 without the dashed edges, they rather define the set of those with the dashed edges meant as possibilities, as usual. However none of the dashed edges can actually appear in our X so by excluding all such, we do not do more than by excluding only the two without the dashed edges. Finally, (1.41) is the loop-free part of X . □

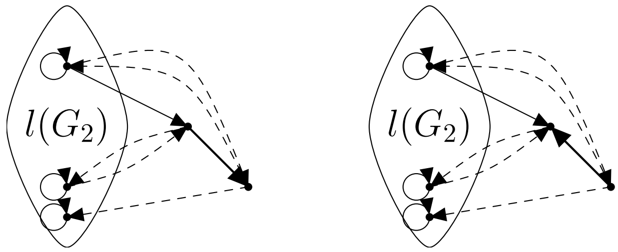


Figure 1.9:

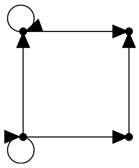


Figure 1.10:

Lemma 1.70. *The following set is definable.*

$$\{G: \ G \text{ is a disjoint union of circles of different sizes}\}. \tag{1.42}$$

Proof. The set of digraphs that are disjoint unions of circles contains those IO graphs that have unique upper-covers (in the set IO). In this set, the digraphs of the form $O_i \dot\cup O_i$ are those that have a unique circle substructure O_i and have twice as many vertices as O_i. We have defined two sets of digraphs, the set of the lemma is just the set of those digraphs of the first set that have no substructures from the second. □

Lemma 1.71. *The following relation is definable.*

$$\{(O^*, G \dot\cup O^*) : G \in \mathcal{D} \text{ and } O^* \text{ is a disjoint union of } |V(G)|\text{-many circles of different sizes such that the smallest has at least } |V(G)|+1 \text{ vertices}\}. \tag{1.43}$$

Proof. First, we define a relation counting the number of circles in O^*, actually we formulate it without the restriction on the sizes of the circles:

$$\{(E_i, O) : O \text{ is a disjoint union of } i \text{ circles}\}. \tag{1.44}$$

The set of O's of this relation was defined in the first sentence of the proof of Lemma 1.70. Let O' denote such a substructure of O that has no circle in it and has a maximal number of vertices with this property. Then $i + |V(O')| = |V(O)|$ holds for the i of (1.44), thus we can conclude with the addition relation defined earlier.

Let O^* be an element of the set defined in Lemma 1.70 and i be the number of its circles. Let X satisfy:

- $|V(X)| = |V(O^*)| + i.$
- The smallest circle in O^* has at least $i+1$ vertices.

- $O^* \sqsubseteq X$.
- X does not have a substructure Y for which
  - $|V(Y)| = |V(O^*)| + 1$, and $Y \sqsupseteq O^*$,
  - Y is loop-free, and
  - Y is not an IO-graph.
- X does not have a substructure Y for which
  - $|V(Y)| = |V(O^*)| + 1$, and $Y \sqsupseteq O^*$,
  - Y has a loop in it, and
  - Y has one of $L_{\rightarrow}$ or $L^T_{\rightarrow}$ (the transpose of $L_{\rightarrow}$) or $c(L_1 \dot\cup E_1)$ as a substructure.

With these properties, X is of the required form $G \dot\cup O^*$. □

Definition 1.72. Let O^* be a digraph that is a disjoint union of circles of different sizes, as usual, and let G be an arbitrary digraph. We introduce the notation $G_N^{O^*}$ for the digraph that is the disjoint union of weakly connected components of G not embeddable into O^*.

Lemma 1.73. *The following relation is definable.*

$$\{(O^*, G, G_N^{O^*}) : G \in \mathcal{D},\ \textit{and } O^* \textit{ is a disjoint union of circles of different sizes}\}$$

Proof. The following conditions suffice.

- $O^* \in (1.42)$,
- $G_N^{O^*} \sqsubseteq G$,
- for all $X \dot\cup O^*$ (using (1.43)), $G_N^{O^*} \sqsubseteq X \dot\cup O^*$ implies $|V(X)| \geq |V(G_N^{O^*})|$, and
- $G_N^{O^*}$ is maximal with the properties above.

The third condition forces $G_N^{O^*}$ to have only such wccs that are not substructures of O^*. Note that the third condition can be encoded into a first-order formula using Lemmas 1.66 and 1.67. □

Lemma 1.74. *The following relation is definable.*

$$\{(O^*, G, G \dot\cup O^*) : (O^*, G \dot\cup O^*) \in (1.43)\} \tag{1.45}$$

Proof. The relation in question is the set of triples (O^*, G, X) for which

1. $(O^*, X) \in (1.43)$,
2. $|V(X)| = |V(O^*)| + |V(G)|$,
3. $X_N^{O^*} = G_N^{O^*}$ (using Lemma 1.73), and
4. $Y \dot\cup O^* \sqsubseteq X$, for all IO-substructures Y of G.

Condition 1 entails $X = G' \,\dot\cup\, O^*$ for some G'. With this notation, what we have to prove is $G = G'$.

Condition 2 adjusts the size of G'.

Before explaining Conditions 3 and 4, we introduce a notation needed. For a digraph G, its wccs fall into two categories, those that are IO-graphs, and those that are not. Let G_{IO} and G_{NIO} stand for the substructures of G consisting of the IO- and non-IO-wccs of G, respectively. To prove $G = G'$, it is enough to show that $G_{\mathrm{IO}} = G'_{\mathrm{IO}}$ and $G_{\mathrm{NIO}} = G'_{\mathrm{NIO}}$ both hold.

Condition 3 forces $G_{\mathrm{NIO}} = G'_{\mathrm{NIO}}$, because we clearly have $X_N^{O^*} \supseteq G'_{\mathrm{NIO}}$ (this is not substructureness, but containment between parts of digraphs, meaning the multiset of wccs of the left side contains that of the right side).

Condition 4 is to ensure $G_{\mathrm{IO}} = G'_{\mathrm{IO}}$, but seeing it serves its purpose is far from trivial. Our goal is to prove that for any wcc W of G_{IO}:

$$\text{the number of wccs isomorphic to } W \text{ in } G'_{\mathrm{IO}} \text{ is greater or equal to the corresponding number of } G_{\mathrm{IO}}. \tag{1.46}$$

If this goal is achieved, we are done as we already have $|V(G_{\mathrm{IO}})| = |V(G'_{\mathrm{IO}})|$. Let us fix an arbitrary wcc W of G_{IO}. We pick a particular $Y = Y^W$ from the many possible Ys of Condition 4. Firstly, let Y^W have a maximal number of vertices (among the IO-substructures of G), and on top of that, let it have a maximal number of wccs isomorphic to W in it. Note that Y^W need not be uniquely determined by these properties. In case it is not, we pick arbitrarily from the ones meeting the requirements. Condition 4 grants $Y^W \,\dot\cup\, O^* \sqsubseteq X$. Let φ be a map from $Y^W \,\dot\cup\, O^*$ to X that certifies this fact. The maximality of the number of vertices of Y^W gives us $G'_{\mathrm{IO}} \subseteq \mathrm{Range}(\varphi)$, and it is easy to see that even if wccs isomorphic to W map into G'_{NIO}, the second maximality property of Y^W implies (1.46). □

Some technical tools follow with another alteration of notation. In [7], the notation ♂ was used slightly differently than in [9]. That is the reason why we replace ♂ with ⚦ for the rest of the section.

Definition 1.75. Let $V(O_n) = \{v_1, \ldots, v_n\}$ and let us define two digraphs with

$$V(⚦_n) := V(O_n) \cup \{u_1, u_2\},\ E(⚦_n) := E(O_n) \cup \{(v_1, u_1), (u_1, u_2)\}, \text{ and}$$

$$V(⚦_n^L) := V(⚦_n),\ E(⚦_n^L) := E(⚦_n) \cup \{(u_2, u_2)\}.$$

Now let m be a different positive integer from n and define $⚦_m$ and $⚦_m^L$ analogously with $V(⚦_m) = V(⚦_m^L) = \{v'_1, \ldots, v'_m, u'_1, u'_2\}$.

Now we are going to deal with pairs of the digraphs just defined, which leaves us $4 = 2\times 2$ cases with respect to the presence of the loops. To avoid the tiresomeness of listing all 4 possibilities all the time, we resort to the following notation. We say, let $(\square, \triangledown) \in \{\emptyset, L\}^2$, and for example, in the case $(\square, \triangledown) = (L, \emptyset)$, we mean $(⚦_n^L, ⚦_m)$ by $(⚦_n^\square, ⚦_m^\triangledown)$, naturally.

Let $(\square, \triangledown) \in \{\emptyset, L\}^2$. We introduce two more types of digraphs with

$$V(⚦_n^\square \to ⚦_m^\triangledown) := V(⚦_n) \cup V(⚦_m),\ E(⚦_n^\square \to ⚦_m^\triangledown) := E(⚦_n^\square) \cup E(⚦_m^\triangledown) \cup \{(u_2, u'_2)\}, \text{and}$$

$$V(⚦_n^\square \leftrightarrow ⚦_m^\triangledown) := V(⚦_n) \cup V(⚦_m),\ E(⚦_n^\square \leftrightarrow ⚦_m^\triangledown) := E(⚦_n^\square \to ⚦_m^\triangledown) \cup \{(u'_2, u_2)\}.$$

Lemma 1.76. *The following relation is definable for all* $(\square, \triangledown) \in \{\emptyset, L\}^2$.

$$\{(E_i, E_j, \male_i^\square, \male_i^\square \dot\cup \male_j^\triangledown, \male_i^\square \to \male_j^\triangledown, \male_i^\square \leftrightarrow \male_j^\triangledown) : i, j > 3, i \neq j\} \tag{1.47}$$

The proof is put in the next subsection for its technical nature.

The following definition introduces a construction of great importance in the remaining half of the proof.

Definition 1.77. Let G be a digraph on n vertices with $V(G) = \{v_1, \dots, v_n\}$, and let $(O^*, G \dot\cup O^*) \in (1.43)$ with $V(O^*) = \{u_i^j : 1 \le j \le n, 1 \le i \le k_j\}$ such that the jth circle O_{k_j} of O^* consists of the vertices $\{u_i^j : 1 \le i \le k_j\}$. Let $C(O^*) = \{O_{k_1}, \dots, O_{k_n}\}$ denote the set of the circles of O^* and let $\alpha : C(O^*) \to V(G)$ be a bijective map. We introduce the notation $G \overset{\alpha}{\leftarrow} O^*$ for the digraph with

$$V(G \overset{\alpha}{\leftarrow} O^*) = V(G \dot\cup O^*) \cup \{w_1, \dots, w_n\}, \text{ and}$$
$$E(G \overset{\alpha}{\leftarrow} O^*) = E(G \dot\cup O^*) \cup \{(u_1^j, w_j) : 1 \le j \le n\} \cup \{(w_j, \alpha(O_{k_j})) : 1 \le j \le n\}.$$

Lemma 1.78. *The following relation is definable.*

$$\{(O^*, G, G \dot\cup O^*, G \overset{\alpha}{\leftarrow} O^*) : (O^*, G \dot\cup O^*) \in (1.43),\ \alpha : C(O^*) \to V(G)\}. \tag{1.48}$$

Proof. As we already defined (1.45), we only need to define the digraphs $G \overset{\alpha}{\leftarrow} O^*$ (using O^*, G and $G \dot\cup O^*$). The relation of the lemma consists of those triples $(O^*, G, G \dot\cup O^*, X)$ for which:

- $|V(X)| = |V(G \dot\cup O^*)| + |V(G)|$,
- $G \dot\cup O^* \sqsubseteq X$,
- $O_i \sqsubseteq O^*$ implies $\male_i \sqsubseteq X$ or $\male_i^L \sqsubseteq X$,
- $O_i, O_j \sqsubseteq O^*$ $(i \neq j)$ implies $\male_i^\square \dot\cup \male_j^\triangledown \sqsubseteq X$, or $\male_i^\square \to \male_j^\triangledown \sqsubseteq X$, or $\male_i^\square \leftrightarrow \male_j^\triangledown \sqsubseteq X$ for some $(\square, \triangledown) \in \{\emptyset, L\}^2$.

Unfortunately, these conditions do not ensure $X = G \overset{\alpha}{\leftarrow} O^*$ yet. That is because, say, the tails of the $\male$s can still get entangled. Additional technical conditions have to be added to avoid this unwanted scenario. For its technical nature, this argument is put in Subsection 1.3.4. □

In the following definition we introduce the soul of our proof: the edge-supporting construction. Before starting to study the long definition, it is worth to read the simplified idea of it, back at the beginning of this subsection.

