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Conical Curves in Constant Curvature Planes

Ph.D. Dissertation

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*To my father's soul and my Brother's soul "Salam"
you will remain in my heart as long as I live.
My work with love and sincerity to your souls.*

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Preface

In a projective-metric space (\mathcal{M}, d) we define

(D₁) a *conical curve* as the set

$$\mathcal{C}_{F, \mathcal{H}}^\varepsilon := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\},$$

where \mathcal{H} is a hyperplane, the *leading hyperplane* or *directrix*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity*. A conical curve is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

For given fixed points F_1, F_2 , the *foci*, and number $a \neq d(F_1, F_2)/2$, the *radius*, we define

(D₂) the *ellipsoid* (*ellipse* in dimension 2) as the set

$$\mathcal{E}_{d; F_1, F_2}^a := \{E : 2a = d(F_1, E) + d(E, F_2)\}, \text{ and}$$

(D₃) the *hyperboloid* (*hyperbola* in dimension 2) as the set

$$\mathcal{H}_{d; F_1, F_2}^a := \{X : 2a = |d(F_1, X) - d(X, F_2)|\},$$

according to $a > d(F_1, F_2)/2$ or $a < d(F_1, F_2)/2$, respectively. Value $2f := d(F_1, F_2)$ is the *eccentricity*, and if the eccentricity vanishes, then the ellipsoid (ellipse) is called *sphere* (*circle*). Further, an ellipsoid (ellipse) or hyperboloid (hyperbola) is called *conical* if it is a conical curve.

According to [15], A. Moór raised the request for determining those Finsler manifolds in which the class of elliptic conical curves coincides with the class of ellipses, or the class of hyperbolic conical curves coincides with the class of hyperbolas. Tamássy and Béteky found in [16, Theorem 2], that the only Finsler space where the class of elliptic conical curves coincides with the class of ellipses is the Euclidean space.

A similar problem was solved by Kurusa in [10, Theorem 6.1], where he proved that the only Minkowski geometry in which either a conical ellipsoid or a conical hyperboloid exists is the Euclidean one. At the end of his paper [10] Kurusa formulated the problem of determining projective-metric spaces in which

- (a) some or all ellipses are conical, or
- (b) some or all hyperbolas are conical.

Kurusa's main result [10, Theorem 6.1] was based on that, by [10, Theorem 4.2 and 4.3], the only Minkowski geometry in which a symmetric conical curve exists is the Euclidean one. Additionally, it is also proved in [10, Theorem 5.1] that the only Minkowski plane in which a quadratic conical curve exists is the Euclidean one. So Kurusa also raised the request to determine the projective-metric spaces in which

- (c) some or all elliptic conical curves are symmetric, or
- (d) some or all hyperbolic conical curves are symmetric, or
- (e) some or all elliptic conical curves are quadratic, or
- (f) some or all hyperbolic conical curves are quadratic.

All these problems are open for curved projective-metric spaces, so it was natural to set the goal of the research to answer Kurusa's request for curved constant curvature spaces. We reached this goal and published the results in [11–13].

Our results are as follow:

Theorem A. *If a conical curve \mathcal{C} in a curved constant curvature plane \mathcal{P} is symmetric, then \mathcal{P} is the sphere and the focus of \mathcal{C} is the pole of the directrix of \mathcal{C} .*

Theorem B. *If a conical curve \mathcal{C} in a curved constant curvature plane \mathcal{P} is quadratic, then \mathcal{P} is the sphere and either the focus of \mathcal{C} is the pole of the directrix of \mathcal{C} or \mathcal{C} is parabolic.*

Theorem C. *If \mathcal{C} is a conical ellipse or a conical hyperbola in a curved constant curvature plane \mathcal{P} , then \mathcal{P} is the sphere and the focus of \mathcal{C} is the pole of the directrix of \mathcal{C} .*

The presentation is based on my papers [11, 12] and [13], but for the sake of a broader view we give precise definitions from the ground up, provide basic theorems for curves and surfaces, and describe thoroughly from both the projective and the differential geometric point of views the spaces used in the text to show the dual nature of the constant curvature spaces.

Chapter 1

Preliminaries and preparations

In this chapter we collect definitions, theorems and some proofs which will be used as auxiliary facts for the next chapters.

Points of \mathbb{R}^n are denoted as A, B, \dots , vectors are \overrightarrow{AB} or $\mathbf{a}, \mathbf{b}, \dots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB} = (A, B)$, \overrightarrow{AB} is the open ray starting from A passing through B , and AB denotes the line through A and B .

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [4, page 243].

Notations $\mathbf{u}_\varphi = (\cos \varphi, \sin \varphi)$ and $\mathbf{u}_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used.

1.1 Basic differential geometry

In this section we provide the basic definitions and theorems of differential geometry that are necessary to understand our results in the next chapter.

1.1.1 Curves

Definition 1.1. A *parameterized differentiable curve* is a differentiable map $\mathbf{p}: \mathcal{I} \rightarrow \mathbb{R}^3$ of an open interval $\mathcal{I} = (a, b)$ of the real line \mathbb{R} into \mathbb{R}^3 .

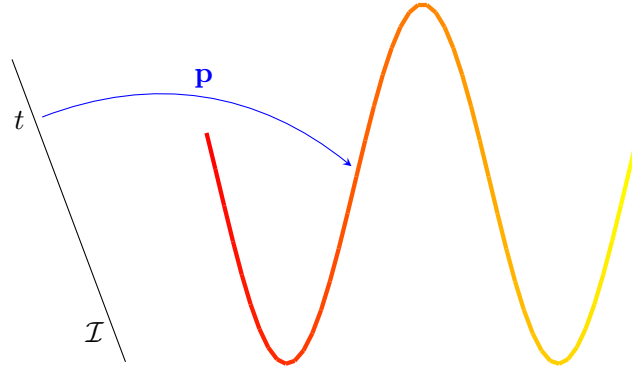


Figure 1.1: Curve and its parameterization

The differentiability means that \mathbf{p} maps each $t \in \mathcal{I}$ into point $\mathbf{p}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such a manner that the functions $x(t), y(t), z(t)$ are differentiable. The variable t is called the *parameter* of the curve.

The vector $\mathbf{p}'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$ is the *tangent vector* of the curve \mathbf{p} at t , and the image set $\mathbf{p} \subset \mathbb{R}^3$ is called the *trace* of \mathbf{p} .

Definition 1.2. A parameterized differentiable curve $\mathbf{p}: \mathcal{I} \rightarrow \mathbb{R}^3$ is said to be *regular* if $\mathbf{p}'(t) \neq 0$ for all $t \in \mathcal{I}$. Then the vector $\mathbf{p}'(t)$ is called the *tangent vector* of \mathbf{p} at $\mathbf{p}(t)$ or at t .

Definition 1.3. The *arc length* of a regular parameterized curve \mathbf{p} from the point $\mathbf{p}(t_0)$ to $\mathbf{p}(t_1)$ is

$$s(t) = \int_{t_0}^{t_1} |\mathbf{p}'(t)| dt, \text{ where } |\mathbf{p}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

A regular parameterized curve \mathbf{p} is said to be *arc length parameterized* if $|\mathbf{p}'(s)| = 1$.

Definition 1.4. Let $\mathbf{p}: \mathcal{I} \rightarrow \mathbb{R}^3$ be a curve parameterized by arc length. The number $|\mathbf{p}''(s)| = \kappa(s)$ is called the *curvature* of \mathbf{p} at $\mathbf{p}(s)$ or simply at s .

1.1.2 Surfaces

Definition 1.5. A subset $\mathcal{S} \subset \mathbb{R}^3$ is a regular surface if for each point $S \in \mathcal{S}$ there exists a neighborhood $\mathcal{V} \subseteq \mathbb{R}^3$ and a map $\mathbf{r}: U \rightarrow \mathcal{V} \cap \mathcal{S}$ of an open set $U \subseteq \mathbb{R}^2$ onto $\mathcal{V} \cap \mathcal{S} \subseteq \mathbb{R}^3$ such that

- (1) the coordinate functions x, y, z of $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ ($(u, v) \in U$), have continuous partial derivatives of all orders;
- (2) the inverse $\bar{\mathbf{r}}: \mathcal{V} \cap \mathcal{S} \rightarrow U$ is well defined and is continuous;
- (3) (*The regularity condition.*) the derivative $\dot{\mathbf{r}}$ is one to one.

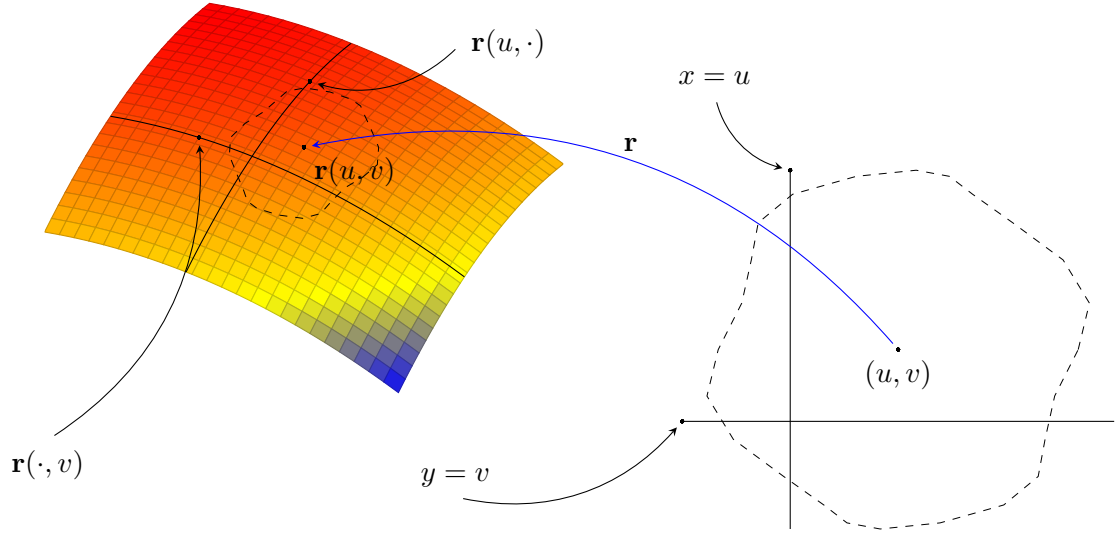


Figure 1.2: Surface and its parameterization

Proposition 1.6. *If $f: \mathcal{U} \rightarrow \mathbb{R}$ is a differentiable function on an open set $\mathcal{U} \subseteq \mathbb{R}^2$, then the graph of f , that is, the subset of \mathbb{R}^3 given by $(x, y, f(x, y))$ for $(x, y) \in \mathcal{U}$, is a regular surface.*

Proposition 1.7. *If $f: \mathcal{U} \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable function and $a \in f(\mathcal{U})$ is a regular value of f , then $f^{-1}(a)$ is a regular surface in \mathbb{R}^3 .*

Example 1.1. The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface by Proposition 1.7 because it is the set $f^{-1}(1)$ where

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

is a differentiable function, and 1 is a regular value of f .

Example 1.2. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface, since it is given by $f^{-1}(1)$, where 1 is a regular value of $f(x, y, z) = -x^2 - y^2 + z^2 - 1$.

Proposition 1.8. *Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a regular surface and $P \in \mathcal{S}$. Then there exists a neighborhood $\mathcal{V} \subseteq \mathcal{S}$ of P such that \mathcal{V} is the graph of a differentiable function which has one of the following three forms $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$.*

Definition 1.9. The set $T\mathcal{S}$ of the tangent vectors of the curves on the surface \mathcal{S} is called the *tangent bundle*. The set $T_P\mathcal{S}$ of tangent vectors $\mathbf{p}'(t) \in T\mathcal{S}$, where $\mathbf{p}(t) = P$, is called the *tangent plane of \mathcal{S} at $P \in \mathcal{S}$* .

Every tangent plane $T_P\mathcal{S}$ is a 2-dimensional vector space. For every tangent vector $\mathbf{v} \in T_P\mathcal{S}$ there are a great many curves \mathbf{p} on the surface \mathcal{S} that satisfies $\mathbf{p}(0) = P$ and $\mathbf{v} = \mathbf{p}'(0)$.

Definition 1.10. A differentiable map $f: \mathcal{S} \rightarrow \mathbb{R}$ is called differentiable *scalar field* on \mathcal{S} . The *differential* $\partial_{\mathbf{v}}f$ of the scalar field f evaluated against the tangent vector $\mathbf{v} \in T_P\mathcal{S}$ is the derivative $(f \circ \mathbf{p})'(0)$, where \mathbf{p} is a curve on the surface \mathcal{S} satisfying $\mathbf{p}(0) = P$ and $\mathbf{v} = \mathbf{p}'(0)$.

We notice that the differential of a scalar field evaluated against a tangent vector does not depend on the choice of the curve chosen in the definition.

Definition 1.11. A differentiable map $X: \mathcal{S} \rightarrow T\mathcal{S}$ is called differentiable *vector field* on \mathcal{S} , if $X(P) \in T_P\mathcal{S}$ for every $P \in \mathcal{S}$. The vector space of the differentiable vector fields on \mathcal{S} is denoted by $T_*\mathcal{S}$.

Definition 1.12. The *Lie-bracket* $[X, Y]$ of two vector fields $X, Y \in T_*\mathcal{S}$ is a linear mapping of scalar fields defined by $f \mapsto [X, Y]f = \partial_X(\partial_Y f) - \partial_Y(\partial_X f)$.

1.1.3 Riemann manifolds

We consider only Riemannian manifolds given on surfaces of the 3-dimensional space.

Definition 1.13. The pair (\mathcal{S}, g) is called a *Riemannian manifold* of dimension 2, if \mathcal{S} is a regular surface and $g: \mathcal{S} \ni P \mapsto g_P$ provides a Euclidean product $g_P: T_P\mathcal{S} \times T_P\mathcal{S} \rightarrow \mathbb{R}$ at every point $P \in \mathcal{S}$ on the corresponding tangent plane $T_P\mathcal{S}$ such that if X and Y are differentiable vector fields on \mathcal{S} , then the function $\mathcal{S} \ni P \mapsto g_P(X(P), Y(P))$ is a smooth function of P . The function g is called a *Riemannian metric* (or *Riemannian metric tensor*).

Every surface with its tangent planes equipped with the Euclidean product $g_P(\mathbf{u}, \mathbf{v}) := \langle \mathbf{u}, \mathbf{v} \rangle$ given by the restriction of the Euclidean product $\langle \cdot, \cdot \rangle$ of the space \mathbb{R}^3 is such a Riemannian manifold of dimension 2. The Riemannian metric given in this way called *inherited Riemannian metric*.

Definition 1.14. The *length of a differentiable curve* $\mathbf{p}: (a, b) \rightarrow \mathcal{S} \subset \mathbb{R}^3$ in a Riemannian manifold (\mathcal{S}, g) is $\ell(\mathbf{p}) := \int_a^b \sqrt{g_{\mathbf{p}(t)}(\dot{\mathbf{p}}(t), \dot{\mathbf{p}}(t))} dt$.

Definition 1.15. The *Riemannian distance function* $d_g: \mathcal{S} \times \mathcal{S} \ni (P, Q) \mapsto d_g(P, Q) \in \mathbb{R}$ on a Riemannian manifold (\mathcal{S}, g) is $\inf_{\mathbf{p} \in \mathcal{C}_{P,Q}} \ell(\mathbf{p})$, where $\mathcal{C}_{P,Q}$ is the set of all the differentiable curve \mathbf{p} in the Riemannian manifold (\mathcal{S}, g) connecting P and Q .

A Riemannian manifold with the Riemannian distance function is a metric space.

Definition 1.16. A bilinear mapping $\nabla: T_*\mathcal{S} \times T_*\mathcal{S} \ni (X, Y) \rightarrow \nabla_X Y \in T_*\mathcal{S}$ is called *affine connection* if for all differentiable functions $f: \mathcal{S} \rightarrow \mathbb{R}$ and for all vector fields $X, Y \in T_*\mathcal{S}$ if $\nabla_{fX} Y = f \nabla_X Y$ (*functional linearity in the first variable*) and $\nabla_X(fY) = \partial_X f Y + f \nabla_X Y$ (*Leibniz rule in the second variable*) hold.

An affine connection is called *torsion-free* if $[X, Y] = \nabla_X Y - \nabla_Y X$ for every $X, Y \in T_*\mathcal{S}$.

Definition 1.17. An affine connection is a *Levi-Civita* connection if it is torsion-free, and *compatible with the Riemannian metric* g , i.e. $\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

There is always a unique Levi-Civita connection that is easy to prove through the *Koszul formula* $2g(\nabla_X Y, Z) = \partial_X(g(Y, Z)) + \partial_Y(g(Z, X)) - \partial_Z(g(X, Y))$.

