On some Isometries and other Preservers

Outline of Ph.D. thesis

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The main part of the dissertation is divided into five chapters, which are followed by a summary (both in English and in Hungarian) and a bibliography. Essentially, the material of each chapter corresponds to an original research article which has been already published in a well-established refereed mathematical journal. These publications are [**8, 9, 11, 12, 13**].

The dissertation contains some results from various branches of preserver problems. The vague formulation of these problems reads as follows. We are given a structure and certain characteristic (a quantity or a relation, etc.) attached to the elements of the underlying structure. The task is to give a precise description of all transformations that preserve the given characteristic. A detailed classification of preserver problems can be found in the monograph of Molnár [**27**] and in the comprehensive survey articles [**20, 22, 32**]. In the dissertation, basically just two kind of preserver problems were considered:

- unitary invariant function preservers of different binary operations between positive operators in Chapters 1-4;
- isometry problems on linear spaces of matrices in Chapter 5.

1. A class of determinant preserving maps for finite von Neumann algebras

In 1897 Frobenius [**6**] proved that if ϕ is a linear map on the matrix algebra $M_n(\mathbb{C})$ of n-by-n complex matrices preserving the determinant, then there are matrices $M, N \in M_n(\mathbb{C})$ such that $\det(MN) = 1$ and ϕ can be written in one of the following forms:

$$\phi(A) = MAN \quad \text{or} \quad \phi(A) = MA^{tr}N$$

where $(.)^{tr}$ denotes transposition of a matrix. It means that the corresponding preservers are only the obvious ones.

In the past decades the aforementioned result of Frobenius has inspired many researchers to deal with different sorts of preserver problems involving various notions of determinant [**2, 4, 5, 18, 29, 35**]. Among others, in [**18**] Huang et al. completely described all maps on the cone $\mathbb{P}_n$ of n-by-n positive definite complex matrices which satisfy the sole property

$$\det(\phi(A) + \phi(B)) = \det(\phi(I))^{1/n} \det(A + B)$$

for all $A, B \in \mathbb{P}_n$.

In Chapter 1 we carried out a similar work concerning maps on the positive definite cone $\mathcal{N}^{++}$ of a finite von Neumann algebra $\mathcal{N}$ with faithful tracial state τ, and thus proved an identical operator algebraic counterpart of the result of Huang et al. If $\mathcal{N}$ is such an algebra, then for an invertible operator $A \in \mathcal{N}$ the Fuglede-Kadison determinant [**7**] Δ associated with τ is defined as

$$\Delta(A) = \exp(\tau(\log \sqrt{A^*A})).$$

Theorem 1. (Gaál, Nayak [**13**])
Let $\mathcal{N}$ be a finite von Neumann algebra with faithful tracial state τ, and let $\phi : \mathcal{N}^{++} \to \mathcal{N}^{++}$ be a bijective map. Then we have

$$\Delta(\phi(A) + \phi(B)) = \Delta(\phi(I)) \cdot \Delta(A + B)$$

for all $A, B \in \mathcal{N}^{++}$ if and only if there is a τ-preserving Jordan *-isomorphism $J : \mathcal{N} \to \mathcal{N}$ and a positive invertible element $T \in \mathcal{N}^{++}$ such that

$$\phi(A) = TJ(A)T, \quad \text{for every } A, B \in \mathcal{N}^{++}.$$

Moreover, if $\mathcal{N}$ is a factor, then ϕ extends to a (τ-preserving) *-automorphism or *-antiautomorphism of $\mathcal{N}$.

Our approach to the solution relied heavily on a generalization of the Minkowski determinant inequality to the setting of von Neumann algebras. Although the inequality itself has been proved several years ago by Arveson [**1**] and also by Hiai and Bourin [**3**], the equality condition has not been treated yet. As that played an important role, we established when

$$\Delta(A + B) = \Delta(A) + \Delta(B)$$

holds for positive invertible operators A, B in the von Neumann algebra $\mathcal{N}$.

2. Norm-additive maps on the positive definite cone of a C^*-algebra

The study of norm-additive maps on an additive semigroup of a normed space is a very active and lively area. As for earlier results on investigations in this directions, especially on function algebras, we mention the series of publications [**14, 15, 16, 17, 36**].

These results attracted some authors to deal with norm-additive maps on different sort of domains, on the cones of positive and positive definite operators. Namely, in [**29**] Nagy completely described the structure of norm-additive maps on positive Schatten p-class operators with respect to the Schatten p-norm. In [**28**] among others Molnár and Szokol managed to determine the structure of maps of the same kind on the positive cone of a standard operator algebra (by what we mean a subalgebra of the algebra of all bounded linear operators containing the finite rank elements).

