ON STABILITY PROPERTIES OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH STEP FUNCTION COEFFICIENTS

Outline of PhD Thesis

LÁSZLÓ SZÉKELY

SUPERVISOR:

DR. LÁSZLÓ HATVANI

DOCTORAL SCHOOL IN MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF SZEGED
FACULTY OF SCIENCE AND INFORMATICS
BOLYAI INSTITUTE

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1 Introduction

In the dissertation we examine two problems related to differential equations with step function coefficients. First, we consider second order linear differential equations, where both elasticity coefficient and damping coefficient are step functions. For such equations, we give sufficient condition on the existence of a small solution, i.e. the existence of such a solution which tends to 0 with respect to x. For the proof of the theorem, as a tool, we need conditions guaranteeing the existence of a small solution of two dimensional systems of linear difference equations. Although, we prove more: we give necessary and sufficient conditions on the existence of a small solution of difference equations of arbitrarily finite dimension.

In the second part of the thesis, we consider the Armellini-Tonelli-Sansone theorem for second order linear differential equations with varying elasticity coefficient. This theorem gives a sufficient condition on that all solutions of such equations are small. We extend this theorem to the so-called half-linear differential equations in the case when the coefficient is a step function. Half-linear differential equations have many important applications (see eg. [3], [4]). For the extension of the Armellini-Tonelli-Sansone theorem to the half-linear case, we need to prove a new theorem on the asymptotic stability of two dimensional systems of linear difference equations. The proof is based on a geometric method which applies also for the nonlinear case.

The dissertation is based on the following papers of the author:

- L. Hatvani, L. Székely, On the existence of small solutions of linear difference equations with varying coefficients, *J. Difference Equ. Appl.*, **12** (2006), No. 8, 837–845.

- L. Hatvani, L. Székely, Asymptotic stability of two dimensional systems of linear difference equations and of second order half-linear differential equations with step function coefficients, *E. J. Qualitative Theory of Diff. Equ.*, **38** (2011), 1–17.

In this outline we use the same notations and numberings (apart from the labels of the formulas) as in the thesis.

Preliminaries

Consider the second order differential equation

$$x'' + a(t)x = 0 \tag{LO}$$

describing the motion of a linear oscillator with varying elasticity coefficient. The Pólya-Sonin theorem (see eg. [19]) says, that if $a : [0, \infty) \to [0, \infty)$ is a monotone non-decreasing function, then all nontrivial solutions of equation (LO) are oscillatory, the maxima of $|x|$, i.e. the size of the amplitudes is non-increasing, and the neighboring maxima of $|x|$, that is the distances between neighboring extrema of x are non-decreasing.

Definition 1.1 *A nontrivial solution x_0 of equation (LO) is called* small *if*

$$\lim_{t\to\infty} x_0(t) = 0.$$

Milloux [16], Prodi [18] and Trevisan [20] proved that if $a : [0, \infty) \to [0, \infty)$ is differentiable and non-decreasing then equation (LO) has at least one small solution if and only if $\lim_{t\to\infty} a(t) = \infty$ holds. Milloux also constructed an example with a step function coefficient a, where not all solutions of equation (LO) were small. Hartman [9] investigated the linear system of differential equations

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \tag{LR}$$

where $\mathbf{x}$ is an m dimensional vector and $\mathbf{A}$ is an $m \times m$ matrix having real continuous entries with domain $[0, \infty)$. He proved the following:

Theorem 1.3 *Suppose that for all solutions* $\mathbf{x}$ *of equation (LR)* $\lim_{t\to\infty} \|\mathbf{x}(t)\| < \infty$ *holds. Then equation (LR) has at least one small solution if and only if*

$$\int^t \operatorname{tr} \mathbf{A}(s)\mathrm{d}s \to -\infty \qquad (t \to \infty).$$

Based on this result, Hartman [9] extended the theorem of Milloux, Prodi and Trevisan to systems of equations, furthermore, he proved that instead of differentiability, it is sufficient to assume the continuity of a.

