Local weak limits of random graphs and parameter continuity

Outline of Ph.D. Thesis

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Introduction

The theory of graph limits is motivated by the aim of understanding the behavior of large graphs. Such graphs can arise from natural models of random graphs, and also in many applications, such as computer science, statistical physics, biological studies, social networks.

In the thesis, we work with the local weak convergence introduced by Benjamini and Schramm. This concept applies well for the graph models that are most widely used in the applications and have unbounded degrees with bounded expected value.

It turned out, that in many cases, it is easier to work with the limit graph as with the large graphs converging to it [2]. One can hope that the examined parameter of the graphs converges for the graph sequence under study, and a proper probabilistic parameter of the limit object can be defined that the parameters of the sequence converge to. One example for such a parameter is the matching ratio. This phenomenon illustrates why graph limit theory can provide a useful approach to various questions about parameters of large graphs. Our work on the matching ratio presented in Chapter 3 fits into the series of studies in this direction.

Local weak convergence of random graphs extends naturally to possibly infinite rooted random graphs. The same questions arise for this wider class: in what sense are infinite random graphs determined by their local structures? Do certain parameters of the graphs converge along a local weak convergent sequence? In Chapter 4, we examine this question for the percolation critical probability, which was originally asked for transitive graphs by Oded Schramm [5]. Typically, however, the natural setting for such locality statements is not the class of transitive graphs, but the class of unimodular random graphs. Therefore, it is natural to investigate the question in the setup of unimodular random graphs and see what the proper notion of critical probability may be from the point of view of locality.

1 Random graphs

In this section, we define the most widely used random graph models. These graphs are examined in Chapter 3. We note that each graph presented here has a directed version and also a generalization to labeled graphs that we define only in the thesis.

1.1 Examples of finite random graphs

The first two examples are finite random graphs. By this concept, we just mean a discrete probability distribution on the countable set of isomorphism classes of finite graphs.

These two graph models have become increasingly important in applications, because they grab important characteristics of real-world networks, such as scale-free degree distributions. This is the reason why in [16], which was motivated by applications of controllability and motivated our work presented in Chapter 3, these graphs were studied.

Definition 1.1 (Random configuration model)**.** *We fix a non-negative integer valued probability distribution ξ. We define the graph G_n in the following way: let $\xi_1, \dots, \xi_n$ be i.i.d. variables with distribution ξ. Given $\xi_1, \dots, \xi_n$ let $\mathcal{E} := \{(k, j) : k \in [n], j \in [\xi_k]\}$ be the* ***set of the half-edges****. Let H be a uniform random perfect matching of the set $\mathcal{E}$ (if $|\mathcal{E}|$ is odd, then put off one half-edge uniformly at random before choosing a perfect matching). The* ***random configuration model*** *is the graph $G_n = G_n(H)$ on $[n]$ given naturally by the random perfect matching H.*

As a special case of the random configuration model, we define the **random d-regular graph** by setting the degree distribution ξ being constant d. This is equivalent with a random graph chosen uniformly at random from the set of graphs on the vertex set $[n]$ with all degrees equal d.

The other investigated graph model, the preferential attachment graph was introduced by Barabási and Albert in [4] and the precise construction was given by Bollobás and Riordan in [10]. There are several versions of the definition of this family of random graphs which have turned out to be asymptotically the same: they all converge to the same infinite limit graph; see [7].

Definition 1.2 (Preferential attachment graphs)**.** *Fix a positive integer r and $\alpha \in [0, 1)$. For each n the* ***preferential attachment graph*** *on n vertices is the random graph $G_n = G^{PA}_{r,\alpha,n}$ on the vertex set $[n]$ defined by the following recursion: let G_1 be the graph with one vertex and no edges. Given G_{n-1} we construct G_n by adding the new vertex n and r new edges with tails n. We choose the heads $w_1, \dots w_r$ of the new edges independently from each other in the following way: with probability α we choose w_j uniformly at random among $[n - 1]$, and with probability $1 - \alpha$ we choose w_j proportional to $\deg_{G_{n-1}}$. Note that each vertex except the starting vertex has out-degree r and each vertex has a random in-degree such that the mean of the average in-degree converges to r as $n \to \infty$.*

1.2 Random rooted graphs

The case of possibly *infinite random graphs* is more difficult. The set of isomorphism classes of locally finite graphs is uncountable and it is not a priori clear, what is the natural topology on it. One way to define a topology on the set of possible infinite graphs is to consider the space of rooted graphs, which is also natural from the point of view of local weak convergence. On this space, the *distance of rooted graphs* can be defined such that two graphs are close, if they are isomorphic in a large neighborhood of the root. We denote by $\mathcal{G}_\star$ the space of the isomorphism classes of locally finite connected rooted graphs with the topology generated by this distance. By a **random rooted graph** we mean a probability distribution on $\mathcal{G}_\star$.

