On some Isometries and other Preservers

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Preface

The current work is intended to serve as the author's doctoral dissertation submitted to the Bolyai Institute, University of Szeged in partial fulfillment of the requirements for the award of the Degree of *Doctor of Philosophy*.

The work is divided into five chapters. Essentially, the material of each chapter corresponds to an original research article which has been already published in a well-established refereed mathematical journal. Four of these articles are written in collaborations with one coauthor, and for one paper I am the sole author. Since each chapter has an own detailed introduction, I find it not necessary to describe the mathematical content here. However, it might be beneficial to tell some words about the topics discussed in the dissertation.

The dissertation does not cover all the scientific papers of the author, it just contains some ideas from various branches of preserver problems. The vague formulation of these problems reads as follows. We are given a structure and certain characteristic (a quantity or a relation, etc.) attached to the elements of the underlying structure. The task is to give a precise description of all transformations that preserve the given characteristic.

A detailed classification of preserver problems can be found in the monograph of Molnár [**65**] and in the comprehensive survey articles [**50, 52, 78**]. Here, just for illustration, we mention only the most fundamental preserver problems such as the *Frobenius problem* on determinant preserving linear maps, the *Banach-Stone theorem* and the celebrated *Mazur-Ulam theorem* on isometries of normed spaces.

The following result of Frobenius characterizes the determinant preserving linear maps.

Theorem. (Froebenius)
Every linear transformation on the set of d-by-d matrices preserving the determinant is of the form

$$\phi(A) = MAN \quad or \quad \phi(A) = MA^{tr}N$$

with some invertible matrices M, N satisfying $\det(MN) = 1$.

The Banach-Stone theorem [**18**] describes the structure of surjective isometries between the spaces of continuous functions.

Theorem. (Banach-Stone)
Let X, Y be compact Hausdorff spaces and let $\phi : C(X) \to C(Y)$ be a surjective isometry between the spaces of continuous functions. Then there exist a homeomorphism $\tau : Y \to X$ and a function $h \in C(Y)$ such that $|h(y)| = 1$ for all $y \in Y$ and

$$\phi(f)(y) = h(y)(f \circ \tau)(y), \quad \text{for all } f \in C(X), y \in Y.$$

The celebrated Mazur-Ulam theorem describes the structure of surjective isometries between real normed spaces.

Theorem. (Mazur-Ulam)
A surjective isometry between any two real normed spaces is affine.

For more on the Mazur-Ulam theorem, including different proof techniques and generalizations, the interested reader is referred to [**55, 57, 74, 89, 90**]. Furthermore, we note that *Wigner's unitary-antiunitary theorem* [**35**] on quantum mechanical symmetry transformations provides another important example of preserver problems.

Theorem. (Wigner)
Let H be a complex Hilbert space and let $\phi : P_1(H) \to P_1(H)$ be an arbitrary function between the set of rank one projections on H. Then ϕ preserves the transition probability (the quantity $\operatorname{Tr} PQ$*) if and only if there exists either a unitary or an antiunitary operator on H such that*

$$\phi(P) = UPU^*, \quad \text{for all } P \in P_1(H).$$

The aforementioned results are not only notable examples but also serve as starting points of the different lines of researches in the present dissertation, as explained in the forthcoming paragraphs.

In this dissertation, basically two kind of preserver problems are considered:

- certain unitary (or just unitary similarity) invariant function preservers of different binary operations on cones of positive operators in Chapters 1-4;
- some isometry problems on matrices in Chapter 5.

The first chapter concerns with infinite dimensional determinant theory, which has been developed by Fuglede and Kadison in 1952. On the set of positive invertible elements in a finite von Neumann algebra, we describe the general form of unital (identity preserving) determinant additive maps. This problem was studied first by Huang et al. in the setting of matrix algebras. Once a result is gained for matrices, it is a natural step to extend it in some way for general operator algebras. Thus, we formulate and present the solution of a completely analogous von Neumann algebraic counterpart of the problem studied by Huang et al. concerning the Fuglede-Kadison determinant. One major step in the proof is isolating the equality case in Minkowski's determinant inequality, which is a rather ambitious task in itself.

The Fuglede-Kadison determinant is a prototype of anti-norms, so the main result in the first chapter can be viewed as a sort characterization of certain anti-norm-additive maps. The problem of studying norm-additive maps between additive semigroups of normed spaces is very close in spirit to the aforementioned one, and was investigated by several authors, especially, on function algebras. Motivated by these results Molnár, Nagy and Szokol obtained structural results concerning that kind of maps on the cones of positive operators, and also on the set of positive Schatten class operators. We follow this line of research in the second chapter, and study the problem in the case where the Schatten norms are considered on the positive definite cone of a C^*-algebra carrying a faithful normalized trace.

The third and the fourth chapters are devoted to the study of certain preserver problems related to different operator means. The most fundamental means are the so-called Kubo-Ando means and the quasi-arithmetic means. Some of them are in an intimate connection with differential geometric structures on the manifold of positive definite matrices, as they arise as geodesic midpoints with respect to suitable Riemann metrics. In this regard, operator means generalize the concept of averaging of positive or nonnegative real numbers.

Means can also be viewed as binary operations, and studying maps which preserve any numerical function of a binary operation has a considerable amount of literature. The third and the fourth chapter discuss some problems in a similar vein. In the third chapter we determine the preservers on the cone of positive invertible operators for unitary invariant norms of quasi-arithmetic means. In addition, some further preserver problems related to quasi-arithmetic means are also considered. In the fourth chapter we describe the structure of those maps on the so-called effect algebra (positive operators which are majorized by the identity) which preserve a unitary invariant norm of any given Kubo-Ando mean.

The remaining part of the dissertation is devoted to the study of isometries on various matrix spaces. Firstly, we determine the structure of linear isometries on the vector space of self-adjoint traceless matrices. The short proof applies a group theoretic scheme called overgroups. This scheme was developed by Dynkin, and works in general as follows. We consider the group of operators under which action the norm is invariant, and determine all possible compact Lie groups lying between this group and the general linear group. If the list contains just a few groups, one can select which could be an isometry group. Secondly, we also give a proof of an old result on the structure of linear isometries of skew-symmetric matrices by Li and Tsing, with some minor revision in the eight dimensional case.

* * *

This was about the content of the dissertation. At the end of this preface, I would also like to take the opportunity to acknowledge my indebtedness to *Professor Lajos Molnár* under whose guidance and permanent encouragement this dissertation was written. I thank him for his time and endless support.

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Contents

1

A class of determinant preserving maps for finite von Neumann algebras

(The results in this chapter are joint with Soumyashant Nayak and have been published in [**29**].)