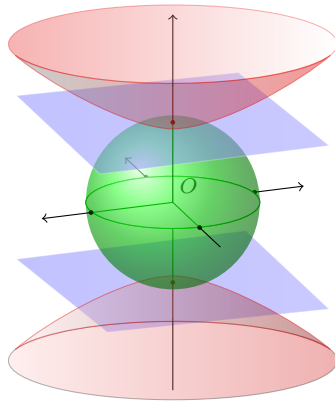
Definition 1.18. The *Riemannian curvature* is the trilinear mapping R of vector fields to vector fields defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

The Riemannian curvature is a tensor, because $R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z$ for every scalar field f and vector fields X, Y, Z , hence $R(X, Y)Z(P)$ depends in fact only on the vectors $X(P), Y(P), Z(P) \in T_P\mathcal{S}$. Further, the expression $\kappa(\mathbf{u}, \mathbf{v}) = \frac{g_P(R(\mathbf{u}, \mathbf{v})\mathbf{v}, \mathbf{u})}{g_P(\mathbf{u}, \mathbf{u})g_P(\mathbf{v}, \mathbf{v}) - g_P^2(\mathbf{u}, \mathbf{v})}$ does not depend on the independent vectors $\mathbf{u}, \mathbf{v} \in T_P\mathcal{S}$.

Definition 1.19. The value $\kappa_P = \kappa(\mathbf{u}, \mathbf{v})$ is called the (*sectional*) *curvature* of (\mathcal{S}, g) at the point $P \in \mathcal{S}$.

1.1.4 Two-dimensional manifolds of constant curvature

It is easy to see that the plane and the sphere with their respective inherited Riemannian metric are surfaces of constant curvature, but there is a third example worth noting.



Let the surface $\mathcal{K}_\kappa^2 \subset \mathbb{R}^3$ of points $\mathbf{p} = (p_1, p_2, p_3)$ satisfying

$$\kappa(p_1^2 + p_2^2) + p_3^2 = 1, \quad (1.1)$$

where $\kappa \in \{1, 0, -1\}$. Equip the surface \mathcal{K}_κ^2 with the Riemannian metric g_κ such that

$$g_{\kappa; \mathbf{p}}: T_{\mathbf{p}}\mathcal{K}_\kappa^2 \times T_{\mathbf{p}}\mathcal{K}_\kappa^2 \ni (\mathbf{x}, \mathbf{y}) \mapsto x_1 y_1 + x_2 y_2 + \kappa x_3 y_3 \quad (1.2)$$

for every point $\mathbf{p} \in \mathcal{K}_\kappa^2$. Then the pairs $(\mathcal{K}_\kappa^2, g_\kappa)$ have constant curvature κ .

If $\kappa \geq 0$, then the Riemannian metric in (1.2) is the inherited metric, and we have the sphere \mathcal{K}_1^2 and two planes \mathcal{K}_0^2 . The Riemannian manifold $(\mathcal{K}_{-1}^2, g_{-1})$ is a different case: both sheets of the hyperboloid \mathcal{K}_{-1}^2 equipped with the Riemannian metric g_{-1} model the hyperbolic plane, but g_{-1} is not the inherited metric.

Theorem 1.20 ([19, Hilbert's theorem]). *There exists no complete regular surface \mathcal{S} of constant negative curvature in \mathbb{R}^3 .*

Then one gets the so-called *projective model* $\bar{\mathcal{K}}_\kappa^2$ of the constant curvature space \mathbb{K}_κ^2 of curvature $\kappa \in \{1, 0, -1\}$ [5], and also the *canonical correspondence*

$$\chi_\kappa: \mathcal{K}_\kappa^2 \ni E \rightarrow \{E, -E\} \in \bar{\mathcal{K}}_\kappa^2 \cong \mathbb{K}_\kappa^2 \quad (1.3)$$

by identifying the points of $\mathcal{K}_\kappa^2 \subset \mathbb{R}^3$ that are symmetric in the origin.

1.2 Projective-metric spaces

Real projective plane \mathbb{P}^2 arises in several different ways.

Considering the real affine plane \mathbb{R}^2 , we call the equivalence sets of the straight lines by parallelism *ideal points*, and add these points to the set of the usual (real) points of \mathbb{R}^2 so that each ideal point becomes a common point of every straight line belonging to that particular ideal point. This extended geometry is the real projective plane.

Another method to construct *real projective plane* \mathbb{P}^2 is to think of the straight lines passing through the origin $(0, 0, 0)$ in \mathbb{R}^3 as *projective points*, and think of the planes passing through the origin $(0, 0, 0)$ in \mathbb{R}^3 as *projective straight lines*.

A more algebraic way is to consider the equivalence classes of the non-vanishing directional vectors by the equivalency relation \sim that relates two non-vanishing directional vectors equivalent \sim if one of them is a scalar multiple of the other one. This leads to the *homogeneous coordinates* which is a coordinatization of the real projective plane \mathbb{P}^2 .

Finally, an intuitive way of considering the real projective plane is to identify diametrical points of the sphere, i.e. these pairs constitute the points of the real projective plane.

A *metric space* is an ordered pair (\mathcal{M}, d) such that \mathcal{M} is a set, the set of *points*, and $d: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ is a *metric*, i.e. for any three points $x, y, z \in \mathcal{M}$ it satisfies $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$, the *triangle inequality*.

If the metric space (\mathcal{M}, d) is that \mathcal{M} is a projective plane \mathbb{P}^2 , or an affine plane $\mathbb{R}^2 \subset \mathbb{P}^2$, or a (not necessarily bounded) proper open convex subset of an affine plane $\mathbb{R}^2 \subset \mathbb{P}^2$, and the metric d is complete, continuous with respect to the usual topology of \mathbb{P}^n , additive on the segments, and the geodesic lines of d are exactly the non-empty intersection of \mathcal{M} with the straight lines, then the pair (\mathcal{M}, d) is called *projective-metric space*¹ [4, p. 115].

Such projective-metric planes are called *elliptic*, *parabolic*, or *hyperbolic*, respectively, according to whether \mathcal{M} is \mathbb{P}^2 , \mathbb{R}^2 , or a proper convex subset of \mathbb{R}^2 . The projective-metric planes of the latter two types are called *straight* [3, p. 1].

¹Determining the projective-metric spaces and studying the individual ones is known as Hilbert's fourth problem.

The geodesics of a projective-metric space of elliptic type have equal lengths, so we can set their length to π by simply multiplying the metric with an appropriate positive constant. Therefore we assume from now on that projective-metric spaces of elliptic type have geodesics of length π .

Let ℓ be a line in (\mathcal{M}, d) and let $P \in \mathcal{M}$ be a point outside of ℓ . A point $P^\perp \in \ell$ is the ℓ -foot of P , if $d(P, X) \geq d(P, P^\perp)$ for every $X \in \ell$. A line ℓ intersecting line ℓ' in a point S is said to be *perpendicular to ℓ* if S is an ℓ -foot of P for every $P \in \ell' \setminus \{S\}$. We denote this relation by $\ell' \perp \ell$, and notice, that \perp is not necessarily a symmetric relation.

Every isometry of (\mathcal{M}, d) is a restriction of a projectivity of the projective space \mathbb{P}^n [3]. They leave perpendicularity invariant, because perpendicularity is determined by the metric.

A set $\mathcal{S} \subset \mathcal{M}$ is called *symmetric about a point C* , if $X \in \mathcal{S}$ if and only if $Y \in \mathcal{S}$, where C is in the metric midpoints of the segment \overline{XY} , i.e. $2d(X, C) = 2d(C, Y) = d(X, Y)$.

1.2.1 Elliptic projective-metric planes

Every *elliptic plane* can be constructed in the following way. Take a Euclidean metric on \mathbb{R}^2 and let $\langle \cdot, \cdot \rangle$ be its Euclidean product. Define the function $\hat{\delta}: \mathcal{S}^2 \times \mathcal{S}^2 \rightarrow \mathbb{R}$ by

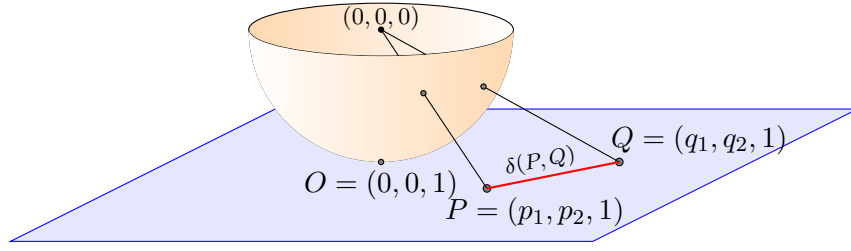
$$\hat{\delta}(\mathbf{x}, \mathbf{y}) = \arccos \langle \mathbf{x}, \mathbf{y} \rangle.$$

This is a metric on \mathcal{S}^2 , and it satisfies the strict triangle inequality, i.e. $\hat{\delta}(A, B) + \hat{\delta}(B, C) = \hat{\delta}(A, C)$ if the points A, B and C are in a hemisphere. Equality happens if and only if B is on the great circle determined by A and C . If the diametrical points are identified and the metric is inherited, then we get an elliptic plane.

The isometries of $(\mathcal{S}^2, \hat{\delta})$ are the restrictions of those isometries of the Euclidean plane $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ that leave \mathcal{S}^2 invariant: the orthogonal transforms $O(2)$. Any orthogonal transform is a product of at most 3 reflections in planes, and any two non-degenerate triangles with pair-wisely equal side-lengths determine one and only one $\hat{\delta}$ -isometry that maps the first of these triangles onto the second one.

Perpendicularity in the elliptic plane is a symmetric relation, because the great circles ℓ' and ℓ are perpendicular to each other if and only if the 2-dimensional planes they span are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

To show that the constructed geometry is an elliptic projective-metric space, we use the *gnomonic projection* [18] $\Gamma_O: \mathcal{S}^2 \rightarrow T_O \mathcal{S}^2$ of the sphere, where $O \in \mathcal{S}^2$ and $T_O \mathcal{S}^2$ is the tangent hyperplane of \mathcal{S}^2 at point O with the projective extension.



Γ_O projects the spherical metric $\hat{\delta}$ to the metric

$$\delta: \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \rightarrow [0, \pi) \quad (P, Q) \mapsto \hat{\delta}(P, Q) = \arccos \left(\frac{\langle P, Q \rangle}{|P||Q|} \right). \quad (1.4)$$

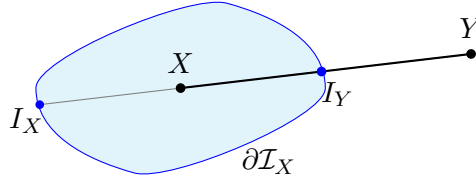
1.2.2 Parabolic projective-metric planes

The most important parabolic projective-metric planes are the *Minkowski planes*². They are constructed in the following way.

Let \mathcal{I} be an open, strictly convex, bounded domain in \mathbb{R}^2 , (centrally) symmetric to the origin. Then the function $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \inf\{\lambda > 0 : (\mathbf{y} - \mathbf{x})/\lambda \in \mathcal{I}\}$$

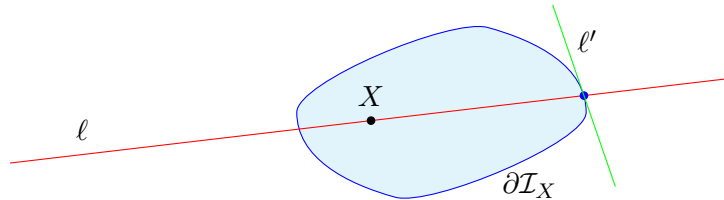
is a metric on \mathbb{R}^2 [4, IV.24], and is called *Minkowski metric on \mathbb{R}^2* . It satisfies the strict triangle inequality, i.e. $d(A, B) + d(B, C) = d(A, C)$ is valid if and only if $B \in \overline{AC}$.



The pair (\mathbb{R}^2, d) is the Minkowski plane, and \mathcal{I} is called the *indicatrix* of it.

The isometries of a Minkowski plane (\mathbb{R}^2, d) are the restrictions of those affine transforms of \mathbb{R}^2 that leave \mathcal{I} invariant.

A line ℓ is perpendicular to a line ℓ' if and only if ℓ' is tangent at the point $\ell \cap \ell'$ to \mathcal{I}_X for a point $X \in \ell$, where \mathcal{I}_X is a translated copy of \mathcal{I} centered at point X .



Perpendicularity in a Minkowski plane is a symmetric relation if and only if the boundary $\partial\mathcal{I}$ of the indicatrix is a Radon curve [14].

²They are also known as normed planes.

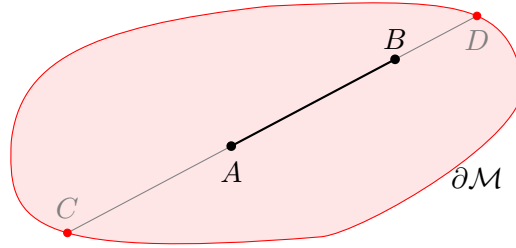
1.2.3 Hyperbolic projective-metric planes

The most important hyperbolic projective-metric planes are the *Hilbert planes*. They are constructed in the following way.

If \mathcal{M} is an open, strictly convex, proper subset of \mathbb{R}^2 , then the function $d: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ defined by

$$d(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2} |\ln(A, B; C, D)|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{M} \cap AB, \end{cases} \quad (1.5)$$

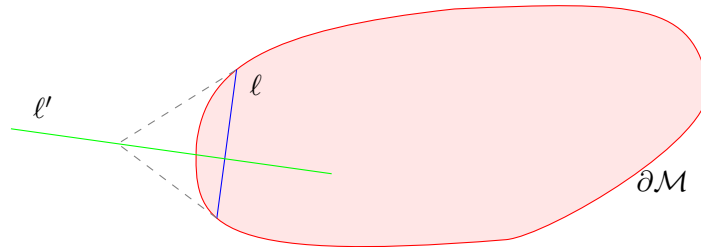
is a *metric* on \mathcal{M} [4, page 297] which satisfies the strict triangle inequality, i.e. $d(A, B) + d(B, C) = d(A, C)$ if and only if $B \in \overline{AC}$.



The pair (\mathcal{M}, d) is the Hilbert plane, \mathcal{M} is its *domain*, and the function d is called the *Hilbert metric* on \mathcal{M} .

The isometries of a Hilbert plane (\mathcal{M}, d) are the restrictions of those projective transforms of \mathbb{P}^2 that leave \mathcal{M} invariant.

A line ℓ in \mathcal{M} is perpendicular to a line ℓ' if and only if ℓ' connects the point $\ell \cap \ell'$ of \mathcal{M} and the intersection of the tangents of \mathcal{M} that touch \mathcal{M} at the points $\partial\mathcal{M} \cap \bar{\ell}$, where $\bar{\ell}$ is the Euclidean line containing ℓ .



Perpendicularity in a Minkowski plane is a symmetric relation if and only if the boundary $\partial\mathcal{M}$ is an ellipse [8].

1.2.4 Constant curvature planes

There are special elliptic, parabolic and hyperbolic projective-metric planes that make Riemannian manifolds.

It is clear that a Minkowski plane is Euclidean if and only if its indicatrix is an ellipse.

It is known [4, (29.3)] that a Hilbert plane is a model of the hyperbolic plane of Bolyai, Lobachevskii and Gauss, if and only if its domain is the interior of an ellipse. Such Hilbert planes are called *Cayley–Klein models* of the hyperbolic plane.

It happens that these have constant curvature, and can also be constructed by the gnomonic projection of the 2-dimensional manifolds $(\mathcal{K}_\kappa^2, g_\kappa)$ [9], where $\kappa \in \{0, \pm 1\}$.

The isometry groups of all these three constant curvature planes are generated by reflections in straight lines. Moreover specifically, we have

Theorem 1.21 ([4]). *Every isometry of each of these three constant curvature planes can be given as a product of at most three reflections in straight lines.*

This statement can be reversed in a sense:

- (1) *If the isometry group is generated by the reflections in a Minkowski plane, then the Minkowski plane is Euclidean.*
- (2) *If the isometry group is generated by the reflections in a Hilbert plane, then the Hilbert plane is a hyperbolic plane.*

1.3 Classes of curves in the Euclidean plane

In the Euclidean plane there are four differently defined classes of curves which however coincide in most of the cases. Here we briefly describe these classes to shade light over the problem considered in the main part of this thesis. We notice though that despite of the usual treatment we present, most parts would give clearer picture if considered in projective geometry.

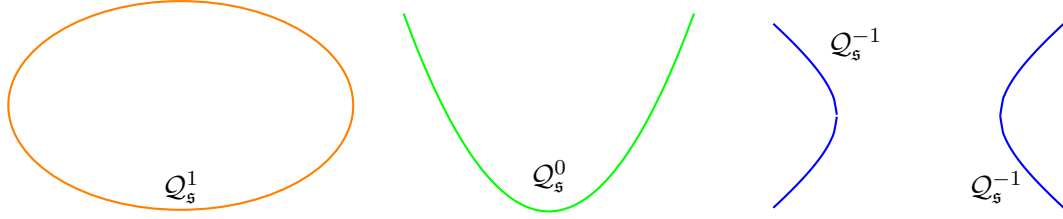
1.3.1 Quadratic curves

The curves presented in this subsection are independent from the metric.

A curve in the plane is called *quadratical*, if it is part of a *quadric*

$$\mathcal{Q}_\mathfrak{s}^\sigma := \left\{ (x, y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases} \right\}, \quad (D_q)$$

where \mathfrak{s} is an affine coordinate system. A quadric is called *ellipse* (*affine circle*), *parabola* and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.



The ellipse is the only bounded conical curve, and it is such that any complete set of parallel lines has a member line which intersect the ellipse in exactly two points.

The parabola is a connected conical curve and that has exactly one complete set of parallel lines such that its every member line intersects the parabola in exactly one point.

The hyperbola is two connected curves (called *branches*) and it is such that exactly two complete sets of parallel lines are such that their every member line, except the one called *asymptote*, intersects the hyperbola in exactly one point.

