Convex and Discrete Geometrical Problems on the Sphere

Outline of Ph.D. thesis

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1 Introduction

The problems discussed in this thesis originate from the topics of convex and discrete geometry. The results belong to three broad categories, namely, maximizing pairwise angles between elements of a pencil of lines, zone-coverings of the unit sphere, and about the volume of random parallelotopes in isotropic measures.

The dissertation is based on the following four papers of the author:

[FNZ18] F. Fodor, M. Naszódi, and T. Zarnócz, *On the volume bound in the Dvoretzky–Rogers lemma*, Pacific J. Math. (2018), accepted for publication. arXiv:1804.03444.

[BFVZ17] A. Bezdek, F. Fodor, V. Vígh, and T. Zarnócz, *On the multiplicity of arrangements of congruent zones on the sphere* (2017), accepted for publication. arXiv:1705.02172.

[FVZ16a] F. Fodor, V. Vígh, and T. Zarnócz, *Covering the sphere by equal zones*, Acta Math. Hungar. **149** (2016), no. 2, 478–489, DOI 10.1007/s10474-016-0613-2. MR3518649

[FVZ16b] F. Fodor, V. Vígh, and T. Zarnócz, *On the angle sum of lines*, Arch. Math. (Basel) **106** (2016), no. 1, 91–100, DOI 10.1007/s00013-015-0847-1. MR3451371

We note that two of the above papers are already published [FVZ16a, FVZ16b], and two are accepted for publication [FNZ18, BFVZ17].

2 On the angle sum of lines

This chapter of the dissertation is based on the paper [FVZ16b].

2.1 Introduction

Consider n lines in the d-dimensional Euclidean space $\mathbb{R}^d$ which all pass through the origin o. What is the maximum $S(n,d)$ of the sum of the pairwise (non-obtuse) angles formed by the lines?

The question was raised by L. Fejes Tóth [FT59] in 1959 for $d = 3$. He also conjectured that in the optimal configuration we have as many identical copies of an orthonormal basis as we can from the lines and possibly an incomplete one (less than d dimensional) if the number of lines is not divisible by d. More precisely, let $n = k \cdot d + m$ $(1 \le m < d)$ be the number of lines, and denote by $x_1, \dots, x_d$ the axes of a Cartesian coordinate system in $\mathbb{R}^d$. The conjectured optimal configuration consists of $k+1$ copies of $x_1, \dots, x_m$ and k copies of $x_{m+1}, \dots, x_d$. The sum of the pairwise angles in this configuration is

$$\left[\binom{d}{2}k^2 + mk(d-1) + \binom{m}{2}\right]\frac{\pi}{2}.$$

L. Fejes Tóth proved the conjecture in 3-dimensional space for $n \le 6$ and gave an upper bound using a recursive formula: $S(n,3) \le n(n-1)\pi/5$. This means that the sum of angles is asymptotically less than $n^2\pi/5$ as $n \to \infty$. In our paper [FVZ16b], we improved this upper bound to $3n^2\pi/16 \approx 0.589 \cdot n^2$, and later Bilyk and Matzke [BM19] further improved it to $\left(\frac{\pi}{4} - \frac{69}{100d}\right)n^2$ as $n \to \infty$. We note that their result for $d = 3$ gives asymptotically less than $0.556 \cdot n^2$ as $n \to \infty$. However, their bound is for general d.

We also mention that this problem has other variants that have been considered, and some of them completely solved. One important example is of the directed lines. Consider n rays emanating from the origin. What is the maximum of the sum of pairwise angles between the rays (vectors)?

Another direction is when instead of angles one consider certain functions of the angles (or Euclidean distances). This direction gives rise to the so-called potentials.

2.2 Results

Our contribution to this problem is summarized in the following theorem.