1.1. Introduction and formulation of the result

In 1897 Frobenius [**20**] proved that if ϕ is a linear map on the matrix algebra $M_d(\mathbb{C})$ of d-by-d complex matrices preserving the determinant, then there are matrices $M, N \in M_d(\mathbb{C})$ such that $\det(MN) = 1$ and ϕ can be written in one of the following forms:

$$\phi(A) = MAN \quad \text{or} \quad \phi(A) = MA^{tr}N$$

where $(.)^{tr}$ denotes transposition of a matrix. It means that the corresponding preservers are only the obvious ones.

In the past decades the result of Frobenius has inspired many researchers to deal with different sorts of preserver problems involving various notions of determinant [**4, 14, 16, 45, 69, 86**]. Among others, in [**45**] Huang et al. completely described all maps on the positive definite cone $\mathbb{P}_d$ of $M_d(\mathbb{C})$ which satisfy the sole property $\det(\phi(A) + \phi(B)) = \det(\phi(I))^{1/d} \det(A + B)$ for all $A, B \in \mathbb{P}_d$. Namely, they managed to prove the following result.

Theorem 1.1. (Huang et al. [**45**])
Suppose that ϕ is a selfmap of $\mathbb{P}_d$ which satisfies that

$$\det(\phi(A) + \phi(B)) = \det(\phi(I))^{1/d} \det(A + B)$$

holds for all $A, B \in \mathbb{P}_d$. Then there is a nonsingular matrix $M \in M_d(\mathbb{C})$ such that either

$$\phi(A) = MAM^* \quad or \quad \phi(A) = MA^{tr}M^*$$

holds for every $A \in \mathbb{P}_d$.

The objective of the current chapter is to carry out a similar work concerning maps on the positive definite cone of a finite von Neumann algebra, and thus prove an identical operator algebraic counterpart of the result of Huang et al.

Our approach to the solution rests heavily on a generalization of the Minkowski determinant inequality to the setting of von Neumann algebras. Note that the usual Minkowski determinant inequality for matrices $A, B \in \mathbb{P}_d$ asserts that

$$\sqrt[d]{\det(A+B)} \geq \sqrt[d]{\det(A)} + \sqrt[d]{\det(B)}$$

with equality if and only if A, B are positive scalar multiples of each other.

In [**3**, Corollary 4.3.3 (i)], Arveson gives a variational proof of a version of the Minkowski determinant inequality in finite von Neumann algebras involving the Fuglede-Kadison determinant. Recently, this result has been subsumed by Bourin and Hiai [**8**, Corollary 7.6], by the study of the antinorm property of a wide class of functionals.

Although the inequality itself has been proved several years ago, the equality conditions are much harder to isolate from these proofs because of limiting arguments and are not explicitly documented. As that will play an important role in our results, we first need to establish when $\Delta(A + B) = \Delta(A) + \Delta(B)$ holds for positive operators A, B in a finite von Neumann algebra. To this end, we give a new proof of the inequality using a generalized version of the Hadamard determinant inequality [**36, 19**]. In this way, we are also able to solve the aforementioned preserver problem concerning Fuglede-Kadison determinants on finite von Neumann algebras.

We begin with some conventions that we shall use throughout this chapter. Let $\mathcal{N}$ denote a finite von Neumann algebra acting on the complex (separable) Hilbert space H and containing the identity operator I. Let τ be a faithful tracial state on $\mathcal{N}$, by which we mean a linear functional $\tau : \mathcal{N} \to \mathbb{C}$ such that for all $A, B \in \mathcal{N}$, we have (i) $\tau(AB) = \tau(BA)$, (ii) $\tau(A^*A) \geq 0$ with equality if and only if $A = 0$, (iii) $\tau(I) = 1$. The cone of positive and invertible positive operators in $\mathcal{N}$ will be denoted by $\mathcal{N}^+$ and $\mathcal{N}^{++}$, respectively.

For an invertible operator $A \in \mathcal{N}$, the Fuglede-Kadison determinant Δ associated with τ is defined as

$$\Delta(A) = \exp(\tau(\log \sqrt{A^*A})).$$

The dependence of Δ on τ is suppressed in the notation and it is to be assumed that a choice of a faithful tracial state has already been made. Although this concept of determinant was developed in [**21**] in the context of type II_1 factors, it naturally extends to finite von Neumann algebras as above.

Now, let us give some examples of Fuglede-Kadison determinants.

(i) The simplest example is $M_d(\mathbb{C})$, the matrix algebra of d-by-d complex matrices. For $A \in M_d(\mathbb{C})$, the Fuglede-Kadison determinant $\Delta(A)$ is given by $\sqrt[d]{|\det(A)|}$ where $\det(.)$ is the usual matrix determinant.

(ii) On $M_2(\mathbb{C})$ the unique faithful tracial state is given by

$$\mathrm{Tr} : M_2(\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \frac{a_{11} + a_{22}}{2}$$

where $a_{ij} \in \mathbb{C}$ $(1 \leq i, j \leq 2)$ denotes the $(i, j)^{\text{th}}$ entry of the matrix A in $M_2(\mathbb{C})$. Denote by $D_2(\mathbb{C})$ the *-subalgebra of diagonal matrices in $M_2(\mathbb{C})$. The von Neumann algebra $M_2(\mathcal{N})$ is also finite and the faithful tracial state on $M_2(\mathcal{N})$ is given by

$$\tau_2(A) = \frac{\tau(A_{11}) + \tau(A_{22})}{2}.$$

If Δ_2 denotes the Fuglede-Kadison determinant on $M_2(\mathcal{N})$ corresponding to τ_2, then one checks that for invertible operators A_1, A_2 in $\mathcal{N}$, we have $\Delta_2(\mathrm{diag}(A_1, A_2)) = \sqrt{\Delta(A_1) \cdot \Delta(A_2)}$.

(iii) Let X be a compact Hausdorff space with a probability Radon measure ν. The space of essentially bounded complex functions $L^\infty(X, \nu)$ acting via left multiplication on $L^2(X, \nu)$ forms an abelian von Neumann algebra. The involution is $f^*(x) := \overline{f(x)}$. A faithful tracial state on $L^\infty(X, \nu)$ is obtained by

$$\tau_\nu(f) = \int_G f(x)\, d\nu(x), \text{ for } f \in L^\infty(X, \nu)$$

and the corresponding Fuglede-Kadison determinant determinant is just the geometric mean

$$\Delta_\nu(f) = \exp\left(\int_G \log(|f(x)|) d\nu(x)\right).$$

Furthermore, let us mention that group von Neumann algebras provide another important class of examples of finite von Neumann algebras (see, for instance, [**82**, §3.2] for details).