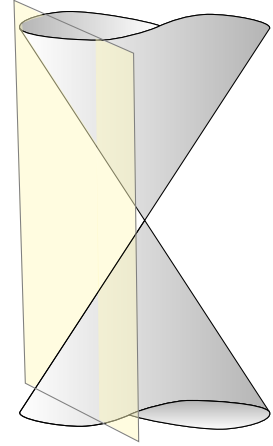
1.3.2 Conic sections

Although it could be discussed in affine, or even better in projective geometry, in this section we present the subject in the Euclidean geometry. Notice however, that the metric is only used to formulate what a circular cone is, i.e. it could be formulated as an elliptic quadratic cone in affine space or as a quadratic cone in projective space.

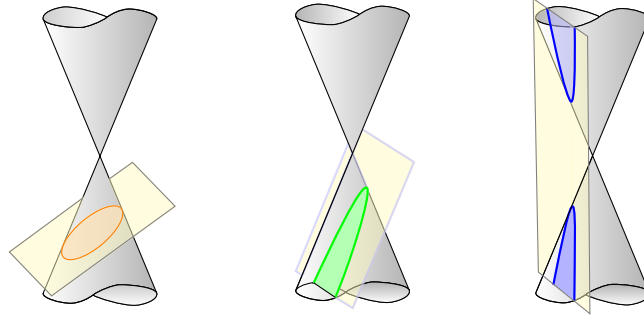
Definition 1.22. Let a point, the *apex*, and two different lines, the *axis* and a *generating line*, be given in the space. Turning the generating line about the axis scours a surface called rotational or circular (*double*) cone (*with two nappes*).

The lines obtained while turning the generating line are called *generating line* too.

Definition 1.23. A curve obtained as the intersection of a plane, the *cutting plane*, with a cone is called *conic section*. A conic section is called *elliptic*, *parabolic*, and *hyperbolic*, if the cutting plane is parallel with 0, 1, and 2 generating line, respectively.



Planes that pass through the apex of the cone intersect the cone in a point, in a generating line or in a pair of generating lines. These are called *degenerate conic sections*.

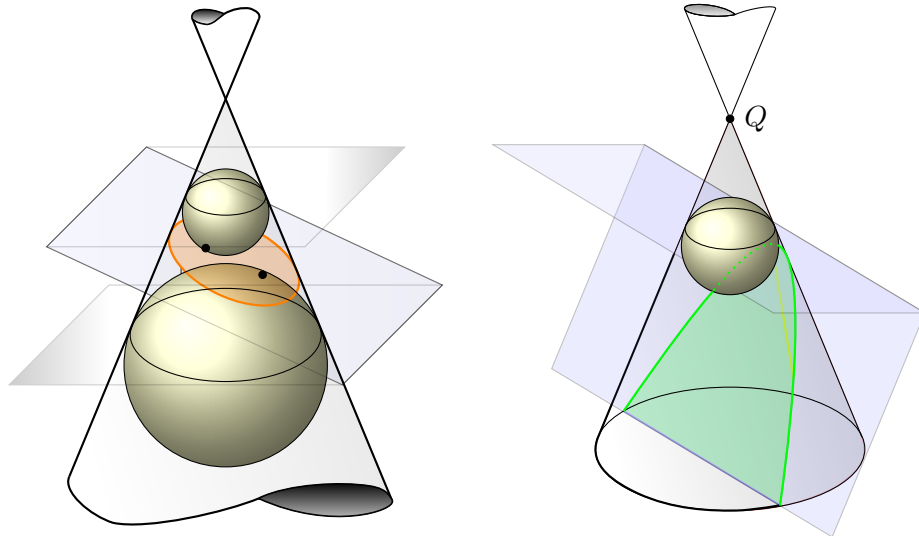


Every non-degenerate elliptic conic section is a bounded closed curve contained in one halves of the cone. The non-degenerate elliptic conic sections are affine ellipses \mathcal{Q}_s^1 , and also metric ellipses \mathcal{E}_{F_1, F_2}^a . Further, every affine ellipse \mathcal{Q}_s^1 can be constructed as a non-degenerate elliptic conic section.

Every non-degenerate parabolic conic section is an unbounded connected curve contained in one halves of the cone. The non-degenerate parabolic conic sections are affine parabolas \mathcal{Q}_s^0 . Further, every affine parabola \mathcal{Q}_s^0 can be constructed as a non-degenerate parabolic conic section.

Every non-degenerate hyperbolic conic section has points in both halves of the cone, so they have two separate unbounded connected curves, the *branches*. The non-degenerate hyperbolic conic sections are affine hyperbolas \mathcal{Q}_s^{-1} , and also metric hyperbolas \mathcal{H}_{F_1, F_2}^a . Further, every affine hyperbola \mathcal{Q}_s^{-1} can be constructed as a non-degenerate hyperbolic conic section.

A very effective method to consider plane sections of a circular cone in Euclidean space is to put Dandelin spheres [17] into the cone. Each Dandelin sphere touches the cone in a circle whose plane intersects the cutting plane of the corresponding conic section in a (maybe ideal) line, the *directrix*. A point of the cutting plane where the cutting plane touches a Dandelin sphere is called *foci*.



1.3.3 Curves defined by sum or difference of distances

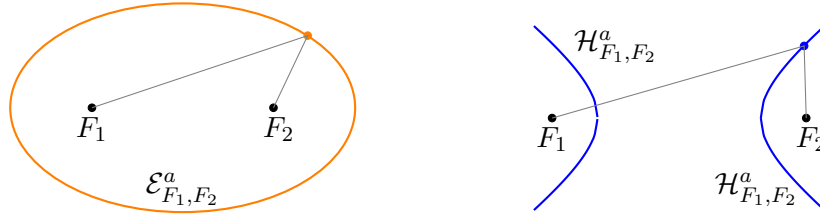
The curves presented in this section are bound to the metric. For now, we stay in Euclidean geometry.

A closed segment $\overline{F_1 F_2}$ of the different points F_1, F_2 are the locus of points P in the plane such that the sum of the distances from P to the two fixed points F_1 and F_2 is the constant $d(F_1, F_2)$.

Definition 1.24. An *ellipse* \mathcal{E}_{F_1, F_2}^a is the locus of points P in the plane such that the sum of the distances from P to the two fixed points F_1 and F_2 , the *foci*, is a constant $a > d(F_1, F_2)$. An ellipse $\mathcal{E}_{C, C}^r$ is called *circle of radius r with center C* .

The closed rays $F_1 F_2 \setminus \overline{F_1 F_2}$ of the different points F_1, F_2 is the locus of points P in the plane such that the absolute value of the difference of the distances from P to the two fixed points F_1 and F_2 is the constant $d(F_1, F_2)$.

Definition 1.25. A *hyperbola* \mathcal{H}_{F_1, F_2}^a is the locus of points P in the plane such that the absolute value of the difference of the distances from P to the two fixed points F_1 and F_2 , the *foci*, is a constant $a < d(F_1, F_2)$.



Every ellipse is an affine ellipse \mathcal{Q}_s^1 , and every affine ellipse \mathcal{Q}_s^1 is the circle $\mathcal{E}_{(0,0),(0,0)}^1$ in the Euclidean metric d defined by the inner product $\langle (x, y), (z, t) \rangle = xz + yt$.

Every hyperbola \mathcal{H}_{F_1, F_2}^a is an affine hyperbola \mathcal{Q}_s^{-1} , and every affine hyperbola \mathcal{Q}_s^{-1} is the hyperbola $\mathcal{H}_{(2,0),(-2,0)}^2$ in the Euclidean metric d .

1.3.4 Curves defined by ratio of distances

In this section we consider curves which are bound to the metric, for now, it is the Euclidean metric.

Definition 1.26. Given a positive number ε , the *numerical eccentricity*, a straight line ℓ , the *directrix*, and a point $F \notin \ell$, the *foci*, in the plane, the *conical curve* $\mathcal{C}_{F, \ell}^\varepsilon$ is the locus of points P in the plane such that $d(F, P) = \varepsilon d(P, \ell)$.

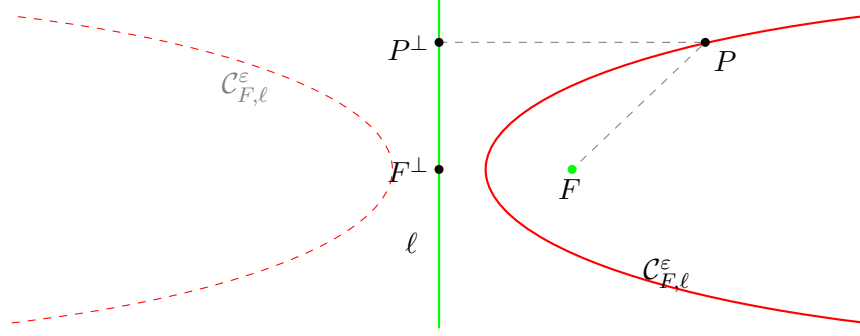
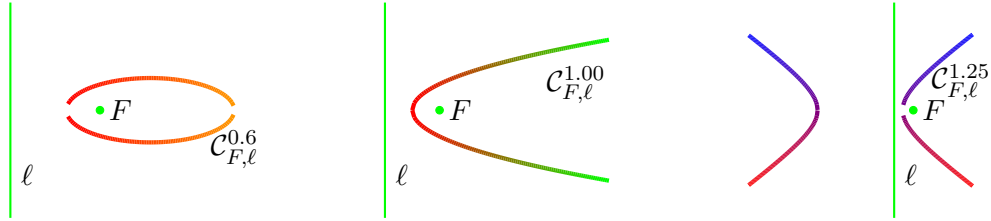


Figure 1.3: Conical curve in Euclidean plane

Definition 1.27. A conical curve is called *elliptic*, *parabolic*, and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$, and $\varepsilon > 1$, respectively.



Every elliptic conical curve is a bounded closed curve contained in one side of the directrix. The elliptic conical curves are affine ellipses \mathcal{Q}_s^1 , metric ellipses \mathcal{E}_{F_1, F_2}^a , and conic sections. Further, except the circles, every metric ellipse \mathcal{E}_{F_1, F_2}^a is an elliptic conical curve.

Every parabolic conical curve is an unbounded curve contained in one side of the directrix. The parabolic conical curves are affine parabolas \mathcal{Q}_s^0 , and conic sections. Further, every affine parabola \mathcal{Q}_s^0 is a parabolic conical curve.

Every hyperbolic conical curve has two separate unbounded connected curves, the *branches*, one-one on both sides of the directrix. The hyperbolic conical curves are affine hyperbolas \mathcal{Q}_s^{-1} , metric hyperbolas \mathcal{H}_{F_1, F_2}^a , and conic sections. Further, every affine hyperbola \mathcal{Q}_s^{-1} is a hyperbolic conical curve.

Chapter 2

Conical curves with given properties

In this chapter we consider conical curves in constant curvature planes. It turns out that some of their usual properties, like symmetry and quadraticity, remains valid only in very special configurations. We prove some of Kurusa's conjectures for curved constant curvature planes:

- (1) no conical curve in the hyperbolic plane can be quadratic;
- (2) no conical curve in the hyperbolic plane can be symmetric;
- (3) if the focus of a conic curve on the sphere is not the pole of the directrix, then the conic can only be quadratic if it is a parabolic, and it can not be symmetric.

2.1 Quadratic conical curves in the hyperbolic plane

As for any pair (F, ℓ) of a point F in \mathcal{D} and an h-line ℓ there exists an isometry ι such that $\iota(\ell)$ goes through the center O of \mathcal{D} , and O is the foot of $\iota(F)$ on $\iota(\ell)$, we can restrict without loss of generality the investigation of conical curves to those conical curves $\mathcal{C}_{F,\ell}^\varepsilon$ in (\mathcal{D}, δ) for which the directrix ℓ is the y -axis, and the focus F is $(f, 0)$, where $f \in (0, 1)$.

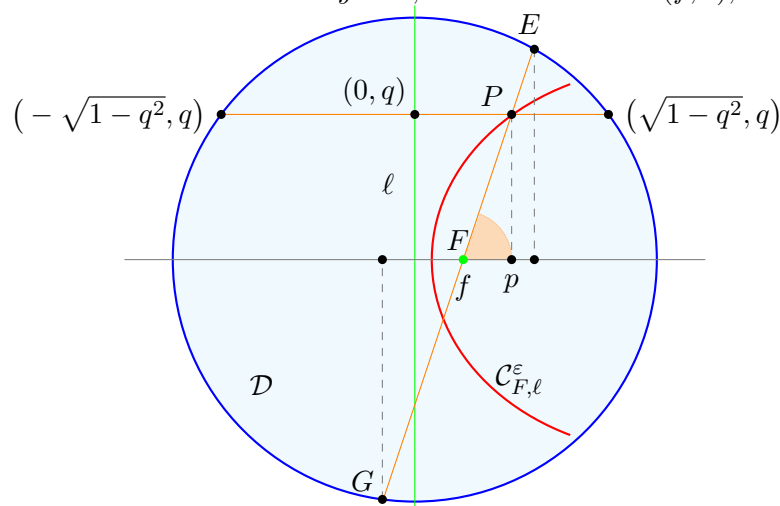


Figure 2.1: Directrix ℓ is through the center of the Cayley–Klein model, the focus F is at $(f, 0)$, where $f \in (0, 1)$.

To calculate the points $P = (p, q)$ on $\mathcal{C}_{F,\ell}^\varepsilon$, we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P = (p, q) \in \mathcal{C}_{F,\ell}^\varepsilon$. It is easy to get that

$$\delta(P, \ell) = \frac{1}{2} \left| \log \left\{ \frac{p + \sqrt{1 - q^2}}{p - \sqrt{1 - q^2}} : \frac{0 + \sqrt{1 - q^2}}{0 - \sqrt{1 - q^2}} \right\} \right|. \quad (2.1)$$

To obtain $\delta(F, P)$, we firstly determine the points $\{E, G\} = \{(x_\pm, y_\pm)\}$, where line FP intersects the unit circle, the border of \mathcal{D} . These points clearly satisfy the equations $x^2 + y^2 = 1$ and $(x - f)q = y(p - f)$. So $(p - f)^2(1 - x^2) = (x - f)^2q^2$, and we obtain $0 = x^2((p - f)^2 + q^2) - 2fq^2x + (f^2q^2 - (p - f)^2)$, hence

$$\begin{aligned} x_\pm &= \frac{fq^2 \pm (p - f)\sqrt{(p - f)^2 + (1 - f^2)q^2}}{(p - f)^2 + q^2}, \\ y_\pm &= \frac{-qf(p - f) \pm q\sqrt{(p - f)^2 + (1 - f^2)q^2}}{(p - f)^2 + q^2}. \end{aligned}$$

Thus, we get

$$\begin{aligned} \delta(F, P) &= \frac{1}{2} \left| \log \left\{ \frac{q^2 + (p - f)^2 + (f(p - f) + \sqrt{(p - f)^2 + (1 - f^2)q^2})}{q^2 + (p - f)^2 + (f(p - f) - \sqrt{(p - f)^2 + (1 - f^2)q^2})} \right. \right. \\ &\quad \left. \left. : \frac{f(p - f) + \sqrt{(p - f)^2 + (1 - f^2)q^2}}{f(p - f) - \sqrt{(p - f)^2 + (1 - f^2)q^2}} \right\} \right| \\ &= \frac{1}{2} \left| \log \left\{ \frac{(fp - 1 - \sqrt{(p - f)^2 + (1 - f^2)q^2})^2}{(1 - f^2)(1 - p^2 - q^2)} \right\} \right|, \end{aligned} \quad (2.2)$$

where we have used the identities

$$\begin{aligned} (f(p - f) + \sqrt{(p - f)^2 + (1 - f^2)q^2})(f(p - f) - \sqrt{(p - f)^2 + (1 - f^2)q^2}) \\ = f^2(p - f)^2 - (p - f)^2 - (1 - f^2)q^2 = -(1 - f^2)(q^2 + (p - f)^2), \end{aligned}$$

and

$$\begin{aligned} (fp - 1 - \sqrt{(p - f)^2 + (1 - f^2)q^2})(fp - 1 + \sqrt{(p - f)^2 + (1 - f^2)q^2}) \\ = (fp - 1)^2 - (p - f)^2 - (1 - f^2)q^2 = (1 - f^2)(1 - p^2 - q^2). \end{aligned}$$

According to (D_1) equations (2.1) and (2.2) give

$$\begin{aligned} (1 - q^2 - p^2) \left(1 + \frac{2p}{\sqrt{1 - q^2} - p} \right)^\epsilon \\ = \frac{(fp - 1 - \sqrt{q^2(1 - f^2) + (p - f)^2})^2}{1 - f^2}, \end{aligned} \quad (2.3)$$

where $\epsilon = \pm\varepsilon$. Figure 2.2 shows how these conical curves look like based on (2.3).