We restrict our attention to *unimodular* random rooted graphs, which is a natural symmetry assumption on the distribution of the random graph. This notion generalizes the features of the most often investigated random graphs, such as invariant random subgraphs of Cayley graphs and local weak limits of finite graphs. There are several equivalent definitions of unimodularity. The following definition is useful in the proofs: finding a proper mass transport, one can prove (in)equalities concerning unimodular graphs.

Definition 1.3 (Unimodular random graphs)**.** *[1, Definition 2.1] We say that a random rooted (labeled) graph (G, o) is **unimodular** if it obeys the **Mass Transport Principle**:*

$$\mathbb{E}_G\left(\sum_{x\in V(\omega)} f(\omega, o, x)\right) = \mathbb{E}_G\left(\sum_{x\in V(\omega)} f(\omega, x, o)\right)$$

for any measurable function f on the space of isomorphism classes of locally finite graphs with an ordered pair of distinguished vertices.

The following classes of unimodular random rooted graphs provide basic examples of the definition and are investigated in the thesis.

Example 1.4 (Finite random graphs)**.** *It is easy to check that a finite graph with a uniform random root is unimodular. It follows by the convexity of the class of unimodular random graphs that every finite random graph with a root chosen uniformly at random is unimodular.*

An important property of unimodular random graphs is that this class is *closed under taking local weak limits* of random graphs. Also, it is an open question [1, Question 10.1] whether the class of unimodular random graphs is strictly larger than the class of sofic measures; i.e., the closure of the set of finite random graphs with a uniformly chosen root under local weak convergence.

Proposition 1.5 (Local weak limits of unimodular graphs [6])**.** *The space of unimodular random rooted graphs is closed under taking local weak limits (Definition 2.1). It follows that random rooted graphs that arise as limits of finite graphs are unimodular.*

Our last example of a unimodular random rooted graph is an interesting object in its own right. As described in Theorem 2.4, it also arises as local weak limit of finite graphs. We will present another important example of a unimodular random rooted graph given as a limit of finite graphs in the next section.

Definition 1.6 (Unimodular Galton–Watson tree)**.** *Let ξ be a non-negative integer valued random variable with $\mathbb{E}\xi < \infty$. The **unimodular Galton–Watson tree** with offspring distribution ξ (denoted by $UGW(\xi)$) is the following random rooted tree with root o. We say that a vertex y is the* child *of the vertex x, if they are adjacent and $dist(y, o) = dist(x, o) + 1$. The distribution of the graph $UGW(\xi)$ is given by the following recursive definition:*

- *The probability that o has $k \geq 0$ children is $\mathbb{P}(\xi = k)$.*
- *For each vertex x the probability that x has $k \geq 0$ children is $\frac{(k+1)\mathbb{P}(\xi=k+1)}{\mathbb{E}\xi}$, independently for each vertex.*

2 Local weak convergence of graph sequences

The local weak convergence of random rooted graphs is basically the weak convergence of their distributions in the space $\mathcal{G}_\star$ of isomorphism classes of connected, locally finite rooted graphs. The following definition is a more convenient description of local weak convergence, and captures the property that the local statistics of the graphs converge to that of the limit graph.

Definition 2.1 (Local weak convergence of graphs). *We say that the sequence (G_n, o_n) of locally finite random rooted graphs* ***converges in the local weak sense*** *to the locally finite connected random rooted graph (G, o) if for any positive integer r the distribution of the ball of radius r around the root in (G_n, o_n) converges to the distribution of the ball of radius r around the root in (G, o).*

Let $\mathcal{G}_\star^D$ be the subspace of $\mathcal{G}_\star$ consisting of the isomorphism classes of connected rooted graphs with degrees bounded above by a fixed constant D. It is not hard to show that $\mathcal{G}_\star^D$ is a compact topological space. It follows that every sequence in $\mathcal{G}_\star^D$ has a convergent subsequence with a limit in $\mathcal{G}_\star^D$. Many known results about convergent sequences have been proven assuming the stronger property that the sequence is in $\mathcal{G}_\star^D$ for some D. For some questions, as in our Chapter 4, basic examples show that they make sense only in this class. However, many natural graph models do not satisfy the uniformly bounded degree property. Nevertheless, all graph sequences we examine have bounded expected average degrees. Such sequences also have a nice behavior from the point of view of local weak convergence, hence the bounded degree assumption can be removed from certain results about convergent sequences, as we will see in Chapter 3.

In the rest of this section, we list the most important examples of local weak convergent sequences for our research.