One of the most remarkable properties of Δ is that it is a group homomorphism of the group of invertible elements in $\mathcal{N}$ into the multiplicative group of positive real numbers. However, there may be several extensions of Δ to the whole of $\mathcal{N}$, here we consider only the *analytic extension* which is defined as follows. For $A \in \mathcal{N}$, let $\sigma(|A|) \subset [0, \infty)$ denote the spectrum of $\sqrt{A^*A}$ and let $\mu = \tau \circ E(.)$ be the probability measure supported on $\sigma(|A|)$, where $E(.)$ stands for the spectral measure of $|A|$. Then we set

$$\Delta(A) := \exp\left(\int_{\sigma(|A|)} \log \lambda \; d\mu(\lambda)\right)$$

with understanding that $\Delta(A) = 0$ whenever

$$\int_{\sigma(|A|)} \log \lambda \; d\mu(\lambda) = -\infty.$$

We abuse notation and denote this extension also by Δ.

Let us turn to the formulation of the main result of the chapter. To do so, we recall the concept of Jordan *-isomorphisms. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras. Then a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called a

(i) Jordan homomorphism if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$, for all $a \in \mathcal{A}$;
(ii) Jordan *-homomorphism if $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$ and Φ is a Jordan homomorphism;
(ii) Jordan *-isomorphism if Φ is a bijective Jordan *-homomorphism.

Now we are in a position to formulate the result of this section which describes certain bijective determinant-additive maps of $\mathcal{N}^{++}$. Recall that a von Neumann algebra with one dimensional center is called a *factor*.

Theorem 1.2. (Gaál, Nayak [**29**])
Let $\mathcal{N}$ be a finite von Neumann algebra with faithful tracial state τ, and let $\phi : \mathcal{N}^{++} \to \mathcal{N}^{++}$ be a bijective map. Then we have

$$\Delta(\phi(A) + \phi(B)) = \Delta(\phi(I)) \cdot \Delta(A + B)$$

for all $A, B \in \mathcal{N}^{++}$ if and only if there is a τ-preserving Jordan *-isomorphism $J : \mathcal{N} \to \mathcal{N}$ and a positive invertible element $T \in \mathcal{N}^{++}$ such that

$$\phi(A) = TJ(A)T, \quad \textit{for every } A, B \in \mathcal{N}^{++}.$$

Moreover, if $\mathcal{N}$ is a factor, then ϕ extends to a (τ-preserving) *-automorphism or *-antiautomorphism of $\mathcal{N}$.

1.2. Proof

In what follows, we present the proof of the result in this chapter. The core ideas are implemented in [**45**] but we must adjust them to the setting of finite von Neumann algebras. First of all, we need to establish the following operator algebraic counterpart of the Minkowski determinant inequality.

THEOREM 1.3. *For positive operators $A, B \in \mathcal{N}^+$, we have*

$$\Delta(A + B) \geq \Delta(A) + \Delta(B). \tag{1.1}$$

Moreover, if B is invertible, then equality holds in (1.1) *if and only if A is a nonnegative scalar multiple of B.*

A proof of the Minkowski determinant inequality (see [**53**, p. 115]) for matrices is based on the 'traditional' Hadamard determinant inequality which states that for a positive definite matrix A in $M_n(\mathbb{C})$, the determinant of A is less than or equal to the product of its diagonal entries and equality holds if and only if A is a diagonal matrix. For a given $A \in M_n(\mathbb{C})$, considering the positive semidefinite matrix $\sqrt{A^*A}$, one may derive from this inequality the geometrically intuitive fact that the volume of an n-parallelepiped with prescribed lengths of edges is maximized when the edges are mutually orthogonal. In our proof, we make use of an 'abstract' Hadamard-type determinant inequality.

Recall that if $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{N}$, then by a *conditional expectation* we mean a unital positive linear map $\Phi : \mathcal{N} \to \mathcal{M}$ which satisfies $\Phi(SAT) = S\Phi(A)T$ for all $A \in \mathcal{N}$ and $S, T \in \mathcal{M}$. Concerning τ-preserving conditional expectations, in [**72**, Theorem 4.1] Nayak has proved the following generalization of the Hadamard determinant inequality:

THEOREM 1.4. (Nayak [**72**])
For a τ-preserving conditional expectation Φ on $\mathcal{N}$ and an invertible positive operator A in $\mathcal{N}$, we have that

$$\Delta(\Phi(A^{-1})^{-1}) \leq \Delta(A) \leq \Delta(\Phi(A))$$

and equality holds in either of the above two inequalities (and hence in both inequalities) if and only if $\Phi(A) = A$.

The forthcoming basic facts concerning Δ will also be needed for the proofs.

- (P1) $\Delta(U) = 1$ for a unitary U in $\mathcal{N}$;
- (P2) $\Delta(AB) = \Delta(A) \cdot \Delta(B)$ for $A, B \in \mathcal{N}$;
- (P3) Δ is norm continuous on the set of invertible elements;
- (P4) $\Delta(\lambda A) = |\lambda|\Delta(A)$ for $\lambda \in \mathbb{C}, A \in \mathcal{N}$;
- (P5) $\lim_{\varepsilon \to 0+} \Delta(A + \varepsilon I) = \Delta(A)$ for a positive operator $A \in \mathcal{N}^+$.

Now we are in a position to prove our first lemma.

Lemma 1.5. *For positive operators $A_1, A_2 \in \mathcal{N}^+$ and $t \in [0,1]$, the following inequality holds:*

$$\Delta(A_1) \cdot \Delta(A_2) \leq \Delta(tA_1 + (1-t)A_2) \cdot \Delta(tA_2 + (1-t)A_1). \tag{1.2}$$

Furthermore, if A_1, A_2 are invertible, then equality holds if and only if either $t \in \{0,1\}$ or $A_1 = A_2$.