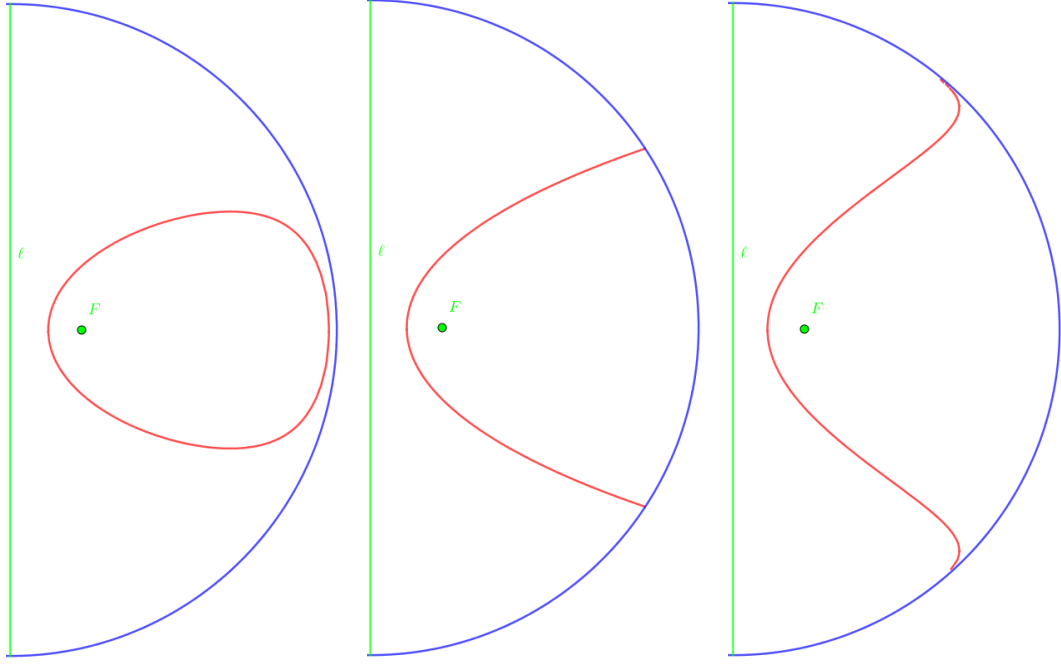


Figure 2.2: An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the Cayley–Klein model of the hyperbolic geometry.

For the sake of later contradiction, we assume from now on that

conical curve $\mathcal{C}_{F,\ell}^\varepsilon$ is quadratic (D_q),

hence it satisfies an equation of the form $\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}x + \bar{e}y + \bar{f} = 0$, where the coefficients are real and $\bar{a} \geq 0$.

As the conical curves $\mathcal{C}_{F,\ell}^\varepsilon$ are symmetric in the x -axis, the quadratic equation should be invariant under changing y to $-y$, so $\bar{b} = \bar{e} = 0$ follows. So the equation is of the form $\bar{a}x^2 + \bar{c}y^2 + \bar{d}x + \bar{g} = 0$, hence $\bar{c} \neq 0$, because otherwise the curve will degenerate into straight lines. So the quadratic equation simplifies to

$$ax^2 + y^2 + bx + c = 0, \quad a \geq 0. \quad (2.4)$$

As conical curve $\mathcal{C}_{F,\ell}^\varepsilon$ is quadratical, we have $q^2 = -ap^2 - bp - c$, $a \geq 0$. Putting this into (2.3) gives the identity

$$\begin{aligned} & (1 - p^2 + ap^2 + bp + c) \left(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c - p}} \right)^\varepsilon \\ &= \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{1 - f^2}. \end{aligned} \quad (2.5)$$

Differentiating this with respect to p gives

$$\begin{aligned}
 & (-2p + 2ap + b) \left(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c - p}} \right)^\varepsilon + \\
 & + \varepsilon (1 - p^2 + ap^2 + bp + c) \left(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c - p}} \right)^{\varepsilon-1} \times \\
 & \times \left(\frac{2}{\sqrt{1 + ap^2 + bp + c - p}} - \frac{2p \left(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1 \right)}{(\sqrt{1 + ap^2 + bp + c - p})^2} \right) \\
 & = \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)}{1 - f^2} \times \\
 & \times \left(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2} \right). \tag{2.6}
 \end{aligned}$$

The exponential multiplier in (2.5) is

$$\begin{aligned}
 & \left(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c - p}} \right)^\varepsilon \\
 & = \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)}.
 \end{aligned}$$

Putting this into (2.6) leads to

$$\begin{aligned}
 & (-2p + 2ap + b) \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)} + \\
 & + \varepsilon \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{1 - f^2} \times \\
 & \times \left(1 + \frac{2p}{\sqrt{1 + ap^2 + bp + c - p}} \right)^{-1} \times \\
 & \times \left(\frac{2}{\sqrt{1 + ap^2 + bp + c - p}} - \frac{2p \left(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1 \right)}{(\sqrt{1 + ap^2 + bp + c - p})^2} \right) \\
 & = \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)}{1 - f^2} \times \\
 & \times \left(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2} \right).
 \end{aligned}$$

This simplifies to

$$\begin{aligned}
 & (-2p + 2ap + b) \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)} + \\
 & + \varepsilon \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2)} + (p - f)^2)^2}{(1 - f^2)(\sqrt{1 + ap^2 + bp + c} + p)} \times \\
 & \times \left(2 - \frac{2p \left(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1 \right)}{\sqrt{1 + ap^2 + bp + c - p}} \right)
 \end{aligned}$$

$$= \frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2})}{1 - f^2} \times \\ \times \left(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}} \right).$$

Dividing by $\frac{(fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2})}{(1 - f^2)(1 - p^2 + ap^2 + bp + c)}$ further simplifies the equation to

$$\left((-2p + 2ap + b) + \varepsilon(\sqrt{1 + ap^2 + bp + c} - p) \left(2 - \frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1)}{\sqrt{1 + ap^2 + bp + c} - p} \right) \right) \times \\ \times (fp - 1 - \sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}) \\ = (1 - p^2 + ap^2 + bp + c) \left(2f - \frac{-(2ap + b)(1 - f^2) + 2(p - f)}{\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2}} \right),$$

i.e.

$$\left((-2p + 2ap + b) + \varepsilon(\sqrt{1 + ap^2 + bp + c} - p) \left(2 - \frac{2p(\frac{2ap+b}{2\sqrt{1+ap^2+bp+c}} - 1)}{\sqrt{1 + ap^2 + bp + c} - p} \right) \right) \times \\ \times ((fp - 1)\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2} + \\ + (ap^2 + bp + c)(1 - f^2) - (p - f)^2) \\ = (1 - p^2 + ap^2 + bp + c) \times \\ \times (2f\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2} + (2ap + b)(1 - f^2) - 2(p - f)).$$

With some rearrangement we obtain

$$2\varepsilon \left(\sqrt{1 + ap^2 + bp + c} - \frac{p(2ap + b)}{2\sqrt{1 + ap^2 + bp + c}} \right) \times \\ \times ((fp - 1)\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2} + \\ + (ap^2 + bp + c)(1 - f^2) - (p - f)^2) \\ = (1 - p^2 + ap^2 + bp + c) \times \\ \times (2f\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2} + (2ap + b)(1 - f^2) - 2(p - f)) - \\ - (-2p + 2ap + b) \times \\ \times ((fp - 1)\sqrt{-(ap^2 + bp + c)(1 - f^2) + (p - f)^2} + \\ + (ap^2 + bp + c)(1 - f^2) - (p - f)^2),$$

and again some rearrangement gives

$$\begin{aligned}
 & 2\varepsilon \left(\sqrt{1+ap^2+bp+c} - \frac{p(2ap+b)}{2\sqrt{1+ap^2+bp+c}} \right) \times \\
 & \quad \times ((fp-1)\sqrt{-(ap^2+bp+c)(1-f^2)} + (p-f)^2 + \\
 & \quad \quad + (ap^2+bp+c)(1-f^2) - (p-f)^2) \\
 & = (2f(1-p^2+ap^2+bp+c) - (-2p+2ap+b)(fp-1)) \times \\
 & \quad \times \sqrt{-(ap^2+bp+c)(1-f^2)} + (p-f)^2 + \\
 & \quad + ((1-p^2+ap^2+bp+c)((2ap+b)(1-f^2) - 2(p-f)) - \\
 & \quad - (-2p+2ap+b)((ap^2+bp+c)(1-f^2) - (p-f)^2)),
 \end{aligned}$$

then

$$\begin{aligned}
 & \varepsilon(2(1+c)+pb) \\
 & = \sqrt{1+ap^2+bp+c} \times \\
 & \quad \times \left(((fb+2a-2)p+2f+2fc+b)\sqrt{-(ap^2+bp+c)(1-f^2)} + (p-f)^2 + \right. \\
 & \quad \quad \left. + f(2-bf-2a)p^2 + 2(a-1+f^2(c+1))p + b + 2f(c+1) \right),
 \end{aligned}$$

so we arrive at

$$\begin{aligned}
 & \varepsilon(2(1+c)+pb) \\
 & = ((fb+2a-2)p+2f(c+1)+b) \times \\
 & \quad \times \sqrt{((p-f)^2 - (ap^2+bp+c)(1-f^2))(1+ap^2+bp+c)} + \\
 & \quad + (f(2-bf-2a)p^2 + 2(a-1+f^2(c+1))p + 2f(c+1)+b) \times \\
 & \quad \times \sqrt{1+ap^2+bp+c}.
 \end{aligned}$$

Squaring this gives

$$\begin{aligned}
 & \varepsilon^2(2(1+c)+pb)^2 - \\
 & \quad - \left(((fb+2a-2)p+2f(c+1)+b)^2 ((p-f)^2 - (ap^2+bp+c)(1-f^2)) + \right. \\
 & \quad \quad \left. + (f(2-bf-2a)p^2 + 2(a-1+f^2(c+1))p + 2f(c+1)+b)^2 \right) \times \\
 & \quad \times (1+ap^2+bp+c) \\
 & = 2((fb+2a-2)p+2f(c+1)+b)(1+ap^2+bp+c) \times \\
 & \quad \times (f(2-bf-2a)p^2 + 2(a-1+f^2(c+1))p + 2f(c+1)+b) \times \\
 & \quad \times \sqrt{(p-f)^2 - (ap^2+bp+c)(1-f^2)}.
 \end{aligned}$$

Squaring again gives the identity of two polynomials

$$\begin{aligned}
 & \varepsilon^4(2(1+c) + pb)^4 + \\
 & + \left(((fb + 2a - 2)p + 2f(c + 1) + b)^2 ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)) + \right. \\
 & \quad \left. + (f(2 - bf - 2a)p^2 + 2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \right)^2 \times \\
 & \quad \times (1 + ap^2 + bp + c)^2 + \\
 & + 2\varepsilon^2(2(1+c) + pb)^2 \times \\
 & \quad \times \left(((fb + 2a - 2)p + 2f(c + 1) + b)^2 ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)) + \right. \\
 & \quad \left. + (f(2 - bf - 2a)p^2 + 2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \right) \times \\
 & \quad \times (1 + ap^2 + bp + c) \\
 & = 4((fb + 2a - 2)p + 2f(c + 1) + b)^2 (1 + ap^2 + bp + c)^2 \times \\
 & \quad \times (f(2 - bf - 2a)p^2 + 2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \times \\
 & \quad \times ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)). \tag{2.7}
 \end{aligned}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of p^{12} are equal, so

$$\begin{aligned}
 & \left((fb + 2a - 2)^2(1 - a(1 - f^2)) + f^2(2 - bf - 2a)^2 \right)^2 a^2 \\
 & = 4(fb + 2a - 2)^2 a^2 f^2 (2 - bf - 2a)^2 (1 - a(1 - f^2)), \tag{2.8}
 \end{aligned}$$

that is equivalent to

$$a^2(fb + 2a - 2)^4 \left(((1 - a(1 - f^2)) + f^2)^2 - 4f^2(1 - a(1 - f^2)) \right) = 0,$$

so either $a = 0$, which is disclosed by $a > 0$, or $1 - a(1 - f^2) = f^2$, i.e. $a = 1$, that implies $b = 0$ in (2.8), or $fb + 2a - 2 = 0$.

Assuming $fb + 2a - 2 = 0$ simplifies (2.7) into

$$\begin{aligned}
 & \varepsilon^4(2(1+c) + pb)^4 + \\
 & + \left((2f(c + 1) + b)^2 ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)) + \right. \\
 & \quad \left. + (2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \right)^2 (1 + ap^2 + bp + c)^2 + \\
 & + 2\varepsilon^2(2(1+c) + pb)^2 \times \\
 & \quad \times \left((2f(c + 1) + b)^2 ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)) + \right. \\
 & \quad \left. + (2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \right) (1 + ap^2 + bp + c) \\
 & = 4(2f(c + 1) + b)^2 (1 + ap^2 + bp + c)^2 \times \\
 & \quad \times (2(a - 1 + f^2(c + 1))p + 2f(c + 1) + b)^2 \times \\
 & \quad \times ((p - f)^2 - (ap^2 + bp + c)(1 - f^2)). \tag{2.9}
 \end{aligned}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of p^8 are equal, so

$$\begin{aligned} & ((2f(c+1) + b)^2(1 - a(1 - f^2)) + 4(a - 1 + f^2(c+1))^2)a^2 \\ &= 4(2f(c+1) + b)^2a^24(a - 1 + f^2(c+1))^2(1 - a(1 - f^2)), \end{aligned} \quad (2.10)$$

that is equivalent to

$$a^2((2f(c+1) + b)^2(1 - a(1 - f^2)) - 4(a - 1 + f^2(c+1))^2)^2 = 0,$$

so either $a = 0$, which is disclosed by $a > 0$, or $(2f(c+1) + b)^2(1 - a(1 - f^2)) = 4(a - 1 + f^2(c+1))^2$. Thus, $(2f^2(c+1) + fb)^2(1 - a(1 - f^2)) = 4f^2(a - 1 + f^2(c+1))^2$, hence $(f^2(c+1) + 1 - a)^2(1 - a(1 - f^2)) = f^2(a - 1 + f^2(c+1))^2$, and therefore $1 - a(1 - f^2) \geq 0$, i.e., $a < 1/(1 - f^2)$. Assuming $f^2(c+1) + (1 - a) \neq 0$, we can write this equation in the form

$$\frac{(f^2(c+1) - (1 - a))^2}{(f^2(c+1) + (1 - a))^2} = \frac{af^2 + 1 - a}{f^2},$$

which, if $c + 1 \neq 0$, leads to

$$\begin{aligned} 1 &> \frac{(f^2(c+1) - (1 - a))^2}{(f^2(c+1) + (1 - a))^2} = \frac{af^2 + 1 - a}{f^2} > 1 && \text{if } a \in (0, 1), \\ 1 &< \frac{(f^2(c+1) - (1 - a))^2}{(f^2(c+1) + (1 - a))^2} = \frac{af^2 + 1 - a}{f^2} < 1 && \text{if } a \in (1, 1/(1 - f^2)), \end{aligned}$$

a clear contradiction. Thus, $c = -1$, and consequently $a = 1$. If $f^2(c+1) + (1 - a) = 0$, then also $f^2(c+1) - (1 - a) = 0$, so $a = 1$.

Thus, in any case, we obtained $a = 1$. This simplifies (2.7) into

$$\begin{aligned} & \varepsilon^4(2(1+c) + pb)^4 + \\ & + \left((fbp + 2f(c+1) + b)^2(f^2 - p(2f + b(1 - f^2)) - c(1 - f^2)) + \right. \\ & \quad \left. + (-bf^2p^2 + 2f^2(c+1)p + 2f(c+1) + b)^2 \right)^2 \times \\ & \quad \times (1 + p^2 + bp + c)^2 + \\ & + 2\varepsilon^2(2(1+c) + pb)^2 \times \\ & \quad \times \left((fbp + 2f(c+1) + b)^2(f^2 - p(2f + b(1 - f^2)) - c(1 - f^2)) + \right. \\ & \quad \left. + (-bf^2p^2 + 2f^2(c+1)p + 2f(c+1) + b)^2 \right) \times \\ & \quad \times (1 + p^2 + bp + c) \\ & = 4(fbp + 2f(c+1) + b)^2(1 + p^2 + bp + c)^2 \times \\ & \quad \times (-bf^2p^2 + 2f^2(c+1)p + 2f(c+1) + b)^2 \times \\ & \quad \times (f^2 - p(2f + b(1 - f^2)) - c(1 - f^2)). \end{aligned} \quad (2.11)$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of p^{12} are equal, so $b^4 f^8 = 0$, hence $b = 0$.

Thus, in any case, we obtained $a = 1$ and $b = 0$. This simplifies (2.11) into

$$\begin{aligned}
& \varepsilon^4 16(1+c)^4 + \\
& + (2f(c+1))^4 \left((f^2 - 2fp - c(1-f^2)) + (fp+1)^2 \right)^2 (1+p^2+c)^2 + \\
& + 2\varepsilon^2 4(1+c)^2 (2f(c+1))^2 \times \\
& \quad \times \left((f^2 - 2fp - c(1-f^2)) + (f(c+1)p+1)^2 \right) (1+p^2+c) \\
& = 4(2f(c+1))^4 (1+p^2+c)^2 (fp+1)^2 (f^2 - 2fp - c(1-f^2)).
\end{aligned} \tag{2.12}$$

Two polynomials can only be equal on a segment, if their corresponding coefficients are pairwise equal. The coefficients of p^8 are equal, so $(2f(c+1))^4(5f^2)^2 = 0$, hence $c = -1$.