The Armellini-Tonelli-Sansone [15] theorem was the first to give a sufficient condition on that *all* solutions of equation (LO) are small with the

following concept. A nondecreasing function $f : [0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} f(t) = \infty$ is called to grow *intermittently* if for every $\varepsilon > 0$ there is a sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ of disjoint intervals such that $a_i \to \infty$ as $i \to \infty$, and

$$\limsup_{i\to\infty} \sum_{k=1}^{i} \frac{b_k - a_k}{b_i} \leq \varepsilon, \qquad \sum_{i=1}^{\infty} (f(a_{i+1}) - f(b_i)) < \infty$$

are satisfied. Roughly speaking, this condition means that the growth of f cannot be located to a set with small measure. If such a sequence does not exist, then f is called to grow *regularly*.

Theorem 1.4 *If a is continuously differentiable and it grows to infinity regularly as $t \to \infty$, then all non-trivial solutions of equation (LO) are small.*

It is important to note that this stability property is weaker than the asymptotic stability of the trivial solution of equation (LO).

The simplest case of intermittent growth is is a monotonously increasing step function. Such equations have an important role for example in the field of control theory thanks to the so-called Bang-Bang principle. Differential equations with step function coefficients can be rewritten as systems of difference equations, thus the proof of theorems on such equations can be deduced to the proof of statements on difference equations.

2 On small solutions of second order linear differential equations with step function coefficients

In Chapter 2 we consider the equation

$$x'' + c(t)x' + a^2(t)x = 0 \tag{LOS}$$

describing the motion of an oscillator where both elasticity coefficient a and damping coefficient c are step functions. Namely, $\{t_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ and

$\{c_n\}_{n=1}^{\infty}$ are real sequences with the following properties:

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n < \ldots; \qquad \lim_{n\to\infty} t_n = \infty,$$
$$a_n > 0, \quad c_n \geq 0 \quad (n = 1, 2, \ldots),$$

furthermore, $a(t) = a_n$ and $c(t) = c_n$ on the interval $[t_n, t_{n-1})$. In the case when $\lim_{n\to\infty} a_n = \infty$ and damping doesn't act, i.e. $c_n = 0$ $(n = 1, 2, \ldots)$, Hatvani [10] proved that there exists at least one small solution of equation (LOS) if $\sum_{n=1}^{\infty} \max\{a_n/a_{n+1} - 1; 0\} < \infty$ holds. It is natural to guess that damping helps weaken this condition and even the condition $\lim_{n\to\infty} a_n = \infty$. In fact, in the main theorem of this chapter we could prove the following.

Theorem 2.2 *Assume that the above conditions on sequences* $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *are satisfied, and let us introduce the notation*

$$\gamma_n := \frac{c_n}{2a_n + c_n}[(2a_n - c_n)(t_n - t_{n-1}) - 2].$$

Suppose, in addition, that

(i) $a_n > c_n/2$ $(n = 1, 2, \ldots)$,

(ii)
$$\sum_{k=1}^{\infty} \left(-\gamma_k + \ln \frac{a_k}{a_{k+1}} \right) = -\infty,$$

(iii) *there is a number* K *such that for arbitrary* n $(n = 1, 2, \ldots)$
$$\sum_{k=1}^{n} \left(-\frac{\gamma_k}{2} + \ln \max\left\{ \frac{a_k}{a_{k+1}}; 1 \right\} \right) < K$$
holds.

Then equation (LOS) has at least one small solution.

Remark 2.3 *Theorem 2.2 is an improvement of Theorem 3.2 in [11], on which paper the dissertation is based.*

Equation (LOS) is equivalent with a two dimensional system of difference equations, therefore for the proof of our theorem we need a sufficient condition guaranteeing the existence of a small solution of such system. We discuss

the problem of finding a sufficient condition guaranteeing the existence of a small solution of two dimensional system of difference equations in a more general manner, namely, we give necessary and sufficient conditions for the existence of such solutions of arbitrarily finite dimensional systems.