Definition 1.79. In this definition, we introduce the *edge-supporting construction.* Let G be a digraph with

$$V(G) = \{v_1, \dots, v_n\}, \text{ and } E(G) = \{e_1, \dots, e_r\}.$$

Note that $r \le n^2$ is necessary. Let p_1 and p_2 be two maps from $E(G)$ to $\{v_1, \dots, v_n\}$ defined by the rule

$$\forall e \in E(G) : e = (v_{p_1(e)}, v_{p_2(e)}).$$

Let us introduce a digraph G_s with

$$V(G_s) := V(G) \cup \{v_1^s, \ldots, v_r^s\}, \text{ and } E(G_s) := E(G) \cup \bigcup_{i=1}^{r} \{(v_{p1(e_i)}, v_i^s), (v_i^s, v_{p2(e_i)})\}.$$

We call the edges added by the big union the *supporting edges*. Let

$$O^* = O_{l_1} \dot{\cup} O_{l_2} \dot{\cup} \cdots \dot{\cup} O_{l_n} \text{ such that } n^2 + n < l_1 < l_2 < \cdots < l_n.$$

Let D_s be a set of integers with

$$|D_s| = r(= |E(G)|), \text{ and } x \in D_s \Rightarrow x > l_n. \tag{1.49}$$

Let s be a bijective map from D_s, satisfying (1.49), to $\{v_1^s, \ldots, v_r^s\}$. Let

$$O_s^* := O^* \dot{\cup} \dot{\bigcup_{x \in D_s}} O_x \text{ with } V(O_s^*) = \{u_i^j : j \in \{l_1, \ldots, l_n\} \cup D_s, \ 1 \leq i \leq j\}.$$

Let $\alpha : C(O^*) \to V(G)$ be a bijective map. We define the digraph $(G \overset{\alpha}{\leftarrow} O^*)_s$ by

$$(G \overset{\alpha}{\leftarrow} O^*)_s := G_s \overset{\beta}{\leftarrow} O_s^*, \text{ where } \beta|_{C(O^*)} := \alpha, \ \beta|_{\{O_x : x \in D_s\}} := \{(O_x, s(x)) : x \in D_s\},$$

and say it is an *edge-supporting digraph for* G.

Remark 1.80. Note that the definition of the edge-supporting digraphs includes a condition for the size of the circles of O^*. That condition is very important here, and was not present in (1.48). We need to be cautious about this later on.

Lemma 1.81. *The following relation is definable.*

$$\{(O^*, G, G \overset{\alpha}{\leftarrow} O^*, (G \overset{\alpha}{\leftarrow} O^*)_s) : (G \overset{\alpha}{\leftarrow} O^*)_s \text{ is an edge-supporting digraph for } G\} \tag{1.50}$$

Proof. The relation in question consists of those quadruples (X_1, X_2, X_3, X_4) for which the highlighted conditions hold. In some cases, there are explanations inserted between the conditions.

- There exists a quadruple $(X_1, X_2, Y, X_3) \in (1.48)$, meaning (X_1, X_2, X_3) is of the form $(O^*, G, G \overset{\alpha}{\leftarrow} O^*)$.

Thus, instead of (X_1, X_2, X_3), we use $(O^*, G, G \overset{\alpha}{\leftarrow} O^*)$ from now on in the proof. Let G have n vertices. Now we are ready to shape O^*.

- $O_i \sqsubseteq O^*$ implies $i > n^2 + n$.

We turn to defining X_4 of the quadruple we started with.

- There exists a quadruple $(W_1, W_2, W_3, X_4) \in (1.48)$, meaning (W_1, X_4) is of the form $(O_s^*, G_s \overset{\beta}{\leftarrow} O_s^*)$.

At this point, O_s^*, G_s, and β are just notations yet, we need additional conditions to make them be like in Definition 1.79.

- $O^* \sqsubseteq O_s^*$

- $O_i \sqsubseteq O_s^*$ implies $i \geq l_1$, where l_1 is the size of the smallest circle of O^*, as before.
- $G \overset{\alpha}{\leftarrow} O^* \sqsubseteq X_4 (= G_s \overset{\beta}{\leftarrow} O_s^*)$.

The following conditions are to shape the supporting edges of our construction according to the definition.

- If $O_i \sqsubseteq O^*$ and $\male_i^L \sqsubseteq X_4$, then there exists $k > l_n$ for which $\male_i^L \leftrightarrow \male_k \sqsubseteq X_4$ holds. Additionally, if l is different from i, k, and $O_l \sqsubseteq O_s^*$, then there exists $\diamond \in \{\emptyset, L\}$ for which $\male_k \mathbin{\dot\cup} \male_l^\diamond \sqsubseteq X_4$ holds.
- If $O_i, O_j \sqsubseteq O^*$, $i \neq j$, and $\male_i^\square \to \male_j^\triangledown \sqsubseteq X_4$ with $(\square, \triangledown) \in \{\emptyset, L\}^2$, then there exists $k > l_n$ for which $\male_i^\square \to \male_k \sqsubseteq X_4$ and $\male_k \to \male_j^\triangledown \sqsubseteq X_4$ both hold. Additionally, if l is different from i, j, k, and $O_l \sqsubseteq O_s^*$, then there exists $\diamond \in \{\emptyset, L\}$ for which $\male_k \mathbin{\dot\cup} \male_l^\diamond \sqsubseteq X_4$ holds.
- If $O_i, O_j \sqsubseteq O^*$, $i \neq j$, and $\male_i^\square \leftrightarrow \male_j^\triangledown \sqsubseteq X_4$ with some $(\square, \triangledown) \in \{\emptyset, L\}^2$, then there exist two different $k_1, k_2 > l_n$ for which all of

  $$\male_i^\square \to \male_{k_1},\ \male_{k_1} \to \male_j^\triangledown,\ \male_j^\triangledown \to \male_{k_2},\ \text{and}\ \male_{k_2} \to \male_i^\square$$

  are substructures of X_4. Additionally, if l is different from i, j, k_i, and $O_l \sqsubseteq O_s^*$, then there exists $\diamond \in \{\emptyset, L\}$ for which $\male_{k_i} \mathbin{\dot\cup} \male_l^\diamond \sqsubseteq X_4$ holds for $i = 1, 2$.
- If $O_k \sqsubseteq O_s^*$ and $k > l_n$, then k is one of the ks or k_is of the previous three conditions.

It is not hard to see that these conditions provide the structure we need. □

We are finally ready to prove our main theorem.

Proof of Theorem 1.52. With (1.50), fix a triple $(G, O^*, (G \overset{\alpha}{\leftarrow} O^*)_s)$, and let n be the number of vertices of G. We need to show that the set of digraphs embeddable into G is definable. Let $X \sqsubseteq (G \overset{\alpha}{\leftarrow} O^*)_s$ and let $(G_X, O_X^*, G_X \overset{\gamma}{\leftarrow} O_X^*)$ be a triple consisting the second, first and third element of a 4-tuple of (1.50) for which the following conditions hold.
Before listing the actual conditions being quite technical, it may be worth summarizing their goal plainly. What they do is link G_X to both X and G the natural way, i. e. making sure that we have $G_X \leq G$ and we leave out the edges that got unsupported taking the substructure X.

- $O_i \sqsubseteq O_X^*$ holds if and only if both $O_i \sqsubseteq O^*$, and $\male_i^\square \sqsubseteq X$ for some $\square \in \{\emptyset, L\}$ hold.
- If $O_i, O_j \sqsubseteq O_X^*$, $i \neq j$, and $(\square, \triangledown) \in \{\emptyset, L\}^2$, then
  - $\male_i^\square \mathbin{\dot\cup} \male_j^\triangledown \sqsubseteq G_X \overset{\gamma}{\leftarrow} O_X^*$ holds if and only if one of the following three holds:
    * $\male_i^\square \mathbin{\dot\cup} \male_j^\triangledown \sqsubseteq X$, or
    * $\male_i^\square \to \male_j^\triangledown \sqsubseteq X$, but the edge is not supported in X, i. e. there exists no $k > l_n$ (where l_n is the size of the largest circle of O^*, as before) for which $\male_i^\square \to \male_k \sqsubseteq X$ and $\male_k \to \male_j^\triangledown \sqsubseteq X$ both hold, or
    * $\male_i^\square \leftrightarrow \male_j^\triangledown \sqsubseteq X$, but none of the two edges is supported in X.

– $\text{⚦}_i^{\square} \to \text{⚦}_j^{\triangledown} \sqsubseteq G_X \overset{\gamma}{\leftarrow} O_X^*$ holds if and only if one of the following two holds
  * $\text{⚦}_i^{\square} \to \text{⚦}_j^{\triangledown} \sqsubseteq X$, and the edge is supported in X, or
  * $\text{⚦}_i^{\square} \leftrightarrow \text{⚦}_j^{\triangledown} \sqsubseteq X$, but only the "$i \to j$" edge is supported in X.

– $\text{⚦}_i^{\square} \leftrightarrow \text{⚦}_j^{\triangledown} \sqsubseteq G_X \overset{\gamma}{\leftarrow} O_X^*$ holds if and only if $\text{⚦}_i^{\square} \leftrightarrow \text{⚦}_j^{\triangledown} \sqsubseteq X$ and both edges are supported in X.

It is not hard to see that $G_X \leq G$ holds indeed, and all embeddable digraphs can be obtained this way. □

1.3.4 The Remaining Technicalities

Definition 1.82. The sum of the number of (both in- and out-)edges for a vertex, not counting the loops, is called the *loop-free degree* of the vertex.

Lemma 1.83. *Let $0 \leq p$ and $1 \leq q$ be two fixed integers. We can define, with finitely many constants added to $(\mathcal{D}, \sqsubseteq)$, the set of digraphs that contain at most p many vertices with loop-free degree at least q each.*

Before the easy proof, note that we can only use this lemma if we have a fixed constant, say $K = 4$, for the whole paper, such that all usage of the lemma restricts to $p, q \leq K$. Otherwise there would be no guarantee we are using finitely many constants at all. Fortunately, $K = 4$ will do for the whole section. At the end of this subsection, we give a more elaborate account on the usage of Lemma 1.83, which then will be behind us.

Proof. Observe that the digraph G has more than p many vertices with at least q loop-free degree each, if and only if it has an at most $(p+1)(q+1)$ element "certificate" substructure with the same property. Hence, by forbidding all those (finitely many) certificates, we define the set we need. □

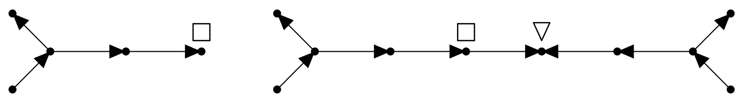


Figure 1.11:

Proof of Lemma 1.76. Let us consider E_i and E_j given. We define the other components of the relation.

We start with $\text{⚦}_i^{\square}$ which is just the digraph X for which

- $|V(X)| = i + 2$.
- $O_i \sqsubseteq X$.
- We use Lemma 1.83 with $p = 1$, and $q = 3$, i. e. X has at most one vertex with loop-free degree at least 3.
- We use Lemma 1.83 with $p = 0$, and $q = 4$ as well.

- The first digraph of Fig. 1.11 is a substructure. The $\square$ symbol is understood naturally, if $\square = L$, then there is a loop there, if $\square = \emptyset$, then there is not.
- Depending on $\square$,
  - if $\square = \emptyset$, then $O_i \dot\cup E_1 \sqsubseteq X$, that is the only cover of O_i among the IO-graphs,
  - if $\square = L$, then $O_i \dot\cup L_1 \sqsubseteq X$, that is definable with Lemma 1.62.

We now start to deal with $\male_i^{\square} \dot\cup \male_j^{\triangledown}$. $O_i \dot\cup O_j$ is the digraph with $i + j$ vertices that is a disjoint union of circles and both O_i and O_j are substructures. $\male_i^{\square} \dot\cup \male_j^{\triangledown}$ is the digraph X for which

- $|V(X)| = |V(\male_i^{\square})| + |V(\male_j^{\triangledown})|$.
- $\male_i^{\square} \sqsubseteq X$, and $\male_j^{\triangledown} \sqsubseteq X$.
- We use Lemma 1.83 with $p = 2$, $q = 3$ and with $p = 0$, $q = 4$.
- Depending on $(\square, \triangledown)$,
  - if $(\square, \triangledown) = (\emptyset, \emptyset)$, then $O_i \dot\cup O_j \dot\cup E_2 \sqsubseteq X$, which is just the digraph Y for which
    - $|V(Y)| = i + j + 2$, and $O_i \dot\cup O_j \sqsubseteq X$,
    - Y has the maximal substructure E_k (among the ones with the previous property).
  - if $(\square, \triangledown) = (L, \emptyset)$ or $(\emptyset, L)$, then $O_i \dot\cup O_j \dot\cup E_1 \dot\cup L_1 \sqsubseteq X$, which is just the digraph Y for which
    - $|V(Y)| = i + j + 2$, and $O_i \dot\cup O_j \sqsubseteq X$,
    - $O_i \dot\cup O_j \dot\cup E_1$, which is the only IO-graph cover of $O_i \dot\cup O_j$, is a substructure,
    - $O_i \dot\cup O_j \dot\cup L_1$ is a substructure, and
    - on two elements, there is no substructure with both a loop and a loop-free edge.
  - if $(\square, \triangledown) = (L, L)$ then $O_i \dot\cup O_j \dot\cup L_2 \sqsubseteq X$.

Now we turn to $\male_i^{\square} \to \male_j^{\triangledown}$, which is just the digraph X for which

- $|V(X)| = |V(\male_i^{\square})| + |V(\male_j^{\triangledown})|$.
- $\male_i^{\square} \sqsubseteq X$, and $\male_j^{\triangledown} \sqsubseteq X$.
- We use Lemma 1.83 with $p = 2$, $q = 3$ and with $p = 0$, $q = 4$.
- The second digraph of Fig. 1.11 is substructure of X.