Thus, by (2.4), if the conical curve $\mathcal{C}_{F,\ell}^\varepsilon$ is quadratic, then it is of the form $x^2 + y^2 = 1$, a clear contradiction that proves the following:

Theorem 2.1 ([11]). *No conical curve of the hyperbolic plane can be quadratic.*

2.2 Symmetric conical curves in the hyperbolic plane

Consider a conical curve $\mathcal{C}_{F,\ell}^\varepsilon$. Let F^\perp be the foot of F on the h-line ℓ , and let C be a point on the h-line FF^\perp different from F^\perp .

It is well known that there are h-isometries that maps C into the center O of \mathcal{D} . Thus we can restrict without loss of generality the investigation of conical curves $\mathcal{C}_{E,\ell}^\varepsilon$ in (\mathcal{D}, δ) to those ones for which $\overline{(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})}$ is the directrix ℓ for some $m \in (-1, 0)$, the center is $O = (0, 0)$, and the focus F is $(f, 0)$, where $f \in (-1, 1) \setminus \{m\}$.

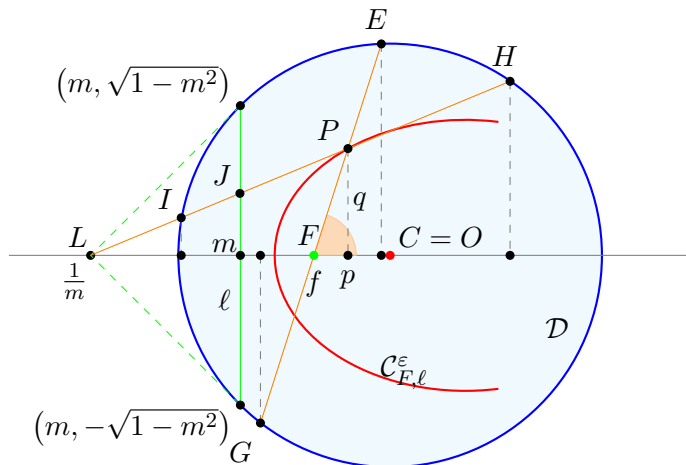


Figure 2.3: Directrix ℓ is $\overline{(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})}$, $\mathcal{C}_{F,\ell}^\varepsilon$ is symmetric in O , the center of the Cayley–Klein model, and the focus F is at $(f, 0)$, where $f \in (-1, 1) \setminus \{m\}$.

To calculate the points $P = (p, q)$ on $\mathcal{C}_{F, \ell}^\varepsilon$, we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P = (p, q) \in \mathcal{C}_{F, \ell}^\varepsilon$. Observe that the line through P orthogonal to ℓ is the one that connects P to L , the intersection of the tangents of \mathcal{D} at the limit points of ℓ . We clearly have $L = (-1/m, 0)$.

To obtain $\delta(P, \ell)$, we firstly determine the points $\{I, H\} = \{(x_\pm, y_\pm)\}$, where line LP intersects the unit circle, the border of \mathcal{D} . These points clearly satisfy the equations $x^2 + y^2 = 1$ and $(x - 1/m)q = y(p - 1/m)$. So $(p - 1/m)^2(1 - x^2) = (x - 1/m)^2q^2$, and we obtain $0 = x^2((p - \frac{1}{m})^2 + q^2) - 2\frac{q^2}{m}x + (\frac{q^2}{m^2} - (p - \frac{1}{m})^2)$, hence

$$x_\pm = \frac{\frac{q^2}{m} \pm (p - \frac{1}{m})\sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}}{(p - \frac{1}{m})^2 + q^2}, \quad y_\pm = \frac{-\frac{q}{m}(p - \frac{1}{m}) \pm q\sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}}{(p - \frac{1}{m})^2 + q^2}.$$

Further, we need the coordinates of point J , where PL intersects ℓ . We clearly have $(m - 1/m)q = y(p - 1/m)$, hence $y = q(m - \frac{1}{m})/(p - \frac{1}{m})$. Thus

$$\begin{aligned} \delta(P, \ell) &= \delta(P, J) \\ &= \frac{1}{2} \left| \log \left\{ \frac{((p - \frac{1}{m})^2 + q^2) + \frac{1}{m}(p - \frac{1}{m}) + \sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}}{((p - \frac{1}{m})^2 + q^2) + \frac{1}{m}(p - \frac{1}{m}) - \sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}} \right. \right. \\ &\quad \left. \left. : \frac{\frac{m-\frac{1}{m}}{p-\frac{1}{m}}((p - \frac{1}{m})^2 + q^2) + \frac{1}{m}(p - \frac{1}{m}) + \sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}}{\frac{m-\frac{1}{m}}{p-\frac{1}{m}}((p - \frac{1}{m})^2 + q^2) + \frac{1}{m}(p - \frac{1}{m}) - \sqrt{(p - \frac{1}{m})^2 + q^2\frac{1-m^2}{m^2}}} \right\} \right|. \end{aligned}$$

So

$$\begin{aligned} \delta(P, \ell) &= \frac{1}{2} \left| \log \left\{ \frac{((p - \frac{1}{m})^2 + q^2) + \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} + \frac{1}{m}(p - \frac{1}{m})\right)}{((p - \frac{1}{m})^2 + q^2) - \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} - \frac{1}{m}(p - \frac{1}{m})\right)} \right. \right. \\ &\quad \left. \left. : \frac{m(p - \frac{1}{m}) + \frac{m-\frac{1}{m}}{p-\frac{1}{m}}q^2 + \sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})}}{m(p - \frac{1}{m}) + \frac{m-\frac{1}{m}}{p-\frac{1}{m}}q^2 - \sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})}} \right\} \right| \\ &= \frac{1}{2} \left| \log \left\{ \frac{((p - \frac{1}{m})^2 + q^2) + \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} + \frac{1}{m}(p - \frac{1}{m})\right)}{((p - \frac{1}{m})^2 + q^2) - \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} - \frac{1}{m}(p - \frac{1}{m})\right)} \times \right. \right. \\ &\quad \left. \left. \times \frac{(p - \frac{1}{m})^2 + (1 - \frac{1}{m^2})q^2 - \frac{1}{m}(p - \frac{1}{m})\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})}}{(p - \frac{1}{m})^2 + (1 - \frac{1}{m^2})q^2 + \frac{1}{m}(p - \frac{1}{m})\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})}} \right\} \right| \\ &= \frac{1}{2} \left| \log \left\{ \frac{((p - \frac{1}{m})^2 + q^2) + \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} + \frac{1}{m}(p - \frac{1}{m})\right)}{((p - \frac{1}{m})^2 + q^2) - \left(\sqrt{(p - \frac{1}{m})^2 + q^2(1 - \frac{1}{m^2})} - \frac{1}{m}(p - \frac{1}{m})\right)} \times \right. \right. \end{aligned}$$

$$\begin{aligned}
 & \times \frac{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\frac{1}{m}(p-\frac{1}{m})}{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\frac{1}{m}(p-\frac{1}{m})} \Bigg\} \Bigg| \\
 &= \frac{1}{2} \left| \log \left\{ \frac{((p-\frac{1}{m})^2+q^2)\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} + ((p-\frac{1}{m})^2+q^2)(1-\frac{1}{m}p)}{((p-\frac{1}{m})^2+q^2)\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} - ((p-\frac{1}{m})^2+q^2)(1-\frac{1}{m}p)} \right\} \right| \\
 &= \frac{1}{2} \left| \log \left\{ \frac{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} + (1-\frac{1}{m}p)}{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} - (1-\frac{1}{m}p)} \right\} \right| \\
 &= \frac{1}{2} \left| \log \left\{ \frac{\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} + (1-\frac{1}{m}p)\right)^2}{(1-p^2-q^2)(\frac{1}{m^2}-1)} \right\} \right|. \tag{2.13}
 \end{aligned}$$

To obtain $\delta(F, P)$, we firstly determine the points $E = (x_1, y_1)$ and $G = (x_2, y_2)$, where line FP intersects the unit circle, the border of \mathcal{D} . These points clearly satisfy the equations $x^2 + y^2 = 1$ and $(x-f)q = y(p-f)$. So $(p-f)^2(1-x^2) = (x-f)^2q^2$, and we obtain $0 = x^2((p-f)^2+q^2) - 2fq^2x + (f^2q^2 - (p-f)^2)$, hence

$$x_{\pm} = \frac{fq^2 \pm (p-f)\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2}, \quad y_{\pm} = \frac{-qf(p-f) \pm q\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2}.$$

Thus, we get

$$\begin{aligned}
 \delta(F, P) &= \frac{1}{2} \left| \log \left\{ \frac{q^2 + (p-f)^2 + (f(p-f) + \sqrt{(p-f)^2 + (1-f^2)q^2})}{q^2 + (p-f)^2 + (f(p-f) - \sqrt{(p-f)^2 + (1-f^2)q^2})} \right. \right. \\
 &\quad \left. \left. : \frac{f(p-f) + \sqrt{(p-f)^2 + (1-f^2)q^2}}{f(p-f) - \sqrt{(p-f)^2 + (1-f^2)q^2}} \right\} \right| \tag{2.14} \\
 &= \frac{1}{2} \left| \log \left\{ \frac{(fp-1 - \sqrt{(p-f)^2 + (1-f^2)q^2})^2}{(1-f^2)(1-p^2-q^2)} \right\} \right|,
 \end{aligned}$$

where we have used the identities

$$\begin{aligned}
 & (f(p-f) + \sqrt{(p-f)^2 + (1-f^2)q^2})(f(p-f) - \sqrt{(p-f)^2 + (1-f^2)q^2}) \\
 &= f^2(p-f)^2 - (p-f)^2 - (1-f^2)q^2 = -(1-f^2)(q^2 + (p-f)^2), \\
 & (fp-1 - \sqrt{(p-f)^2 + (1-f^2)q^2})(fp-1 + \sqrt{(p-f)^2 + (1-f^2)q^2}) \text{ in this section} \\
 &= (fp-1)^2 - (p-f)^2 - (1-f^2)q^2 = (1-f^2)(1-p^2-q^2).
 \end{aligned}$$

According to (D_1) equations (2.13) and (2.14) give

$$\left(\frac{\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})} + (1-\frac{1}{m}p)\right)^2}{(1-p^2-q^2)(\frac{1}{m^2}-1)} \right)^{\epsilon} = \frac{(fp-1 - \sqrt{q^2(1-f^2) + (p-f)^2})^2}{(1-f^2)(1-q^2-p^2)}, \tag{2.15}$$

where $\epsilon \in \{\varepsilon, -\varepsilon\}$. Figure 2.2 shows how these conical curves look like based on (2.15) with $\epsilon = \varepsilon$.

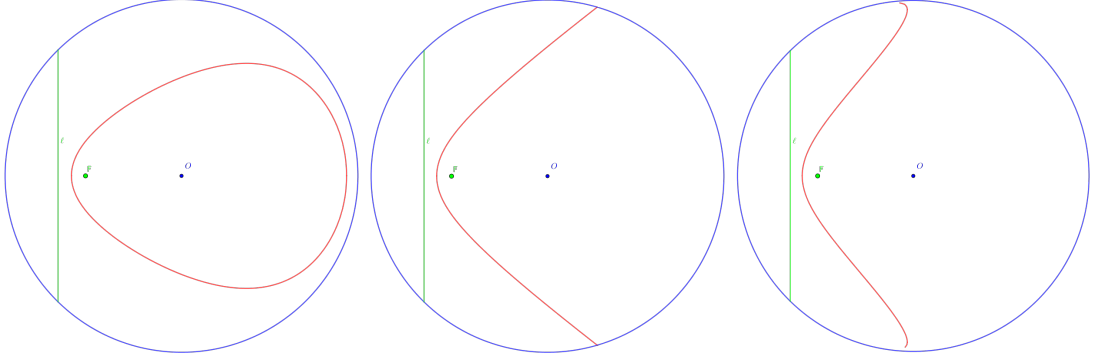


Figure 2.4: An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the Cayley–Klein model of the hyperbolic geometry.

For the sake of later contradiction, we assume from now on that

conical curve $\mathcal{C}_{F,\ell}^\varepsilon$ is symmetric in a point C .

Such a point of symmetry C clearly is on the h-line FF^\perp , where F^\perp is the foot of F on the h-line ℓ . So we can restrict without loss of generality the investigation of symmetric conical curves $\mathcal{C}_{F,\ell}^\varepsilon$ in (\mathcal{D}, δ) to those ones for which directrix ℓ is $(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})$ for some $m \in (-1, 1)$, the center is $O = (0, 0)$, and the focus F is $(f, 0)$, where $f \in (m, 1)$. Thus we can use the formulas given in the previous section.

As the conical curve is symmetric in the x -axis, and it is symmetric in point O , it is symmetric about the y -axis too, so, substituting $-p$ into p , equation (2.15) gives

$$\left(\frac{\left(\sqrt{\left(p + \frac{1}{m}\right)^2 + q^2 \left(1 - \frac{1}{m^2}\right)} + \left(1 + \frac{1}{m}p\right) \right)^2}{(1-p^2-q^2)\left(\frac{1}{m^2} - 1\right)} \right)^\epsilon = \frac{(fp+1 + \sqrt{q^2(1-f^2) + (p+f)^2})^2}{(1-f^2)(1-q^2-p^2)}, \quad (2.16)$$

Dividing (2.15) with this equation and taking the square root result in

$$\left(\frac{\sqrt{\left(p - \frac{1}{m}\right)^2 + q^2 \left(1 - \frac{1}{m^2}\right)} + \left(1 - \frac{1}{m}p\right)}{\sqrt{\left(p + \frac{1}{m}\right)^2 + q^2 \left(1 - \frac{1}{m^2}\right)} + \left(1 + \frac{1}{m}p\right)} \right)^\epsilon = \frac{fp-1 - \sqrt{q^2(1-f^2) + (p-f)^2}}{fp+1 + \sqrt{q^2(1-f^2) + (p+f)^2}}, \quad (2.17)$$

If $q = 0$, then (2.15) gives

$$\left(\frac{\left(\frac{1}{m} - 1\right)(1+p)}{\left(\frac{1}{m} + 1\right)(1-p)} \right)^\epsilon = \frac{(fp-1 - |p-f|)^2}{(1-f^2)(1-p^2)} = \begin{cases} \frac{(1-f)(1+p)}{(1+f)(1-p)}, & \text{if } p > f, \\ \frac{(1+f)(1-p)}{(1-f)(1+p)}, & \text{if } p < f. \end{cases}$$

After some rearrangement this becomes

$$\frac{1 \pm f}{1 \mp f} \left(\frac{1-m}{1+m} \right)^\epsilon = \left(\frac{1+p}{1-p} \right)^{-\epsilon \pm 1}, \quad \text{where } \pm 1 = \frac{p-f}{|p-f|}. \quad (2.18)$$

If p is a solution of these equations, then the symmetry in O implies, that $-p$ is also a solution of (2.18).

Suppose the two solutions $\pm p$ belong to the equation with the same sign of ε . If $\pm p$ are on the same side of f , then $p = 0$, a contradiction follows. If $\pm p$ are on different sides of f , then

$$\frac{1+f}{1-f} \left(\frac{1-m}{1+m} \right)^\varepsilon = \left(\frac{1+p}{1-p} \right)^{-\varepsilon+1}, \quad \text{and} \quad \frac{1-f}{1+f} \left(\frac{1-m}{1+m} \right)^\varepsilon = \left(\frac{1-p}{1+p} \right)^{-\varepsilon-1} = \left(\frac{1+p}{1-p} \right)^{\varepsilon+1}.$$

Division and multiplication of the first equation by the second one give

$$\frac{1+f}{1-f} = \left(\frac{1-p}{1+p} \right)^\varepsilon \quad \text{and} \quad \left(\frac{1-m}{1+m} \right)^\varepsilon = \frac{1-p}{1+p}, \quad (2.19)$$

respectively. Substituting the second equation into the first one results in

$$\frac{1+f}{1-f} = \left(\frac{1-m}{1+m} \right)^{\varepsilon^2}. \quad (2.20)$$

Note that the sign of ε in (2.19) is irrelevant regarding the solutions. If $\varepsilon = 1$, the second equation gives $f = -m$, hence O is the midpoint of the segment of F and its foot on ℓ , so $p = 0$, a contradiction.

Suppose the two solutions $\pm p$ belong to the equation with different signs of ε . If $\pm p$ are on the same side of f , then

$$\frac{1 \pm f}{1 \mp f} \left(\frac{1-m}{1+m} \right)^\varepsilon = \left(\frac{1+p}{1-p} \right)^{-\varepsilon \pm 1}, \quad \text{and} \quad \frac{1 \pm f}{1 \mp f} \left(\frac{1-m}{1+m} \right)^{-\varepsilon} = \left(\frac{1-p}{1+p} \right)^{\varepsilon \pm 1}$$

follow, where $\pm 1 = \frac{p-f}{|p-f|} = \frac{-p-f}{|p+f|}$. Division and multiplication of the first equation by the second one give

$$\left(\frac{1-m}{1+m} \right)^{\pm \varepsilon} = \frac{1+p}{1-p}, \quad \text{and} \quad \frac{1 \pm f}{1 \mp f} = \left(\frac{1+p}{1-p} \right)^\varepsilon \quad (2.21)$$

respectively. Rearranging the first equation, then substituting it into the second one results in

$$\left(\frac{1 \mp m}{1 \pm m} \right)^\varepsilon = \frac{1+p}{1-p}, \quad \text{and} \quad \frac{1 \pm f}{1 \mp f} = \left(\frac{1 \mp m}{1 \pm m} \right)^{\varepsilon^2}, \quad (2.22)$$

respectively. If $\varepsilon = 1$, the second equation gives $f = -m$, hence O is the midpoint of the segment of F and its foot on ℓ , so $p = 0$, a contradiction.