Proof. Consider the unitary operator $U \in M_2(\mathcal{N})$ given by

$$U := \begin{pmatrix} \sqrt{t}I & \sqrt{1-t}I \\ \sqrt{1-t}I & -\sqrt{t}I \end{pmatrix}.$$

Observe that

$$U^* \mathrm{diag}(A_1, A_2) U = \begin{pmatrix} tA_1 + (1-t)A_2 & \sqrt{t(1-t)}(A_1 - A_2) \\ \sqrt{t(1-t)}(A_1 - A_2) & tA_2 + (1-t)A_1 \end{pmatrix}.$$

Clearly, $\Phi_2(U^* \mathrm{diag}(A_1, A_2) U) = \mathrm{diag}(tA_1 + (1-t)A_2, tA_2 + (1-t)A_1)$. Using Theorem 1.4 and property (P5) concerning Δ, we get that

$$\begin{aligned} \sqrt{\Delta(A_1)} \cdot \sqrt{\Delta(A_2)} = \Delta_2(U^* \mathrm{diag}(A_1, A_2) U) \leq \\ \Delta_2(\Phi_2(U^* \mathrm{diag}(A_1, A_2) U)) = \\ \sqrt{\Delta(tA_1 + (1-t)A_2)} \cdot \sqrt{\Delta(tA_2 + (1-t)A_1)}. \end{aligned}$$

If A_1, A_2 are invertible, then $U^* \mathrm{diag}(A_1, A_2) U$ is also invertible and equality holds if and only if $\sqrt{t(1-t)}(A_1 - A_2) = 0$, that is, either $t \in \{0,1\}$ or $A_1 = A_2$. ☐

Next we continue with the following lemma.

Lemma 1.6. *Let n be a positive integer. For positive operators $A_1, A_2, \ldots, A_n \in \mathcal{N}^+$, the following inequality holds:*

$$\Delta\left(\frac{A_1 + \cdots + A_n}{n}\right) \geq (\Delta(A_1) \cdots \Delta(A_n))^{1/n}. \tag{1.3}$$

Furthermore, if $A_1, \ldots, A_n$ are invertible, then equality holds if and only if $A_1 = A_2 = \ldots = A_n$.

Proof. First we prove (1.3) by induction for $n = 2^k$ with $k \in \mathbb{N}$ and then employ a standard argument to complete the proof. Choosing $t = 1/2$ in Lemma 1.5, we obtain immediately for $A_1, A_2 \in \mathcal{N}^+$ that

$$\Delta\left(\frac{A_1 + A_2}{2}\right) \geq (\Delta(A_1) \cdot \Delta(A_2))^{1/2}. \tag{1.4}$$

Further if A_1, A_2 are invertible, equality holds if and only if $A_1 = A_2$. This completes the case $n = 2$.

Now assume that (1.3) is satisfied for $n = 2^{k-1}$ along with the equality condition. For $A_1, A_2, \dots, A_{2^k} \in \mathcal{N}^+$, we define

$$B_1 := \frac{A_1 + \cdots + A_{2^{k-1}}}{2^{k-1}}, \quad B_2 := \frac{A_{2^{k-1}+1} + \cdots + A_{2^k}}{2^{k-1}}.$$

From (1.4), we infer that

$$\Delta\left(\frac{B_1 + B_2}{2}\right) \geq (\Delta(B_1)\Delta(B_2))^{1/2}. \tag{1.5}$$

Furthermore, the induction hypothesis furnishes

$$\Delta(B_1) \geq (\Delta(A_1)\cdots\Delta(A_{2^{k-1}}))^{1/2^{k-1}}, \tag{1.6}$$

$$\Delta(B_2) \geq (\Delta(A_{2^{k-1}+1})\cdots\Delta(A_{2^k}))^{1/2^{k-1}}. \tag{1.7}$$

Combining (1.5), (1.6) and (1.7), we conclude that

$$\Delta\left(\frac{A_1 + \cdots + A_{2^k}}{2^k}\right) \geq (\Delta(A_1)\cdots\Delta(A_{2^k}))^{1/2^k}. \tag{1.8}$$

If $A_1, \dots, A_{2^k}$ are invertible, then so are B_1, B_2, and equality holds if and only if $B_1 = B_2$, $\quad A_1 = \cdots = A_{2^{k-1}}$ and $A_{2^{k-1}+1} = \cdots = A_{2^k}$ or, in other words, if and only if $A_1 = \cdots = A_{2^k}$. Thus, for n a power of 2, we have established (1.3) along with the equality condition.

Next we consider an arbitrary positive integer m. Let k be a positive integer such that $2^{k-1} \leq m < 2^k$. For $A_1, \dots, A_m \in \mathcal{N}^+$, we define the positive operator $B := (A_1 + \cdots + A_m)/m$. It follows that

$$\Delta(B) = \Delta\left(\frac{A_1 + \cdots + A_m + (2^k - m)B}{2^k}\right) \geq$$
$$(\Delta(A_1)\cdots\Delta(A_m))^{1/2^k}(\Delta(B))^{1-m/2^k}$$

and using property (P5), we conclude that

$$\Delta(B)^{m/2^k} \geq (\Delta(A_1)\cdots\Delta(A_m))^{1/2^k}$$

implying

$$\Delta(B) \geq (\Delta(A_1)\cdots\Delta(A_m))^{1/m}.$$

If $A_1, \dots, A_m$ are invertible, then so is B and equality holds if and only if $A_1 = A_2 = \ldots = A_m = B$. □

By means of the forthcoming theorem, we will derive the Minkowski determinant inequality with just a little effort.

Theorem 1.7. *For a positive operator $A \in \mathcal{N}^+$ and real number $t \geq 0$, the following inequality holds:*

$$\Delta(tI + A) \geq t + \Delta(A) \tag{1.9}$$

with equality if and only if either $t = 0$ or A is a nonnegative scalar multiple of the identity.

Proof. Let A be an invertible positive operator such that $\Delta(A) = 1$. For $p, q \in \mathbb{N}$, an application of Lemma 1.6 ensures that

$$\Delta\left(\frac{pI + qA}{p + q}\right) \geq \sqrt[p+q]{\Delta(I)^p \Delta(A)^q} = 1.$$

Thus, it holds $\Delta((p/q)I + A) \geq p/q + 1$ with equality if and only if $A = I$. Approximating with strictly positive rational numbers, we have by property (P5) that

$$\Delta(tI + A) \geq t + 1, \qquad \text{for } t \geq 0. \tag{1.10}$$

Note that for an invertible operator $A \in \mathcal{N}^{++}$ the operator $B := (1/\Delta(A))A$ is also an invertible positive operator satisfying the additional property $\Delta(B) = 1$. As $\Delta(tI + B) \geq t + 1$, for $t \geq 0$ substituting $s = t\Delta(A)$, we get the desired inequality

$$\Delta(sI + A) \geq s + \Delta(A), \qquad \text{for } s \geq 0. \tag{1.11}$$

Next we derive conditions for the case of equality in (1.11). Note that for a particular value of s under consideration, if $s/\Delta(A)$ is rational, then equality holds in (1.11) if and only if $B = I$ or, equivalently, if $A = \Delta(A)I$. For $s > 0$, if $\Delta(sI + A) = s + \Delta(A)$, using (1.11) repeatedly along with the multiplicativity of Δ, we get that

$$\begin{aligned} s^2 + \Delta(A)(2s + \Delta(A)) = (s + \Delta(A))^2 = \Delta(sI + A)^2 &= \Delta(s^2 I + A(2sI + A)) \\ \geq s^2 + \Delta(A)\Delta(2sI + A) &\geq s^2 + \Delta(A)(2s + \Delta(A)). \end{aligned}$$