On small solutions of difference equations

We consider the following nonautonomous system of difference equations:

$$\mathbf{x}_{n+1} = \mathbf{M}_n \mathbf{x}_n, \qquad n = 0, 1, 2, \ldots, \tag{DE}$$

where $\mathbf{x}_n \in \mathbb{R}^m$ is a column vector, $m \in \mathbb{N}$ and $\mathbf{M}_n \in \mathbb{R}^{m \times m}$ is an $m \times m$ matrix having real entries. A nontrivial $\{\mathbf{x}_n\}_{n=0}^{\infty}$ solution of this equation is called *small* if $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{0}$. Our aim is to extend Hartman's theorem on linear system of differential equations to linear systems of difference equations. As the first result of this section, we could prove the following:

Theorem 2.8 ([11]) *Suppose that the finite limit*

$$\prod_{n=0}^{\infty} \|\mathbf{M}_n\| < \infty$$

exists. Then,

(a) *for every solution* $\{\mathbf{x}_n\}_{n=0}^{\infty}$ *of (DE) the sequence* $\{\|\mathbf{x}_n\|\}_{n=0}^{\infty}$ *has a finite limit as* $n \to \infty$;

(b) *the infinite product* $\prod_{n=0}^{\infty} |\det \mathbf{M}_n|$ *is convergent; moreover,*

(c) *there exists at least one small solution of (DE) if and only if*

$$\prod_{n=0}^{\infty} |\det \mathbf{M}_n| = 0.$$

With this result we weaken the sufficient conditions on the existence of the limit of the solutions' norm given by Peil and Patterson [17] and Elbert [7]. In addition, we extend Elbert's method of proof from two dimension to arbitrary dimension m and we give a new proof to the theorem of Peil and Patterson.

One can easily see, that $\prod_{n=0}^{\infty} \|\mathbf{M}_n\| < \infty$ is not necessary for the existence of the limit of the norm of all solutions of (DE). A simple example can be constructed to show that this property is not essential from the point of view of the existence of a small solutions. In the main theorem of this section, by using a geometric method of proof we show that $\prod_{n=0}^{\infty} |\det \mathbf{M}_n| = 0$ is necessary and sufficient to have at least one small solution if we require only the boundedness of the sequence $\| \prod_{n=p}^{q} \mathbf{M}_n \|$ $(0 \le p \le q)$.

Theorem 2.9 ([11]) *Suppose that there is a $K \in \mathbb{R}$ such that for every $p, q \in \mathbb{N}$, $(0 \le p \le q)$ we have*

$$\left\| \prod_{n=p}^{q} \mathbf{M}_n \right\| \le K.$$

Then there exists at least one small solution of (DE) if and only if

$$\prod_{n=0}^{\infty} |\det \mathbf{M}_n| = 0.$$

With an example we show that condition $\left\| \prod_{n=p}^{q} \mathbf{M}_n \right\| \le K$ in Theorem 2.9 cannot be replaced by $\left\| \prod_{n=0}^{k} \mathbf{M}_n \right\| \le K$ $(k = 0, 1, 2, \dots)$. The question that this can condition be replaced by $\prod_{n=0}^{k} \|\mathbf{M}_n\| \le K$ $(k = 0, 1, 2, \dots)$ has remained open here.

In Section 2.2, with the aid of theorem 2.9 we prove Theorem 2.2, i.e. we show that under the given conditions differential equation (LOS) with step function coefficients has at least one small solution.