In Chapter 2 we considered the problem in the setting of C^*-algebra equipped with a faithful normalized trace τ, and gave a complete description of the structure of those maps on the positive definite cone $\mathcal{A}^{++}$ of a unital C^*-algebra which satisfy the sole property

$$\|\phi(a) + \phi(b)\|_p = \|a + b\|_p$$

for all $a, b \in \mathcal{A}^{++}$, where $\|a\|_p = \tau(|a|^p)^{1/p}$ is the Schatten p-norm of the element a.

Theorem 2. (Gaál [**8**])
Let $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ be a bijective transformation and $p > 1$. Then

$$\|\phi(a) + \phi(b)\|_p = \|a + b\|_p, \quad \text{for all } a, b \in \mathcal{A}^{++}$$

if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ and a positive invertible element $d \in \mathcal{A}^{++}$ such that

$$\phi(a) = dJ(a)d, \quad \text{for every } a \in \mathcal{A}^{++},$$

and J and d satisfy

$$\|dJ(a)d\|_p = \|a\|_p, \quad \text{for } a \in \mathcal{A}^{++}. \tag{1}$$

Moreover, if $\mathcal{A}$ is a finite von Neumann factor, then ϕ extends to either an algebra *-automorphism or an algebra *-antiautomorphism of $\mathcal{A}$.

After publishing our paper [**8**] it turned out that the condition appearing in (1) is equivalent to d being central [**26**].

3. Preserver problems related to quasi-arithmetic means

In Chapter 3 we considered quasi-arithmetic means of positive definite matrices. The quasi-arithmetic mean $M_{f,t} \colon \mathbb{P}_n^2 \to \mathbb{P}_n$ generated by f with weight $t \in [0, 1]$ is defined by

$$M_{f,t}(A, B) = f^{-1}(tf(A) + (1 - t)f(B)). \tag{2}$$

We remark that this operation is an extension of the quasi-arithmetic means generated by f from positive numbers to positive definite matrices. The most fundamental quasi-arithmetic means are the *log-Euclidean mean* and the *λ–power mean* (which is a generalization both of the arithmetic and the harmonic means). Their common weight is $1/2$, and their generating functions are given by $x \mapsto \log(x)$ and $x \mapsto x^\lambda$ with some real number $\lambda \neq 0$, respectively.

In Chapter 3, we discussed three preserver problems related to quasi-arithmetic means.

Problem A. Describe the corresponding homomorphisms with respect to quasi-arithmetic means.

We pointed out that such maps have no general structure, however, those transformations that are automorphisms with respect to the mean $M_{f,t}$ for all $t \in [0,1]$ have a straightforward description. Then we turned to the investigation of the following problem.

Problem B. Determine the structure of transformations which preserve norms of quasi-arithmetic means.

This question has not been answered yet for general quasi-arithmetic means, but for the corresponding results on operator means we refer to the papers [**28, 31**]. In Chapter 3, we solved Problem B under certain quite general conditions.

Theorem 3. (Gaál, Nagy [**12**])
Suppose that $|\lim_{x\to 0} f(x)| = \infty$ *and that* $f(]0,\infty[)$ *is any of the sets* $\mathbb{R}$, $]0,\infty[$. *Moreover, let* $t \in]0,1[$ *be a fixed real number and* $N(.)$ *be a unitary invariant norm on* $M_n(\mathbb{C})$. *In addition, assume that* $\phi\colon \mathbb{P}_n \to \mathbb{P}_n$ *is a bijection with*

$$N(M_{f,t}(\phi(A),\phi(B))) = N(M_{f,t}(A,B)) \tag{3}$$

for all $A, B \in \mathbb{P}_n$. *Then there is a unitary matrix* $U \in M_n(\mathbb{C})$ *such that either*

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{P}_n$.

Our goal concerning the result was to cover as many of the most fundamental quasi-arithmetic means, the log-Euclidean mean and the λ–power means, as we can. Clearly, the generating functions of the first and, in the case $\lambda < 0$, the second type of means (hence also that of the harmonic mean) satisfy the conditions of Theorem 3. However, this is not the case with those functions of λ–power means for $\lambda > 0$. Concerning the latter operations, we proved the following somewhat partial result.