Finally, $\male_i^{\square} \leftrightarrow \male_j^{\triangledown}$ is defined with the analogues of the conditions for $\male_i^{\square} \to \male_j^{\triangledown}$. $\square$

The end of the proof of Lemma 1.78 To exclude the possible entanglement of the ⚦s, it is enough to forbid two types of digraphs as substructures. To introduce these two, first, we need two circles O_i and O_j with

$$i \neq j, \quad V(O_i) = \{u_1, \ldots, u_i\}, \text{ and } V(O_j) = \{v_1, \ldots, v_j\}.$$

Then we define P_i and $P^{\triangledown}_{i,j}$ (where, as usual, $\triangledown \in \{\emptyset, L\}$) to be

$$V(P_i) = V(O_i) \cup \{u', u''\}, \quad E(P_i) = E(O_i) \cup \{(u_1, u'), (u_1, u'')\},$$

and

$$V(P_{i,j}) = V(O_i) \cup V(O_j) \cup \{u', v', w\},$$
$$E(P_{i,j}) = E(O_i) \cup E(O_j) \cup \{(u_1, u'), (u', w), (v_1, v'), (v', w)\},$$

with the usual, minor modification

$$V(P^L_{i,j}) = V(P_{i,j}), \quad E(P^L_{i,j}) = E(P_{i,j}) \cup \{(w, w)\}.$$

First, we define P_i (using O_i). Actually, the very first, we define P'_i, that is just P_i minus the vertex u''. This is easy as P'_i is the digraph one below ⚦$_i$ such that it has O_i in it as a substructure but it does not equal $O_i \mathbin{\dot\cup} E_1$. Now the sought $P_i = X$ can be defined by the following properties:

- X covers P'_i,
- $O_i \sqsubseteq X$,
- X has exactly the same three-element substructures as P_i, and
- if X covers Y such that $O_i \sqsubseteq Y$, then $Y = P'_i$.

Second, we define $P^{\triangledown}_{i,j}$ (using O_i and O_j). It is the only digraph X for which

- $|V(X)| = |V(O_i)| + |V(O_j)| + 3$,
- $O_i \mathbin{\dot\cup} O_j \sqsubseteq X$,
- ⚦$^{\triangledown}_i$, ⚦$^{\triangledown}_j$ $\sqsubseteq X$, and
- we use Lemma 1.83 with $p = 2$, $q = 3$.

Finally, to conclude the proof, we add two conditions to the ones already listed at the beginning of the proof:

- $O_i \sqsubseteq O^*$ implies $P_i \not\sqsubseteq X$,
- $O_i, O_j \sqsubseteq O^*$ $(i \neq j)$ implies $P^{\triangledown}_{i,j} \not\sqsubseteq X$ (for both $\triangledown \in \{\emptyset, L\}$).

□

It is worth counting how big constants the usage of Lemma 1.83 requires. By its proof, it is clear that its usage with the pair (p, q) requires constants of size at most $(p+1)(q+1)$. Looking back, we see that we used the lemma for the pairs $(0, 4), (1, 4)$, and $(2, 3)$. Hence the answer to our question is 12, that is just $\max\{5, 10, 12\}$.

1.4 Table of Notations

Notation	Number of item it is in	Page number
$\leq$		8
$\sqsubseteq$		8
$\dot{\cup}$		8
$\lhd$, $\unlhd$	1.33	19
$\equiv$, $\equiv^C_G$, $=^C_G$	1.33	19
$(.)_{\mathcal{CD}}$	1.47	26
$♂_n$	1.26	17
$♂^L_n$	1.26	17
$♂^L_{i,j}$	1.27	17
$♂_i \to ♂_j$	1.39	23
$⚦_m$, $⚦^L_m$	1.75	41
$⚦^\square_n \to ⚦^\triangledown_m$	1.75	41
$⚦^\square_n \leftrightarrow ⚦^\triangledown_m$	1.75	41
A		11
$\mathbf{A}$		11
(A, α, B)		9
c	1.56	34
$\mathcal{CD}$		9
$\mathcal{CD}'$	1.3	10
$\mathcal{D}$		7
$\mathcal{D}'$	1.4	12
$\mathfrak{E}$	1.16	15
$\mathfrak{E}_+$	1.20	16
$E(G)$		8
$\mathbf{E}_1$	1.3	10
E_n	1.11	14
$F_\alpha(n, m)$	1.43	25
$\mathcal{F}(n, m)$	1.43	25
E_n	1.11	14
$\mathbf{f}_1$, $\mathbf{f}_2$	1.3	10
F_n	1.11	14
G^T		8
$G^{O^*}_N$	1.72	40
$G \to l(G)$	1.64	35
$G \overset{\alpha}{\leftarrow} O^*$	1.77	42
$(G \overset{\alpha}{\leftarrow} O^*)_s$	1.79	42
$G \overset{v}{\leftarrow} O^*_n$	1.37	23
$\hom(A, B)$		9
$\mathbf{I}_2$	1.3	10

Notation	Number of item it is in	Page number
I_n	1.12	14
IO-graph, IO	1.57	34
l	1.56	34
$\mathfrak{L}$	1.16	15
$L_\rightarrow$	1.63	35
L_n	1.12	14
$L(.)$	1.13	14
$\mathcal{L}(.)$	1.13	14
$\mathfrak{M}$	1.19	15
$M(.)$	1.14	14
$\mathcal{M}(.)$	1.14	14
$\mathcal{O}$	1.17	15
$\mathcal{O}_\cup$	1.19	15
O_n	1.12	14
$\mathcal{O}_n^\rightarrow$	1.24	17
O_n^*	1.23	16
$O_{n,L}^*$	1.29	17
$O_{n,L}$	1.28	17
$O_{i,i}$	1.41	24
$O_{i\rightarrow j}$	1.30	19
ob($\mathcal{C}$)		9
$V(G)$		8
$\mathcal{W}$	1.32	19

Chapter 2

On Finite Generability of Clones of Finite Posets

2.1 Introduction

Let F be a set of operations on a set A. We call F a *clone* if it is closed under composition and contains the projections. A subset of a clone is called a *subclone* if it is closed under composition and contains the projections. On a set A the subclones of the clone of all operations of A form a lattice, the *lattice of clones on* A.

A *generating set of a clone* F is a subset of F from which every element of F is obtained by the use of composition and projections. A *clone is finitely generated* if it has a finite generating set. In this chapter we study certain clones related to finite posets. Our main goal is to decide if these clones are finitely generated.

We say that an n-ary operation f on A *preserves* a k-ary relation R on A, if by applying f componentwise to any $r_1, \dots, r_n \in R$ the resulting k-tuple also is in R. Clearly, for any set of relations S on A, the set of operations that preserve all of the relations of S is a clone. The operations that preserve the one element subsets of their base sets are called *idempotent*.

Let P be a partially ordered set, a *poset* for short. An operation f on the base set of P is called *monotone* if f preserves the ordering $\leq$ of P. Then we also say that P *admits* the operation f. For a finite poset P, let $\mathcal{C}(P)$ and $\mathcal{I}(P)$ denote the clone of monotone operations of P and the clone of idempotent monotone operations of P, respectively. We call $\mathcal{C}(P)$ *the clone of* P and $\mathcal{I}(P)$ *the idempotent clone of* P.

A clone is called *maximal* if it is a coatom in the lattice of clones. In [13] Rosenberg proved that there are only six types of maximal clones in the lattice of clones on a finite set. Later the clones of five types of them were shown to be finitely generated. The clones of the sixth type are the clones of bounded posets. A poset is *bounded* if it has a smallest and a largest element. On the finite generability of clones of bounded posets only partial results were obtained so far.

An n-ary operation f, $n \geq 3$, is a *near unanimity operation* if it satisfies the identities

$$f(x, y, \dots, y) = f(y, x, \dots, y) = \cdots = f(y, y, \dots, x) = y.$$

Notice that the near unanimity operations are idempotent. It is well known that on a finite set any clone that contains an n-ary near unanimity operation is finitely generated. In [1]

Demetrovics, Hannák and Rónyai proved that by deleting any convex subset of a finite lattice we obtain a poset whose clone contains a near unanimity operation. A *fence* is a finite poset of height 1 whose covering graph is a path. The *linear sum* $P+Q$ *of two posets* P *and* Q is the poset whose base set is the union of the base sets of P and Q, and whose ordering is defined by $a \leq b$ iff either $a \in P$ and $b \in Q$ or $a \leq b$ in P or $a \leq b$ in Q. Let $\mathbf{k}$ denote the k-element antichain. If F is a fence, then $\mathbf{1} + \mathbf{2} + F + \mathbf{2} + \mathbf{1}$ is a called a *locked fence*. Fences and locked fences also admit a near unanimity operation. It is easy to see that the class of finite posets whose clones contain near unanimity operations is closed under retract and finite product. A *retract* of a poset P is a poset R that is isomorphic to the image of a unary monotone operation f on P where $f^2 = f$.

It is an open question if besides the finite bounded posets that admit a near unanimity operation there are other types of finite bounded posets whose clones are finitely generated. If we drop the boundedness condition in this question, then the answer is positive. A *crown* is a poset of height 1 whose covering graph is a cycle. In [2] Demetrovics and Rónyai proved that the clone of any crown is finitely generated. It is well known, on the other hand, that the idempotent clone of any crown contains only projections, hence its clone does not contain a near unanimity operation.

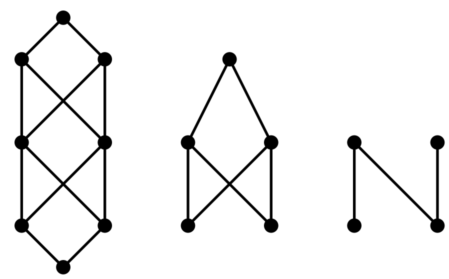


Figure 2.1: Posets T, H, and N

In his famous paper [14] Tardos proved that the clone of the eight element poset T in Figure 2.1 is not finitely generated. His result was generalized by Zádori in [18]. A finite poset P is *series-parallel* if the four element poset N in Figure 2.1 is not an induced subposet of P. In [18] it was proved that for a series-parallel poset P, $\mathcal{C}(P)$ is finitely generated if and only if none of the posets T, H in Figure 2.1 and the dual of H are retracts of P. A natural question arises: is it true that if the clone of a finite poset is finitely generated, then the clone of any of its retracts is finitely generated. We are not able to answer even the simpler question: is it true that if T or H is a retract of a finite poset P, then $\mathcal{C}(P)$ is non-finitely generated.

The aim of this chapter is to establish the non-finitely generated (or finitely generated) property for clones of posets in new classes of finite posets. We think that such results eventually may lead to a characterization of finite posets with non-finitely generated clones.

In Section 2.2 we exhibit an infinite family of finite (bounded) posets which are not series-parallel and have non-finitely generated clones. Hence we get to new examples of non-finitely generated maximal clones. Let A_n be the poset obtained from the Boolean lattice with n atoms by removing its greatest element, and B_n the dual of A_n. Let $C_{m,n} = A_m + \mathbf{2} + B_n$ (see Fig. 2.2). We shall prove that if $m, n \geq 2$, then $\mathcal{C}(C_{m,n})$ and $\mathcal{I}(C_{m,n})$ are non-finitely generated. An analogous proof shows that $\mathcal{C}(\mathbf{2}+B_n)$ and $\mathcal{I}(\mathbf{2}+B_n)$ where

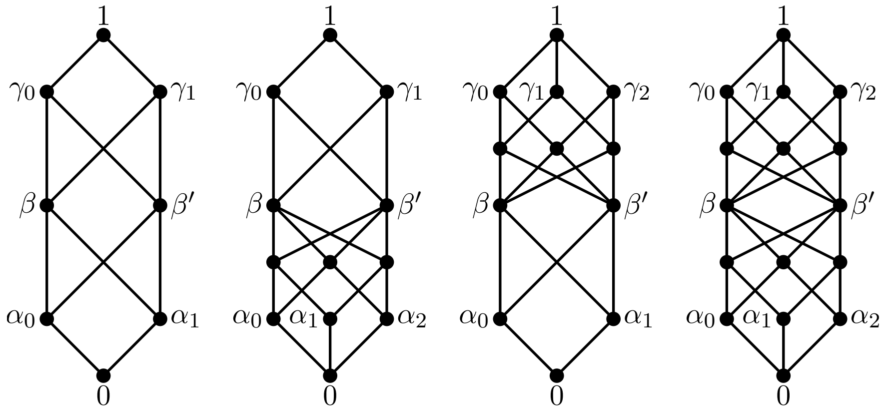


Figure 2.2: The posets $C_{2,2}$, $C_{3,2}$, $C_{2,3}$, and $C_{3,3}$

$n \geq 2$ are not finitely generated. We note that each of the posets $C_{m,n}$ where $m, n \geq 2$ retracts onto T, and each of the $\mathbf{2} + B_n$ where $n \geq 2$ retracts onto H.

For any integer $k \geq 2$, let C_k denote the $2k$-element crown. Let D_k denote the poset $\mathbf{1} + \mathbf{2} + C_k + \mathbf{2} + \mathbf{1}$. These posets were introduced by McKenzie in [11] under the name of *locked crowns*. To settle the finite generability question for $\mathcal{C}(D_k)$ when $k \geq 3$ seems difficult and needs essentially new ideas beyond the scope of the ones in Tardos's seminal paper [14]. The poset D_2 is series-parallel and hence, by [18], its clone is non-finitely generated. When $k \geq 3$, then D_k is not series-parallel and it is not known whether $\mathcal{C}(D_k)$ is finitely generated or not. Our investigations in this direction led to the results in Section 2.3.