On small solutions of nonlinear difference equations

In the final section of Chapter 2 we examine the possible extensions of Theorem 2.9 to nonlinear systems of difference equations. We consider the difference equation

$$\mathbf{x}_{n+1} = \mathbf{f}(n, \mathbf{x}_n) \qquad n = 0, 1, 2, \dots, \tag{ND}$$

where $m \in \mathbb{N}$, $\mathbf{x}_n \in \mathbb{R}^m$ is a column vector, and functions $\mathbf{f}(n, \cdot)$ have the following properties for all $n \in \mathbb{N}_0$:

$$\mathbf{f}(n, \cdot) : \mathrm{D}_n \subset \mathbb{R}^m \to \mathbb{R}^m, \qquad \operatorname{ran} \mathbf{f}(n, \cdot) \subset \mathrm{D}_{n+1},$$

$$\mathbf{f}(n,0)=0, \qquad \mathbf{f}(n,\cdot)\in C^1(\mathrm{D}_n),$$

where D_n is a convex domain ($n=0,1,\ldots$). Define

$$\mathbf{F}(q,p\,;\cdot):=\mathbf{f}(q,\cdot)\circ\ldots\circ\mathbf{f}(p,\cdot) \qquad (0\le p\le q, \quad p,q\in\mathbb{N}_0),$$

furtehrmore let $F^j(q,p\,;\cdot):\mathrm{D}_p\to\mathbb{R}$ ($j=1,\ldots,m$) be the jth component of function $\mathbf{F}(q,p\,;\cdot)$, i.e.

$$\mathbf{F}(q,p\,;\mathbf{x})=\begin{pmatrix}F^1(q,p\,;\mathbf{x})\\ \vdots \\ F^m(q,p\,;\mathbf{x})\end{pmatrix}.$$

With the aid of a Lyapunov function, Karsai, Graef and Li [14] gave a sufficient condition for such equations to have at least one small solution. Currently, with the application of our topological method of proof, we could only conclude such result which is a consequence of their theorem. Since the conditions in our theorem are based only on the right hand side of equation (ND), furthermore its proof is analogous to the one of Theorem 2.9, therefore we present this result as well.

Theorem 2.12 *Suppose that there exists a closed ball H_0 around the origin and a number $K>0$, such that for all $p,q\in\mathbb{N}_0$ $(0\le p\le q)$, $j=1,\ldots,m$ and $\mathbf{x}\in\mathrm{H}_0$*

$$\big\|\operatorname{grad}F^j(q,p\,;\mathbf{x})\big\|\le K$$

holds, furthermore

$$\lim_{n\to\infty}\int_{H_0}\left|\det F'(n,0;\mathbf{x})\right|\,\mathrm{d}\mathbf{x}=0.$$

Then equation (ND) has at least one small solution.

3 On stability of second order half-linear differential equations with step function coefficients

In Chapter 3 we consider the half-linear second order differential equation

$$x''|x'|^{n-1}+q(t)|x|^{n-1}x=0, \qquad n\in\mathbb{R}^+, \tag{FD}$$

which is an important generalization of the second order differential equation (LO) and was introduced by Imre Bihari [1] and Árpád Elbert [5]. They called it half-linear because its solution set is homogeneous, but it is not additive. To this equation Bihari [2] proved an Armellini-Tonelli-Sanone-type theorem, namely, he proved that the trivial solution of (LO) is asymptotically stable with respect to x if coefficient q is continuously differentiable and tends "regularly" to infinity as $t \to \infty$. Such result for this equation with irregularly (or intermittently) growing coefficients was unknown until the appearance of our paper [12]. In Chapter 3 we give a sufficient condition on the asymptotic stability of the trivial solution with respect to x in the case when coefficient q is the most typically intermittently growing, that is when q is a step function. In the proof of our theorem we could successfully replace the method used for the linear case ($n = 1$ in (FD)) to a geometric technique which does not require linearity. What is more, this new method of proof allows us to sharpen the known results for the linear case. Therefore, our results not just include, but even sharpen the Armellini-Tonelli-Sanone-type theorems of Elbert [6, 8] for linear differential equations with step function coefficients, thus we first introduce this method to linear systems of difference equations.