Theorem 4. (Gaál, Nagy [**12**])
Suppose that $f(x) = x^\lambda$ $(x > 0)$ *with some scalar* $\lambda > 0$ *and let* $t \in]0,1[$ *be a fixed real number. Denote by* $\|.\|_\infty$ *the spectral norm on* $M_n(\mathbb{C})$. *In addition, assume that* $\phi\colon \mathbb{P}_n \to \mathbb{P}_n$ *is a bijection satisfying*

$$\|M_{f,t}(\phi(A),\phi(B))\|_\infty = \|M_{f,t}(A,B)\|_\infty \tag{4}$$

for all $A, B \in \mathbb{P}_n$. *Then there is a unitary matrix* $U \in M_n(\mathbb{C})$ *such that either*

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{P}_n$.

If $\mathcal{A}$ is a C^*-algebra, then for any $t \in [0,1]$ and for every $a, b \in \mathcal{A}^{++}$ one can define the object $M_{f,t}(a,b)$ by the formula (2). Now suppose that f is increasing and smooth. Then for all $a, b \in \mathcal{A}^{++}$ and $t \in [0,1]$ the element $\Gamma_{a,b}(t)$ can be defined as

$$\Gamma_{A,B} : [0,1] \to \mathcal{A}^{++}, \quad t \mapsto M_{f,1-t}(A,B)$$

Given a norm $|||.|||$ on $\mathcal{A}$, we shall say that a map ϕ of $\mathcal{A}^{++}$ preserve the norm $|||.|||$ of the geodesic correspondence whenever

$$|||\Gamma_{\phi(a),\phi(b)}(t)||| = |||\Gamma_{a,b}(t)||| \tag{5}$$

holds for all $t \in [0,1]$ and for every $a, b \in \mathcal{A}^{++}$. Our third problem can be formulated in the following way.

Problem C. Characterize those maps of $\mathcal{A}^{++}$ which preserve a norm of the geodesic correspondence.

Problem C was partially solved in [**34**, Theorem 4.1] by Szokol et. al for $\mathcal{A} = M_n(\mathbb{C})$ in the case of the Schatten p-norm. Observe that our statement in Theorem 3 is much more stronger than their corresponding result in the following respects:

1) we consider quite general quasi-arithmetic means, not just the ones whose generating function is log;
2) we require that (5) should hold not for all but *one* fixed $t \in [0,1]$;
3) we consider an arbitrary unitary invariant norm, not necessarily $\|.\|_p$.

Our further contribution to Problem C is the following.

Theorem 5. (Gaál, Nagy [**12**])
Let $\mathcal{A}$ be a C^*-algebra such that a faithful normalized trace τ on $\mathcal{A}$ can be given. Fix a number $p \geq 1$ and assume that $f = \log$. If $\phi \colon \mathcal{A}^{++} \to \mathcal{A}^{++}$ is a bijective map which preserves the Schatten p-norm of the geodesic correspondence, then there exist a Jordan *-isomorphism J of $\mathcal{A}$ and a central element $c \in \mathcal{A}^{++}$ such that

$$\phi(a) = cJ(a), \quad \textit{for all } a \in \mathcal{A}^{++}.$$

Moreover, c and J satisfy

$$\tau(cJ(x)J(y)) = \tau(xy), \quad \textit{for every } x, y \in \mathcal{A}.$$

4. Norm preservers of Kubo-Ando means of Hilbert space effects

The result of Chapter 4 concerned the set $E(H)$ of Hilbert space effects on H, by which we mean the operator interval

$$[0, I] = \{X \in B(H) \ : \ X = X^*, \ 0 \leq X \leq I\}.$$

A binary operation on the set of positive operators is a mean in the Kubo-Ando sense [**19**] if it possesses the following properties. For any A, B, C, D and sequences $(A_n), (B_n)$, we have

(i) $I\sigma I = I$;
(ii) if $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$;
(iii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$.

Here, the symbol $\downarrow$ refers for monotone decreasing convergence in the strong operator topology. If σ is a Kubo-Ando mean, then its transpose $\tilde{\sigma}$ is defined as $A\tilde{\sigma}B := B\sigma A$. One of the main result of the beautiful Kubo-Ando theory tells us that if $\dim H = d < \infty$, then there is a d-monotone function f_σ (called the generating function of σ) such that σ admits the explicit form

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all positive operators A, B with A being invertible. If the Hilbert space H is infinite dimensional, the associated generating function is operator monotone. Further the function f_σ due to the property (iv) uniquely determines the mean σ.