Our first example provides important examples in Section 4. It may be surprising at first sight that the limit of the balls in a regular tree is not the regular tree.

Example 2.2 (Limit of the balls in the 3-regular tree). *Let $\mathbb{T}_3$ denote the 3-regular infinite tree and let $G_n = B_{\mathbb{T}_3}(v, n)$ be the ball of radius n around a vertex of $\mathbb{T}_3$. Then the graphs G_n converge to an infinite tree Λ with a random root, referred to it as the* ***canopy tree****. The vertex set of the connected graph Λ can be partitioned into countably many sets: let $L(0)$ be the set of vertices with degree 1, and for each $k > 0$, let $L(k)$ be the set of vertices that are at distance k from $L(0)$. Each set $L(j)$ in the partition has infinitely many vertices and each vertex in the set $L(j)$ is connected to one vertex in $L(j+1)$ and to two vertices in $L(j-1)$. The root of Λ is a vertex in $L(j)$ with probability 2^{-j-1}.*

Our next non-trivial example is a convergent sequence of infinite random graphs, which also shows that the bounded degree assumption in Chapter 4 is natural. This result was partially published in the case $\mathbb{E}X > 1$ in Beringer, Pete and Timár [8] and extended later to the present generality.

Proposition 2.3. *Given a non-negative integer valued random variable X, denote by $\hat{X}$ the positive integer valued random variable with distribution $\mathbb{P}(\hat{X} = k) = \frac{\mathbb{P}(X=k-1)}{k\,\mathbb{E}(\frac{1}{X+1})}$ for $k \geq 1$. Let $UGW_\infty(\hat{X})$ be the unimodular Galton–Watson tree with offspring distribution $\hat{X}$, conditioned to be infinite. If $\mathbb{E}X_n \geq 1$ and $\mathbb{E}X \geq 1$, then $UGW_\infty(\hat{X}_n) \to UGW_\infty(\hat{X})$ in the local weak sense iff $X_n \to X$ in distribution.*

The next two convergent graph sequences will be important in Chapter 3.

Let G_n be the random graph on n vertices given by *the random configuration model* with degree distribution ξ with $\mathbb{E}(\xi^2) < \infty$. Then G_n converge to $UGW(\xi)$ in the local weak sense as $n \to \infty$; see [1, Example 10.2], [11, Theorem 3.15]. It is standard, that in the special case of *random d-regular graphs*, the local weak limit of the sequence is the infinite d-regular tree $\mathbb{T}_d$.

Berger, Borgs, Chayes and Saberi [7] showed that the sequence of *preferential attachment graphs* with fixed parameters r and α converges in the local weak sense to the Pólya-point graph (defined in [7]) with the same parameters.

It turned out that these two latter models converge also in a stronger sense. If we define the sequence (G_n) on a common probability space with the given marginals for each n but with an arbitrary joint distribution, then the sequence converges *almost surely in the local weak sense* to the same limiting distribution. The directed versions of the above models, which are important in applications, converge to the natural directed versions of the limit graphs, even in this stronger sense. We proved Part 2) of Theorem 2.4 using the local weak convergence of preferential attachment graphs in [7] and showing a strong concentration of the probabilities in the definition of the local weak convergence. Part 3) follows from the generalization of [14, Proposition 2.2] to the unbounded degree case.

Theorem 2.4. *1) [11, Theorem 3.28] The random configuration model with degree distribution ξ with $\mathbb{E}(\xi^p) < \infty$ for some $p > 2$ converges to the unimodular Galton–Watson tree $UGW(\xi)$ almost surely in the local weak sense for any joint distribution of the sequence.*

2) [9] The preferential attachment graphs converges to the Pólya-point graph almost surely in the local weak sense for any joint distribution of the sequence.

3) [9] The directed versions of both models converge almost surely in the local weak sense to the directed versions of the limit graphs.

3 The matching ratio of large graphs

There is an important parameter in control theory which is closely related to the directed matching ratio of the network, as shown in the paper of Liu, Slotine and Barabási [16]. Informally, the controllability parameter of a network is defined as the minimum number N_D of nodes needed to control a network, e.g., the number of nodes that can shift molecular networks of the cell from a malignant state to a healthy state. In [16], it was showed that the proportion $n_D = N_D/|V(G)|$ of nodes needed to control a finite network G equals one minus the **directed matching ratio**, i.e., the relative size of the maximal size directed matching, where a directed matching is a subset of directed edges such that each vertex has in- and out-degree at most 1. This makes possible to prove results on n_D by proving the corresponding statement for the directed matching ratio.