If $\pm p$ are on different sides of f , we may suppose that $p > f$, and so $-p < f$. Then

$$\frac{1+f}{1-f} \left(\frac{1-m}{1+m} \right)^\varepsilon = \left(\frac{1+p}{1-p} \right)^{-\varepsilon+1}, \quad \text{and} \quad \frac{1-f}{1+f} \left(\frac{1-m}{1+m} \right)^{-\varepsilon} = \left(\frac{1-p}{1+p} \right)^{\varepsilon-1}$$

follows, where $\frac{p-f}{|p-f|} = \frac{p+f}{|p+f|}$. Division and multiplication of the first equation by the second one give

$$\frac{1+f}{1-f} \left(\frac{1-m}{1+m} \right)^\varepsilon = 1, \quad \text{and} \quad 1 = \left(\frac{1+p}{1-p} \right)^{1-\varepsilon} \quad (2.23)$$

respectively. This is obviously a contradiction.

Thus we have either $\varepsilon \in (0, 1)$ or $\varepsilon \in (1, \infty)$, the elliptic or the hyperbolic case, respectively. Further, we have for the pair of points either (2.19) and (2.20), or (2.21) and (2.22), respectively. Moreover, $\pm p$ falls in different sides of f , and in the same side of f , respectively.

Theorem 2.2 ([12]). *No conical curve of the hyperbolic plane can be symmetric.*

Proof. Assume first the elliptic case, so that $\varepsilon \in (0, 1)$.

Rearrangement of (2.17) gives

$$\left(1 + \frac{-\frac{2}{m}p}{\sqrt{(p+\frac{1}{m})^2 + q^2(1-\frac{1}{m^2})} + (1+\frac{1}{m}p)} \right)^\varepsilon = \frac{2fp}{fp+1+\sqrt{q^2(1-f^2)+(p+f)^2}} - 1, \quad (2.24)$$

This is contradictory, because $p \rightarrow 0$ causes the left-hand side to tend to 1, and the right-hand side to tend to -1 .

Assume now the hyperbolic case, so that $\varepsilon > 1$.

Rearrangement of (2.17) gives for $p^2 + q^2 \rightarrow 1$ that

$$\left(\frac{|1-\frac{1}{m}p| + (1-\frac{1}{m}p)}{|1+\frac{1}{m}p| + (1+\frac{1}{m}p)} \right)^\varepsilon = \frac{fp-1-|fp-1|}{fp+1+|fp+1|}, \quad (2.25)$$

Since $|fp| < 1$, the right-hand side vanishes, hence also the left-hand side vanishes, so $\frac{1}{m}p > 1$. This is a contradiction because it can not be valid for both p and $-p$.

The contradictions prove the statement. \square

2.3 Quadratic conical curves on the sphere

Let \hat{O} be the polar of the great circle $\hat{\ell}$ on the \mathcal{S}^2 . Let \hat{F} be in the half sphere \mathcal{S}_O^2 of $\hat{\ell}$ that contains \hat{O} . Let \hat{P} be on the half circle \mathcal{G}_O^2 of the great circle of \hat{O} and \hat{F} that is contained by \mathcal{S}_O^2 . Fix the coordinate system in the plane of \mathcal{G}_O^2 such that $(0, 0)$ is the center of \mathcal{G}_O^2 , $\hat{O} = (0, 1)$, $\hat{F} = (\cos \varphi, \sin \varphi)$ and $\hat{P} = (\cos \varpi, \sin \varpi)$ for some $\varphi \in (0, \pi/2)$

and $\varpi \in (-\pi/2, \pi/2)$. Then $\hat{P} \in \hat{\mathcal{C}}_{\delta; \hat{F}, \hat{\ell}}^\varepsilon$ if and only if

$$\varepsilon = \begin{cases} \frac{\varpi - \varphi}{\pi/2 - \varpi}, & \text{if } \varpi \in (\varphi, \pi/2), \\ \frac{\varphi - \varpi}{\pi/2 - \varpi}, & \text{if } \varpi \in (0, \varphi), \\ \frac{\varphi - \varpi}{\pi/2 + \varpi}, & \text{if } \varpi \in (-\pi/2, 0]. \end{cases}$$

If $\varpi \in (\varphi, \pi/2)$, then $\frac{\varpi - \varphi}{\pi/2 - \varpi} = \frac{\pi/2 - \varphi}{\pi/2 - \varpi} - 1$ is a strictly monotone increasing function of ϖ that vanishes when $\varpi = \varphi$ and tends to infinity when $\varpi \rightarrow \pi/2$.

So there is exactly one $\varpi \in (\varphi, \pi/2)$ for which $\hat{P} \in \hat{\mathcal{C}}_{\delta; \hat{F}, \hat{\ell}}^\varepsilon$.

If $\varpi \in (0, \varphi)$, then $\frac{\varphi - \varpi}{\pi/2 - \varpi} = 1 - \frac{\pi/2 - \varphi}{\pi/2 - \varpi}$ is a strictly monotone decreasing function of ϖ that tends to $\frac{\varphi}{\pi/2}$ when $\varpi \rightarrow 0$ and vanishes when $\varpi = \varphi$.

If $\varpi \in (-\pi/2, 0]$, then $\frac{\varphi - \varpi}{\pi/2 + \varpi} = \frac{\pi/2 + \varphi}{\pi/2 + \varpi} - 1$ is a strictly monotone decreasing function of ϖ that tends to infinity when $\varpi \rightarrow \pi/2$ and is $\frac{\varphi}{\pi/2}$ when $\varpi \rightarrow 0$.

Thus there is exactly one $\varpi \in (-\pi/2, \varphi)$ for which $\hat{P} \in \hat{\mathcal{C}}_{\delta; \hat{F}, \hat{\ell}}^\varepsilon$.

Let $\mathcal{C}_{F, \ell}^\varepsilon := \Gamma_{\hat{O}}(\hat{\mathcal{C}}_{\delta; \hat{F}, \hat{\ell}}^\varepsilon)$, $O := \Gamma_{\hat{O}}(\hat{O})$, $F := \Gamma_{\hat{O}}(\hat{F})$, and $\ell := \Gamma_{\hat{O}}(\hat{\ell})$. Choose the coordinate system so that $O = (0, 0, 1)$ and $F = (f, 0, 1)$, where $f > 0$. Figure 2.5 shows what we have on the plane $\mathcal{P} := T_{\hat{O}}\mathcal{S}^2 = \{(x, y, z) : z = 1\}$.

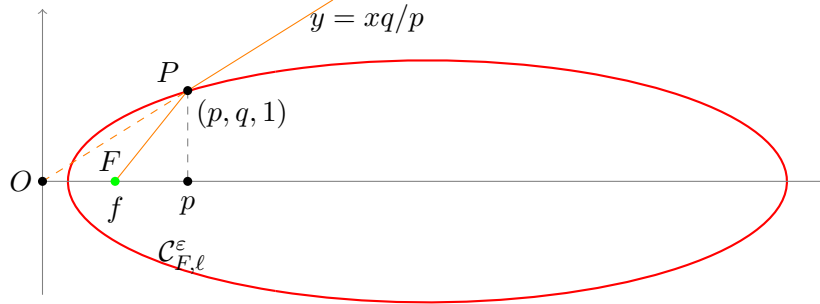


Figure 2.5: Projected conical curve $\mathcal{C}_{F, \ell}^\varepsilon$, if the directrix ℓ is in the infinity and the focus F is at $(f, 0)$, where $f > 0$.

To calculate the points $(p, q, 1) = P = \Gamma_{\hat{O}}(\hat{P})$ of $\mathcal{C}_{F, \ell}^\varepsilon$ we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P \in \mathcal{C}_{F, \ell}^\varepsilon$. Observe that the line through O and P is the gnomonic image of the great circle that is perpendicular to $\hat{\ell}$ and going through $\Gamma_{\hat{O}}^{-1}(P)$. Thus, by (1.4), we have

$$\delta(P, \ell) = \frac{\pi}{2} - \delta(P, O) = \frac{\pi}{2} - \arccos \frac{1}{\sqrt{p^2 + q^2 + 1}}. \quad (2.26)$$

For the distance of P from the focus we obtain from (1.4) that

$$\delta(P, F) = \delta(P, (f, 0, 1)) = \arccos \frac{pf + 1}{\sqrt{f^2 + 1} \sqrt{p^2 + q^2 + 1}}. \quad (2.27)$$

According to (D₁) equations (2.26) and (2.27) give that

$$\varepsilon \left(\frac{\pi}{2} - \arccos \frac{1}{\sqrt{p^2 + q^2 + 1}} \right) = \arccos \frac{pf + 1}{\sqrt{f^2 + 1} \sqrt{p^2 + q^2 + 1}} \quad (2.28)$$

is the equation of $\mathcal{C}_{F,\ell}^\varepsilon$. Figure 2.6 shows how $\mathcal{C}_{F,\ell}^\varepsilon$ looks like for different values of ε .

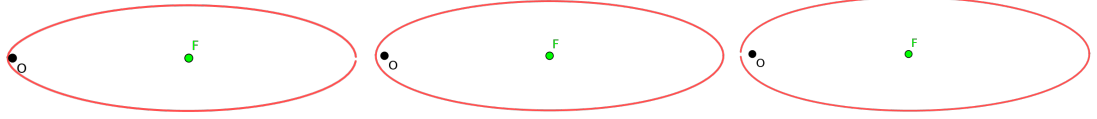


Figure 2.6: An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in the projected model of the sphere.

The parabolic conical curves (i.e. $\varepsilon = 1$) are quadratic because taking the cosine of (2.28) results in

$$\sqrt{1 - \frac{1}{p^2 + q^2 + 1}} = \left| \frac{pf + 1}{\sqrt{f^2 + 1} \sqrt{p^2 + q^2 + 1}} \right|,$$

the square of which is the clearly quadratic equation $(p^2 + q^2)(f^2 + 1) = pf + 1$.

To find all the quadratic conical curves,

from now on we assume that $\mathcal{C}_{F,\ell}^\varepsilon$ is quadratic,

hence satisfies an equation of the form $\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}x + \bar{e}y + \bar{f} = 0$, where the coefficients are real and $\bar{a} \geq 0$.

As every conical curve $\mathcal{C}_{F,\ell}^\varepsilon$ is symmetric in the x -axis, the quadratic equation should be invariant under changing y to $-y$, so $\bar{b} = \bar{e} = 0$ follows. Then the equation is of the form $\bar{a}x^2 + \bar{c}y^2 + \bar{d}x + \bar{g} = 0$, hence $\bar{c} \neq 0$, because otherwise the curve will degenerate into straight lines. So the quadratic equation simplifies to

$$ax^2 + y^2 + bx + c = 0, \quad a \geq 0. \quad (2.29)$$

As this is an ellipse, because it is bounded and intersect line OP in exactly two point, we deduce that

$$a > 0 \quad \text{and} \quad b^2 > 4ac. \quad (2.30)$$

Thus, for a point P of $\mathcal{C}_{F,\ell}^\varepsilon$ we have $q^2 = -ap^2 - bp - c$. Putting this into (2.28) gives the identity

$$\varepsilon \left(\frac{\pi}{2} - \arccos \frac{1}{\sqrt{(1-a)p^2 - bp + 1 - c}} \right) = \arccos \frac{(pf + 1)/\sqrt{f^2 + 1}}{\sqrt{(1-a)p^2 - bp + 1 - c}}. \quad (2.31)$$

Differentiating this with respect to p gives

$$\begin{aligned} & \varepsilon \frac{-\frac{2(1-a)p-b}{2((1-a)p^2-bp+1-c)^{3/2}}}{\sqrt{1-\frac{1}{(1-a)p^2-bp+1-c}}} \\ &= -\frac{-\frac{(2(1-a)p-b)(1+fp)}{2\sqrt{1+f^2}((1-a)p^2-bp+1-c)^{3/2}} + \frac{f}{\sqrt{1+f^2}\sqrt{(1-a)p^2-bp+1-c}}}{\sqrt{1-\frac{(1+fp)^2}{(1+f^2)((1-a)p^2-bp+1-c)}}}. \end{aligned} \quad (2.32)$$

Simplifying the ratios and multiplying both sides by $2(1-c-bp+(1-a)p^2)$ give the equivalent form

$$\begin{aligned} & \varepsilon \frac{2(1-a)p-b}{\sqrt{(1-a)p^2-bp-c}} \\ &= \frac{-(2(1-a)p-b)(1+fp) + 2f((1-a)p^2-bp+1-c)}{\sqrt{((1-a)p^2-bp+1-c)(1+f^2)-(1+fp)^2}}. \end{aligned} \quad (2.33)$$

After the additions and multiplications are completed this becomes

$$\varepsilon \frac{2(1-a)p-b}{\sqrt{(1-a)p^2-bp-c}} = \frac{-(fb+2(1-a))p+2f(1-c)+b}{\sqrt{((1-a)p^2-bp+1-c)(1+f^2)-(1+fp)^2}}. \quad (2.34)$$

Multiplying both sides of (2.34) by the product of the denominators and squaring gives

$$\begin{aligned} & \varepsilon^2(2(1-a)p-b)^2((1-a(1+f^2))p^2-(2f+b(1+f^2))p+(f^2-c(1+f^2))) \\ &= ((fb+2(1-a))p-(b+2f(1-c)))^2((1-a)p^2-bp-c). \end{aligned} \quad (2.35)$$

This equation is valid on an interval of p , so the coefficients of the polynomials on the sides are equal, hence

$$\begin{aligned} (p^4) \quad & 4\varepsilon^2(1-a)^2(1-a(1+f^2)) = (1-a)(fb+2(1-a))^2 \\ (p^3) \quad & 4\varepsilon^2((1-a)^2(2f+b(1+f^2))+b(1-a)(1-a(1+f^2))) \\ &= b(fb+2(1-a))^2+2(1-a)(b+2f(1-c))(fb+2(1-a)) \\ (p^2) \quad & \varepsilon^2(b^2(1-a(1+f^2))+4b(1-a)(2f+b(1+f^2))+ \\ &+4(1-a)^2(f^2-c(1+f^2))) \\ &= -c(fb+2(1-a))^2+2b(b+2f(1-c))(fb+2(1-a))+ \\ &+ (1-a)(b+2f(1-c))^2 \\ (p^1) \quad & 4\varepsilon^2(b(1-a)(f^2-c(1+f^2))+b^2(2f+b(1+f^2))) \\ &= b(b+2f(1-c))^2-2c(b+2f(1-c))(fb+2(1-a)) \\ (p^0) \quad & \varepsilon^2b^2(f^2-c(1+f^2)) = -c(b+2f(1-c))^2, \end{aligned}$$

where $\varepsilon, f > 0$ are fixed, and $a > 0$, $b^2 > 4ac$ by (2.30).

If $b = 0$, then (p^4) implies $a = 1$, and (p^0) implies $c = 0$ or $c = 1$. In both cases $\mathcal{C}_{F,\ell}^\varepsilon$ is empty by (2.29), and this contradiction proves $b \neq 0$.

If $c = 0$, then (p^0) implies $b = 0$, a contradiction, so $c \neq 0$.

If $a = 1$, then (p^3) gives $b = 0$, a contradiction, so $a \neq 1$.

Dividing (p^4) with $1 - a$ gives

$$4\varepsilon^2(1-a)(1-a(1+f^2)) = (fb + 2(1-a))^2, \quad (2.36)$$

Multiplying the sides of (2.36) with the opposite sides of (p^0) , respectively, we obtain

$$\begin{aligned} -4c(1-a)(b+2f(1-c))^2(1-a(1+f^2)) \\ = b^2(f^2 - c(1+f^2))(fb + 2(1-a))^2. \end{aligned} \quad (2.37)$$

Multiplying (2.36) with b and subtracting the result from (p^3) give, after a light simplification, that

$$2\varepsilon^2(1-a)(2f + b(1+f^2)) = (b + 2f(1-c))(fb + 2(1-a)). \quad (2.38)$$

The right-hand side of the square of (2.38) multiplied by $-c$ is the product of the right-hand sides of (p^0) and (2.36), so we get

$$-4c(1-a)(2f + b(1+f^2))^2 = b^2(f^2 - c(1+f^2))(1-a(1+f^2)). \quad (2.39)$$

Multiplying (2.38) by $2b$ and subtracting the product from (p^2) give

$$\begin{aligned} \varepsilon^2(b^2(1-a(1+f^2)) + 4(1-a)^2(f^2 - c(1+f^2))) \\ = (1-a)(b + 2f(1-c))^2 - c(fb + 2(1-a))^2. \end{aligned} \quad (2.40)$$

Multiplying (2.36) by c and adding to this give

$$\begin{aligned} \varepsilon^2((b^2 + 4c(1-a))(1-a(1+f^2)) + 4(1-a)^2(f^2 - c(1+f^2))) \\ = (1-a)(b + 2f(1-c))^2. \end{aligned} \quad (2.41)$$

Multiplying this with c and adding to the product of (p^0) and $(1-a)$ result in

$$c(b^2 + 4c(1-a))(1-a(1+f^2)) + (1-a)(b^2 + 4c(1-a))(f^2 - c(1+f^2)) = 0,$$

hence

$$b^2 + 4c(1-a) = 0 \quad \text{or} \quad 1 = a + c. \quad (2.42)$$

Add (p^1) times $1 - a$, (2.41) times $-b$ and (2.38) times $2c(1 - a)$. The result is

$$4(b^2 + c(1 - a))(1 - a)(2f + b(1 + f^2)) = b(b^2 + 4c(1 - a))(1 - a(1 + f^2)). \quad (2.43)$$

• Assume that $1 = a + c$ fulfills in (2.42).