Note that properties (i)-(ii) ensure that the set $E(H)$ is closed under the binary operation σ defined by the above purely axiomatic way and therefore it is a well-defined operation also on that structure. Thus, Problems A-B in the previous section make sense regarding the effect algebra $E(H)$, too.

As for Problem A, the automorphisms of $E(H)$ with respect to the geometric and the harmonic mean was determined by Šemrl [**33**]. In Chapter 4 we investigated Problem B on $E(H)$ for symmetric norms and general Kubo-Ando means. A norm N is called symmetric if

$$N(AXB) \leq ||A||N(X)||B||$$

is satisfied by any A, B, X, where $||.||$ denotes the operator norm. Several norms appearing in matrix theory, including the Schatten and the Ky Fan norms are symmetric. Moreover, we mention that every symmetric norm N on $B(H)$ is unitarily invariant. Furthermore, in the case where $\dim H < \infty$ these latter two properties are equivalent.

THEOREM 6. (Gaál, Nagy [**11**])
Let σ be a Kubo-Ando mean, and suppose that f_σ is strictly concave and either $f_\sigma(0) = 0$ or $\tilde{f}_\sigma(0) = 0$ is fulfilled. Furthermore, let N be a symmetric norm. Then the bijection $\phi\colon E(H) \to E(H)$ has the property

$$N(\phi(A)\sigma\phi(B)) = N(A\sigma B), \qquad \text{for all } A, B \in E(H) \tag{6}$$

if and only if there exists either a unitary or an antiunitary operator U on H such that

$$\phi(A) = UAU^*, \qquad \text{for every } A \in E(H).$$

We remark that the assumptions $f_\sigma(0) = 0$ and $\tilde{f}_\sigma(0) = 0$ are the same as assuming $A\sigma 0 = 0$ and $0\sigma A = 0$ for every positive operator A, respectively.

As for recent investigations regarding Problem A-B on the whole set of positive operators rather than $E(H)$, the reader can consult with the series of publications [**10, 23, 24, 25, 28**].

5. Isometry groups of self-adjoint traceless and skew-symmetric matrices

In the paper [**30**] titled *"Isometries of the spaces of self-adjoint traceless operators"* Nagy proved that the isometries on the real vector space of n-by-n self-adjoint traceless matrices H_n^0 are automatically surjective, whence linear up to a translation, by the celebrated Mazur-Ulam theorem. Furthermore, the complete description of the structure of linear isometries with respect to a large class of unitary similarity invariant norm, more precisely, with respect to any Schatten p-norm $\|.\|_p$ was given whenever $n \neq 3$.

In the first part of Chapter 5, we determined the isometry group of any *unitary invariant* norm on H_n^0, which was posed in the paper [**30**] as an open problem. In addition, it turned out that we need to require only some weaker invariance property: the norm in question must be invariant just under *unitary similarity transformations*. At the same time, we completed the former work of Nagy where the case $n = 3$ has not been treated yet.

In the formulation of the forthcoming results, $PSU(n)$ denotes the image of $SU(n)$ in $GL(n^2-1, \mathbb{R})$ acting on H_n^0 via the adjoint representation

$$\mathrm{Ad} : SU(n) \to GL(n^2-1, \mathbb{R}), \quad U \mapsto \mathrm{Int}_U(.).$$

The group $PSU(n)$ clearly preserves the trace form $\langle A, B\rangle = \operatorname{Tr} AB$ and this it embeds in the orthogonal group of H_n^0. Let us denote by $O(n^2-1, \mathbb{R})$ this group, and let $(.)^{tr}$ be the transpose map.

If H is a subgroup of G, then we shall write $H \leq G$. Whenever the group G is generated by subgroups $H_1, \dots H_s$ and elements $g_1, \dots g_t$, we write just $\langle H_1, \ldots, H_s, g_1, \ldots, g_t\rangle$ in order to denote the generated group. Assume now further that G is embedded into the general linear group $GL(V)$ for some vector space V. Then $C(G)$ and $\mathcal{N}(G)$ denote its centralizer and its normalizer, respectively.

Theorem 7. (Gaál, Guralnick [**9**])
Let $V := H_n^0$ and assume that $n \geq 3$. Let $\mathcal{K}$ be a compact Lie group satisfying $PSU(n) \leq \mathcal{K} \leq GL(V)$. Then one of the following happens:

(a) $\mathcal{K} \leq \mathcal{N}(PSU(n)) = \langle PSU(n), GL(1,\mathbb{R}), (.)^{tr}\rangle$;
(b) $SO(V) \leq \mathcal{K} \leq \mathcal{N}(SO(V)) = \langle O(V), GL(1,\mathbb{R})\rangle$.