Hence we conclude that $\Delta(2sI + A) = 2s + \Delta(A)$. Pick a real number r from the interval $]0, s[$. By virtue of (1.11), we deduce that

$$\begin{aligned} 2s + \Delta(A) = \Delta(2sI + A) &= \Delta((s + r)I + (s - r)I + A) \\ &\geq (s + r) + \Delta((s - r)I + A) \\ &\geq (s + r) + (s - r) + \Delta(A) = 2s + \Delta(A). \end{aligned}$$

As $(s+r)+\Delta((s-r)I+A) = 2s+\Delta(A)$, we have that $\Delta((s-r)I+A) = s-r+\Delta(A)$ for all $r \in]0, s[$. We may choose r such that $(s - r)/\Delta(A)$ is rational and thus conclude that A is a scalar multiple of the identity. Hence equality holds in (1.11) if and only if either $s = 0$ or A is a scalar multiple of the identity.

Next we consider the case when A is not necessarily invertible. For some $t > 0$ and any $s \in]0, t]$, define $A_s := sI + A$. As documented in property (P5) for Δ, we have $\lim_{\varepsilon\to 0+} \Delta(A_\varepsilon) = \Delta(A)$. Thus,

$$\Delta(A_t) = \Delta\left(\frac{t}{2}I + A_{t/2}\right) \geq \frac{t}{2} + \Delta(A_{t/2}) \geq \sum_{i=1}^{k} \frac{t}{2^i} + \Delta(A_{t/2^k}).$$

Taking the limit $k \to \infty$, we conclude that

$$\Delta(tI + A) = \Delta(A_t) \geq \sum_{i=1}^{\infty} \frac{t}{2^i} + \Delta(A) = t + \Delta(A).$$

If A is a scalar multiple of the identity, equality trivially holds. If $\Delta(tI+A) = t + \Delta(A)$, we must have $\Delta(A_t) = t/2 + \Delta(A_{t/2})$. Therefore, the operator $A_{t/2}$ is a scalar multiple of the identity, whence so is A. □

Proof of Theorem 1.3. If B is invertible, then by (1.9) we infer that

$$\Delta(I + B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) \geq 1 + \Delta(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$$

with equality if and only if $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = \lambda I$ with some $\lambda \geq 0$. Using the multiplicative property of the determinant Δ, we conclude that $\Delta(A + B) \geq \Delta(A) + \Delta(B)$ with equality if and only if $A = \lambda B$ with some $\lambda \geq 0$.

If B is not invertible, we consider the invertible positive operator $B_\varepsilon := B + \varepsilon I$ ($\varepsilon > 0$). Then we have $\Delta(A + B_\varepsilon) \geq \Delta(A) + \Delta(B_\varepsilon)$ and taking the limit $\varepsilon \to 0+$, we see that $\Delta(A + B) \geq \Delta(A) + \Delta(B)$, as required. □

Next we continue with the proof of Theorem 1.2. First we paraphrase an auxiliary lemma, which is taken from [**21**] and usually termed as the *Dixmier-Fuglede-Kadison* differential rule. Note that a functional l, defined on an algebra, is called *tracial* whenever $l(AB) = l(BA)$ is satisfied for every A, B from the domain of l.

Lemma 1.8. (Lemma 2., [**21**])
Let $\mathcal{B}$ be a complex Banach algebra with a continuous tracial linear functional l. Let $f(\lambda)$ be a holomorphic function on a domain $\Omega \subseteq \mathbb{C}$ which is bounded by a curve Γ, and let $\gamma : [0, 1] \to \mathcal{B}$ be a differentiable family of operators such that the spectrum of $\gamma(t)$ lies in Ω as t varies. Then the function $f \circ \gamma$ is differentiable and it holds

$$l((f \circ \gamma)'(t))) = l(f'(\gamma(t)) \cdot \gamma'(t)), \quad \textit{for } t \in]0, 1[.$$

By particular choice of the holomorphic function in the last lemma, we easily deduce the following pertinent corollary.

COROLLARY 1.9. *For invertible positive operators $A, B \in \mathcal{N}^{++}$, the function $g : [0,1] \to \mathbb{R}$ defined by $g(t) = \Delta(tA + (1-t)B)$ is differentiable at* $0+$ *and* $g'(0+) = \Delta(B)(\tau(B^{-1}A) - 1)$.

PROOF. Let $\gamma : [0,1] \to \mathcal{N}^{++}$ be the line segment joining A and B, that is, $\gamma(t) = tA + (1-t)B$. Clearly, the curve γ is continuously differentiable with derivative $\gamma'(t) = A - B$ for all $t \in [0,1]$. If $\varepsilon > 0$ is such that $\varepsilon I \leq A$ and $\varepsilon I \leq B$, we have that $\varepsilon I \leq \gamma(t)$ for any $t \in [0,1]$. Thus, we may choose a domain $\Omega \subseteq \mathbb{C}$ not containing 0 that is bounded by a curve Γ which surrounds the spectra of $\gamma(t)$ for all $t \in [0,1]$ and does not wind around 0. On the domain Ω, the function $f = \log$ is holomorphic. Define $G(t) = \tau((\log \circ \gamma)(t))$. Using Lemma 1.8, we get that

$$G'(0+) = \tau(B^{-1}(A - B)) = \tau(B^{-1}A - I) = \tau(B^{-1}A) - 1.$$

As $g(t) = \exp G(t)$, we conclude that

$$g'(0+) = \exp(G(0)) \cdot G'(0+) = \Delta(B)(\tau(B^{-1}A) - 1).$$

□

Now we are in a position to prove the main result of the chapter.