Our main motivation was the two further observations in [16], which are the following. First, simulations run on both real networks and network models suggested that the matching ratio is mainly determined by the degree sequence of the graph; more precisely, if the edges are randomized in a way that does not change the degrees, then the matching ratio does not change significantly. Second, arguments based on methods from statistical physics and numerical results suggested that for the most widely used families of scale-free networks, the directed matching ratio converges to a constant when the size of the network tends to infinity. The models that were most relevant in [16] are the so-called scale-free networks, which are known to exhibit several characteristics, such as a power-law degree decay, of the networks observed in real-world applications. Our aim was to give rigorous mathematical proofs of these observations of [16], by extending the result of Elek and Lippner [15] on the convergence of the matching ratio. By the connection between the matching ratio and the controllability parameter, our results translate to theorems concerning controllability of networks.

The results of this chapter have been published in Beringer and Timár [9].

3.1 Concentration of the matching ratio

Our theorems presented in this section extends the series of results on the concentration of certain parameters of random graphs. Previous results made use of the independence in the examined graphs [3, 11], but the definitions of the models in Theorems 3.1 and 3.2 give rise to more dependence which we had to deal with.

Part 1) of Theorem 3.1 shows the concentration for randomized graphs with the in- and out-degrees left unchanged. This is the result that was observed through simulations in [16], which motivated our work in this direction. Part 2) of the theorem shows that a very similar concentration phenomenon holds even after a randomizing that does not require the in- and out-degrees to be unchanged but only the total degree to remain the same for every vertex. In particular, Theorem 3.1 shows that if a graph sequence satisfies that the empirical second moment of the degree sequence is $o(n)$ with probability tending to 1 (as $n \to \infty$), then the directed matching ratios of the graphs with randomized edges are strongly concentrated around their mean with high probability. The most widely used random graph models, defined in Section 1, satisfy this property.

Theorem 3.1 (Concentration of the matching ratio of the random configuration model)**.** *1) Let G be a random directed graph on n vertices given by the random configuration model with a fixed sequence of in- and out-degrees with $e(G)$ edges. Then for all $\varepsilon > 0$, the directed matching ratio $m(G)$ of G satisfies*

$$\mathbb{P}\left(|m(G) - \mathbb{E}(m(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

2) Let G be a random directed graph on n vertices given by the random configuration model with a fixed sequence of degrees with $e(G)$ edges. Then for all $\varepsilon > 0$, the directed matching ratio $m(G)$ of G satisfies

$$\mathbb{P}\left(|m(G) - \mathbb{E}(m(G))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{8e(G)}\right\}.$$

Preferential attachment graphs also satisfy a strong concentration phenomenon. We note that Theorem 3.2 does not follow from Theorem 3.1, since the orientations of the edges of the preferential attachment graph are given naturally by the recursive definition, and differ significantly from the independent random orientation.

Theorem 3.2 (Concentration of the matching ratio of preferential attachment graphs)**.** *Let G_n be a random graph sequence obtained by the preferential attachment rule with parameter r. Then $m(G_n)$ is concentrated around its expected value: for any $\varepsilon > 0$ we have*

$$\mathbb{P}\left(|m(G_n) - \mathbb{E}(m(G_n))| > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2 n}{8r^2}\right\}.$$

3.2 Convergence of the matching ratio

We present our results on the convergence of the directed matching ratio for convergent sequences of *random directed graphs* in this section. This convergence is understood in the stronger sense of almost sure convergence, as we will see. For a fixed *deterministic undirected* graph sequence that is locally convergent when a uniform root is taken, the convergence of the matching ratio is proved by Elek and Lippner in [15] in the uniformly bounded degree case and by Bordenave, Lelarge and Salez in [12] in the unbounded case. To prove the results of Liu, Slotine and Barabási in [16], we need to generalize these results for *directed random* graphs. Our first theorem shows the convergence of the *mean* of the matching ratio for this more general setting.

Theorem 3.3. *Let G_n be a sequence of random finite (directed) graphs that converges to the random (directed) rooted graph (G, o) that has finite expected degree. Then*

$$\lim_{n\to\infty} \mathbb{E}(m(G_n)) = m_E(G, o),$$

*where $m_E(G, o) := \sup_M \mathbb{P}_G(o \in V^{(-)}(M))$ is the **(expected) matching ratio** of (G, o), defined as a supremum taken over all random (directed) matchings M of G such that M is almost surely a (directed) matching of (G, o) and the distribution of the labeled graph of (G, M, o) is unimodular. The notation $V^{(-)}(M)$ stands for the vertex set of the matching M in the undirected case and for the set of vertices with positive in-degree in M in the directed case.*

The proof of Theorem 3.3 follows the method of [15]. The main differences to that proof come from the lack of uniform bound on the degrees. We handle the unbounded degrees by defining a random matching $M(T)$ without long augmenting paths as *factor of IID*. For graphs with unbounded degrees, Lemma 4.1 of [15] does not apply, hence we have to proceed through Lemma 3.4.