Then $b^2 + 4a(1 - c) = b^2 + 4a^2 > 0$, and it is also easy to show that $f^2 - c(1 + f^2) = -(1 - a(1 + f^2))$. Further, (2.37) gives $4c^2(b + 2f(1 - c))^2 = b^2(fb + 2c)^2$, hence

$$\begin{aligned} 0 &= b^2(fb + 2c)^2 - 4c^2(b + 2f(1 - c))^2 \\ &= (b(fb + 2c) - 2c(b + 2f(1 - c)))(b(fb + 2c) + 2c(b + 2f(1 - c))) \\ &= f(b^2 - 4c(1 - c))(b(fb + 2c) + 2c(b + 2f(1 - c))). \end{aligned}$$

Since $b^2 > 4c(1 - c)$ by (2.30), we obtain

$$-b(fb + 2c) = 2c(b + 2f(1 - c)). \quad (2.44)$$

From (p^3) we obtain

$$\begin{aligned} 4\varepsilon^2 c(c(2f + b(1 + f^2)) - b(f^2 - c(1 + f^2))) \\ = b(fb + 2c)^2 + 2c(fb + 2c)(b + 2f(1 - c)). \end{aligned} \quad (2.45)$$

The right-hand side of this equation vanishes by (2.44), so we arrive at

$$c(2f + b(1 + f^2)) = b(f^2 - c(1 + f^2)). \quad (2.46)$$

Since $1 - a = c$, (2.38) reads

$$2\varepsilon^2 c(2f + b(1 + f^2)) = (b + 2f(1 - c))(fb + 2c). \quad (2.47)$$

Multiplying this with $2c$ and using (2.44) results in

$$4\varepsilon^2 c^2(2f + b(1 + f^2)) = 2c(b + 2f(1 - c))(fb + 2c) = -b(fb + 2c)^2. \quad (2.48)$$

Using (2.46) and then (2.36) this gives

$$4\varepsilon^2 cb(f^2 - c(1 + f^2)) = -(fb + 2c)^2 b = 4\varepsilon^2 c(1 - (1 - c)(1 + f^2))b. \quad (2.49)$$

Thus $f^2 - c(1 + f^2) = 1 - (1 - c)(1 + f^2)$, so, by (2.42) and (2.46), we have

$$c = \frac{f^2}{1 + f^2}, \quad a = \frac{1}{1 + f^2}, \quad \text{and} \quad b = \frac{-2f}{1 + f^2}. \quad (2.50)$$

This, however gives $b^2 = 4ac$ that contradicts (2.30).

- Assume that $1 \neq a + c$ and $b^2 + 4c(1 - a) = 0$ fulfills in (2.42).

As a first consequence, we get from (2.43) that

$$b^2 + c(1 - a) = 0 \quad \text{or} \quad 2f + b(1 + f^2) = 0. \quad (2.51)$$

In the former case the assumption implies $c(1 - a) = 0$, so either $c = 0$ or $a = 1$, which was already closed out, so we deduce $b = \frac{-2f}{1+f^2}$. From this (2.39) implies immediately that $0 = (f^2 - c(1 + f^2))(1 - a(1 + f^2))$, so

$$c = \frac{f^2}{1 + f^2} \quad \text{or} \quad a = \frac{1}{1 + f^2}. \quad (2.52)$$

Then, by the assumption we respectively obtain that

$$a = \frac{4c + b^2}{4c} = \frac{2 + f^2}{1 + f^2} \quad \text{and} \quad c = \frac{-b^2}{4(1 - a)} = \frac{-1}{1 + f^2}. \quad (2.53)$$

In the first case we get

$$b^2 - 4ac = \frac{4f^2}{(1 + f^2)^2} - 4 \frac{2 + f^2}{1 + f^2} \frac{f^2}{1 + f^2} = \frac{4f^2}{(1 + f^2)^2} (1 - (2 + f^2)) = \frac{-4f^2}{1 + f^2} < 0$$

that contradicts (2.30), so we deduce

$$c = \frac{-1}{1 + f^2}, \quad a = \frac{1}{1 + f^2}, \quad \text{and} \quad b = \frac{-2f}{1 + f^2}. \quad (2.54)$$

With these values (p^0) gives $\varepsilon = 1$. (p^1) gives also $\varepsilon = 1$.

Thus the second case in (2.42) implies a contradiction, while it follows from the first case of (2.42) that the conical curve is parabolic.

The contradiction means that the system of equations (p^0) – (p^4) does not have a solution, so the polynomials of the sides in (2.35) are different, hence the conical curves in this case are not quadratic.

Theorem 2.3 ([13]). *A conical curve on the sphere is quadratic if and only if either the focus is the pole of the directrix, or the focus is not the pole of the directrix, but the conical curve is parabolic, i.e. $\varepsilon = 1$.*

2.4 Symmetric conical curves on the sphere

Firstly we notice that the conical curve on the sphere is a hypersphere, hence symmetric if the focus is the pole of the directrix, so we assume for the sake of a later contradiction that

$$\hat{F} \text{ is not the pole of } \hat{\ell}, \text{ and } \hat{C}_{\delta; \hat{F}, \hat{\ell}}^{\varepsilon} \text{ is symmetric in a point } \hat{C}.$$

Such a point of symmetry \hat{C} clearly is on the great circle of $\hat{F}\hat{F}^\perp$, where \hat{F}^\perp is the unique foot of \hat{F} on the great circle $\hat{\ell}$.

Take the gnomonic projection $\Gamma_{\hat{C}}$. Let $\mathcal{C}_{F,\ell}^\varepsilon := \Gamma_{\hat{C}}(\hat{\mathcal{C}}_{\delta;\hat{F},\hat{\ell}}^\varepsilon)$, $P := \Gamma_{\hat{C}}(\hat{P})$ and $P^\perp := \Gamma_{\hat{C}}(\hat{P}^\perp)$ for any point P , and $\ell := \Gamma_{\hat{C}}(\hat{\ell})$. Choose the coordinate system so that $C = (0, 0, 1)$, $F = (f, 0, 1)$, and $\ell = \{(x, y, z) : x = m \wedge z = 1\}$. Figure 2.7 shows what we have on the plane $\mathcal{P} := T_{\hat{C}}\mathcal{S}^2 = \{(x, y, z) : z = 1\}$.

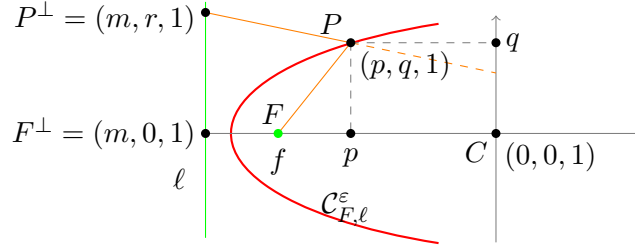


Figure 2.7: Projected conical curve $\mathcal{C}_{F,\ell}^\varepsilon$, if the directrix ℓ is parallel to the y -axis and the focus F is at $(f, 0)$, where $f < 0$.

The advantage of taking the gnomonic projection $\Gamma_{\hat{C}}$ is that $\hat{\mathcal{C}}_{\delta;\hat{F},\hat{\ell}}^\varepsilon$ is symmetric about \hat{C} in the spherical meaning if and only if $\mathcal{C}_{F,\ell}^\varepsilon$ is symmetric about C in the Euclidean meaning.

Since $\hat{F}^\perp \hat{P}^\perp \hat{P}$ is a right triangle on the sphere, the cosine rule for the spherical triangle [21] gives $\cos \hat{\delta}(\hat{F}^\perp, \hat{P}) = \cos \hat{\delta}(\hat{F}^\perp, \hat{P}^\perp) \cos \hat{\delta}(\hat{P}^\perp, \hat{P})$. Then (1.4) gives

$$\cos(\hat{\delta}(\hat{F}^\perp, \hat{P})) = \frac{\langle \hat{F}^\perp, \hat{P} \rangle}{|\hat{F}^\perp| |\hat{P}|}, \quad \cos(\hat{\delta}(\hat{F}^\perp, \hat{P}^\perp)) = \frac{\langle \hat{F}^\perp, \hat{P}^\perp \rangle}{|\hat{F}^\perp| |\hat{P}^\perp|}, \quad \cos(\hat{\delta}(\hat{P}^\perp, \hat{P})) = \frac{\langle \hat{P}^\perp, \hat{P} \rangle}{|\hat{P}^\perp| |\hat{P}|},$$

so we obtain $\langle \hat{F}^\perp, \hat{P} \rangle |\hat{P}^\perp|^2 = \langle \hat{F}^\perp, \hat{P}^\perp \rangle \langle \hat{P}^\perp, \hat{P} \rangle$, i.e.

$$(mp + 1)(m^2 + r^2 + 1) = (m^2 + 1)(mp + rq + 1).$$

This equation is equivalent to equation $r(r(mp + 1) - q(m^2 + 1)) = 0$ that gives

$$r = \frac{q(m^2 + 1)}{mp + 1}. \quad (2.55)$$

Thus, by (1.4), we have

$$\begin{aligned} \delta(P, \ell) &= \delta(P, (m, r, 1)) \\ &= \arccos \frac{mp + rq + 1}{\sqrt{m^2 + r^2 + 1} \sqrt{p^2 + q^2 + 1}} = \arccos \frac{(mp + 1) \sqrt{m^2 + r^2 + 1}}{(m^2 + 1) \sqrt{p^2 + q^2 + 1}} \\ &= \arccos \frac{\sqrt{(mp + 1)^2 + q^2(m^2 + 1)}}{\sqrt{m^2 + 1} \sqrt{p^2 + q^2 + 1}}, \end{aligned} \quad (2.56)$$

where we used (2.55) and its predecessor. For the distance of P from the focus we have (2.27).

According to (D_1) equations (2.56) and (2.27) give

$$\varepsilon \arccos \frac{\sqrt{(mp+1)^2 + q^2(m^2+1)}}{\sqrt{m^2+1}\sqrt{p^2+q^2+1}} = \arccos \frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+q^2+1}}. \quad (2.57)$$

Figure 2.8 shows how these conical curves look like by (2.28).

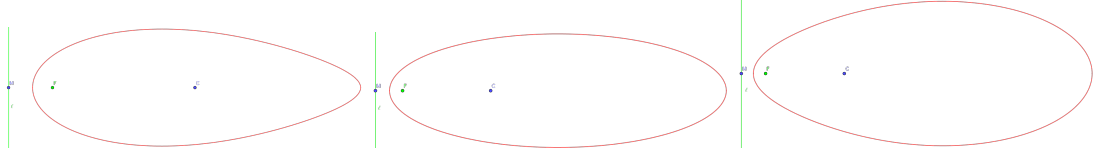


Figure 2.8: An elliptic ($\varepsilon = 0.90$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conical curve in projected model of the sphere.

We now that there exist exactly two solutions of (2.57) for $q = 0$, and by the symmetry these are $\pm p_0$. Thus $\pm p_0$ satisfies

$$\varepsilon \arccos \frac{|mp+1|}{\sqrt{m^2+1}\sqrt{p^2+1}} = \arccos \frac{pf+1}{\sqrt{f^2+1}\sqrt{p^2+1}}. \quad (2.58)$$

Let $m = \tan \mu$, $f = \tan \varphi$, and $p_0 = \tan \varpi$. Substituting these values into (2.58) results in

$$\varepsilon \arccos \frac{|1 \pm \tan \mu \tan \varpi|}{\sqrt{1 + \tan^2 \mu} \sqrt{1 + \tan^2 \varpi}} = \arccos \frac{1 \pm \tan \varphi \tan \varpi}{\sqrt{1 + \tan^2 \varphi} \sqrt{1 + \tan^2 \varpi}}, \quad (2.59)$$

i.e.

$$\varepsilon \arccos |\cos \mu \cos \varpi \pm \sin \mu \sin \varpi| = \arccos(\cos \varphi \cos \varpi \pm \sin \varphi \sin \varpi), \quad (2.60)$$

hence by the angle sum and difference identities [20] we get

$$\varepsilon \arccos |\cos(\mu \mp \varpi)| = \arccos(\cos(\varphi \mp \varpi)), \quad (2.61)$$

Thus, $\varepsilon \mu - \varphi = \pm \varpi(1 - \varepsilon)$, hence $\varpi(1 - \varepsilon) = 0$. Since $\varpi \neq 0$, we get $\varepsilon = 1$, so $\mu = \varphi$ that is a contradiction.

In sum, we have proved the following theorem.

Theorem 2.4 ([13]). *A conical curve on the sphere is symmetric if and only if the focus is the pole of the directrix.*

2.5 Conical ellipses and conical hyperbolas

As every ellipse and every hyperbola in the hyperbolic plane is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem 2.2 implies the following.

Theorem 2.5. *There is no conical ellipse or conical hyperbola in the hyperbolic plane.*

As every ellipse and every hyperbola on the sphere is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem 2.4 implies the following.

Theorem 2.6. *Every conical ellipse and every conical hyperbola on the sphere is a circle.*

Bibliography

- [1] V. I. ARNOLD, *Real algebraic geometry*, Unitext, vol. **66**, Springer, Heidelberg, 2013. Translated from the 2009 Russian original by Gerald G. Gould and David Kramer, Edited and with a foreword by Ilia Itenberg, Viatcheslav Kharlamov and Eugenii I. Shustin, La Matematica per il 3+2.
- [2] F. JR. AYRES, *Projective geometry*, Schaum's outline series, Schaum Pub. Co., 1967.
- [3] H. BUSEMANN, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955.
- [4] H. BUSEMANN, and P. J. KELLY, *Projective geometry and projective metrics*, Academic Press Inc., New York, N. Y., 1953.
- [5] J. W. CANNON, W. J. FLOYD, R. KENYON, AND W. R. PARRY, Hyperbolic geometry, *Flavors of geometry*, Math. Sci. Res. Inst. Publ., vol. **31**, Cambridge Univ. Press, Cambridge, 1997, pp. 59–115.
- [6] M. P. DO CARMO, *Differential geometry of curves & surfaces*, Dover Publications, Inc., Mineola, NY, 2016.
- [7] R. CASSE, *Projective geometry: an introduction*, Oxford University Press, Oxford, 2006.
- [8] P. J. KELLY AND L. J. PAIGE, Symmetric perpendicularity in Hilbert geometries, *Pacific J. Math.* **2** (1952), 319–322, available at <https://projecteuclid.org/euclid.pjm/1103051777>.
- [9] Á. KURUSA, Support theorems for totally geodesic Radon transforms on constant curvature spaces, *Proc. Amer. Math. Soc.* **122** (1994), no. 2, 429–435, doi: [10.2307/2161033](https://doi.org/10.2307/2161033).
- [10] Á. KURUSA, Conics in Minkowski geometries, *Aequationes Math.* **92** (2018), no. 5, 949–961, doi: [10.1007/s00010-018-0592-1](https://doi.org/10.1007/s00010-018-0592-1).
- [11] A. M. MAHDI, Quadratic conics in hyperbolic geometry, *Int. J. Geom.* **8** (2019), no. 2, 60–69, available at <https://ijgeometry.com/wp-content/uploads/2019/09/60-69.pdf>.
- [12] A. M. MAHDI, Symmetric conics in hyperbolic geometry, *Glob. J. Adv. Res. Class. Mod. Geom.* **8** (2019), no. 2, 49–56, available at <http://geometry-math-journal.ro/pdf/Volume8-Issue2/1.pdf>.