PROOF OF THEOREM 1.2. Let us begin with the necessity part. Observe that the transformation $\psi : \mathcal{N}^{++} \to \mathcal{N}^{++}$ defined by

$$\psi(A) = \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}$$

is unital. From the multiplicative property of Δ, we see that

$$\Delta(\psi(A) + \psi(B)) = \Delta(A + B), \quad \text{for } A, B \in \mathcal{N}^{++}. \tag{1.12}$$

Plugging $A = B$ into (1.12), we deduce that $\Delta(\psi(A)) = \Delta(A)$ for every $A \in \mathcal{N}^{++}$ and thus for a positive real number $\lambda > 0$ we obtain

$$\Delta(\psi(A) + \psi(\lambda A)) = \Delta(A + \lambda A) = \Delta(A) + \Delta(\lambda A) = \Delta(\psi(A)) + \Delta(\psi(\lambda A)).$$

An application of Theorem 1.3 entails that $\psi(\lambda A) = \mu\psi(A)$ for some $\mu > 0$. As noted earlier, $\Delta(\psi(A)) = \Delta(A)$ which implies $\lambda = \mu$ meaning that ψ is positive homogeneous. For $A, B \in \mathcal{N}^{++}$ and any real number $t \in]0,1[$, we get that

$$\begin{aligned}\Delta(t\psi(A) + (1-t)\psi(B)) &= \Delta\left(\psi(tA) + \psi((1-t)B)\right) \\ &= \Delta(tA + (1-t)B).\end{aligned}$$

For $s \in \{0, 1\}$, as $\Delta(\psi(A)) = \Delta(A)$, $\Delta(\psi(B)) = \Delta(B)$, it follows that

$$\Delta(s\psi(A) + (1-s)\psi(B)) = \Delta(sA + (1-s)B).$$

Putting this all together, we have

$$\Delta(t\psi(A) + (1-t)\psi(B)) = \Delta(tA + (1-t)B), \quad \text{for } t \in [0, 1].$$

Taking the derivative of both sides with respect to the variable t at $0+$, and using Corollary 1.9, one finds that

$$\tau(\psi(B)^{-1}\psi(A)) = \tau(B^{-1}A), \quad \text{for all } A, B \in \mathcal{N}^{++}. \tag{1.13}$$

The right hand side of (1.13) is additive in the variable A. As B runs through the whole of $\mathcal{N}^{++}$, substituting $X = \psi(B)^{-1}$, it follows from (1.13) that for all $A, C, X \in \mathcal{N}^{++}$, we must have

$$\tau(X\psi(A + C)) = \tau(X\psi(A)) + \tau(X\psi(C)),$$

or, equivalently,

$$\tau(X[\psi(A + C) - (\psi(A) + \psi(C))]) = 0.$$

Since a self-adjoint operator X in $\mathcal{N}$ could be written as the difference of two invertible positive operators $X + (\|X\| + \varepsilon)I, (\|X\| + \varepsilon)I$ for $\varepsilon > 0$, we further have that

$$\tau(X[\psi(A + C) - (\psi(A) + \psi(C))]) = 0$$

for all $A, C \in \mathcal{N}^{++}$ and all self-adjoint operators X in $\mathcal{N}$. Choosing $X = \psi(A + C) - (\psi(A) + \psi(C))$ and using the faithfulness of the tracial state τ, we conclude that

$$\psi(A + C) - (\psi(A) + \psi(C)) = 0, \quad \text{for all } A, C \in \mathcal{N}^{++}.$$

Therefore, the transformation ψ is an additive bijection. The structure of such maps is described in [**5**]. According to [**5**, Lemma 8], there exists a Jordan *-isomorphism $J : \mathcal{N} \to \mathcal{N}$ such that $\psi(A) = J(A)$ for all $A \in \mathcal{N}^{++}$. The desired τ-preserving property also follows from (1.13). Setting $T := \sqrt{\phi(I)}$ completes the necessity part.

It remains to prove the sufficiency. It is well-known that for a Jordan *-homomorphism J on $\mathcal{N}$ and a continuous function f defined on the spectrum of a self-adjoint operator A in $\mathcal{N}$, we have $J(f(A)) = f(J(A))$. As J is assumed to be τ-preserving, we observe that $\Delta(J(A)) = \Delta(A)$. With these considerations in mind and by the multiplicative property of Δ, we finally conclude the following: if there is a τ-preserving Jordan *-homomorphism

$J : \mathcal{N} \to \mathcal{N}$ and an operator $T \in \mathcal{N}^{++}$ such that $\Phi(A) = TJ(A)T$ for all $A \in \mathcal{N}^{++}$, then we must have

$$\Delta(\phi(A+B)) = \Delta(T)\Delta(J(A+B))\Delta(T) =$$
$$\Delta(T^2)\Delta(A+B) = \Delta(\phi(I)) \cdot \Delta(A+B).$$

As for the statement concerning finite factors, note that finite factors are simple rings (see, for instance, [**47**, Corollary 6.8.4]). According to a celebrated result of Herstein [**41**, Theorem I], in such a case every Jordan *-isomorphism is necessarily implemented by either a *-automorphism or a *-antiautomorphisms of the underlying algebra. □

2

Norm-additive maps on the positive definite cone of a C^*-algebra

2.1. Introduction and formulation of the result

(The result in this chapter has been published in [**24**].)

The study of norm-additive maps on an additive semigroup of a normed space is a very active and lively area. As for earlier results on investigations in this directions, especially on function algebras, we mention the series of publications [**37, 38, 43, 44, 87**].

These results inspired some authors to deal with norm-additive maps on different sort of domains, on the cones of positive (either positive definite or positive semidefinite) operators. Namely, in [**69**] Nagy completely described the structure of norm-additive maps on positive Schatten p-class operators with respect to the Schatten p-norm. In [**66**] among others Molnár and Szokol managed to determine the structure of maps of the same kind on the positive cone of a standard operator algebra (by what we mean a subalgebra of the algebra of all bounded linear operators containing the finite rank elements). Motivated by the aforementioned results, we consider this problem in the setting of C^*-algebra equipped with a faithful normalized trace τ. More precisely, our goal is to give a complete description of the structure of those maps on the positive definite cone $\mathcal{A}^{++}$ of a unital C^*-algebra which satisfy the sole property

$$\|\phi(a) + \phi(b)\|_p = \|a + b\|_p$$

for all $a, b \in \mathcal{A}^{++}$ where $\|a\|_p = \tau(|a|^p)^{1/p}$ is the Schatten p-norm of the element a. We note that on a C^*-algebra $\|.\|_p$ is indeed a norm in the case where $p \geq 1$ (see below).