Theorem 2.1. *Let $l_1, \ldots, l_n$ be lines in $\mathbb{R}^3$ which all pass through the origin. If we denote by φ_{ij} the angle formed by l_i and l_j, then*

$$\sum_{1 \le i < j \le n} \varphi_{ij} \le \begin{cases} \frac{3}{2}k^2 \cdot \frac{\pi}{2}, & \text{if} \quad n = 2k, \\ \frac{3}{2}k(k+1) \cdot \frac{\pi}{2}, & \text{if} \quad n = 2k+1. \end{cases}$$

We first investigated the planar case. The following theorem had probably been known prior to our work, but we could not find an explicit proof in the literature so we decided to include one. We say that a pencil of lines is balanced if for every line the number of other lines making a positive angle (smaller than $\pi/2$) and the number of other lines making a negative angle differ by at most one.

Theorem 2.2. *Let $l_1, \ldots, l_n$ be lines in $\mathbb{R}^2$ which all pass through the origin. If we denote by φ_{ij} the angle formed by l_i and l_j, then*

$$\sum_{1 \le i < j \le n} \varphi_{ij} \le \begin{cases} k^2 \cdot \frac{\pi}{2}, & \text{if} \quad n = 2k, \\ k(k+1) \cdot \frac{\pi}{2}, & \text{if} \quad n = 2k+1. \end{cases}$$

Equality holds if, and only if, $l_1, \ldots, l_n$ is balanced.

The main idea of the proof of Theorem 2.2 is that if we have a perpendicular pair of lines, then the pair can be rotated freely without changing the total sum of the pairwise angles.

Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors and φ the angle between them. Then for the 3-dimensional case we first define the function

$$I : [0, \pi/2] \to \mathbb{R}, \quad I(\varphi) := \frac{1}{4\pi} \int_{S^2} \varphi_*^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2) \mathrm{d}\mathbf{u},$$

where $\varphi_*^{\mathbf{u}}(\mathbf{v}_1, \mathbf{v}_2)$ is the angle between the perpendicular components of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to $\mathbf{u}$, or the complement of that angle (to π), whichever is smaller. Note that the function I, in fact, depends only on φ and not on the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ themselves. $I(\varphi)$ is the average angle of the orthogonal projections of the lines to a plane with normal vector $\mathbf{u}$.

Next, we show, with the help of two lemmas, that $I(\varphi) \ge 2\varphi/3$ for all $\varphi \in [0, \pi/2]$. The first lemma states that this holds at the end points of the domain, that is, for $\varphi = 0$ and $\varphi = \pi/2$.

Lemma 2.4. *With the notation introduced above,*

$$I(0) = 0 \quad \text{and} \quad I(\pi/2) = \pi/3.$$

The second lemma shows that I is concave. The combination of the two statements clearly proves our claim.

Lemma 2.5. *The function $I(\varphi)$ is concave on $[0, \pi/2]$, and*

$$I(\varphi) \geq 2\varphi/3 \qquad \textit{for} \qquad 0 \leq \varphi \leq \pi/2. \tag{1}$$

From these results our main theorem follows directly. Since the average of the sum of the pairwise angles of the projections (the average taken with respect to the normal vector of the projecting plane) is at least 2/3 times the sum of the original angles, there exist a $\mathbf{u}_0$ such that if we project the lines to the plane with normal vector $\mathbf{u}_0$ then the sum of the angles formed by the projections is greater than 2/3 times the sum of the angles formed by the lines. Finally we know the optimum for the planar case and hence Theorem 2.1 holds.

3 Covering the sphere by equal zones

This chapter of the dissertation is based on the paper [FVZ16a].

3.1 Introduction

Let S^2 be the unit sphere in the 3-dimensional Euclidean space $\mathbb{R}^3$ centered at the origin o. The spherical distance between two points $x, y \in S^2$ is defined as the length of the shorter geodesic arc connecting x and y. We define a zone Z of half-width w as the parallel domain of radius w of a great circle C, which is the set of point on S^2 whose spherical distance from C is at most w. The main problem investigated in this section is the following.