Lemma 3.4. *Let (G, o) be a unimodular graph with distribution μ and finite expected degree. Then for any $\varepsilon > 0$ and any n there is a δ such that if a measurable event H satisfies $\mu(H) < \delta$, then $\mu(H^n) < \varepsilon$, where $H^n := \{(\omega, x) : (\omega, o) \in H, dist_\omega(o, x) \leq n\}$.*

The main result of this section, the almost sure convergence of the (directed) matching ratio of the examined graph models follows from Theorems 3.3 and 2.4. As a special case of Part 1), we get that the directed matching ratio of the directed random configuration model converges almost surely to a constant.

Theorem 3.5 (Almost sure convergence of the matching ratio)**.** *1) Let G_n be a sequence of undirected finite graphs defined on a common probability space that converges almost surely in the local weak sense and let $\overrightarrow{G_n}$ be a sequence of random directed graphs obtained from G_n by giving each edge a random orientation independently. Then $m(\overrightarrow{G_n})$ converges almost surely to the constant $\lim_{n\to\infty} \mathbb{E}(m(\overrightarrow{G_n}))$.*

2) Let G_n be the sequence of random directed graphs given by the preferential attachment rule. Then $m(G_n)$ converges almost surely to the constant $\lim_{n\to\infty} \mathbb{E}(m(G_n))$.

4 Percolation critical probabilities and unimodular random graphs

The notion of local weak convergence was originally introduced for sequences of *finite* graphs. However, the definition applies also for sequences of *infinite* random rooted graphs, and the same question arises naturally: do certain parameters of infinite graphs converge along local weak convergent sequences? In this chapter, we examine percolation critical probabilities, originally defined for deterministic infinite graphs.

There are several definitions of the critical probability for percolation on the lattices $\mathbb{Z}^d$, which have turned out to be equivalent not only on $\mathbb{Z}^d$, but also in the more general context of arbitrary transitive graphs. In Section 4.2, we investigate the relationship between these different definitions when the graph G is an extremal unimodular random graph, which is the natural extension of transitivity to the disordered setting.

The continuity of percolation critical probability is conjectured by Oded Schramm [5] in the class of transitive graphs:

Conjecture 4.1 (Schramm)**.** *If G_n is a sequence of vertex-transitive infinite graphs such that G_n converges locally to G and $\sup_n p_c(G_n) < 1$ then $p_c(G_n) \to p_c(G)$ holds.*

The conjecture has been proven for some special for some special cases but not in full generality.

Typically, however, the natural setting for such locality statements is not the class of transitive graphs, but the class of unimodular random graphs. Therefore, in Section 4.3, we investigate Schramm's locality conjecture in the setup of unimodular random graphs and see what the proper notion of critical probability may be from the point of view of locality.

The results of this chapter have been published in Beringer, Pete and Timár [8].

4.1 Generalizations of percolation critical probabilities

We examine Bernoulli bond **percolation** on a graph G which is the following random subgraph: each edge is present (open) with probability p and removed (closed) with probability $1-p$ independently. We fix a root o of G and denote by $\mathcal{C}_o$ the connected component of o in the percolation subgraph. The notations $\mathbb{P}_p$ and $\mathbb{E}_p$ stand for the probability and expectation with respect to percolation with parameter p.

A fundamental and long studied question in percolation theory is the value of the **critical probabilities** $p_c = \sup\{p : \mathbb{P}_p(|\mathcal{C}_o| = \infty) = 0\}$ and $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$. These quantities have natural generalizations to *extremal* unimodular random graphs. In this section, we define the generalized critical probabilities p_c, p_T, $\tilde{p}_c$, p_T^a, and $\tilde{p}_c^a$; somewhat simplistically saying, the first three will be quenched versions of the quantities mentioned above, while the last two will be annealed versions.

Let (G, o) be an **extremal** unimodular random graph with distribution μ; i.e., μ cannot be written as a non-trivial convex combination of unimodular measures. In this case, the critical probability $p_c(\omega)$ of an instance ω of (G, o) is μ-almost surely a constant and the same holds for p_T (see [1], Section 6.). Thus, the following definitions make sense for extremal unimodular graphs. Using the notation $\mathbb{P}_p^\omega$ and $\mathbb{E}_p^\omega$ for the probability and expectation with respect to percolation with parameter p on the fixed realization ω of the random graph G, we define

$$\begin{aligned} p_c &= \inf\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) > 0\right) = 1\right\} \\ &= \sup\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) = 0\right) = 1\right\} \end{aligned}$$

and

$$\begin{aligned} p_T &= \sup\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty\right) = 1\right\} \\ &= \inf\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) = \infty\right) = 1\right\}. \end{aligned}$$