On asymptotic stability of difference equations

First, we investigate the asymptotic stability of the trivial solution of the linear system of difference equations (DE) in the case when it is two dimensional. It is well-known that if $\prod_{n=0}^{\infty} \|\mathbf{M}_n\| = 0$, then all solutions of equation (DE) tend to zero as $n \to \infty$. Elbert [7] gave a sufficient condition for the asymptotic stability under the assumptions: (i) $\prod_{n=0}^{\infty} \max\{\|\mathbf{M}_n\|, 1\} < \infty$, (ii) $0 < \prod_{n=0}^{\infty} \|\mathbf{M}_n\|$, (iii) $\prod_{n=0}^{\infty} \max\{\det|\mathbf{M}_n|, 1\} < \infty$. His proof was based on estimation of the norm of some special matrices and a "tricky" decomposition of matrices $\mathbf{M}_n$.

To investigate equation (DE), we define a difference equation (DE') on the plane which has the same stability properties as equation (DE). The construction of this equation is based on the polar factorization theorem (see

eg. [13, p. 188]). Let

$$\mathbf{x}_{n+1} = \|\mathbf{M}_n\| \begin{pmatrix} 1 & 0 \\ 0 & d_n \end{pmatrix} \begin{pmatrix} \cos\omega_n & -\sin\omega_n \\ \sin\omega_n & \cos\omega_n \end{pmatrix} \mathbf{x}_n, \qquad \text{(DE')}$$
$$0 \le d_n \le 1, \qquad n = 0, 1, 2, \dots,$$

where d_n and ω_n $(n = 0, 1, 2, \dots)$ can be calculated from matrices $\mathbf{M}_0, \dots, \mathbf{M}_n$. In the main theorem of this section we show that conditions (i) – (iii) of Elbert's theorem can be weakened.

Theorem 3.3 ([12]) *Suppose that* $\limsup_{n\to\infty} \prod_{k=0}^{n} \|\mathbf{M}_k\| < \infty$. *If*

$$\sum_{n=0}^{\infty} \min\{1 - d_n, 1 - d_{n+1}\} \sin^2 \omega_{n+1} = \infty,$$

then the zero solution of difference equation (DE') is asymptotically stable.

The extension of the Armellini-Tonelli-Sansone theorem to second order half-linear differential equations with step function coefficients

The main result of this chapter is the following:

Theorem 3.5 ([12]) *Let* $n > 1$ *and*

$$0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots, \qquad \lim_{k\to\infty} t_k = \infty,$$
$$0 < q_0 \le q_1 \le \ldots \le q_k \le q_{k+1} \le \ldots, \qquad \lim_{k\to\infty} q_k = \infty.$$

Then all non-trivial solutions of equation

$$x''|x'|^{n-1} + q_k|x|^{n-1}x = 0 \qquad (t_k \le t < t_{k+1},\ k = 0, 1, \ldots)$$

are small, if

$$\sum_{k=0}^{\infty} \min\left\{1 - \frac{q_k}{q_{k+1}}, 1 - \frac{q_{k+1}}{q_{k+2}}\right\} \left| S\left(q_{k+1}^{\frac{1}{n+1}}(t_{k+2} - t_{k+1})\right)\right|^{n+1} = \infty.$$

The function S appearing in the theorem is the so-called generalized sine function, that is, the solution of the initial value problem

$$\begin{cases} S''|S'|^{n-1} + S|S|^{n-1} = 0, \\ S(0) = 0, \quad S'(0) = 1. \end{cases}$$

Note, that S satisfies the identity $|S(\Phi)|^{n+1} + |S'(\Phi)|^{n+1} \equiv 1$. The proof is similar to the one of Theorem 3.3, but due to the appearance of the generalized trigonometric functions we have to modify our estimations. The main difficulty is that exact addition formulae for these functions are unknown.

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