Problem 3.1 (L. Fejes Tóth [FT73]). *For a given n, find the smallest number w_n such that one can cover S^2 with n zones of half-width w_n. Find also the optimal configurations of zones that realize the optimal coverings.*

It is an analogous question to this one (also posed by L. Fejes Tóth) when we do not require the zones to be congruent, and we seek the minimum of the sum of the widths of the zones needed to cover S^2. This minimum is conjectured to be π. A similar and somewhat more general question in this topic that has been considered in the literature is the following: What is the minimum of the sum of the half-widths of n not necessarily congruent zones that can cover a spherically convex disc on S^2? All of these questions are similar in nature to the famous plank problem of Tarski.

In 1932 Tarski posed the original problem [Tar32], which later became known as the plank problem. He conjectured that if a convex body $K \subset \mathbb{R}^d$ is covered by a finite number of planks, then the sum of their widths is no less than the minimal width of K. The original conjecture was proved by Bang [Ban50, Ban51]. A plank in this context means the part of $\mathbb{R}^d$ between two parallel hyperplanes. The width of such a plank is the distance between its supporting hyperplanes.

Let K be a convex body, h_K its support function and $u \in \mathbb{R}^d$ a unit vector. The width of K in the direction u is $h_K(u) + h_K(-u)$ and the minimal width of K is the minimum of these widths

$$w_K = \min_{u \in \mathbb{S}^{d-1}} h_K(u) + h_K(-u)$$

Considering the case of the sphere covering by zones, before our work the only known general lower bound was a trivial one: The sum of the areas of the zones must be at least 4π, so the common half-width of the zones needs to be at least $\arcsin(1/n)$. This trivial lower bound is of course not sharp in case $n \geq 2$, since any two zones intersect, so their contribution to the covering (starting with the second one) cannot be their whole area. We note, that the problem was solved for $n = 3, 4$ zones by Rosta [Ros72] and Linhart [Lin74], respectively. We consider a covering as it is being built up zone by zone and investigate the contribution of each zone (which is less than its area) to the covering. Estimating the area of the intersection of two zones, depending on the half-width and angle, we give an upper bound for the contribution of each zone, and, in turn, a lower bound for w_n which is better than the trivial one.

We note that after our work Jiang and Polyanskii [JP17] proved the original conjecture of L. Fejes Tóth, thus completely solving the original problem. However, Lemma 3.3 was used by Steinerberger subsequently to estimate the overlap of n zones of $1/(2n)$ width, which has a strong connection to s-Riesz energies.

3.2 Intersection of two zones

Let $2F(w, \alpha)$ denote the area of intersection of two zones with w radius and making an angle of α.

Lemma 3.3. *Let $0 \leq w \leq \pi/4$ and $2w \leq \alpha \leq \pi/2$. Then*

$$\begin{aligned} F(w,\alpha) = 2\pi &+ 4\sin w \arcsin\left(\frac{1-\cos\alpha}{\cot w \sin\alpha}\right) + 4\sin w \arcsin\left(\frac{1+\cos\alpha}{\cot w \sin\alpha}\right) \\ &- 2\arccos\left(\frac{\cos\alpha - \sin^2 w}{\cos^2 w}\right) - 2\arccos\left(\frac{-\cos\alpha - \sin^2 w}{\cos^2 w}\right). \end{aligned} \tag{2}$$

Moreover, $F(w, \alpha)$ is a monotonically decreasing function of α in the interval $[0, \pi/2]$.

This lemma helps us estimate the contribution of a zone to the covering. The closer the zone is to an earlier zone the smaller its contribution (which is the area covered by only this new zone) is.