As an immediate consequence, we obtained the following answer to the question posed by Nagy.

Corollary 8. *Assume that $n \geq 3$. If $\mathcal{K}$ is the isometry group of any unitary similarity invariant norm on H_n^0, then we have the following possibilities:*

(a) $\mathcal{K} = \langle PSU(n), \mathbb{Z}/2, (.)^{tr}\rangle$;
(b) $\mathcal{K} = O(n^2 - 1, \mathbb{R})$.

The second result of Chapter 5 concerned the set $K_n(\mathbb{R})$ of n-by-n skew-symmetric matrices over the field of real numbers. Denote $PSO(n,\mathbb{R})$ the image of $SO(n,\mathbb{R})$ under the adjoint representation

$$\mathrm{Ad} : SO(n,\mathbb{R}) \to GL(K_n(\mathbb{R})), \quad Q \mapsto \mathrm{Int}_Q(.).$$

Theorem 9. (Gaál, Guralnick [**9**])
Let $W := K_n(\mathbb{R})$. Assume that $n \geq 4$. Let $\mathcal{K}$ be a compact Lie group with the property that $PSO(n,\mathbb{R}) \leq \mathcal{K} \leq GL(W)$. Then one of the following occurs:

(a) $n > 4$, $n \neq 8$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(n,\mathbb{R})) = \langle PO(n,\mathbb{R}), GL(1,\mathbb{R})\rangle;$$

(b) $n = 4$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(4,\mathbb{R})) = \langle PO(4,\mathbb{R}), C(PSO(4,\mathbb{R}))\rangle;$$

(c) $n = 8$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(8,\mathbb{R})) = \langle PSO(8,\mathbb{R}), GL(1,\mathbb{R}), S_3\rangle;$$

(d) $SO(W) \leq \mathcal{K} \leq \mathcal{N}(SO(W)) = \langle O(W), GL(1,\mathbb{R})\rangle$.

Define the involution $A^* := \psi(A)$ by

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Applying our result on the description of $PSO(n, \mathbb{R})$ overgroups, one could recover the following result by Li and Tsing provided that $n \neq 8$.

THEOREM 10. (Li, Tsing [**21**])
Let $L : K_n(\mathbb{R}) \to K_n(\mathbb{R})$ be a linear map. Then the following conditions are equivalent:

(a) *L is an isometry with respect to any orthogonal congruence invariant norm on $K_n(\mathbb{R})$ which is not a constant multiple of the Frobenius norm;*
(b) *there exist a real number $\eta \in \{-1, 1\}$ and an orthogonal matrix $Q \in O(n, \mathbb{R})$ such that one of the following hold:*
   (i) *$L(X) = \eta Q X Q^{-1}$ for every $X \in K_n(\mathbb{R})$;*
   (ii) *$n = 4$ and $L(X) = \eta Q \psi(X) Q^{-1}$ for every $X \in K_n(\mathbb{R})$.*

In the case where $n = 8$, the corresponding isometry group can be larger.

THEOREM 11. (Gaál, Guralnick)
Let $\mathcal{K}$ be the isometry group of an orthogonal congruence invariant norm on $K_8(\mathbb{R})$ which is not a constant multiple of the Frobenius norm. Then one of the following holds:

(a) *$\mathcal{K} = \langle PO(8, \mathbb{R}), \mathbb{Z}/2 \rangle$;*
(b) *$\mathcal{K} = \langle PSO(8, \mathbb{R}), \mathbb{Z}/2, S_3 \rangle$.*

Conversely, both of the groups (a) and (b) are isometry groups of certain orthogonal congruence invariant norms on $K_8(\mathbb{R})$.

Bibliography

[1] W. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642.

[2] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London Math. Soc. **62** (2000), 917–924.

[3] J. Bourin and F. Hiai, *Anti-norms on finite von Neumann algebras*, Publ. Res. Inst. Math. Sci. **51** (2015), 207–235.

[4] G. Dolinar and P. Šemrl, *Determinant preserving maps on matrix algebras*, Linear Algebra Appl. **348** (2002), 189–192.

[5] M.L. Eaton, *On linear transformations which preserve the determinant*, Illinois J. Math. **13** (1969), 722–727.

[6] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, J. Reine Angew. Math. (1897), 994–1015.

[7] B. Fuglede and R.V. Kadison, *Determinant theory in finite factors*, Ann. of Math. **55** (1952), 520–530.