We call an n-ary monotone operation f on a poset *ascending* if it is greater than or equal to some projection, that is there is an i such that $f(x_1, \ldots, x_n) \geq x_i$ for all $(x_1, \ldots, x_n)$. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of (ascending) idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. Another interesting consequence of our result is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations. Our result does not extend to half bounded finite posets: we prove that the clone of ascending idempotent operations of H is finitely generated but, as we mentioned above, the clone of H is not finitely generated.

Our investigations on the clone of D_k led us to seemingly simpler problems. Unfortunately, these problems turned out to be difficult ones, as well. For example, we are not able to decide whether the clone of ascending idempotent operations of $\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{1}$ is finitely generated. Per se, it also remains an open question whether the clone of D_k, $k \geq 3$, is finitely generated.

2.2 Classes of Finite Posets with Non-Finitely Generated Clones

In this section we shall prove that the clones and the idempotent clones of the posets $C_{m,n}$, $A_n + \mathbf{2}$ and $\mathbf{2} + B_n$ where $m, n \geq 2$ are not finitely generated. We require some basic definitions to proceed.

For two posets O and P, the partial mappings $f : O \rightharpoonup P$ are called *P-colorings of O*. If f is a P-coloring of a poset O, then we call the pair (O, f) a *P-colored poset*. The P-colored poset (O, f) is called *P-extendible* if there exists a fully defined monotone extension of f to O. We say that a poset O' is *contained* in an other poset O if the ordering relation of O' is contained in the ordering relation of O. A P-colored poset (O, f) is called a *P-obstruction* if (O, f) is not extendible, but for all posets O' properly contained in O, $(O', f|_{O'})$ is extendible. An *obstruction is trivial* if it has two elements or, equivalently, has no non-colored elements. We note that if O is connected, then in the preceding definition it suffices to take those O' that are obtained from O by deleting a single covering edge. Clearly, every finite non-extendible colored poset contains an obstruction. Later throughout the text, we frequently use the notation $P \setminus S$ where P is a poset and either S is a subset of elements of P or S only contains a covering edge of P. In these cases, $P \setminus S$ denotes the poset remaining from P after removing the elements of S and all edges incident with the elements in S, or removing the covering edge of S from P, respectively.

First we describe the B_n-obstructions. By Proposition 1.12 and Theorem 2.2 in [19] each non-trivial B_n-obstruction consists of a single non-colored element that is covered by the colored elements of the obstruction. By taking into account the definition of obstruction we have the following.

Theorem 2.1. *Every non-trivial B_n-obstruction consists of a single non-colored element that is covered by the colored elements of the obstruction. The colors of the colored elements form an antichain in B_n such that their intersection does not exist in B_n and the intersection of all but any one of them does exist in B_n.*

Observe that the number of colored elements of a non-trivial B_n-obstruction is at most n, and if the set of colors of a B_n-obstruction is contained in the set of coatoms of B_n, then it is equal to it. It also follows that the set of colors of any B_n-obstruction with n-colored elements is equal to the set of coatoms of B_n. We need the following result, see Theorem 3.3 in [19].

Theorem 2.2. *Let P be a finite poset and B a poset whose obstructions have at most one non-colored element. Let $P' = P + B$. Then every non-trivial P'-obstruction is in one of the following form:*

(i) a P-obstruction in which every maximal element is colored,

(ii) a B-obstruction in which every minimal element is colored, or

(iii) it is obtained from a P-obstruction (O, f) such that to each non-colored maximal element of (O, f) we glue a B-obstruction with a non-colored minimal element at its minimal element, possibly identifying some colored maximal elements of the same color after the gluing.

We note that part (i) is a special case of part (iii), when the P-obstruction (O, f) has only colored maximal elements. For a more interesting use of part (iii) we provided an example in Figure 2.3.

We remark that the obstructions of the two element antichain $\{\beta, \beta'\}$ are the colored fences whose only colored elements are their two endpoints colored by β and β', respectively.

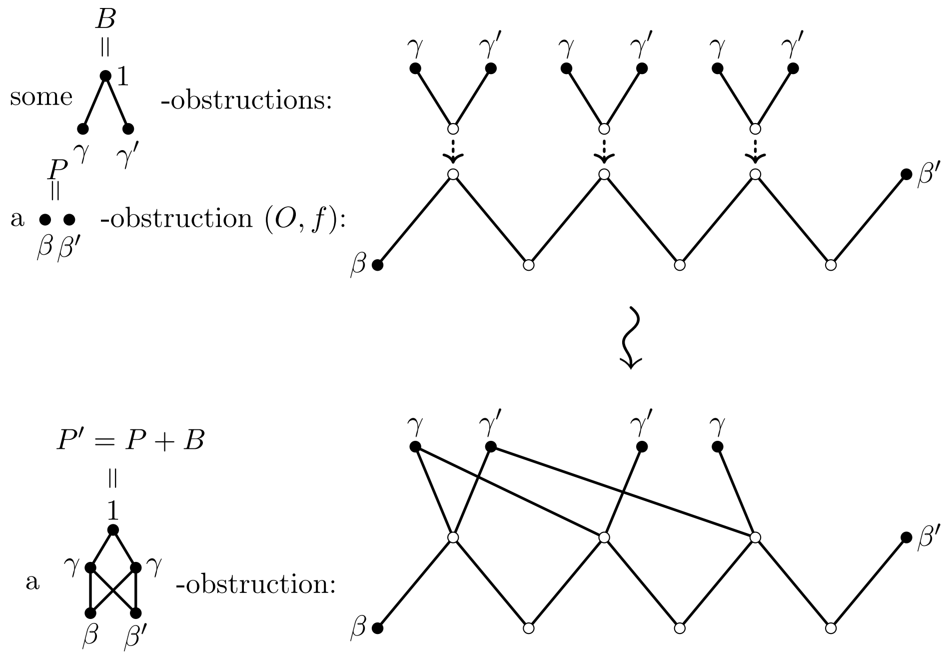


Figure 2.3: An example of construction (iii) in Theorem 2.2

By this remark, the preceding two theorems and their dual, we obtain a description of the $C_{m,n}$-obstructions. From now on, we refer to the members of the two element antichain in the definition (in the middle) of $C_{m,n}$ as β and β'.

Corollary 2.3. *Every non-trivial $C_{m,n}$-obstruction is obtained from a colored fence (O, f) whose endpoints are colored by β and β' such that to each non-colored maximal element of (O, f) we glue a non-trivial B_n-obstruction and to each non-colored minimal element of (O, f) we glue a non-trivial A_m-obstruction, possibly identifying some colored maximal elements of the same color and some colored minimal elements of the same color after the gluing.*

Now, we are set to prove the main theorem of the section. Our proof is analogous to that of Tardos, hence we advise the reader to consult Tardos's original paper [14] before getting into the proof of our theorem.

Theorem 2.4. *If $m,\ n \geq 2$, then the clone of $C_{m,n}$ and the idempotent clone $C_{m,n}$ are non-finitely generated.*

Proof. First we prove that the clone of $C_{m,n}$ is not finitely generated. For every $k \geq 4$ we shall define a relation R such that all $[k/2]$-ary monotone operations of $C_{m,n}$ preserve R but there is a monotone operation of $C_{m,n}$ that does not preserve R. Then, clearly, for every $k \geq 4$, $\mathcal{C}(C_{m,n})$ is not generated by the $[k/2]$-ary operations. Thus, $\mathcal{C}(C_{m,n})$ is not finitely generated.

The relation R is defined by the help of the poset Q in Figure 2.4. For every $k \geq 4$, poset Q consists of

(i) the fence $y, w_1, w_2, \ldots, w_{2k-1}, y'$,

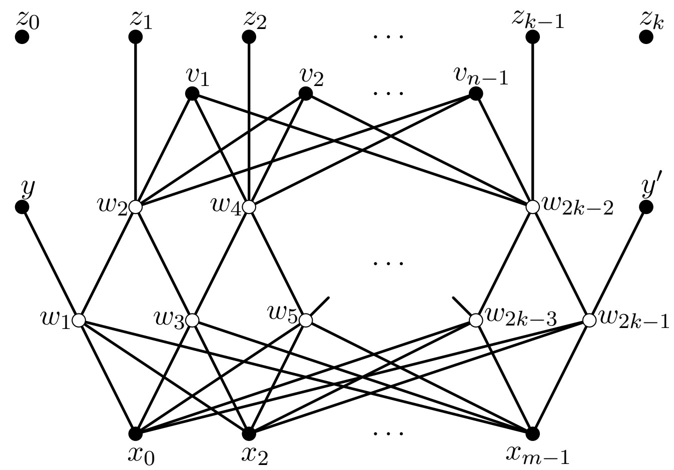


Figure 2.4: Poset Q

(ii) the minimal elements $x_0, x_2, \ldots, x_{m-1}$ that are all lower covers of the minimal elements $w_1, w_3, \ldots, w_{2k-1}$ of the fence,

(iii) the maximal elements $v_1, v_2, \ldots, v_{n-1}$ that all cover the maximal elements $w_2, w_4, \ldots, w_{2k-2}$ of the fence,

(iv) the maximal elements $z_1, z_2, \ldots, z_{k-1}$ such that z_i uniquely covers w_{2i} for $1 \leq i \leq k-1$, and

(v) two isolated elements z_0 and z_k.

Suppose f is a partial map from Q to $C_{m,n}$ whose domain is the set of extremal elements of Q. For every $0 \leq j \leq k$ we set $f_j(z_i) = f(z_{i+j})$ for all $0 \leq i \leq k$ where the indices are meant modulo $k+1$, and $f_j(x) = f(x)$ where x is extremal and $x \neq z_0, \ldots, z_k$.

Now, we define R_i to be the $(m+n+k+2)$-ary relation that consists of those partially defined maps f on Q whose domains are the set of extremal elements of Q, $(Q \setminus \{e\}, f_j)$ is extendible for every $0 \leq j \leq k$ and covering edge e of Q, and (Q, f_i) is extendible. We note that the R_i are preserved by the monotone operations of $C_{m,n}$. Let $R = \cup_{i=0}^{k} R_i$. We conceive each element $f \in R$ as an $(m+n+k+2)$-tuple (a column vector) of the form

$$(f(x_0), \ldots, f(x_{m-1}), f(y), f(y'), f(z_0), \ldots, f(z_k), f(v_1), \ldots, f(v_{n-1})). \tag{2.1}$$

First, we prove that the $[k/2]$-ary operations of $C_{m,n}$ preserve R. This follows from the fact that for any $[k/2]$ elements in R there is an i such that R_i contains all of these elements. To prove this we show that any element f of R is contained by $k-1$ of the R_i. Suppose that f is in R but not in any of R_{i_0}, R_{i_1} and R_{i_2} where i_0, i_1 and i_2 are pairwise different indices. This implies that $(Q \setminus \{z_0, z_k\}, f_{i_0})$ is an obstruction. Hence - by the use of Corollary 2.3, the second remark after Theorem 2.1 and its dual - up to a symmetry of $C_{m,n}$

$$f_{i_0}(x_0) = \alpha_0, \ldots, f_{i_0}(x_{m-1}) = \alpha_{m-1},\ f_{i_0}(y) = \beta, f_{i_0}(y') = \beta',$$

$$f_{i_0}(z_1) = \cdots = f_{i_0}(z_{k-1}) = \gamma_0,\ f_{i_0}(v_1) = \gamma_1, \ldots, f_i(v_{n-1}) = \gamma_{n-1}$$

where the α_j are the atoms of A_m, $\{\beta, \beta'\} = \mathbf{2}$ is the two element antichain in the middle of $C_{m,n}$, and the γ_l are the coatoms of B_n. We similarly have

$$f_{i_1}(z_1) = \cdots = f_{i_1}(z_{k-1}) = \gamma_0 \text{ and } f_{i_2}(z_1) = \cdots = f_{i_2}(z_{k-1}) = \gamma_0.$$

So by the definition of the f_i, $f_{i_0}(z_0) = f_{i_0}(z_k) = \gamma_0$ also holds. Hence f is not in any of the R_i, which contradicts $f \in R$. Thus, f is contained by $k-1$ of the R_i. Therefore, for any choice of $[k/2]$ elements in R there exists a j such that R_j contains them. Hence, any $[k/2]$-ary monotone operation of $C_{m,n}$ preserves R.

$$
\begin{array}{c|cccccc|cccccc|c}
 & \alpha_0 & \alpha_0 & & \cdots & & & & & & \cdots & & \alpha_0 & \alpha_0 \\
 & \vdots & \vdots & & \ddots & & & & & & \ddots & & \vdots & \vdots \\
m+2 \text{ rows} & \alpha_{m-1} & \alpha_{m-1} & & \cdots & & & & & & \cdots & & \alpha_{m-1} & \alpha_{m-1} \\
 & \beta & \beta & & \cdots & \beta & \beta & 1 & 1 & & \cdots & & 1 & \beta \\
 & \beta' & \beta' & & \cdots & \beta' & \beta' & 1 & 1 & & \cdots & & 1 & \beta' \\
\hline
 & 1 & \gamma_0 & \gamma_0 & \cdots & \gamma_0 & \gamma_0 & 1 & \beta & \gamma_0 & \cdots & \gamma_0 & \beta' & \gamma_0 \\
 & \gamma_0 & 1 & \gamma_0 & \cdots & \gamma_0 & \gamma_0 & \beta' & 1 & \beta & \cdots & \gamma_0 & \gamma_0 & \gamma_0 \\
 & \gamma_0 & \gamma_0 & 1 & \cdots & \gamma_0 & \gamma_0 & \gamma_0 & \beta' & 1 & \cdots & \gamma_0 & \gamma_0 & \gamma_0 \\
k+1 \text{ rows} & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & \gamma_0 & \gamma_0 & \gamma_0 & \cdots & 1 & \gamma_0 & \gamma_0 & \gamma_0 & \gamma_0 & \cdots & 1 & \beta & \gamma_0 \\
 & \gamma_0 & \gamma_0 & \gamma_0 & \cdots & \gamma_0 & 1 & \beta & \gamma_0 & \gamma_0 & \cdots & \beta' & 1 & \gamma_0 \\
\hline
 & \gamma_1 & \gamma_1 & & \cdots & & & & & & \cdots & & \gamma_1 & \gamma_1 \\
n-1 \text{ rows} & \gamma_2 & \gamma_2 & & \cdots & & & & & & \cdots & & \gamma_2 & \gamma_2 \\
 & \vdots & \vdots & & \ddots & & & & & & \ddots & & \vdots & \vdots \\
 & \gamma_{n-1} & \gamma_{n-1} & & \cdots & & & & & & \cdots & & \gamma_{n-1} & \gamma_{n-1}
\end{array}
$$

Figure 2.5: The matrix defining g

Let g be the partial function from $C_{m,n}^{2(k+1)}$ to $C_{m,n}$ defined by the $(k+m+n+2)\times(2k+3)$-matrix in Figure 2.5 such that for each row g assigns the $(2k+3)$-th component to the $2(k+1)$-tuple determined by the first $2(k+1)$ components of the row. As we mentioned earlier, we conceive each element $f \in R$ as a column vector of the form (2.1). Notice then that the first $2(k+1)$ columns of the matrix in Figure 2.5 are in R, and the last column is not in R. We shall prove that the colored poset $(C_{m,n}^{2(k+1)}, g)$ is extendible. Then any extension of g is a monotone $2(k+1)$-ary operation of $C_{m,n}$ that does not preserve R, which concludes the proof of the first part of the theorem.

So it remains to prove that $(C_{m,n}^{2(k+1)}, g)$ is extendible. Suppose that $(C_{m,n}^{2(k+1)}, g)$ is not extendible. Then it contains an obstruction (O, g'). We invoke Corollary 2.3, the first remark after Theorem 2.1 and its dual. Since g is monotone on its domain, (O, g') is obtained by adding some suitable colored elements to a colored fence whose endpoints are colored by β and β', respectively. In particular, the endpoints colored by β and β' are maximal in O, for otherwise one of these elements would be below an element colored by γ_0, which is impossible by the definition of g. As the set of colors of (O, g') is determined by g, each minimal non-colored element of the fence has a lower cover colored by α_i for all $0 \leq i \leq m-1$ and each maximal non-colored element of the fence has an upper cover colored by a γ_j for all $0 \leq j \leq n-1$. Observe that all rows with a last component γ_0

from the matrix occur in (O, g') as γ_0-colored elements. Indeed, if the l-th one of them was missing, then the l-th projection of O would be an extension of g'. Let a_i, $1 \leq i \leq t$, be the sequence of γ_0-colored elements in (O, g') where a_i covers the i-th maximal non-colored element in the fence of non-colored elements of (O, g'). Let (a_j, γ_0) the row of the matrix that occurs last in the sequence (a_i, γ_0) $1 \leq i \leq t$. Say, (a_j, γ_0) is the s-th row of the matrix. Then the $s-1$-th and the $s+1$-th rows of the matrix occur preceding (a_j, γ_0) in the sequence (a_i, γ_0), $1 \leq i \leq t$. Hence there is a subsequence of consecutive elements of (a_i, γ_0), $1 \leq i \leq t$ such that none of the $s-1$-th, s-th and $s+1$-th rows occur in it except the first and the last members that coincide with the $s-1$-th and $s+1$-th rows in some order. Here the indices $s-1$, s and $s+1$ are considered modulo $k+1$. Then, the colored poset whose base poset is O and whose coloring is the restriction of the $(s+k+1)$-th projection to the colored elements of O is a non-extendible colored poset, a contradiction. Thus we have proved that the clone of $C_{m,n}$ is non-finitely generated.

In order to prove that the idempotent clone of $C_{m,n}$ is not finitely generated it suffices to prove that the partial function g given by the matrix in Figure 2.5 has a totally defined idempotent monotone extension. First, we extend g by adding the constant γ_0 row to the matrix to obtain a new partial function. The same proof as in the preceding paragraph gives that the partial function defined in this way is extendible. Let $\hat{g}$ be any monotone extension of it onto $C_{m,n}^{2(k+1)}$. We claim that the restriction of $\hat{g}$ onto the diagonal tuples must be a projection. The map $\hat{g}$ restricted to the diagonal elements where g is defined and to the constant γ_0-tuple is clearly a projection. The value of $\hat{g}$ on the constant β-tuple must equal β by the definition of g and by the monotonicity of $\hat{g}$. Similarly, on the constant β'-tuple the value of $\hat{g}$ is β'. Then the values of $\hat{g}$ are uniquely determined on the remaining diagonal elements by the monotonicity of $\hat{g}$. Hence, $\hat{g}$ is a projection restricted to the diagonal, so $\hat{g}$ is an idempotent extension of g. □

Let Q_2 denote the poset Q from the preceding proof for the parameters $m = n = 2$. We note that Q_2 is the poset used by Tardos in his original proof. By using Q_2 instead of Q for defining R for any $m, n \geq 2$, a similar but a bit simpler proof can be given to prove that the clone of $C_{m,n}$ is non-finitely generated. We have opted for the present proof, since it easily carries over to prove that the idempotent clone of $C_{m,n}$ is non-finitely generated and, in particular, to prove that the idempotent clone of $\mathbf{2} + B_n$ is non-finitely generated.

Let Q' be the poset obtained from poset Q in the preceding proof by deleting the minimal elements $x_0, \ldots, x_{m-1}$. Then Q' is used to get a proof of the following theorem. The proof follows mutatis mutandis of the preceding proof, hence we omit it.

Theorem 2.5. *If $n \geq 2$, then the clones $\mathcal{C}(\mathbf{2}+B_n)$ and $\mathcal{I}(\mathbf{2}+B_n)$ are non-finitely generated.*

We note that a similar claim holds for the poset $A_n + \mathbf{2}$ if $n \geq 2$. We shall see by Corollary 2.8 in the next section that if the idempotent clone of a finite bounded poset is finitely generated, then its clone is also finitely generated. By this result, the first part of Theorem 2.4 implies its second part. We do not know a general result by which the second part of Theorem 2.5 follows from its first part.

2.3 The Clone of Ascending Idempotent Operations

Recall that a monotone operation of a poset is ascending if it is greater than or equal to some projection. Clearly, the ascending idempotent monotone operations form a subclone in the clone of a poset. In this section we prove a theorem that reduces the finite generability of the clone of a finite bounded poset to the finite generability of the clone of its ascending idempotent operations. We prove that a similar theorem does not hold for half bounded posets. Let D_k denote the poset $\mathbf{1}+\mathbf{2}+C_k+\mathbf{2}+\mathbf{1}$ where C_k is the $2k$-element crown. We sketch a possible way to prove that the clone of monotone ascending idempotent operations of D_k, $k \geq 3$, is non-finitely generated. To decide if $\mathcal{C}(D_k)$, $k \geq 3$, is finitely generated looks further away. An approach like the ones in Tardos's paper and in the proof of Theorem 2.4 does not seem to work since the shapes of the D_k-obstructions are too unwieldy due to the fact that the shapes of the C_k-obstructions are too unwieldy, cf. Theorem 2.2.

We call the clone of the ascending idempotent operations of a poset the *reduced idempotent clone* of the poset. The reduced idempotent clone of P is denoted by $\mathcal{I}_r(P)$. The following theorem gives indication how ascending idempotent operations play a role in the generability of the clone of a bounded poset.

Theorem 2.6. *The clone of a finite bounded poset is generated by its ascending idempotent operations and the unary constant operations* 0 *and* 1*.*

Proof. Let P be a finite bounded poset. It suffices to prove that for any monotone n-ary $f : P^n \to P$ there exists an ascending idempotent monotone $(n+2)$-ary f_I such that $f_I(0,1,x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$. We define f_I as follows:

$$f_I(y_1,y_2,x_1,x_2,\ldots,x_n) := \begin{cases} 1 \text{ if } y_1 \neq 0 \text{ and } y_2 = 1, \\ f(x_1,\ldots,x_n) \text{ if } y_1 = 0 \text{ and } y_2 = 1, \\ y_1 \text{ otherwise.} \end{cases} \tag{2.2}$$

Now it is clear that f_I is idempotent, monotone, moreover

$$f_I(0,1,x_1,\ldots,x_n) = f(x_1,\ldots,x_n) \text{ and } f_I(y_1,y_2,x_1,x_2,\ldots,x_n) \geq y_1.$$

□

The preceding theorem has the following corollaries.

Corollary 2.7. *If the reduced idempotent clone of a finite bounded poset is finitely generated, then its clone is also finitely generated.*

Corollary 2.8. *If the idempotent clone of a finite bounded poset is finitely generated, then its clone is also finitely generated.*

The first part of Theorem 2.4 and Corollary 2.7 immediately yield the following.

Corollary 2.9. *If $m, n \geq 2$, then the clone $\mathcal{I}_r(C_{m,n})$ is non-finitely generated.*

We also note that the first part of Theorem 2.4 and Corollary 2.8 implies the second part of Theorem 2.4.

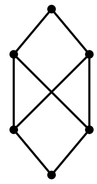


Figure 2.6: The poset $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$

We do not know if the converse of Corollary 2.7 is true. The poset $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is a candidate for a counterexample. It is well known that $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ admits a 5-ary near unanimity operation, so its clone and idempotent clone are finitely generated. On the other hand, a near unanimity operation on a poset of more than one elements is never ascending. So if the reduced idempotent clone of $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is yet finitely generated, the usual near unanimity argument does not work to prove it. Nevertheless, we are able to prove for a finite bounded poset P that $\mathcal{C}(P)$ is finitely generated if and only if an appropriate subclone of $\mathcal{I}_r(P)$ is finitely generated. For a finite bounded poset P, let $\mathcal{D}(P)$ denote the clone generated by the ascending idempotent operations defined in the proof of Theorem 2.6.

Corollary 2.10. *For a finite bounded poset P, $\mathcal{C}(P)$ is finitely generated if and only if $\mathcal{D}(P)$ is finitely generated.*

Proof. If $\mathcal{D}(P)$ is finitely generated, then $\mathcal{C}(P)$ is finitely generated by the proof of Theorem 2.6. For the converse suppose that $\mathcal{C}(P)$ has a finite generating set and is generated by the operations $f^1,\dots,f^k$. Let $f_I^1,\dots,f_I^k$ be the corresponding ascending idempotent operations defined in the proof of Theorem 2.6.

Now we prove that for any monotone operation g, g_I is a composition of $f_I^1,\dots,f_I^k$, hence $\mathcal{D}(P)$ is generated by $f_I^1,\dots,f_I^k$. The operation g is a composition of the operations $f^1 = f_I^1(0,1,\dots),\dots,f^k = f_I^1(0,1,\dots)$ where the $\dots$ within the parentheses stands for a suitable number of variables. By replacing 0 with the variable y_1 and 1 with the variable y_2 in this composition, we get to a composition g' of $f_I^1,\dots,f_I^k$. By the definition in (2.2), it is now easy to check that $g' = g_I$. □

Another interesting corollary of Theorem 2.6 is as follows.

Corollary 2.11. *If the clone of a finite bounded poset is finitely generated, then it is generated by three elements: an ascending idempotent operation and the constant operations 0 and 1.*

Proof. Let P be a finite bounded poset such that $\mathcal{C}(P)$ is generated by the operations $f^1,\dots,f^k$. Then let $f_I^1,\dots,f_I^k$ be the corresponding ascending idempotent operations defined in the proof of Theorem 2.6. Then $f_I^1,\dots,f_I^k$ and the 0 and 1 constant operations generate $\mathcal{C}(P)$. Finally, in this generating set we replace $f_I^1,\dots,f_I^k$ by a composition f of them such that $f_I^1,\dots,f_I^k$ are obtained from f by identifying variables. Such an f is defined by replacing two members - say, an m-ary s and an n-ary t - in the sequence $f_I^1,\dots,f_I^k$ by the operation $s(t(x_1,\dots,x_n),\dots,t(x_{(m-1)n+1},\dots,x_{mn}))$ and by iterating this process until we get to a one element sequence of operations. □

It looks as an interesting and non-trivial problem to give some tractable characterization of the finite posets P such that the idempotent operations of P and the constant operations of P together generate the clone of P. A finite connected poset P with this property must satisfy the *fixed point property*, that is, every monotone unary operation on P has a fixed point. In this respect, we note that H is a finite connected poset that has the fixed point property, but it is not hard to prove that the idempotent operations and constant operations of H do not generate the clone of H. Our next theorem yields the weaker consequence that the ascending idempotent operations and the constant operations of H do not generate the clone of H.

We shall prove that the reduced idempotent clone of H is finitely generated. On the other hand, by Theorem 2.5 the clone of H is not finitely generated. This shows that Corollary 2.7 does not extend to the class of the half bounded posets. Just to compare, the idempotent clone of H is not finitely generated, also by Theorem 2.5.

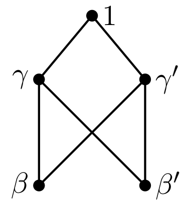


Figure 2.7: Poset H with labeling

Theorem 2.12. *The reduced idempotent clone of H is finitely generated.*

This theorem is an immediate consequence of the next two lemmas. We are going to prove that any idempotent operation that is greater than or equal to the first projection is a composition of 4-ary operations of such a type. The whole argument works for the other operations of the reduced idempotent clone analogously.

Let Ir_1 denote the set of the operations in $\mathcal{I}_r(H)$ that are greater than or equal to the first projection π_1, and let $Ir_{1,n}$ be the n-ary part of Ir_1. Next we define some basic operations in Ir_1. Our proof is based on the observation that all members of Ir_1 are built as compositions from these operations.

We say that $f \in Ir_{1,n}$ *jumps* to q at $x \in H^n$ if $\pi_1(x) < f(x) = q$. We define the smallest operations in $Ir_{1,n}$ that jump to a certain value at a certain element. For $a \in H$, let $\underline{a}$ denote the m-tuple, each of whose components equals a, where m will be clear from the context throughout. Let $z = (z_1, \dots, z_n)$ be an arbitrary element of H^n.

For any z with $z_1 < 1$, $z \not\leq \underline{\gamma}$ and $z \not\leq \underline{\gamma'}$ we define

$$g_1^z(x) := \begin{cases} 1 \text{ if } z \leq x, \\ \pi_1(x) \text{ otherwise.} \end{cases}$$

For any z with $z_1 < \gamma$ and $z \not\leq \underline{\gamma'}$ we define

$$g_\gamma^z(x) := \begin{cases} \gamma \text{ if } z \leq x \text{ and } \pi_1(x) = z_1, \\ 1 \text{ if } z \leq x \text{ and } \pi_1(x) = \gamma', \\ \pi_1(x) \text{ otherwise.} \end{cases}$$

The operation $g^z_{\gamma'}$ is defined analogously to g^z_γ. It is easy to see that $g^z_y \in Ir_{1,n}$ for every possible values of y and z. Notice that g^z_y is the smallest operation in $Ir_{1,n}$ that jumps to y at z.

We define a binary operation denoted by $\vee$ on H that is almost a compatible join semilattice operation:

$$x \vee y = \begin{cases} x \text{ if } \{x, y\} = \{\beta, \beta'\}, \\ \text{the least upper bound of } x \text{ and } y \text{ otherwise.} \end{cases}$$

Obviously, $\vee \in Ir_{1,2}$. Moreover, $\vee$ is associative, not commutative, though.

Lemma 2.13. *For any $f \in Ir_{1,n}$ and $x \in H^n$ we have*

$$f(x) = \bigvee \{ g^z_y(x) : \quad f \text{ jumps to } y \text{ at } z \; \}$$

where the order of joinands on the right hand side is chosen arbitrarily.

Proof. On one hand for each $x \in H^n$ if $y = f(z) > \pi_1(z)$, then $g^z_y(x)$ takes on a value between $\pi_1(x)$ and $f(x)$. On the other hand, for each x where f jumps $g^x_{f(x)}(x) = f(x)$, so the join on the right hand side of the equality in the claim equals $f(x)$. If f does not jump at x, then $g^z_y(x) = \pi_1(x)$ for all of the g^z_y on the right hand side, and so the join equals $\pi_1(x)$. □

By Lemma 2.13, it suffices to exhibit a finite generating set for the operations g^z_y to finish our proof. The following lemma yields us a generating set of 4-ary operations. We note that the operations g^z_y are defined only under some stipulations for the values of the parameters y and z, see definition.

Lemma 2.14. *Let $n \geq 5$. Let $y \in H$ and $z = (z_1, \ldots, z_n) \in H^n$ such that the n-ary operation g^z_y is defined. Then there exist i, j and $k \neq i, j, 1$ such that for the 4-tuple $z' = (z_1, z_i, z_j, z_k)$ and the $(n-1)$-tuple $z'' = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$, the 4-ary operation $g^{z'}_y$ and the $(n-1)$-ary operation $g^{z''}_y$ are defined, and*

$$g^z_y(x) = g^{(z_1,y,y)}_y(x_1, g^{z'}_y(x'), g^{z''}_y(x''))$$

where $x' = (x_1, x_i, x_j, x_k) \in H^4$ and $x'' = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in H^{n-1}$.

Proof. First, we consider the case when $y = 1$. Then $z_1 < 1$, $z \not\leq \underline{\gamma}$ and $z \not\leq \underline{\gamma'}$. If $z_i = 1$ for some i, then let $j = i$ and choose k to be different from 1 and i. If for all i, $z_i \neq 1$, then there are two components of z such that one of them equals γ and the other does γ'. Then we choose i, j and k such that $z_i = \gamma$, $z_j = \gamma'$ and k is different from $1, i, j$. In both cases, we take z' and z'' as in the claim. Notice that for the tuples z' and z'', $g^{z'}_1$ and $g^{z''}_1$ are defined. Moreover,

$$z \leq x \text{ iff } (z' \leq x' \text{ and } z'' \leq x'').$$

Thus if $z \leq x$, then $g^{z'}_1(x') = 1$ and $g^{z''}_1(x') = 1$, hence

$$g^{(z_1,1,1)}_1(x_1, g^{z'}_1(x'), g^{z''}_1(x'')) = g^{(z_1,1,1)}_1(x_1, 1, 1) = 1 = g^z_1(x).$$

For the case when $z \not\leq x$, we may assume that $x_1 < 1$, since otherwise both sides of the equality in the claim equal 1. Now if, for example, $z' \not\leq x'$, then $g_1^{z'}(x') = x_1 < 1$. This yields

$$g_1^{(z_1,1,1)}(x_1, g_1^{z'}(x'), g_1^{z''}(x'')) = g_1^{(z_1,1,1)}(x_1, x_1, g_1^{z''}(x'')) = x_1 = g_1^{z}(x),$$

which concludes our proof for the case $y = 1$.

For the remaining part of the proof, we assume without loss of generality that $y = \gamma$. Then $z_1 < \gamma$ and $z \not\leq \underline{\gamma'}$. We may assume that $z_1 = \beta$. Now, there exists an i such that $z_i = 1$ or $z_i = \gamma$. We put $j = i$ and choose k different from 1 and i. We take z' and z'' as in the claim. Then $g_\gamma^{z'}$ and $g_\gamma^{z''}$ are defined, and

$$z \leq x \text{ iff } (z' \leq x' \text{ and } z'' \leq x'').$$

We split the rest of the proof in three cases.

In the first case we assume that $z \leq x$ and $x_1 = \beta$. Then we have that $g_\gamma^{z'}(x') = \gamma$ and $g_\gamma^{z''}(x'') = \gamma$, hence

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(\beta, \gamma, \gamma) = \gamma = g_\gamma^{z}(x).$$

In the second case we assume that $z \leq x$ and $x_1 = \gamma'$. Now we have that $g_\gamma^{z'}(x') = 1$ and $g_\gamma^{z''}(x'') = 1$, and hence

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(\gamma', 1, 1) = 1 = g_\gamma^{z}(x).$$

For the third case we assume that none of the conditions

$$(z \leq x \text{ and } \ x_1 = \beta) \text{ and } (z \leq x \text{ and } x_1 = \gamma')$$

hold. This implies that if $z \leq x$, then $x_1 = \gamma$ or $x_1 = 1$, and it is clear in both cases that both sides of the equality in the claim equal x_1. Hence we have to consider only $z \not\leq x$. Then, for example, $z'' \not\leq x''$ and $g_\gamma^{z''}(x'') = x_1$. This yields

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), x_1) = x_1 = g_\gamma^{z}(x),$$

which concludes the proof. □

Finally, we delineate some ideas on the question if $\mathcal{I}_r(D_k)$ is finitely generated. We proceed with a straightforward lemma on general clones. A *homomorphism from a clone C to a clone D* is a map that preserves the projections and commutes with composition of operations. A clone D is a *homomorphic image* of a clone C if there is an onto homomorphism from C to D.

Lemma 2.15. *If a clone is finitely generated, then its homomorphic images are also finitely generated.*

Let P be a finite poset. A subset U of P is called an *up-set* of P, if for any $a \in U$, $b \in P$ and $a \leq b$ we have $b \in U$. We note that every n-ary monotone ascending idempotent operation of an up-set U of P extends to an n-ary monotone ascending idempotent operation on P. Indeed, by taking an appropriate projection on $P^n \setminus U^n$ yields an extension. Moreover, any up-set U of P is preserved by all monotone ascending operations of P, hence $\mathcal{I}_r(U)$ is a homomorphic image of $\mathcal{I}_r(P)$ via the restriction homomorphism. So by Lemma 2.15 we get the following.

Corollary 2.16. *If the reduced idempotent clone of a finite poset P is finitely generated, then the reduced idempotent clone of any up-set of P is finitely generated.*

We mentioned above that we are not able to decide whether $\mathcal{I}_r(\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1})$ is finitely generated. By the preceding corollary - as $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is an up-set in D_k - a negative answer would yield that $\mathcal{I}_r(D_k)$ is non-finitely generated. We note that D_2 is series-parallel and T is a retract of it, and hence $\mathcal{C}(D_2)$ is non-finitely generated. So by Corollary 2.7, $\mathcal{I}_r(D_2)$ is non-finitely generated. Nevertheless, it remains open whether $\mathcal{I}_r(D_k)$ and $\mathcal{C}(D_k)$ are finitely generated if $k \geq 3$.

Summary

This thesis is about two problems both concerning partially ordered sets, shortly, *posets*. Though being connected by the type of their main objects of interest, i.e. posets, the two problems are unrelated.

The first problem, which fills up Chapter 1, is *first-order definability in substructure and embeddability orderings*. It is based on three papers of the author [7–9], where the research questions seem like logic: try to grasp the expressive power of a certain first-order language in a given structure. To get answers, we use basic, finite, combinatorial thinking, no more. What looms behind the problems though, is the symmetries of some particular, complicated, infinite posets. This research is, in fact, a continuation of a series of papers by Jaroslav Ježek and Ralph McKenzie [3–6], published in 2009-2010. Beyond the author of this thesis, others have picked up on this topic [12, 15–17].

Let us go into detail a little more. Let $\mathcal{D}$ be the set of (the isomorphism types of) finite directed graphs, shortly, digraphs. For two digraphs $G, G' \in \mathcal{D}$, let $G \leq G'$ denote that G is *embeddable* into G', that is we can get G from G' by leaving out vertices and edges. Equivalently, there exists an injective map from G to G' preserving the edges. An ostensibly similar notion follows. Let $G \sqsubseteq G'$ denote that G is a *substructure* of G', that is we can get G from G' by leaving out vertices only. Equivalently, there exists an injective map from G to G' preserving both edges and non-edges (i. e. the absence of edges). What we have so far is two partially ordered sets: $(\mathcal{D}; \leq)$ and $(\mathcal{D}; \sqsubseteq)$. In the first chapter of the thesis, we investigate the expressive power of the first-order language of partially ordered sets for these two particular posets (see Figs. 2.8 and 2.9).

Probably, the most natural question is elementwise definability. Can you identify every single element in either $(\mathcal{D}; \leq)$ or $(\mathcal{D}; \sqsubseteq)$ with a first-order formula in the language of posets? This is where symmetries, i. e. automorphisms, come into play. Say, in a poset P, the element p is taken by an automorphism to some different p'. Then, naturally, first-order formulas cannot distinguish p from p' as they share the exact same structural properties in P.

With regard to both the automorphisms and definability, $(\mathcal{D}; \leq)$ is a much easier nut to crack. Therefore, we start Chapter 1 with the embeddability ordering. The automorphism that sends G to its transpose G^T, that is just reversing all edges, is easy to discover. Consequently, the strongest we can prove, in terms of elementwise definability, is that the set $\{G, G^T\}$ is first-order definable for every digraph $G \in \mathcal{D}$. Indeed, this is proven in the thesis. Using this theorem, we can show that there is no other nontrivial automorphism, pointing to a strong, back-and-forth connection between the definability we investigate and the automorphisms. So far, what we have settled is the definability of finite subsets of

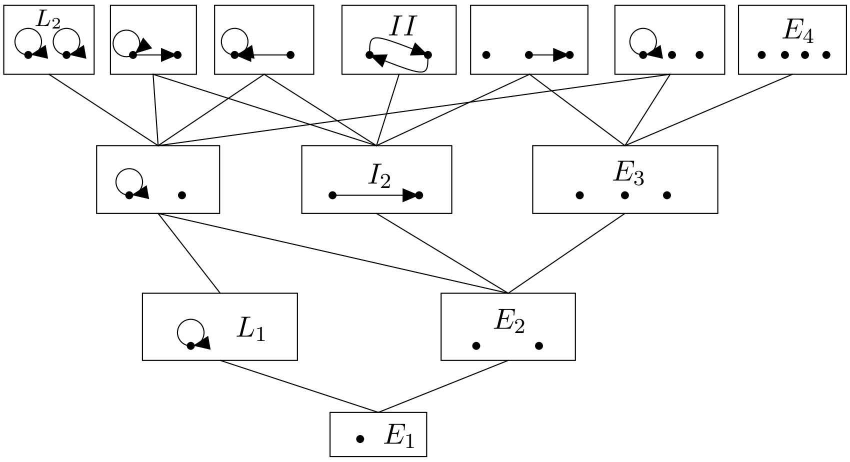


Figure 2.8: The initial segment of the Hasse diagram of the embeddability ordering, $(\mathcal{D}; \leq)$.

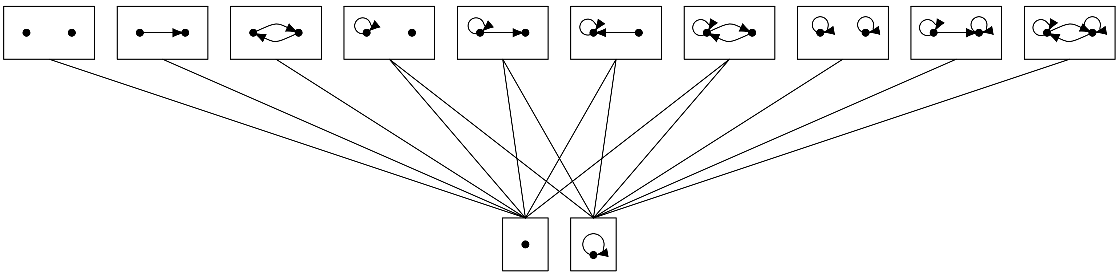


Figure 2.9: The initial segment of the Hasse diagram of the substructure ordering, $(\mathcal{D}; \sqsubseteq)$.

$(\mathcal{D}; \leq)$: a finite subset $S \subset \mathcal{D}$ is first-order definable if and only if $G \in S$ implies $G^T \in S$. Hence, to move forward, we must ask about infinite subsets.

As a famous statement in model theory reveals, there is no first-order formula defining the set of weakly connected digraphs in their own first-order language. Surprisingly, we do have such a formula in our language. What we show in fact is that with the addition of just a single constant, a digraph that is not isomorphic to its transpose, the whole second-order language of directed graph is expressible in our language. Technically, what we do is go on path laid by Ježek and McKenzie in [6]. We define a new language, say, $\mathcal{L}$. The language $\mathcal{L}$ is seemingly much stronger than the first-order language in question. Nonetheless we show that, in fact, it possesses the same expressive power. $\mathcal{L}$ belongs to a concrete small category, consisting of directed graphs as objects and maps between them as morphisms. To prove that $\mathcal{L}$ is indeed expressible with our language, we somehow 'model' the workings of this category in $(\mathcal{D}; \leq)$ using the first-or language of posets.

The second part of Chapter 1 examines the *substructure* ordering, $(\mathcal{D}; \sqsubseteq)$. Here, we are faced with something new right away. Unprecedented in the line of this topic, we find nontrivial automorphisms. Though we present a conjecture for the automorphism group, it is unproven at the moment. Our conjecture is that the automorphism group is isomorphic to a 768-element group, $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$, with a given α in the semidirect product. We have already seen that there is a strong connection between the expressive power of the first-order language of posets and their automorphism groups. Does this mean that the uncertain automorphism group blocks us from getting any definability result? Though this could very well be the case, fortunately, it is not. What we show is that with the addition of finitely many constants, the first-order language of $(\mathcal{D}; \sqsubseteq)$ can express that of $(\mathcal{D}; \leq)$. This statement carries weight only because, at this point, we've already established that the first-order language of $(\mathcal{D}; \leq)$ is very strong. Finding a minimal list of these constants is almost equivalent to determining the automorphism group. Hence such a minimal list is not provided. A possible, far-from-minimal list consists of the digraphs of at most 12 elements. As a corollary (to the definability statement), we get that the automorphism group is finite—the best we can prove as for now. It might seem odd that we do not 'know' what constant digraphs we use in our proof. This is because some of our arguments go the following way. Some properties of digraphs can be told by saying something about them *locally*. For example, one can test if a digraph has a non-loop edge by the set of its (at most) 2-element substructures. Far more complicated properties can be tested in this way. It would get overwhelmingly tedious to list all the digraphs that are used in this manner. And even if we did so, though we would get a much more concrete list, it would still be quite far from minimal. Therefore, analyzing this particular proof to get a minimal list seems hopeless (at least to the author).

Chapter 2 investigates a completely different problem, still having posets as main players in it. A set of finitary operations is called a clone if it contains all projections and is closed under superposition (composition). In this thesis, we always assume the base set of our operations to be finite. Clearly, the set of all operations (on a finite base set) is a clone. The largest (with respect to inclusion) clones that are smaller than this one are called maximal clones. Ivo G. Rosenberg, in a classical result [13], classified the maximal clones into six classes. For five of the six classes it has been shown that the clones of these classes are finitely generated. The unsettled class is the class of clones consisting of the monotone operations of bounded partial orders, that is posets having both least and largest elements. Some partial results have already been obtained. Monotone clones of at most seven element posets are proven to be finitely generated and so are posets with a monotone near unanimity operation. In a brilliant paper [14] from 1986, Gábor Tardos shows that the clone of a particular eight element poset is not finitely generated. This was the first proof showing a maximal clone to be not finitely generated. In a 1993 paper [18], László Zádori generalized Tardos's result by describing all series parallel posets having not finitely generated clones. Since Zádori, up until recently no one found non-finitely generated maximal clones, though one may conjecture that there are a lot of them. We present the recent paper [10] finding new such clones in Chapter 2. The author submerged in this topic as a PhD student guided by his second supervisor, the professor Zádori just mentioned. Miklós Maróti, the first supervisor of the author, also joined. The three of them wrote the paper [10] that comes up with a new family of finite bounded posets whose clones of monotone operations are not finitely generated and suggests some directions where, the authors think, this research might evolve in the future.

In the first part of the chapter, we present this new family of finite bounded posets whose clones of monotone operations are not finitely generated. Let $\mathbf{k}$ denote the k-element antichain. Let A_n be the poset obtained from the Boolean lattice with n atoms by removing its greatest element, and B_n the dual of A_n. Let $C_{m,n} = A_m + \mathbf{2} + B_n$.

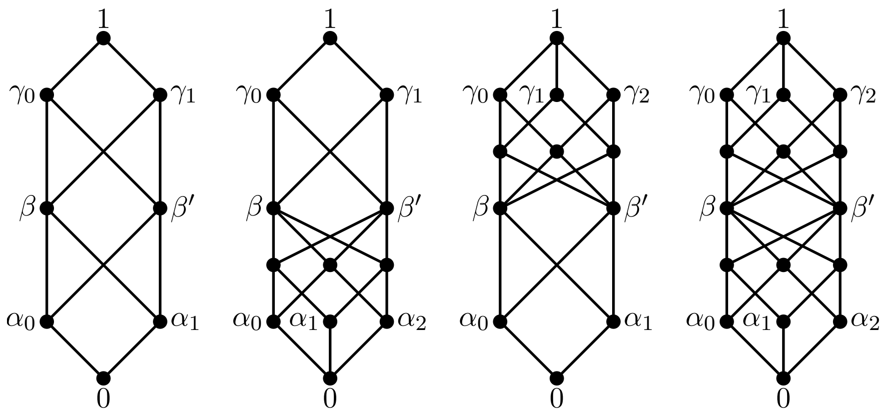


Figure 2.10: The posets $C_{2,2}$, $C_{3,2}$, $C_{2,3}$, and $C_{3,3}$

We prove that if $m, n \geq 2$, then the clones and idempotent clones of $C_{m,n}$ are non-finitely generated, where by the idempotent clone, as usual, we mean the clone of those monotone operations that satisfy the identity $f(x, \ldots, x) = x$. The proofs of these results are analogues of those in the famous paper of Tardos.

Another interesting family of finite posets from the finite generability point of view is the family of locked crowns. To decide whether the clone of a locked crown of at least six elements is finitely generated or not, one needs to go beyond the scope of Tardos's proof. Although our investigations are not complete in this direction, they led to the results in the second part of the chapter.

We call a monotone operation ascending if it is greater than or equal to some projection. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of ascending idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. We provide an example of a half bounded finite poset whose clone of ascending idempotent operations is finitely generated but whose clone is not finitely generated. Another interesting consequence of our result is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations.

Összefoglaló

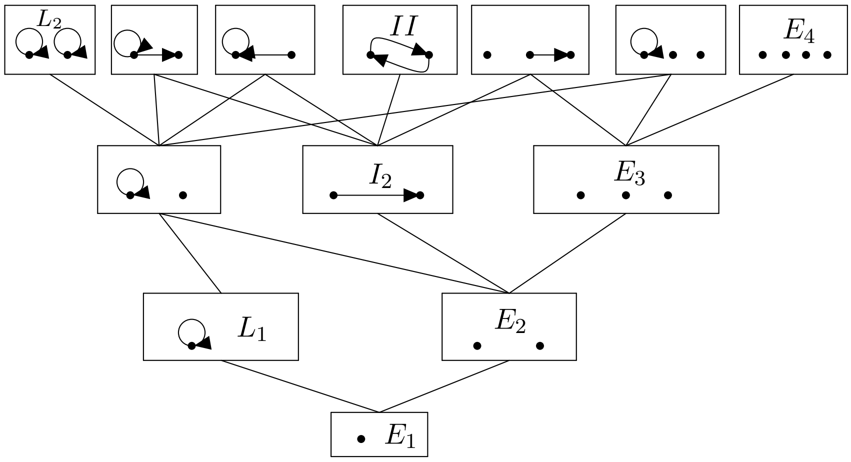
Az értekezés két részbenrendezett halmazokkal kapcsolatos problémával foglalkozik. Habár az érdeklődésük tárgyát képező struktúrák típusa (részbenrendezések) összeköti őket, a két probléma ezen túl nem kapcsolódik egymással.

Az első probléma, ami az első fejezetet tölti ki, *elsőrendű definiálhatóság részstruktúra- és beágyazás-részbenrendezésekben.* Az első fejezet a szerző három cikkén alapul [7–9], melyekben az alapkérdések logikának tűnnek: próbáljuk megfogni a kifejező erejét adott struktúrák elsőrendű nyelveinek. Hogy válaszokat kapjunk, nem használunk mást, mint alapvető kombinatorikus gondolkodást. A problémák mögött azonban ott húzódnak végtelen, bonyolult részbenrendezett halmazok szimmetriái. Ez a kutatás, valójában, Jaroslav Ježek és Ralph McKenzie egy 2009-2010-es cikksorozatának [3–6] folytatása. Az értekezés szerzőjén túl mások is felkapták ezt a témát [12, 15–17].

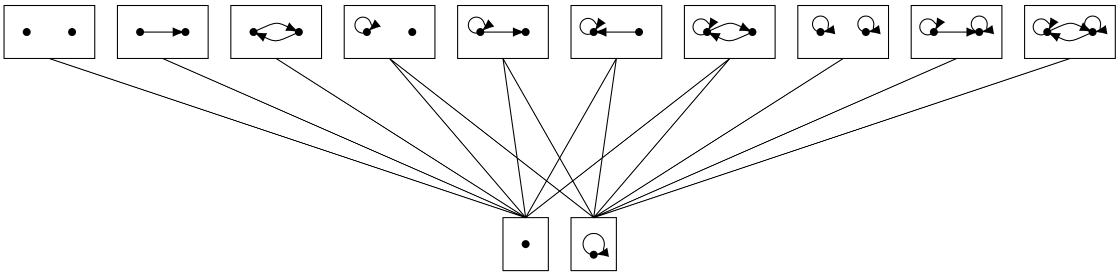
Menjünk bele a részletekbe egy kicsit. Jelölje $\mathcal{D}$ a véges irányított gráfok (izomorfiatípusainak) halmazát. Két irányított gráf, $G, G' \in \mathcal{D}$, esetén jelölje $G \leq G'$ azt, hogy G *beágyazható* G'-be, azaz G-t megkaphatjuk G'-ből csúcsok és élek elhagyásával. Más szóval, létezik egy $G \to G'$ injektív leképezés, amely megtartja az éleket. Egy látszólag hasonló fogalom következik. Jelölje $G \sqsubseteq G'$ azt, hogy G *részstruktúrája* G'-nek, azaz G-t megkaphatjuk G'-ből csak csúcsok elhagyásával. Más szóval, létezik egy $G \to G'$ injektív leképezés, amely megtartja az éleket és a „nem-éleket” (az élek hiányát) is. Eddig két részbenrendezésünk van: $(\mathcal{D}; \leq)$ és $(\mathcal{D}; \sqsubseteq)$ (ld. 2.11. és 2.12. ábrák). Az értekezés első fejezetében ezen részbenrendezések esetén vizsgáljuk a részbenrendezett halmazok elsőrendű nyelvének kifejező erejét.

Valószínűleg a legtermészetesebb kérdés az elemenkénti definiálhatóság kérdése. Tudjuk-e definiálni a $(\mathcal{D}; \leq)$ vagy $(\mathcal{D}; \sqsubseteq)$ részbenrendezések elemeit elsőrendű formulákkal (a részbenrendezett halmazok nyelvén)? Itt jönnek be a szimmetriák, más néven automorfizmusok. Tegyük fel, hogy, egy P részbenrendezett halmazban, a p elemet egy automorfizmus egy tőle különböző p'-be visz. Ekkor, természetesen, elsőrendű formulák nem tudják megkülönböztetni p-t és p'-t, hiszen ugyanazok a strukturális jellemzőik a P-n belül.

Az automorfizmusok és a definiálhatóság tekintetében is sokkal könnyebben kezelhető $(\mathcal{D}; \leq)$, így a beágyazás-részbenrendezéssel kezdjük az 1. fejezetet. Jelölje G^T a G transzponáltját, azt a gráfot, melyet G-ből az élek megfordításával kapunk. Az automorfizmust, ami G-t G^T-be képezi, könnyű felfedezni. Ennélfogva, a legerősebb állítás, amiben az elemenkénti definiálhatóság szempontjából reménykedhetünk az, hogy a $\{G, G^T\}$ halmaz minden G irányított gráf esetén definiálható. Ezt bizonyítjuk az értekezésben. Ezt a tételt használva, bebizonyítjuk, hogy nincs más nemtriviális automorfizmus, rámutatva az erős oda-vissza kapcsolatra a definiálhatóság és az automorfizmusok között. Eddig odáig jutottunk, hogy



2.11. ábra. A $(\mathcal{D}; \leq)$ részbenrendezés aljának Hasse-diagramja.



2.12. ábra. A $(\mathcal{D}; \sqsubseteq)$ részbenrendezés Hasse-diagramjának alja.

$(\mathcal{D}; \leq)$ véges részhalmazainak definiálhatóságát lezártuk: Egy véges $S \subset \mathcal{D}$ részhalmaz akkor és csak akkor elsőrendű-definiálható, ha minden $G \in S$ esetén $G^T \in S$ is teljesül. Tehát ahhoz, hogy tovább menjünk, végtelen részhalmazok definiálhatóságát kell vizsgálnunk.

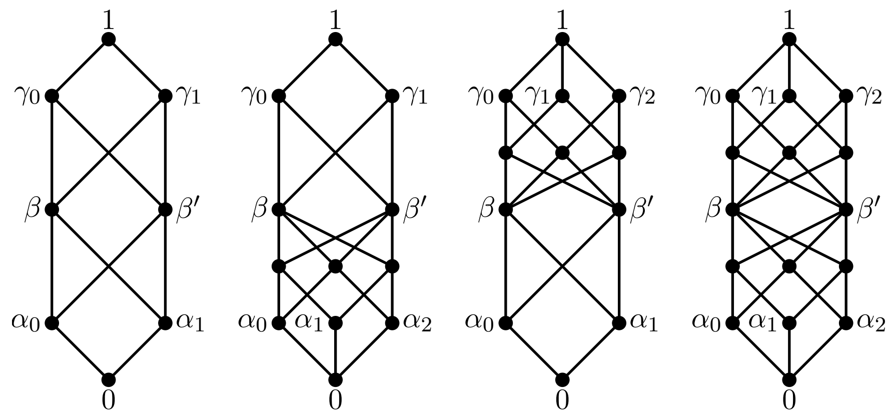
Ahogy egy híres modellelméleti állítás mutatja, nincs olyan elsőrendű formula az irányított gráfok nyelvén, mely definiálná a gyengén-összefüggő irányított gráfok halmazát. Meglepő módon, az általunk vizsgált elsőrendű nyelven van ilyen formula. Még azt is megmutatjuk, hogy egy konstans (egy konkrét, transzponáltjával nem izomorf irányított gráf) hozzáadásával az irányított gráfok teljes másodrendű nyelve kifejezhető az általunk vizsgált nyelv segítségével. Egy olyan utat járunk be, melyet Ježek és McKenzie fektettek le a [6] dolgozatban. Egy új nyelvet definiálunk, jelölje most $\mathcal{L}$. Az $\mathcal{L}$ nyelv látszólag sokkal erősebbnek tűnik, mint az általunk vizsgált elsőrendű nyelv. Ennek ellenére megmutatjuk, hogy valójában ugyanolyan erős. $\mathcal{L}$ egy konkrét kis kategóriához tartozik, melynek objektumai irányított gráfok, morfizmusai pedig éltartó leképezések közöttük. Ahhoz, hogy megmutassuk, hogy $\mathcal{L}$ valóban kifejezhető az általunk vizsgált nyelv segítségével, valamilyen módon „modellezzük" ennek a kategóriának a működését a $(\mathcal{D}; \leq)$ struktúra elsőrendű nyelvének segítségével.

Az első fejezet második fele a $(\mathcal{D}; \sqsubseteq)$ *részstruktúra*-részbenrendezést vizsgálja. Itt rögtön valami újdonsággal találjuk szembe magunkat. Ennek a kutatási témának a történetében eddig példátlan módon, nemtriviális automorfizmusokat találunk. Habár prezentálunk egy sejtést az automorfizmuscsoportra, bizonyítani nem tudjuk azt. A sejtésünk az, hogy a $(\mathcal{D}; \sqsubseteq)$ részbenrendezés automorfizmuscsoportja a 768-elemű $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$ csoporttal izomorf, ahol α egy adott hatás.

Már láttuk, hogy erős kapcsolat van az általunk vizsgált elsőrendű nyelvek kifejező ereje és az automorfizmusok között. A kérdés, hogy a bizonytalan automorfizmuscsoport jelen esetben megakadályozza-e, hogy a definiálhatóságról állításokat bizonyítsunk. Bár könnyen lehetne, szerencsére nem ez a helyzet. Azt mutatjuk meg, hogy véges sok konstans hozzáadásával a $(\mathcal{D}; \sqsubseteq)$ elsőrendű nyelve ki tudja fejezni a $(\mathcal{D}; \leq)$ elsőrendű nyelvét. Ennek a tételnek csak azért van súlya, ezen a ponton, mert a $(\mathcal{D}; \leq)$ elsőrendű nyelvéről már korábbról tudjuk, hogy nagyon erős. Ehhez a tételhez megtalálni konstansoknak egy minimális listáját majdnem ekvivalensnek tűnik az automorfizmuscsoport meghatározásával. Ennélfogva, nem tudunk egy ilyen minimális listát adni. Egy lehetséges, de minimálistól távol álló lista állhat a legfeljebb 12 elemű irányított gráfokból. A definiálhatósági eredmény következményeképpen azt kapjuk, hogy az automorfizmuscsoport véges — ez a legjobb, amit jelen pillanatban bizonyítani tudunk. Furcsának tűnhet, hogy nem „tudjuk”, hogy milyen konstansokat használunk a bizonyításunk során. Ez azért van, mert néhány esetben a következő gondolatmenetet használjuk. Bizonyos tulajdonságait az irányított gráfoknak jellemezni lehet *lokális* módon. Például meg tudjuk mondani tartalmaz-e az irányított gráf egy nem-hurok élet a 2-elemű részstruktúrái alapján. Sokkal bonyolultabb tulajdonságokat is el lehet így mondani. Nagyon fáradtságos és hosszú volna listázni azokat a konstansokat, melyeket ilyen módon használunk a bizonyításunk során. Még ha meg is tennénk, akkor sem kapnánk egy minimális listát (habár a listánk sokkal konkrétabb lenne). Emiatt, ennek a konkrét bizonyításnak az analizálásával reménytelennek tűnik egy minimális konstanslista megállapítása (legalábbis a szerző számára).

A második fejezet egy teljesen másik problémát vizsgál, melyben azért továbbra is részbenrendezett halmazok játsszák a főszerepet. Véges műveletek egy halmazát klónnak nevezzük, ha tartalmazza az összes projekciót és zárt a kompozícióra (függvényösszetétel). Az értékezésben mindig feltesszük, hogy a műveleteink alaphalmaza véges. Világos módon, az összes művelet (egy véges alaphalmazon) klón. Tartalmazásra nézve a legnagyobbakat, melyek kisebbek ennél, maximális klónoknak nevezzük. Ivo G. Rosenberg, egy klasszikus eredményben [13], klasszifikálta a maximális klónokat, hat osztályra bontva őket. Ezek közül öt esetén meg lett mutatva, hogy az ezekben lévő klónok végesen generáltak. A hatodik, kérdéses osztály a korlátos részbenrendezett halmazok monoton klónjainak osztálya. Azokat a részbenrendezéseket nevezzük korlátosnak, melyeknek van legnagyobb és legkisebb eleme. Néhány részeredmény már ismert. Ismert, hogy a legfeljebb 7-elemű részbenrendezések monoton klónjai végesen generáltak, továbbá azoké is, melyeknek van monoton többségi művelete. Egy zseniális dolgozatban [14], 1986-ban, Tardos Gábor megmutatta, hogy egy adott 8-elemű részbenrendezés klónja nem végesen generált. Ez volt az első alkalom, hogy egy maximális klónról kiderült, hogy nem végesen generált. Egy 1993-as dolgozatban Zádori László általánosította Tardos eredményét, karakterizálva azon soros párhuzamos részbenrendezett halmazokat, melyek klónja nem végesen generált. Azóta, egészen mostanáig, senki nem talált nem végesen generált maximális klónokat, annak ellenére, hogy azt sejthetjük, hogy sok ilyen van. A 2. fejezetben egy olyan friss dolgozatot [10] mutatunk be, mely új ilyen klónokat talál. Az értekezés szerzőjét, PhD hallgató korában, második témavezetője, a fent említett Zádori professzor vezette be témába. Maróti Miklós, az első témavezetője is csatlakozott a kutatáshoz. Hárman írták a fent említett [10] dolgozatot, mely korlátos részbenrendezések egy új családjáról mutatja meg, hogy nem végesen generált a klónjuk, továbbá néhány új irányt is javasol, amerre a szerzők szerint ezek a kutatások fejlődhetnek.

A fejezet első felében bemutatjuk ezt az új családot, mely tagjainak klónjai nem végesen generáltak. Jelölje $\mathbf{k}$ a k-elemű antiláncot. Legyen A_n az a részbenrendezés, melyet úgy kapunk az n-atomú Boole-hálóból, hogy elhagyjuk a legnagyobb elemét. Jelölje B_n az A_n duálisát. Legyen $C_{m,n} = A_m + \mathbf{2} + B_n$.



2.13. ábra. The posets $C_{2,2}$, $C_{3,2}$, $C_{2,3}$, and $C_{3,3}$

Bizonyítjuk, hogy ha $m, n \geq 2$, akkor $C_{m,n}$ klónja és idempotens klónja is nem végesen generált. Itt, a szokásos módon, idempotens klón alatt azon monoton műveletek klónját értjük, melyek teljesítik az $f(x, \dots, x) = x$ azonosságot. A tétel bizonyítása Tardos bizonyításának analogonja.

Egy másik érdekes családja a korlátos részbenrendezett halmazoknak a zárt koronák családja. Ezek esetén az akadályok leírása reménytelennek tűnik, így a Tardos-bizonyítás nem átvihető. Habár az ebbe az irányba tett vizsgálataink még hiányosak, a fejezet második felében ismertetünk néhány eredményünket.

Egy monoton műveletet felszállónak nevezünk, ha nagyobb vagy egyenlő valamelyik projekciónál. Bebizonyítjuk, hogy a korlátos részbenrendezések klónjait generálják bizonyos felszálló idempotens műveletek és a 0, 1 konstans műveletek. Következésképpen, ha egy korlátos részbenrendezett halmaz felszálló idempotens műveleteinek klónja végesen generált, akkor a klónja is végesen generált. Mutatunk egy példát egy félig-korlátos részbenrendezett halmazra, melynek felszálló idempotens klónja ugyan végesen generált, de a klónja nem az. Egy másik érdekes következménye az eredményeinknek, hogy ha egy véges korlátos részbenrendezésnek a klónja végesen generált, akkor három elem generálja, egy (konkrét) felszálló idempotens művelet, és a 0, 1 konstans műveletek.

Publications of the Author

Research papers covered in the thesis

- Á. Kunos, *Definability in the embeddability ordering of finite directed graphs*, Order **32**/1 (2015) 117-133
- Á. Kunos, *Definability in the Embeddability Ordering of Finite Directed Graphs, II*, Order **36**/2 (2019) 291-311
- Á. Kunos, M. Maróti, L. Zádori, *On Finite Generability of Clones of Finite Posets*, Order **36**/3, (2019) 653-666
- Á. Kunos, *Definability in the substructure ordering of finite directed graphs*, accepted at Order and already appeared online

Research papers outside the thesis

- G. Czédli and Á. Kunos, *Geometric constructibility of cyclic polygons and a limit theorem*, Acta Sci. Math. (Szeged) **81** (2015), 643-683
- G. Czédli, G. Gyenizse, Á. Kunos, *Symmetric embeddings of free lattices into each other*, Algebra Universalis **80**:11 (2019)
- Á. Kunos, M. Maróti, L. Zádori, *Critical relations of crowns in critical times of coronavirus depression*, accepted at Order

Other papers

- Á. Kunos, *Amikor érdemes általánosítani*, Polygon, XX. köt., 2. sz., 2012. máj., 33-48.
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