3.3 A lower bound for the minimal width

Since the contribution of a zone to the covering depends on its proximity to other zones, we needed to estimate how close n given points on the sphere can be to one another (this is equivalent to asking how close the zones can be, as the distance of two zones is the distance of their poles). For $n \geq 3$, let d_n denote the maximum of the minimal pairwise (spherical) distances of n points on the unit sphere S^2. Finding d_n is a long-standing problem of discrete geometry leading us back to the famous Tammes-problem. For a few values of n the exact value of d_n is known, for others, we are going to use estimations.

László Fejes Tóth [FT72] proved the following upper bound for d_n:

$$d_n \leq \tilde{\delta}_n := \arccos\left(\frac{\cot^2\left(\frac{n}{n-2}\frac{\pi}{6}\right) - 1}{2}\right), \tag{3}$$

For $n \geq 13$ Robinson [Rob61] improved this bound, let his bound be denoted by δ_n, $d_n^* := \min\{\pi/2, d_n\}$ and let

$$\delta_n^* := \begin{cases} d_n^* & \text{for } 3 \leq n \leq 14 \text{ and } n = 24, \\ \delta_n & \text{otherwise.} \end{cases} \tag{4}$$

We also need a lower bound on d_n and for a saturated point set a simple bound is immediate:

$$\frac{2}{\sqrt{n}} \leq d_n^* \leq \delta_n^*.$$

For $0 \leq \alpha \leq \pi/2$ and $n \geq 3$ we introduce $f(w, \alpha) = 4\pi \sin w - 2F(w, \alpha)$ and

$$G(w, n) = 4\pi \sin w + \sum_{i=2}^{n} f(w, \delta_{2i}^*).$$

Lemma 3.5. *For a fixed $n \geq 3$, the function $G(w, n)$ is continuous and monotonically increasing in w in the interval $[0, \delta_{2n}^*/3]$. Furthermore, $G(0, n) = 0$ and $G(\delta_{2n}^*/3, n) \geq 4\pi$.*

All of the above leads us to our main theorem.

Theorem 3.6. *For $n \geq 3$, let w_n^* denote the unique solution of the equation $G(w, n) = 4\pi$ in the interval $[0, \delta_{2n}^*/3]$. Then $\arcsin(1/n) < w_n^* \leq w_n$.*

4 On the multiplicity of arrangements of congruent zones on the sphere

This chapter of the dissertation is based on the paper [BFVZ17].

In this section we examine arrangements of equal zones on S^{d-1} from the point of view of multiplicity. The multiplicity of an arrangement is the maximum number of zones the points of the sphere belong to. We seek to minimize the multiplicity for given d and n as a function of the common width of the zones. It is clear that for $n \geq d$, the multiplicity of any arrangement with n equal zones is at least d and at most n. Notice that in the Fejes Tóth configuration the multiplicity is exactly n, that is, maximal.

In particular, if $d = 3$ and $n \geq 3$, then the multiplicity of any covering is at least 3. Our first result is a very slight strengthening of this simple fact for the case when $n \geq 4$.

Theorem 4.1. *Let $n \geq 1$ be an integer, and let S^2 be covered by the union of n congruent zones. If each point of S^2 belongs to the interior of at most two zones, then $n \leq 3$. If, moreover, $n = 3$, then the three congruent zones are pairwise orthogonal.*

Now we want to find upper bounds on the multiplicity. For this, we need the following definitions.

Let $\alpha : \mathbb{N} \to (0, 1]$ be a positive real function with $\lim_{n\to\infty} \alpha(n) = 0$. For a positive integer $d \geq 3$, let $m_d = \sqrt{2\pi d} + 1$. Let $k : \mathbb{N} \to \mathbb{N}$ be a function that satisfies the limit condition

$$\limsup_{n\to\infty} \alpha(n)^{-(d-1)} \left(\frac{e\, C_d^*\, n\, \alpha(n)}{k(n)} \right)^{k(n)} = \beta < 1, \tag{5}$$

where C_d^* is a suitable constant depending only on the dimension.

Theorem 4.2. *For each positive integer $d \geq 3$, and any real function $\alpha(n)$ described above, for sufficiently large n, there exists an arrangement of n zones of spherical half-width $m_d\alpha(n)$ on S^{d-1} such that no point of S^{d-1} belongs to more than $k(n)$ zones.*

The following statement provides the wanted upper bound on the multiplicity of coverings of the d-dimensional unit sphere by n congruent zones.

Theorem 4.3. *For each positive integer $d \geq 3$, there exists a positive constant A_d such that for sufficiently large n, there is a covering of S^{d-1} by n zones of half-width $m_d \frac{\ln n}{n}$ such that no point of S^{d-1} belongs to more than $A_d \ln n$ zones.*

We note that Theorem 4.3 and an implicit version of Theorem 4.2 were proved by Frankl, Nagy and Naszódi for the case $d = 3$, see Theorem 1.5 and Theorem 1.6 [FNN18] and also the proof of Theorem 1.5 therein. They provided two independent proofs, one of which is a probabilistic argument and the other one uses the concept of VC-dimension. Our proofs of Theorems 4.2 and 4.3 are based on the probabilistic argument of Frankl, Nagy and Naszódi [FNN18], which we modified in such a way that it works in all dimensions. In the course of the proof we also give an upper estimate for the constant A_d whose order of magnitude is $O(d)$.

Below we list some more interesting special cases according to the size of the function $\alpha(n)$.

Corollary 4.4. *With the same hypotheses as in Theorem 4.2, the following statements hold.*

i) If $\alpha(n) = n^{-(1+\delta)}$ for some $\delta > 0$, then $k(n) = const.$. Moreover, if $\delta > d-1$, then $k(n) = d$.

ii) If $\alpha(n) = \frac{1}{n}$, then $k(n) = B_d \frac{\ln n}{\ln \ln n}$ for some suitable constant B_d.

There is an obviously large gap between the lower and upper bounds for the multiplicity. The problem of finding the minimum multiplicity for zone coverings of S^{d-1} remains open.

5 On the volume bound in the Dvoretzky–Rogers lemma

This chapter of the dissertation is based on the paper [FNZ18].

5.1 Introduction and results

We say that a measure μ is an *isotropic measure* if it is a probability measure on $\mathbb{R}^d$ with the following two properties. First its inertia tensor is the identity matrix

$$\int_{\mathbb{R}^d} x \otimes x \, \mathrm{d}\mu(x) = \mathrm{Id}_d, \tag{6}$$

and its center of mass of μ is at the origin, that is,

$$\int_{\mathbb{R}^d} x \, \mathrm{d}\mu(x) = 0. \tag{7}$$

The Dvoretzky–Rogers lemma states that one may select a d-subset of any isotropic vector set in $\mathbb{R}^d$ such that the subset is well spread out, which means that the volume of the spanned parallelepiped is large. Consequently the determinant is at least $\sqrt{d!/d^d}$. The selection method here is deterministic.

On the other hand we can choose the d vectors randomly then compute the expectation of the square of the resulting determinant. This has been worked out by Pivovarov [Piv10]. We extended this result to a wider class of measures to obtain the improved lower bound of Pełczyński and Szarek [PS91] on the maximum of the volume of the spanned parallelotope and also we give a probabilistic interpretation of the volume bound in the Dvoretzky–Rogers lemma.

The result of Pivovarov is in the following lemma.

Lemma 5.1 (Pivovarov [Piv10], Lemma 3)**.** *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the isotropic measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$. Assume that $x_1, \dots, x_d$ are linearly independent with probability* 1*. Then*

$$\mathbb{E}([\det(x_1, \dots, x_d)]^2) = d!. \tag{8}$$

Our extension allows us to apply it for discrete isotropic measures.

Lemma 5.2. *Let $x_1, \dots, x_d$ be independent random vectors distributed according to the measures $\mu_1, \dots, \mu_d$ in $\mathbb{R}^d$ satisfying* (6)*. Assume that $\mu_i(\{0\}) = 0$ for i=1,..., d. Then* (8) *holds.*

The geometric motivation in studying isotropic measures is the celebrated theorem of John [Joh48].

Theorem 5.3. *Let K be a convex body in $\mathbb{R}^d$. Then there exists a unique ellipsoid of maximal volume contained in K. Moreover, this maximal volume ellipsoid is the d-dimensional unit ball B^d if and only if there exist vectors $u_1, \dots, u_m \in \mathrm{bd}K \cap S^{d-1}$ and (positive) real numbers $c_1, \dots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_d, \tag{9}$$

and

$$\sum_{i=1}^{m} c_i u_i = 0. \tag{10}$$

If a set of unit vectors $(u_1, \dots u_m)$ along with positive constants satisfies the two conditions in John's theorem then we say those vectors form a John decomposition of the identity. If the vectors are the contact points of a convex body with its John ellipsoid than one can always find a subset of the vectors $\{x_1, \dots, x_k\} \subseteq \{u_1, \dots u_m\}$ and weights $c_1, \dots, c_k > 0$ such that together they form a John decomposition of the identity. The classical lemma of Dvoretzky and Rogers stated that in a John decomposition of the identity we can always find d vectors such that they are not too far from an orthonormal system.

Lemma 5.4 (Dvoretzky–Rogers lemma [DR50]). *Let $u_1, \dots, u_m \in S^{d-1}$ and $c_1, \dots, c_m > 0$ such that* (9) *holds. Then there exists an orthonormal basis $b_1, \dots, b_d$ of $\mathbb{R}^d$ and a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with $x_j \in \operatorname{lin}\{b_1, \dots, b_j\}$ and*

$$\sqrt{\frac{d-j-1}{d}} \leq \langle x_j, b_j \rangle \leq 1 \tag{11}$$

for $j = 1, \dots, d$.

If we consider the parallelotope P spanned by the selected d vectors $x_1, \dots, x_d$ then its volume is bounded from below:

$$(\mathrm{Vol}\,(P))^2 = [\det(x_1, \dots, x_d)]^2 \geq \frac{d!}{d^d}. \tag{12}$$

Our main results in this topic are the following two Theorems, the first of which is essentially the same as Pelczyński and Szarek's [PS91], however with a probabilistic approach, proof and interpretation.

Theorem 5.5. *Let $u_1, \dots, u_m \in S^{d-1}$ be unit vectors satisfying* (9) *with some $c_1, \dots, c_m > 0$. Then there is a subset $\{x_1, \dots, x_d\} \subset \{u_1, \dots, u_m\}$ with*

$$[\det(x_1, \dots, x_d)]^2 \geq \gamma(d, \overline{m}) \cdot \frac{d!}{d^d},$$

where $\gamma(d, \overline{m}) = \frac{\overline{m}^d}{d!} \binom{\overline{m}}{d}^{-1}$, and $\overline{m} = \min\{m, d(d+1)/2\}$.

Moreover, for $\gamma(d, \overline{m})$, we have

(i) $\gamma(d, \overline{m}) \geq \gamma(d, d(d+1)/2) \geq 3/2$ for any $d \geq 2$ and $m \geq d$. And $\gamma(d, d(d+1)/2)$ is monotonically increasing, and $\lim_{d\to\infty} \gamma(d, d(d+1)/2) = e$.

(ii) Fix a $c > 1$, and consider the case when $m \leq cd$ with $c \geq 1 + 1/d$. Then

$$\gamma(d, m) \geq \gamma(d, \lceil cd \rceil) \sim \sqrt{\frac{c-1}{c}} \left(\frac{c-1}{c}\right)^{(c-1)d} e^d, \quad \text{as } d \to \infty.$$

(iii) Fix an integer $k \geq 1$, and consider the case when $m \leq d + k$. Then

$$\gamma(d, m) \geq \gamma(d, d+k) \sim \frac{k! e^k}{\sqrt{2\pi}} \frac{e^d}{(d+k)^{k+1/2}}, \quad \text{as } d \to \infty.$$

The geometric interpretation of this theorem is the following. If K is a convex polytope with n facets, and B^d is the maximal volume ellipsoid in K, then the number of contact points $u_1, \ldots, u_m$ in John's theorem is at most $m \leq n$. Thus, it yields a simplex in K of not too small volume, with one vertex at the origin.

The following statement provides a lower bound on the probability that d independent, identically distributed random vectors selected from $\{u_1, \ldots, u_m\}$ according to the distribution determined by the weights $\{c_1, \ldots, c_m\}$ has large volume.

Proposition 5.6. *Let $\lambda \in (0, 1)$. With the notations and assumptions of Theorem 5.5, if we choose the vectors $x_1, \ldots, x_d$ independently according to the distribution $\mathbb{P}(x_\ell = u_i) = c_i/d$ for each $\ell = 1, \ldots, d$ and $i = 1, \ldots, m$, then with probability at least $(1 - \lambda)e^{-d}$, we have that*

$$[\det(x_1, \ldots, x_d)]^2 \geq \lambda \gamma(d, \overline{m}) \cdot \frac{d!}{d^d}.$$

In particular, consider $k = 1$ in 5.5 (iii), that is the case when K is the regular simplex whose inscribed ball is B^d. Then after doing the necessary calculations we get that the bound provided by our theorem is sharp in this case.

References

[Ban51] T. Bang, *A solution of the "plank problem."*, Proc. Amer. Math. Soc. **2** (1951), 990–993.

[Ban50] T. Bang, *On covering by parallel-strips*, Mat. Tidsskr. B. **1950** (1950), 49–53.

[BM19] D. Bilyk and R. W. Matzke, *On the Fejes Tóth problem about the sum of angles between lines*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 51–59.

[DR50] A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 192–197.

[FT72] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer-Verlag, Berlin-New York, 1972.

[FT59] L. Fejes Tóth, *Über eine Punktverteilung auf der Kugel*, Acta Math. Acad. Sci. Hungar **10** (1959), 13–19.

[FT73] L. Fejes Tóth, *Research Problems: Exploring a Planet*, Amer. Math. Monthly **80** (1973), no. 9, 1043–1044.

[FNN18] N. Frankl, J. Nagy, and M. Naszódi, *Coverings: variations on a result of Rogers and on the epsilon-net theorem of Haussler and Welzl*, Discrete Math. **341** (2018), no. 3, 863–874.

[JP17] Z. Jiang and A. Polyanskii, *Proof of László Fejes Tóth's zone conjecture*, Geom. Funct. Anal. **27** (2017), no. 6, 1367–1377.

[Joh48] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y., 1948, pp. 187–204.

[Lin74] J. Linhart, *Eine extremale Verteilung von Grosskreisen*, Elem. Math. **29** (1974), 57–59.

[PS91] A. Pełczyński and S. J. Szarek, *On parallelepipeds of minimal volume containing a convex symmetric body in* $\mathbf{R}^n$, Math. Proc. Cambridge Philos. Soc. **109** (1991), no. 1, 125–148.

[Piv10] P. Pivovarov, *On determinants and the volume of random polytopes in isotropic convex bodies*, Geom. Dedicata **149** (2010), 45–58.

[Rob61] R. M. Robinson, *Arrangements of 24 points on a sphere*, Math. Ann. **144** (1961), 17–48.

[Ros72] V. Rosta, *An extremal distribution of three great circles*, Mat. Lapok **23** (1972), 161–162 (1973).

[Ste18] S. Steinerberger, *Well-distributed great circles on* $\mathbb{S}^2$, Discrete Comput. Geom. **60** (2018), no. 1, 40–56.

[Tar32] A. Tarski, *Remarks on the degree of equivalence of polygons*, Parametr. **2** (1932), 310–314.