Before moving on, we gather some basic notions and set some necessary notation. Throughout the sequel $\mathcal{A}$ denotes a C^*-algebra with unit e. Two elements $a, b \in \mathcal{A}$ commute if $ab = ba$. The *center* of $\mathcal{A}$ is the set of all elements that commute with all other elements. If the center of $\mathcal{A}$ is one-dimensional, then we call it a *factor*. The symbol $\mathcal{A}_{sa}$ stands for the self-adjoint part of $\mathcal{A}$. The set of positive and invertible positive elements are denoted by $\mathcal{A}^+$ and $\mathcal{A}^{++}$, respectively. By a *projection* p of $\mathcal{A}$, we mean a self-adjoint idempotent, that is, an element which satisfies $p^2 = p = p^*$. We shall assume that $\mathcal{A}$ admits at least one *faithful normalized trace* τ, by what we mean a positive linear functional on $\mathcal{A}$ satisfying (i) $\tau(ab) = \tau(ba)$, for all $a, b \in \mathcal{A}$; (ii) $\tau(a^*a) = 0$ if and only if $a = 0$; (iii) $\tau(e) = 1$. Fundamental examples of such algebras are the finite von Neumann algebras (see Chapter 1) and the irrational rotational algebras.

The result of the chapter reads as follows.

Theorem 2.1. (Gaál [**24**])
Let $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ be a bijective transformation and $p > 1$. Then

$$\|\phi(a) + \phi(b)\|_p = \|a + b\|_p, \quad \text{for all } a, b \in \mathcal{A}^{++}$$

if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ and a positive invertible element $d \in \mathcal{A}^{++}$ such that

$$\phi(a) = dJ(a)d, \quad \text{for every } a \in \mathcal{A}^{++},$$

and J and d satisfy

$$\|dJ(a)d\|_p = \|a\|_p, \quad \text{for } a \in \mathcal{A}^{++}. \tag{2.1}$$

Moreover, if $\mathcal{A}$ is a finite von Neumann factor, then ϕ extends to either an algebra *-automorphism or an algebra *-antiautomorphism of $\mathcal{A}$.

We remark that after publishing our paper [**24**] it is shown in [**64**] by Molnár that the condition appearing in (2.1) is equivalent to d being central.

2.2. Proof

This section is devoted to the proof of the theorem. We invoke a variant of Minkowski's inequality in C^*-algebras, which plays an inevitable role in our investigations. It is known that the inequality

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p \tag{2.2}$$

is satisfied for all $a, b \in \mathcal{A}$, see for instance [**17, 73**]. Therefore, the function

$$a \mapsto \tau(|a|^p)^{1/p} = \|a\|_p$$

defines a true norm on $\mathcal{A}$. Indeed, it is clearly positive homogeneous and $\|a\|_p = 0$ holds if and only if $a = 0$ because of the faithfulness of τ. Next we derive the condition of the case of equality in (2.2) for positive invertible elements.

PROPOSITION 2.2. *Let $\mathcal{A}$ be a C^*-algebra with faithful normalized trace. For $a, b \in \mathcal{A}^{++}$, we have*

$$\|a + b\|_p = \|a\|_p + \|b\|_p$$

if and only if a is a positive scalar multiple of b.

PROOF. Take $a, b \in \mathcal{A}^{++}$ and calculate

$$\|a + b\|_p^p = \tau((a + b)^p) = \tau((a + b)(a + b)^{p-1}) =$$
$$\tau(a(a + b)^{p-1}) + \tau(b(a + b)^{p-1}) = \|a(a + b)^{p-1}\|_1 + \|b(a + b)^{p-1}\|_1 \leq$$
$$\|a\|_p \cdot \|(a + b)^{p-1}\|_{\frac{p}{p-1}} + \|b\|_p \cdot \|(a + b)^{p-1}\|_{\frac{p}{p-1}} =$$
$$(\|a\|_p + \|b\|_p) \cdot \|(a + b)^{p-1}\|_{\frac{p}{p-1}} = (\|a\|_p + \|b\|_p) \cdot (\|a + b\|_p^{p-1})$$

where the inequality follows from Hölder's inequality [**49**] in which we have equality if and only if

$$a^p = \lambda(a + b)^p, \quad b^p = \lambda'(a + b)^p$$

with some real numbers $\lambda, \lambda' > 0$. This yields that a is a positive scalar multiple of b. □

In the forthcoming lemma, we present a characterization of the usual order on $\mathcal{A}^{++}$ what we shall also need (cf. [**66**, Lemma]).

LEMMA 2.3. *For $a, b \in \mathcal{A}^{++}$, we have $a \leq b$ if and only if*

$$\|a + x\|_p \leq \|b + x\|_p, \quad \text{for } x \in \mathcal{A}^{++}. \tag{2.3}$$

PROOF. The necessity part is easy because the Schatten p-norm $\|.\|_p$ is unitary invariant [**73**], whence monotone [**8**] on $\mathcal{A}^{++}$, by which we mean that the condition $a \leq b$ implies $\|a\|_p \leq \|b\|_p$.

As for the sufficiency, we derive a formula for the Gâteaux differential of the function $y \mapsto \|y\|_p^p$ at the footpoint $x \in \mathcal{A}^{++}$ in the direction $a \in \mathcal{A}^{++}$. Using Lemma 1.8, we easily get that

$$\frac{\partial}{\partial t}\|x + ta\|_p^p\bigg|_{t=0+} = p \cdot \tau(x^{p-1}a). \tag{2.4}$$

Let us now turn to verifying that (2.3) implies $a \leq b$. The assumption (2.3) yields that

$$\frac{\|(1/t)a + x\|_p - \|x\|_p}{1/t} = \|a + tx\|_p - \|tx\|_p \leq$$
$$\|b + tx\|_p - \|tx\|_p = \frac{\|(1/t)b + x\|_p - \|x\|_p}{1/t}, \quad \text{for } t > 0.$$

Taking the limit $t \to +\infty$, we infer from (2.4) via the chain rule that

$$\tau(x^{p-1}a) \leq \tau(x^{p-1}b)$$

for all $x \in \mathcal{A}^{++}$ or, equivalently,

$$\tau(y(b-a)) \geq 0 \tag{2.5}$$

is satisfied for all $y \in \mathcal{A}^{++}$. By the continuity of the trace τ, the inequality (2.5) holds for all $y \in \mathcal{A}^{+}$, too. Since every self-adjoint element can be decomposed as the difference of its positive and negative parts, there are $(b-a)_+, (b-a)_- \in \mathcal{A}^+$ with $(b-a)_+(b-a)_- = 0$ such that $b - a = (b-a)_+ - (b-a)_-$. Thus, taking $y = (b-a)_-$ in (2.5) we conclude that $\tau((b-a)_-^2) \leq 0$, that is, $a \leq b$. □

Now, we are in a position to present the proof of the main result of the chapter. Recall that in a C^*-algebra the Thompson metric (or, in another words, part metric) between $a, b \in \mathcal{A}^{++}$ is defined by

$$d_T(a,b) = \|\log(a^{-1/2}ba^{-1/2})\|$$

and the surjective isometries with respect to this metric are of the form

$$\psi(a) = d(qJ(a) + q^{\perp}J(a^{-1}))d \tag{2.6}$$

where $d \in \mathcal{A}^{++}$, q is a central projection and J is a Jordan *-isomorphism of $\mathcal{A}$, see [**39**, Theorem 4].