- [13] A. M. MAHDI, Conics on the sphere, *Int. J. Geom.* **9** (2020), no. 2, 5–14, available at <https://ijgeometry.com/wp-content/uploads/2020/>.
- [14] J. RADON, Über eine besondere Art ebener konvexer Kurven, *Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl.* **68** (1916), 123–128 (German).
- [15] L. TAMÁSSY, and K. BÉLTEKY, On the coincidence of two kinds of ellipses in Minkowskian spaces and in Finsler planes, *Publ. Math. Debrecen* **31** (1984), no. 3-4, 157–161.
- [16] L. TAMÁSSY, and K. BÉLTEKY, On the coincidence of two kinds of ellipses in Riemannian and in Finsler spaces, *Topics in differential geometry, Vol. I, II (Debrecen, 1984)*, Colloq. Math. Soc. János Bolyai, vol. **46**, North-Holland, Amsterdam, 1988, pp. 1193–1200.
- [17] WIKIPEDIA, Dandelin spheres, available at https://en.wikipedia.org/wiki/Dandelin_spheres.
- [18] WIKIPEDIA, Gnomonic projection, available at https://en.wikipedia.org/wiki/Gnomonic_projection.
- [19] WIKIPEDIA, Hilbert's theorem (differential geometry), available at [https://en.wikipedia.org/wiki/Hilbert's_theorem_\(differential_geometry\)](https://en.wikipedia.org/wiki/Hilbert's_theorem_(differential_geometry)).
- [20] WIKIPEDIA, List of trigonometric identities, available at https://en.wikipedia.org/wiki/List_of_trigonometric_identities.
- [21] WIKIPEDIA, Spherical trigonometry: Cosine rules and sine rules, available at https://en.wikipedia.org/wiki/Spherical_trigonometry#Cosine_rules_and_sine_rules.
- [22] J. A. WOLF, *Spaces of constant curvature*, Sixth, AMS Chelsea Publishing, Providence, RI, 2011.
- [23] WOLFRAMALPHA, Solving equation:, available at [https://www.wolframalpha.com/input/?i=x\(1-f^2\)sqrt{1-\(x-f\)^2}=x^2%2B2f^2-f^4-4xf%2B2xf^3-\(fx-1\)|x-f|sqrt{2-f^2}](https://www.wolframalpha.com/input/?i=x(1-f^2)sqrt{1-(x-f)^2}=x^2%2B2f^2-f^4-4xf%2B2xf^3-(fx-1)|x-f|sqrt{2-f^2}).
- [24] WOLFRAMALPHA, Solving equation:, available at [https://www.wolframalpha.com/input/?i=\(1-f^2\)\(sqrt{1-\(x-f\)^2}%2Bx\)^2=\(fx-1-\(f-x\)sqrt{2-f^2}\)^2](https://www.wolframalpha.com/input/?i=(1-f^2)(sqrt{1-(x-f)^2}%2Bx)^2=(fx-1-(f-x)sqrt{2-f^2})^2).
- [25] WOLFRAMALPHA, Solving equation:, available at [https://www.wolframalpha.com/input/?i=\(1-f^2\)\(sqrt{1-3\(x-f\)^2}%2Bx\)^2=\(fx-1-\(f-x\)sqrt{4-3f^2}\)^2](https://www.wolframalpha.com/input/?i=(1-f^2)(sqrt{1-3(x-f)^2}%2Bx)^2=(fx-1-(f-x)sqrt{4-3f^2})^2).

Notations

\mathbb{R} the set of real numbers

\mathbb{R}^n the n -dimensional real space

\mathbb{P}^n the n -dimensional projective space

points A, B, \dots or \mathbf{a} if the origin is fixed,

vectors \overrightarrow{AB} or \mathbf{a} ,

\overline{AB} open segment with endpoints A and B

\overrightarrow{AB} open ray starting from A passing through B

AB the line through points A and B ;

(A, B) open segment with endpoints A and B

$(A, B; C)$ the affine ratio of the collinear points A, B and C

$(A, B; C, D)$ the cross ratio of the collinear points A, B and C, D

\mathcal{S}^{n-1} the sphere in the n -dimensional Euclidean space

Summary

According to [15], A. Moór raised the request for determining those Finsler manifolds in which the class of elliptic conical curves coincides with the class of ellipses, or the class of hyperbolic conical curves coincides with the class of hyperbolas. Tamássy and Béteky found in [16, Theorem 2], that the only Finsler space where the class of elliptic conical curves coincides with the class of ellipses is the Euclidean space.

A similar problem was solved by Kurusa in [10, Theorem 6.1], where he proved that the only Minkowski geometry in which either a conical ellipsoid or a conical hyperboloid exists is the Euclidean one. At the end of his paper [10] Kurusa formulated the problem of determining projective-metric spaces in which

- (a) some or all ellipses are conical, or
- (b) some or all hyperbolas are conical.

Kurusa's main result [10, Theorem 6.1] was based on that, by [10, Theorem 4.2 and 4.3], the only Minkowski geometry in which a symmetric conical curve exists is the Euclidean one. Additionally, it is also proved in [10, Theorem 5.1] that the only Minkowski plane in which a quadratic conical curve exists is the Euclidean one. So Kurusa also raised the request to determine the projective-metric spaces in which

- (c) some or all elliptic conical curves are symmetric, or
- (d) some or all hyperbolic conical curves are symmetric, or
- (e) some or all elliptic conical curves are quadratic, or
- (f) some or all hyperbolic conical curves are quadratic.

All the requests remained unanswered for curved projective-metric spaces.

The purpose of my research was to answer Kurusa's requests for the curved constant curvature spaces. In the hyperbolic case there was no surprise, but it turned out in the spherical case that there are some very special conical curves.

The results in this dissertation are the fruit of my research. They were published in my papers [11], [12], and [13].

In a projective-metric space (\mathcal{M}, d) we define

(D_1) a *conical curve* as the set

$$\mathcal{C}_{F, \mathcal{H}}^\varepsilon := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\},$$

where \mathcal{H} is a hyperplane, the *leading hyperplane* or *directrix*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity*. A conical curve is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

For given fixed points F_1, F_2 , the *foci*, and number $a \neq d(F_1, F_2)/2$, the *radius*, we define

(D_2) the *ellipsoid* (*ellipse* in dimension 2) as the set

$$\mathcal{E}_{d; F_1, F_2}^a := \{E : 2a = d(F_1, E) + d(E, F_2)\}, \text{ and}$$

(D_3) the *hyperboloid* (*hyperbola* in dimension 2) as the set

$$\mathcal{H}_{d; F_1, F_2}^a := \{X : 2a = |d(F_1, X) - d(X, F_2)|\}.$$

By the triangle inequality, $a > d(F_1, F_2)/2$ for the ellipsoid, and $a < d(F_1, F_2)/2$ for the hyperboloid. Value $2f := d(F_1, F_2)$ is the *eccentricity*, and if the eccentricity vanishes, then the ellipsoid (ellipse) is called *sphere* (*circle*). Further, an ellipsoid (ellipse) or hyperboloid (hyperbola) is called *conical* if it is a conical curve.

Unlike the above definitions quadraticity is defined with no reference for the metric. We define

(D_4) a *quadratical curve* as the set of points satisfying the equation of the form

$$\mathcal{Q}_s^\sigma := \left\{ (x, y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases} \right\}, \quad (D_q)$$

for some affine coordinate system \mathfrak{s} .

Quadratical curves are also called *quadric*, and a quadric is called *elliptic*, *parabolic*, and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.

A set $\mathcal{S} \subset \mathcal{M}$ in a projective-metric space (\mathcal{M}, d) is called *symmetric about a point* $C \in \mathcal{M}$, if $X \in \mathcal{S}$ if and only if $Y \in \mathcal{S}$, where C is in the metric midpoint of the segment \overline{XY} , i.e. $2d(X, C) = 2d(C, Y) = d(X, Y)$.

Section 2.1, "Quadratic conical curves in the hyperbolic plane", is based on my paper [11].

We say that a conical curve is quadratic if it fits on a quadric (D_q), i.e. satisfies an equation of the form $\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}x + \bar{e}y + \bar{f} = 0$, where the coefficients are real and $\bar{a} \geq 0$.

Theorem S.1 ([11]). *No conical curve of the hyperbolic plane can be quadratic.*

The proof is a long and tricky calculation that determines the coefficients of the quadratic equation and leads to contradiction.

Section 2.2, "Symmetric conical curves in the hyperbolic plane", is based on my paper [12].

We say that a conical curve is symmetric if it is symmetric in a point.

Theorem S.2 ([12]). *No conical curve of the hyperbolic plane can be symmetric.*

For the proof we first determine with some tiring calculations the exponential equation of a conical curve in the Cayley–Klein model and then investigate the tangents of such curve if the conical curve is symmetric in the center of the model. Then properly chosen points on the conical curve show that the symmetry causes contradictions, hence proving the theorem.

Section 2.3, "Quadratic conical curves on the sphere", and Section 2.4, "Symmetric conical curves on the sphere", are based on the paper [13].

Theorem S.3 ([13]). *A conical curve on the sphere is quadratic if and only if either the focus is the pole of the directrix, or the focus is not the pole of the directrix, but the conical curve is parabolic, i.e. $\varepsilon = 1$.*

The proof is a long and tricky calculation that determines the coefficients of the quadratic equation and leads to contradiction.

Theorem S.4 ([13]). *A conical curve on the sphere is symmetric if and only if the focus is the pole of the directrix.*

Section 2.5, "Conical ellipses and conical hyperbolas", contains some direct consequences of the above results.

As every ellipse and every hyperbola in the hyperbolic plane is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem S.2 implies the following.

Theorem S.5. *There is no conical ellipse or conical hyperbola in the hyperbolic plane.*

As every ellipse and every hyperbola on the sphere is symmetric, every conical ellipse and every conical hyperbola is a symmetric conical curve, hence Theorem S.4 implies the following.

Theorem S.6. *Every conical ellipse and every conical hyperbola on the sphere is a circle.*

Magyar nyelvű összefoglaló

(Summary in Hungarian)

Tamássy Lajos [15] cikke szerint Moór Artúr vetette fel azon Finsler sokaságok meghatározásának problémáját, amelyekben az elliptikus kúpszeletek osztálya megegyezik az ellipszisek osztályával, vagy a hiperbolikus kúpszeletek osztálya megegyezik a hiperbolák osztályával. Az első esetet Tamássy és Béteky tisztázta [16, Theorem 2]: az egyetlen Finsler tér, amelyben az elliptikus kúpszeletek osztálya megegyezik az ellipszisek osztályával az euklideszi tér.

Hasonló probléma megoldását adta Kurusa Árpád is, amikor igazolta [10, Theorem 6.1], hogy az Euklidészi az egyetlen olyan Minkowski sík, amelyben egy kúpmetaszt-görbe ellipszis vagy egy kúpmetaszt-görbe hyperbola. A [10] cikke végén Kurusa felvetette azon projektív-metrikus terek meghatározásának problémáját, melyekben

- (a) néhány vagy minden ellipszis kúpmetaszt-görbe, vagy
- (b) néhány vagy minden hyperbolas kúpmetaszt-görbe.

Kurusa [10, Theorem 6.1] eredményének bizonyítása azon alapult, hogy a [10, Theorem 4.2 and 4.3] tételek szerint az Euklidészi az egyetlen olyan Minkowski sík, amelyben van szimmetrikus kúpmetaszt-görbe. Továbbá a [10, Theorem 5.1] eredményben Kurusa azt is igazolta, hogy az Euklidészi az egyetlen olyan Minkowski sík, amelyben van kvadratikus kúpmetaszt-görbe. Ezekből kiindulva Kurusa felvetette azon projektív-metrikus terek meghatározásának problémáját, melyekben

- (c) néhány vagy minden elliptikus kúpmetaszt-görbe szimmetrikus, vagy
- (d) néhány vagy minden hiperbolikus kúpmetaszt-görbe szimmetrikus, vagy
- (e) néhány vagy minden elliptikus kúpmetaszt-görbe kvadratikus, vagy
- (f) néhány vagy minden hiperbolikus kúpmetaszt-görbe kvadratikus.

Minden Kurusa által felvetett kérdés megválaszolatlanul maradt a görbült projektív-metrikus terekre.

Kutatásaim célja az volt, hogy meghatározzam Kurusa kérdéseire a megfelelő görbült konstans görbületű tereket. A hiperbolikus tér esetén nem volt meglepetés, de a gömbfelületen kiderült, hogy van néhány nagyon speciális kúpmetaszt-görbe.

Ebben a disszertációban kutatásaimnak a [11], [12], és [13] cikkekben már publikált eredményei kerülnek bemutatásra.

Egy (\mathcal{M}, d) projektív-metrikus térben definíció szerint

(D_1) egy *kúpmetszet-görbe* nem más mint a

$$\mathcal{C}_{F, \mathcal{H}}^\varepsilon := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\} \text{ halmaz,}$$

ahol \mathcal{H} egy hipersík, ami a *vezérhipersík* vagy *direktrix*, $F \notin \mathcal{H}$ egy pont, ami a *fókusz* és $\varepsilon > 0$ egy szám, a *numerikus excentricitás*. Egy kúpmetszet-görbéről azt mondjuk, hogy *elliptikus*, *parabolikus* vagy *hiperbolikus*, ha rendre $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$.

Az F_1, F_2 pontok, a *fókuszok*, továbbá egy $a \neq d(F_1, F_2)/2$ szám, a *sugár*, esetén

(D_2) *ellipszoidnak* (kettő dimenzióban *ellipszisnek*) hívjuk a

$$\mathcal{E}_{d; F_1, F_2}^a := \{E : 2a = d(F_1, E) + d(E, F_2)\},$$

(D_3) *hiperboloidnak* (kettő dimenzióban *hiperbolának*) hívjuk a

$$\mathcal{H}_{d; F_1, F_2}^a := \{X : 2a = |d(F_1, X) - d(X, F_2)|\},$$

halmazt annak megfelelően, hogy rendre $a > d(F_1, F_2)/2$ vagy $a < d(F_1, F_2)/2$. A $2f := d(F_1, F_2)$ érték az *excentricitás*, és ha az excentricitás eltűnik, akkor az ellipszoidot (ellipszist) gömbfelületnek (körvonalnak) hívjuk. Továbbá, egy ellipszoid (ellipszis) vagy hiperboloid (hiperbola) kúpmetszett, ha kúpmetszet-görbe.

Eltérően a fenti definícióktól a kvadratikuság definíciója nem hivatkozik a metrikára.

(D_4) Egy *kvadratikus görbe* azon pontok halmaza, amelyek valamely \mathfrak{s} affin koordináta-rendszerben teljesítenek egy

$$\mathcal{Q}_{\mathfrak{s}}^\sigma := \left\{ (x, y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases} \right\}, \quad (D_q)$$

egyenletet.

A kvadratikus görbéket *kvadrikáknak* is nevezik, és egy kvadrikát *elliptikusnak*, *parabolikusnak* vagy *hiperbolikusnak* hívunk, ha rendre $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$.

Egy projektív-metrikus (\mathcal{M}, d) tér $\mathcal{S} \subset \mathcal{M}$ részhalmazát egy $C \in \mathcal{M}$ pontra nézve *szimmetrikusnak* mondunk, ha $X \in \mathcal{S}$ akkor és csak akkor, ha $Y \in \mathcal{S}$, ahol C az \overline{XY} szakasz metrikus középpontja, vagyis $2d(X, C) = 2d(C, Y) = d(X, Y)$.

Az 2.1 szakasz, „Kvadratikus, kúpmetszet-görbék a hiperbolikus síkon”, a [11] cikkemen alapul.

Egy kúpmetszet-görbét kvadratikusnak mondunk ha egy (D_q) kvadrikára illeszkedik, vagyis teljesít egy $\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}x + \bar{e}y + \bar{f} = 0$ egyenletet, ahol az együtthatók valós számok és $\bar{a} \geq 0$.

Theorem O.1 ([11]). *A hiperbolikus síkon kúpmetaszt-görbe nem lehet kvadrátikus.*

A bizonyítás egy hosszú és trükkös számítás, amely meghatározza a másodlagos egyenlet együtthatóit és ellentmondáshoz vezet.

A 2.2 szakasz, „Szimmetrikus kúpmetaszt-görbék a hiperbolikus síkon”, a [12] cikkemen alapul.

Egy kúpmetaszt-görbét szimmetrikusnak neveziünk, ha szimmetrikus egy pontra nézve.

Theorem O.2 ([12]). *Hiperbolikus síkon kúpmetaszt-görbe nem lehet szimmetrikus.*

Ennek igazolására először fárasztó számítással meghatározzuk a kúpmetaszt-görbe exponenciális egyenletét a Cayley–Klein modellben, majd megvizsgáljuk az ilyen görbe érintőit, feltételezve, hogy a kúpmetaszt-görbe szimmetrikus a modell centrumára nézve. Ezután a kúpmetaszt-görbe megfelelően megválasztott pontjai azt mutatják, hogy a szimmetria ellentmondásokat idéz elő, ezáltal bizonyítva a tételt.

Az 2.3 szakasz, „Kvadrátikus kúpmetaszt-görbék a gömbfelületen”, és az 2.4 szakasz, „Szimmetrikus kúpmetaszt-görbék a gömbfelületen”, alfejezetek a [13] cikkemen alapulnak.

Theorem O.3 ([13]). *A gömbfelületen egy kúpmetaszt-görbe pontosan akkor kvadrátikus, ha a fókuszpontja a direktrix pólusa vagy nem a direktrix pólusa, de a kúpmetaszt-görbe parabolikus, vagyis $\varepsilon = 1$.*

Theorem O.4 ([13]). *A gömbfelületen egy kúpmetaszt-görbe pontosan akkor szimmetrikus, ha a fókuszpontja a direktrix pólusa.*

A 2.5 szakasz, „Kúpmetaszt-ellipszisek és kúpmetaszt-hiperbolák”, a fenti eredmények egyenes következményeit tartalmazza.

Mivel minden ellipszis és minden hiperbola szimmetrikus a hiperbolikus síkon, ezért minden kúpmetaszt-ellipszis és kúpmetaszt-hiperbola szimmetrikus kúpmetaszt-görbe, tehát a O.4. Tételtől következik a következő.

Theorem O.5. *Hiperbolikus síkon egyetlen ellipszis vagy hiperbola se kúpmetaszt-görbe.*

Mivel a gömbfelszínen minden ellipszis és minden hiperbola szimmetrikus, így minden kúpmetaszt-ellipszis és kúpmetaszt-hiperbola szimmetrikus kúpmetaszt-görbe, és így a O.4. Tételtől következik a.

Theorem O.6. *A gömbfelületen minden kúpmetaszt-ellipszis és kúpmetaszt-hiperbola körvonal.*

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Szeged, 2020

AHMED MOHSIN MAHDI

Declaration of Authorship

I, AHMED MOHSIN MAHDI, declare that this thesis titled, '*Conical Curves in Constant Curvature Planes*' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:
