Global stability for the delayed logistic map

Outline of Ph.D. Thesis

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One of the most well-known nonlinear maps is the logistic map $[0,1] \ni x \mapsto ax(1-x) \in \mathbb{R}$ with parameter $a > 0$. In this thesis we are studying the global stability of the delayed version of the logistic map, more precisely we consider the delayed logistic difference equation

$$x_{n+1} = ax_n(1 - x_{n-d}),$$

or equivalently the $(d+1)$-dimensional map

$$F_d : \mathbb{R}^{d+1} \ni u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{d+1} \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ au_{d+1}(1-u_1) \end{pmatrix} \in \mathbb{R}^{d+1},$$

where $a > 0$ and $d \in \mathbb{N}$. The thesis is based on papers [1, 2] of the author, where we study the cases $d = 1$ and $d = 2$, respectively.

Despite the fact that we demonstrate our method only on a specific equation, i.e., on the delayed logistic map, we believe that it can be applied or extended to other similar maps, for instance the Ricker map (see [3]) or the Pielou map (see [4]). So the delayed logistic map can be considered also as a case study for the method developed in the thesis, since the whole argument could be repeated with slight and straightforward modifications to the aforementioned maps as well.

It is well known that for $a \in (0,1]$ the origin is the unique fixed point of F_d in $[0,1]^{d+1}$, which is locally stable and $\lim_{n\to\infty} F_d^n(u) = 0$ for every $u \in [0,1]^{d+1}$. For $a > 1$ a nontrivial fixed point $u_A = (A, A, \dots, A)$ with $A = 1 - \frac{1}{a}$ appears in $[0,1]^{d+1}$. There exists an $a_0 > 1$, depending on d such that this fixed point is locally asymptotically stable for $a \in (1, a_0)$, and unstable for $a > a_0$. At $a = a_0$ a Neimark–Sacker bifurcation takes place. We show that the following conjecture is true for $d \in \{1, 2\}$.

Conjecture. *The nontrivial fixed point u_A is locally stable and $\lim\limits_{n\to\infty} F_d^n(u) = u_A$ for $a \in (1, a_0]$ and $u \in S^d$, where*

$$S^d = \Big\{ u \in [0,1)^d \times (0,1) : a^k u_{d+1} \prod_{j=1}^{k} (1 - u_j) < 1, \\ k \in \{1, 2, \dots, d\} \Big\}.$$

Here, S^d contains exactly those $(x_1, x_2, \dots, x_{d+1}) \in \mathbb{R}_+^{d+1}$ for which $x_n > 0$ for every $n > d+1$. The conjecture can be formulated so that local stability implies global stability for the fixed point u_A. This is satisfied for several problems, see e.g., [5, 6, 4, 3, 7], but it is not true in general, see e.g., [8].

For smaller parameter values, more precisely for $a \in \left(1, \frac{d+2}{d+1}\right]$ with $d = 1$ and $d = 2$ we give purely analytical proofs of the conjecture. However, in the main part of the thesis, for larger a the proof of the global stability is a combination of analytical and rigorous computer-aided tools. First, we construct analytically an attracting neighborhood $\mathcal{M}$ of the nontrivial fixed point u_A, i.e., we show that $\lim_{n\to\infty} F_d^n(u) = u_A$ for every $u \in \mathcal{M}$. Then, by applying reliable numerical tools, it is shown that for every $u \in S^d$ the iterates $F_d^n(u)$ eventually enter $\mathcal{M}$, i.e., there exists an $n_0 = n_0(u)$ such that $F_d^{n_0}(u) \in \mathcal{M}$. Consequently, all points of S^d belong to the region of attraction of u_A. Here, reliable means that all possible numerical errors are controlled by using interval arithmetic techniques. Therefore, the computer-assisted part also provides mathematically rigorous statements.

As a first approach to construct an attracting neighborhood around u_A we use a standard linearization technique for parameter values further from a_0. However, the attracting neighborhood obtained via linearization shrinks to the fixed point as a tends to a_0. Therefore, for parameter values a close to a_0 this neighborhood is not big enough for computer use in the second part of the method and we need another approach to construct an attracting neighborhood for these parameter values.

For parameter values $a < a_0$ close to a_0 we use the normal form of the Neimark–Sacker bifurcation. More precisely, in case $d = 1$, with smooth and invertible maps we transform the map into the form

$$w \mapsto \lambda w + c_1 w^2 \overline{w} + R_2,$$

where c_1 is the Lyapunov-coefficient and $R_2 = O\left(|w|^4\right)$ denotes the higher-order terms. Then we show that there exists a $\rho_0 > 0$ such that

$$\left|\lambda w + c_1 w^2 \overline{w} + R_2\right| < |w|$$

for every $w \in \mathbb{C}$ with $0 < |w| \leq \rho_0$, which guaratees that B_{ρ_0} is inside the attracting neighborhood. Since we need the size of the constructed neighborhood $\mathcal{M}$ for computer use, it is not enough to determine only the lower-order terms during the normal form transformation, like we would do in a regular bifurcation analysis. These lower-order terms only assure the existence of such a sufficiently small neighborhood, whose size is not explicitly determined by them. Therefore, it is essential during the transformation to trace the higher-order terms and to estimate them as well as possible, in order to obtain a sufficiently big neighborhood $\mathcal{M}$.

For the case $d = 2$ our aim is to adapt the Neimark–Sacker bifurcational normal form technique. However, we need new ideas, since $F_3(u)$ is three-dimensional, and thus the adaptation of the method is not that straightforward. The novelty of this thesis is an explicit construction of a relatively large attracting neighborhood of the nontrivial fixed point of the three-dimensional logistic map by using center manifold techniques and the Neimark–Sacker bifurcational normal form.

To this end we carry out an approximate version of the center manifold reduction. We consider the fourth-order polynomial approximation $\phi(z)$ of the center manifold and the set

$$T(r, C) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le r, \ |y - \phi(z)| \le C|z|^5\}$$

around $y = \phi(z)$, where r and C are some positive constants. The appropriate shape of $T(r, C)$ assures the adaptability of the two-dimensional technique to higher dimension.

First, we investigate the y-directional dynamics in $T(r, C)$. Using the property that solutions close to the fixed point decay exponentially to the center manifold we show that $T(r, C)$ is conditionally invariant in direction y. After that the z-directional dynamics in $T(r, C)$ is investigated by using the Neimark–Sacker bifurcational normal form technique. Exploiting the special shape of $T(r, C)$ we can show that in an appropriate coordinate system the transformed z coordinate is strictly decreasing during the iteration, similar to the two-dimensional case. Finally, combining the y- and z-directional dynamics, we obtain that $T(r, C)$ is inside the region of attraction of the fixed point.

However, $T(r, C)$ is clearly not a proper neighborhood of the origin in $\mathbb{C} \times \mathbb{R}$. Therefore, we define the set

$$\tilde{T}(\hat{r}, K) = \{(z, y) \in \mathbb{C} \times \mathbb{R} : \ |z| \le \hat{r}, \ |\phi(z) - y| \le K\}$$

for some $\hat{r} > 0$ and $K > 0$. By using the exponential y-

directional attractivity of $T(r, C)$ we show that $\tilde{T}(\hat{r}, K)$ is also in the region of attraction of the fixed point. So this proper neighborhood can be used in the second part of the method.

Finally, we describe the computer-assisted part of our method for $d = 1$ and $d = 2$, respectively. We associate the delayed logistic map with a directed graph reflecting the behavior of the map up to a given resolution. More precisely, we cover S^d with finitely many $(d + 1)$-dimensional small cubes. Considering these cubes as vertices of a graph we introduce a directed graph, which, to a certain extent, describes the behavior of map F_d on these cubes. Therefore, we convert the issue of examining infinitely many points into a finite graph problem, which can be handled by computer. To construct the edges of this graph we use reliable numerical methods in order to handle the rounding errors of the computer. We show with the help of this graph that the iterates of every point from S^d enter the neighborhood constructed before, and the proof of Conjecture is completed for $d \in \{1, 2\}$.

As some final remarks we emphasize that the computational part is more and more compute-intensive and time-consuming as we get closer to the fixed point, so it is of crucial importance to construct with analytical tools a neighborhood which is relatively large. On the other hand the analytical part becomes more and more cumbersome as we aim to obtain higher precision during the estimations and there is an upper limit to analytically obtainable size of the attracting neighborhood which cannot be exceeded. Moreover, in higher dimension it is also essential to choose r and C (where the one can be enlarged only at the expense of the other) in $T(r, C)$ appropriately since they also have a great impact on the speed of the computer-aided part.

Note that the aforementioned Ricker and Pielou maps with delay $d = 2$ essentially differ only in that they are not polynomial maps. Hence, only a slight modification would be necessary in the estimations. However, the main question is whether the analytically obtained neighborhood is large enough for the computer-aided part of the method. These two maps along with the logistic map would also be interesting for larger delay, i.e., $d > 2$. We believe that the analytical part could be extended using only natural modifications. However, the computer-aided part can be critical in these cases, since the increasing dimension causes an exponentially growing graph.

It also would be interesting to prove the existence of the unique invariant closed curve around the nontrivial fixed point for parameter values larger than the critical value. However, this question is substantially different from the one studied in this thesis.

References

[1] J. Dudás. Global stability for the 2-dimensional logistic map. *Journal of Difference Equations and Applications*, 25(2):179–201, 2019.

[2] J. Dudás and T. Krisztin. Global stability for the three-dimensional logistic map. *Nonlinearity*, 34(2):894–938, 2021.

[3] F. A. Bartha, Á. Garab, and T. Krisztin. Local stability implies global stability for the 2-dimensional Ricker map. *Journal of Difference Equations and Applications*, 19(12):2043–2078, 2013.

[4] E. Camouzis and G. Ladas. *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjectures*. Chapman and Hall/CRC, 2008.

[5] E. Liz. Local stability implies global stability in some one-dimensional discrete single-species models. *Discrete and Continuous Dynamical Systems. Series B*, 7(1):191–199, 2007.

[6] E. Liz, V. Tkachenko, and S. Trofimchuk. Global stability in discrete population models with delayed-density dependence. *Mathematical Biosciences*, 199(1):26–37, 2006.

[7] F. A. Bartha and Á. Garab. Necessary and sufficient condition for the global stability of a delayed discrete-time

single neuron model. *Journal of Computational Dynamics*, 1(2):213–232, 2014.

[8] V. J. López and E. Parreño. L.a.s. and negative Schwarzian derivative do not imply g.a.s. in Clark's equation. *Journal of Dynamics and Differential Equations*, 28(2):339–374, 2016.