It may happen that although $\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty$ for μ-almost every ω, the expectation of these quantities with respect to μ is infinite. This provides a second natural extension of p_T to unimodular random graphs defined using the average size of $\mathcal{C}_o$:

$$\begin{aligned} p_T^a &= \sup\left\{p : \mathbb{E}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) < \infty\right\} \\ &= \inf\left\{p : \mathbb{E}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) = \infty\right\}. \end{aligned}$$

It follows from the definitions that $p_c \geq p_T \geq p_T^a$. It is known that $p_c = p_T$ in the case of transitive graphs. For extremal unimodular random graphs (even with sub-exponential volume growth), the three critical probabilities can differ, as our examples show.

The last two critical probabilities are generalizations of a quantity introduced by Duminil-Copin and Tassion [13] for transitive graphs. Let (G, o) be a rooted graph, $S \in \mathcal{S}(G)$ be a finite subgraph containing the root, and define

$$\phi_p(S) := \sum_{e \in \partial_E S} p \, \mathbb{P}_p(o \overset{S}{\longleftrightarrow} e^-)\,,$$

the expected number of open edges on the boundary of S such that there is an open path from o to e^- in S. In [13], the following critical probability was defined:

$$\begin{aligned} \tilde{p}_c &:= \sup\{p : \text{there is an } S \in \mathcal{S}(G) \text{ s.t. } \phi_p(S) < 1\} \\ &= \inf\{p : \phi_p(S) \geq 1 \text{ for all } S \in \mathcal{S}(G)\}\,. \end{aligned} \tag{4.1}$$

It was proven in [13] that transitive graphs satisfy $p_c = \tilde{p}_c$. In fact, $\tilde{p}_c$ was designed to address the question of locality of the critical probability.

How to generalize this definition to unimodular random graphs is not a priori clear. The simplest way to define a similar critical probability seems to be a quenched version: find a suitable $S_\omega \in \mathcal{S}(\omega)$ for almost every configuration ω. For a subgraph $S \in \mathcal{S}(\omega)$ denote by

$$\phi_p^\omega(S) := \sum_{e \in \partial_E S} p \, \mathbb{P}_p^\omega \left(o \underset{S}{\overset{\omega, p}{\longleftrightarrow}} e^- \right)$$

the expected number of open edges on the boundary of S in ω such that there is an open path from o to e^- in the percolation on ω with parameter p. Then let

$$\tilde{p}_c := \sup\left\{p : \mu\left(\left\{\omega : \exists S_\omega \in \mathcal{S}(\omega) \text{ s.t. } \phi_p^\omega(S_\omega) < 1\right\}\right) = 1\right\}. \tag{4.2}$$

The following critical probability is another natural extension of the definition of $\tilde{p}_c$, an annealed version of $\tilde{p}_c$:

$$\tilde{p}_c^a := \sup\{p : \exists r \text{ such that } \mathbb{E}\left(\phi_p^\omega(B_\omega(o, r))\right) < 1\}.$$

4.2 Relationship of the critical probabilities of unimodular random graphs

In this section, we examine the relationship between these different generalizations. The one sentence summary is that $p_c = \tilde{p}_c$ always holds, but otherwise almost anything can happen, unless the random graph satisfies some very strong uniformity conditions; one that we call "uniformly good" suffices for most purposes.

We start with our positive results, i.e., with the results that show coincidence of certain critical probabilities.

Our first theorem is indispensable to the proofs in the chapter. One direction of the proof depends on new ideas, while the other direction is a slight modification of the proof in [13] for our settings.

Theorem 4.2. *If G is a bounded degree unimodular random rooted graph, then $p_c(G) = \tilde{p}_c(G)$.*

One advantage of the definition of $\tilde{p}_c$ for transitive graphs is that it enables one to check whether a certain p is under $\tilde{p}_c$ using a finite witness. This characteristic makes the next definition natural. This assumption captures important features of transitive graphs, which imply a strong connection between the two critical probabilities $\tilde{p}_c$ and $\tilde{p}_c^a$.

Definition 4.3. *We say that a bounded degree unimodular random graph G is* ***uniformly good*** *if for any $p < p_c$ there exists a positive integer $r(p)$ such that $\mu_G(\{\omega : \exists S_\omega \subseteq B_\omega(o, r(p)), o \in S_\omega \text{ s.t. } \phi_p^\omega(S_\omega) < 1\}) = 1$.*

Uniformly good unimodular graphs satisfy the following exponential decay of $\phi_p(B_\omega(o,r))$ in r. As a corollary, we get that $p_c \leq \tilde{p}_c^a$ for uniformly good graphs.

Lemma 4.4. *Let G be a bounded degree unimodular random graph. G is uniformly good if and only if for all $p < p_c$ there are constants $c = c(p) < 1$ and $R(p)$ such that if $r \geq R(p)$, then $\phi_p^\omega(B) \leq c^r$ for almost every ω and every finite $B \supseteq B_\omega(o,r)$.*

Corollary 4.5. *If G is a uniformly good unimodular graph, then $p_c \leq \tilde{p}_c^a$.*

If a uniformly good unimodular graph has uniform sub-exponential growth, then the critical probabilities p_c, p_T, p_T^a coincide, as in the transitive case.

Definition 4.6. *We say that a unimodular graph G has* ***uniform sub-exponential volume growth*** *if for any $c < 1$ and $\varepsilon > 0$ there is an R such that $\mathbb{P}_G\left(\omega : |B_\omega(o,r)|c^r < \varepsilon\right) = 1$ for any $r > R$.*

Corollary 4.7. *If G is a uniformly good unimodular graph with uniform sub-exponential volume growth, then $p_c = p_T = p_T^a$.*

$\tilde{p}_c = p_c$	bounded degree
$p_c \geq p_T \geq p_T^a$	always
$p_c = p_T^a$	bounded degree uniformly good with sub-exp. growth
$p_c > p_T$	example with polynomial growth
$p_T > p_T^a$	example with polynomial growth
$p_c \leq \tilde{p}_c^a$	bounded degree uniformly good
$p_c < \tilde{p}_c^a$	a bounded degree uniformly good example
$p_c > \tilde{p}_c^a$	a not uniformly good example

Table 1: Relationship of the critical probabilities

As we mentioned at the beginning of this subsection, without the assumption of uniform goodness, the critical probabilities can differ. The easiest example is the canopy tree (see Example 2.2), which is not uniformly good, has no uniform sub-exponential volume growth and satisfies $p_c = 1 > \frac{1}{\sqrt{2}} = p_T^a = \tilde{p}_c^a$. Also, we constructed an example that shows that the inequality in Corollary 4.5 can be strict for uniformly good graphs. We also found examples with polynomial growth, that provide strong inequalities between p_c, p_T and p_T^a. These show that omitting only one of the conditions of Corollary 4.7 makes the statement already false.

Our results on the relationship of the critical probabilities are summarized in Table 1.

4.3 Locality of the critical probability

In this section, we investigate the extension of Schramm's conjecture for unimodular random graphs:

Question 4.8. *Does $p_c(G_n)$ converge to $p_c(G)$ if G_n are unimodular random graphs, $G_n \to G$ in the local weak sense and* $\sup p_c(G_n) < 1$*?*

First we note that, as a corollary of Proposition 2.3, locality holds for unimodular Galton-Watson trees with bounded degrees, but not in general; this shows that it is natural to restrict one's attention to bounded degree unimodular random graphs.

In Section 4.3.1, we give conditions which imply $\lim p_c(G_n) = p_c(G)$. However, our examples show that there are sequences of unimodular random graphs such that $G_n \to G$ but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$. These examples indicate a negative answer to Question 4.8; i.e., Schramm's conjecture does not hold in the generality of unimodular random graphs, although many such statements, formulated originally for transitive graphs, extend to this class.

4.3.1 Lower semicontinuity and continuity

The quantity $\phi_p(S)$ can be used to give a short proof of the lower semicontinuity of $p_c(G)$ in the local topology of transitive graphs: that is, $\liminf p_c(G_n) \geq p_c(G)$ holds. It can be shown in a similar way that this inequality is also true for $\tilde{p}_c^a$ and unimodular graphs.

Proposition 4.9. *Let G_n and G be unimodular random graphs with uniformly bounded degrees. If G_n converges to G then* $\liminf_{n\to\infty} \tilde{p}_c^a(G_n) \geq \tilde{p}_c^a(G)$.

We show in Proposition 4.10 that under certain restrictions on the graphs G and G_n the convergence $\lim p_c(G_n) = p_c(G)$ holds; i.e., we give assumptions that imply a positive answer to Question 4.8.

Proposition 4.10. *Let G be a uniformly good unimodular random graph. Furthermore, let G_n be unimodular random graphs converging to G in the local weak sense, in a* uniformly sparse *way: there is a positive integer k such that for each n there is a coupling ν_n of μ_G and μ_{G_n} such that $G \subseteq G_n$ and there is a sequence of positive integers $r_n \to \infty$ that satisfies $|(E(G_n) \setminus E(G)) \cap B_{G_n}(o, r_n)| \leq k$ ν_n-almost surely. Then*

$$\lim_{n\to\infty} p_c(G_n) = p_c(G).$$

In the thesis, we present an example where this proposition applies.

In the quite special setting of unimodular trees of uniform sub-exponential growth, the assumption of uniformly sparse convergence from Proposition 4.10 can be relaxed. The following proposition gives further examples of uniformly good unimodular graphs, while the convergence part will be used in Proposition 4.12.

Proposition 4.11. *If G is a bounded degree unimodular random tree with uniformly sub-exponential volume growth, then all five critical percolation densities equal 1, and G is uniformly good.*

If G_n is a sequence of bounded degree unimodular random graphs with uniformly sub-exponential volume growth and girth tending to infinity, then $p_c(G_n)$, $\tilde{p}_c(G_n)$, $\tilde{p}_c^a(G_n)$ all tend to 1.

4.3.2 Counterexamples

In the thesis, we present graph sequences that provide counterexamples for Question 4.8, i.e., they indicate that unimodular graphs do not satisfy Schramm's conjecture in general. The first two examples show that both of the conditions in Proposition 4.10 are necessary. We omit the details of the examples here, and only describe the most important features of them.

The first example is a sequence of invariant random subgraphs G_n of a Cayley graph, converging to an invariant subgraph G of the same Cayley graph in a uniformly sparse way. In this example, $\lim p_c(G_n) < p_c(G)$ and the limit graph G is not uniformly good.

The second example is a sequence G_n of invariant random subgraphs of $\mathbb{Z}^5$ converging to $\mathbb{Z}^2$ such that $\lim p_c(G_n) < p_c(\lim G_n)$. In this example, the limit graph is uniformly good, but the convergence is not uniformly sparse. In the thesis, we describe also a more general version of this example.

In the third example, we show a sequence with $p_c(\lim G_n) < \lim p_c(G_n) < 1$. Each G_n is a quasi-transitive subgraph of $\mathbb{Z}^2$ which can be viewed as a random invariant subgraph of $\mathbb{Z}^2$ (hence unimodular), and G_n converges to $\mathbb{Z}^2$. In this example, $\lim G_n$ and each G_n satisfy the conditions of Corollaries 4.5 and 4.7, thus $p_c = p_T = p_T^a$ and also $\tilde{p}_c^a(G) < \lim \tilde{p}_c^a(G_n) < 1$. This shows that none of the generalizations of the critical probabilities can possibly satisfy the extension of Schramm's conjecture for unimodular graphs in general. This example also shows that the inequality of the lower semicontinuity in Proposition 4.9 may be strict even when invariant subgraphs G_n of $\mathbb{Z}^2$ converge to $\mathbb{Z}^2$.

4.4 On transitive graphs of cost 1

A corollary to our positive results is that if G is a transitive graph of sub-exponential volume growth, then there exists a sequence of invariant bi-Lipschitz spanning subgraphs G_n such that $p_c(G_n) \to 1$.

Proposition 4.12. *If G is a transitive amenable graph, then there is a sequence of invariant random subgraphs G_k which satisfies the following: each G_k is a bi-Lipschitz (in particular, connected) spanning subgraph of G, the girth of G_k tends to infinity and G_k locally converges to an invariant random spanning tree $\mathcal{T}$ with at most two ends.*

If G is a unimodular transitive graph with sub-exponential volume growth then $p_c(G_k) \to 1$.

Our proposition may be thought of as a strengthening of the simple fact that groups of sub-exponential growth have cost 1, as our Lemma 4.13 shows. The **cost of a group** $\mathbb{G}$ is defined as half of the infimum of the expected degrees of its invariant connected spanning graphs. The **cost of a transitive graph** Γ may be defined similarly, over $\mathbb{G}$-invariant random connected spanning *sub*graphs, where $\mathbb{G} \leq Aut(\Gamma)$ is a vertex-transitive subgroup of graph-automorphisms that is usually fixed implicitly.

Lemma 4.13. *If Γ is a Cayley graph of $\mathbb{G}$, and there exists a sequence of $\mathbb{G}$-invariant connected spanning subgraphs $G_k \subset \Gamma$ with $p_c(G_k) \to 1$, then the cost of Γ, hence of $\mathbb{G}$, is 1.*

We do not know if the assumption of the lemma is equivalent of cost 1; i.e., the converse of Lemma 4.13 holds.

Having a $\mathbb{G}$-invariant connected spanning graph $\mathcal{T}$ with $p_c(\mathcal{T}) = 1$ is equivalent with the amenability of $\mathbb{G}$, which implies that the cost of $\mathbb{G}$ is 1. However, we found a sequence of invariant bi-Lipschitz subgraphs $G_k \subset \mathbb{T}_3 \times \mathbb{Z}$ which shows that $p_c(G_k) \to 1$ does not imply amenability.

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