[8] M. Gaál, *Norm-additive maps on the positive definite cone of a C^*-algebra*, Results Math. (to appear)

[9] M. Gaál and R.M. Guralnick, *On isometry groups of self-adjoint traceless and skew-symmetric matrices*, Linear Algebra Appl. **541** (2018), 211–220.

[10] M. Gaál and G. Nagy, *A characterization of unitary-antiunitary similarity transformations via Kubo-Ando means*, Ann. Math. (to appear)

[11] M. Gaál and G. Nagy, *Maps on Hilbert space effect algebras preserving norms of operator means*, Acta Sci. Math. (Szeged) **84** (2018), 201–208.

[12] M. Gaál and G. Nagy, *Preserver problems related to quasi-arithmetic means of invertible positive operators*, Integr. Equ. Oper. Theory **90**:7 (2018).

[13] M. Gaál and S. Nayak, *On a class of determinant preserving maps for finite von Neumann algebras*, J. Math. Anal. Appl. **464** (2018), 317–327.

[14] O. Hatori G. Hirasawa and T. Miura, *Additively spectral-radius preserving surjections between unital semisimple commutative Banach algebras*, Cent. Eur. J. Math. **8** (2010), 597–601.

[15] O. Hatori, S. Lambert, A. Luttman, T. Miura, T. Tonev, and R. Yates, *Spectral preservers in commutative Banach algebras*, in Function Spaces in Modern Analysis, K. Jarosz, Edt. Contemporary Mathematics vol. 547, American Mathematical Society, 2011, pp. 103–124.

[16] M. Hosseini and J.J. Font, *Norm-additive in modulus maps between function algebras*, Banach J. Math. Anal. **8** (2014), 79–92.

[17] M. Hosseini and J.J. Font, *Real-linear isometries and jointly norm-additive maps on function algebras*, Mediterr. J. Math. **13** (2016), 1933–1948.

[18] H. Huang, C.N. Liu, P. Szokol, M.C. Tsai and J. Zhang, *Trace and determinant preserving maps of matrices*, Linear Algebra Appl. **507** (2016), 373–388.

[19] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.

[20] C.K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly **108** (2001), 591–605.

[21] C.K. Li and N.K. Tsing, *Duality between some linear preserver problems. III. c-spectral norms and (skew)-symmetric matrices with fixed singular values*, Linear Algebra Appl. **143** (1991), 67–97.

[22] C.K. Li and N.K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. **162-164** (1992), 217–235.

[23] L. Molnár, *Maps preserving general means of positive operators*, Electron. J. Linear Algebra **22** (2011), 864–874.

[24] L. Molnár, *Maps preserving the geometric mean of positive operators*, Proc. Amer. Math. Soc. **137** (2009), 1763–1770.

[25] L. Molnár, *Maps preserving the harmonic mean or the parallel sum of positive operators*, Linear Algebra Appl. **430** (2009), 3058–3065.

[26] L. Molnár, *Quantum Rényi relative entropies on density spaces of C^*-algebras: their symmetries and their essential difference*, submitted.

[27] L. Molnár, Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces, Springer, 2007.

[28] L. Molnár and P. Szokol, *Transformations preserving norms of means of positive operators and nonnegative functions*, Integr. Equ. Oper. Theory **83** (2015), 271–290.

[29] G. Nagy, *Determinant preserving maps: an infinite dimensional version of a theorem of Frobenius*, Linear Multilinear Algebra **65** (2017), 351–360.

[30] G. Nagy, *Isometries of the spaces of self-adjoint traceless operators*, Linear Algebra Appl. **484** (2015), 1–12.

[31] G. Nagy, *Preservers for the p-norm of linear combinations of positive operators*, Abstr. Appl. Anal. **2014** (2014), Article ID 434121, 9 pages.

[32] S. Pierce et al., *A survey of linear preserver problems*, Linear Multilinear Algebra **33** (1992), 1–129.

[33] P. Šemrl, *Symmetries of Hilbert space effect algebras*, J. London Math. Soc. **88** (2013), 417–436.

[34] P. Szokol, M.C. Tsai and J. Zhang, *Preserving problems of geodesic-affine maps and related topics on positive definite matrices*, Linear Algebra Appl. **483** (2015), 293–308.

[35] V. Tan and F. Wang, *On determinant preserver problems*, Linear Algebra Appl. **369** (2003), 311–317.

[36] T. Tonev and R. Yates, *Norm-linear and norm-additive operators between uniform algebras*, J. Math. Anal. Appl. **357** (2009), 45–53.