Proof. First we verify that ϕ is positive homogeneous. To this end, note that ϕ preserves the norm of elements in $\mathcal{A}^{++}$. Let $r > 0$ be a real number. For any $a \in \mathcal{A}^{++}$, we compute

$$\|\phi(ra) + \phi(a)\|_p = (r+1)\|a\|_p =$$
$$r\|a\|_p + \|a\|_p = \|\phi(ra)\|_p + \|\phi(a)\|_p.$$

By Proposition 2.2, the above equality yields that $\phi(ra)$ is a positive scalar multiple of $\phi(a)$. Referring to the aforementioned norm preserving property, this scalar must be r and thus we obtain the positive homogeneity of ϕ.

Applying Lemma 2.3, we conclude that ϕ is an order automorphism on the cone $\mathcal{A}^{++}$. We have learnt before that ϕ is positive homogenous, as well.

It is pointed out in [**56**] that positive homogenous order automorphisms are isometries with respect to the Thompson metric. Indeed, this follows from the formula [**1**, Proposition]

$$d_T(a,b) = \log\max\{M(a/b), M(b/a)\}$$

where $M(x/y) = \inf\{t > 0 \ : \ x \le ty\}$ for $x, y \in \mathcal{A}^{++}$. It follows that ϕ is of the form as described in (2.6). Since ϕ is positive homogeneous, the term $dq^{\perp}J(a^{-1})d$ cannot show up in (2.6). Therefore, we have $q = e$ and thus $\phi(a) = dJ(a)d$ with some $d \in \mathcal{A}^{++}$ and Jordan *-isomorphism J. Since any $x \in \mathcal{A}^{++}$ can be obtained as the sum of two elements a, b from the same class, we get

$$\|dJ(x)d\|_p = \|dJ(a+b)d\|_p =$$
$$\|dJ(a)d + dJ(b)d\|_p = \|a+b\|_p = \|x\|_p,$$

as wanted.

As for the statement concerning finite von Neumann factors, it was noted in Chapter 1 that by a celebrated result of Herstein any Jordan *-isomorphism onto a factor is either an algebra *-isomorphism or an algebra *-antiisomorphism. Further we have also learnt that algebra isomorphisms and algebra antiisomorphism onto finite factors are necessarily trace preserving. Since Jordan *-isomorphism are compatible with the continuous function calculus, for any $a \in \mathcal{A}^{++}$ the equality $\|J(a)\|_p = \|a\|_p$ holds, too.

Up to here we have proved

$$\|dJ(a)d\|_p = \|a\|_p = \|J(a)\|_p$$

which gives us that $\|dxd\|_p = \|x\|_p$ is satisfied for all $x \in \mathcal{A}^{++}$. Substitute $e + tb$ into the place of x. Then we have

$$\|d^2 + tdbd\|_p^p = \|e + tb\|_p^p, \qquad \text{for } b \in \mathcal{A}^{++}.$$

Considering the series expansion of the right hand side, the coefficient of t is $p \cdot \tau(b)$. It means that the derivative of the left hand side at $t = 0$ is the same. Applying Lemma 1.8 for the left hand side, we infer

$$\tau(d^{2p}b) = \tau(b), \qquad \text{for } b \in \mathcal{A}^{++}$$

which yields $d = e$. □

2.3. Final remarks

After publishing our paper [**24**] Tsai, Wong and Zhang determined the structure of norm-additive maps between positive parts of Hagerup's L^p spaces [**88**].

3

Preserver problems related to quasi-arithmetic means

(The results in this chapter are joint with Gergő Nagy and have been published in [**28**].)

3.1. Introduction and formulation of the results

Means form a fundamental concept in mathematics, originally they are introduced for the averaging of real numbers. A mean $M\colon I^2 \to I$ on an interval I is defined as a binary operation satisfying the inequalities $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$ for $x, y \in I$. Such objects have been intensively studied for a long time by many researchers, their investigation forms a broad field of mathematics. Among them, quasi-arithmetic means are one of the most basic quantities and they are particularly important. As for means of other objects, in [**48**] Kubo and Ando established the theory of operator means which are certain operations on the cone of positive operators on a Hilbert space. In the finite dimensional case, that notion reduces to means of positive semidefinite matrices which are widely used and investigated in several branches of mathematics.

In this chapter, we consider another sort of means, namely the quasi-arithmetic means of positive definite matrices. In order to introduce them, we fix some notation. Let $d \geq 2$ be a given integer. We denote by $\mathbb{H}_d$, $\mathbb{H}_d^+$ and $\mathbb{P}_d$ the linear space of Hermitian matrices and the cones of positive semidefinite and positive definite matrices, respectively. Fix a strictly monotone continuous function $f\colon]0, \infty[\to \mathbb{R}$. Then the quasi-arithmetic mean $M_{f,t}\colon \mathbb{P}_d^2 \to \mathbb{P}_d$ generated by f with weight $t \in [0, 1]$ is defined by

$$M_{f,t}(A, B) = f^{-1}(tf(A) + (1 - t)f(B)), \qquad \text{for } A, B \in \mathbb{P}_d. \tag{3.1}$$

We remark that this operation is an extension of the quasi-arithmetic means generated by f from positive numbers to positive definite matrices. The most fundamental quasi-arithmetic means are the *log-Euclidean mean* and the *λ–power mean* (which is a generalization both of the arithmetic and the harmonic means). Their common weight is $1/2$, and their generating functions are given by $x \mapsto \log(x)$ and $x \mapsto x^\lambda$ with some real number $\lambda \neq 0$, respectively.

Means are algebraic operations but also play a central role in the study of differential geometric structures on the positive definite cone $\mathbb{P}_d$. Some of them are in an intimate connection with certain geodesics in $\mathbb{P}_d$. For example, according to [**22**, Theorem 3.1] in the case where f is increasing and smooth, there is a connection on the manifold $\mathbb{P}_d$ such that any geodesic $\Gamma\colon [0,1] \to \mathbb{P}_d$ for it is given by $\Gamma(t) = M_{f,t}(A,B)$ with $A, B \in \mathbb{P}_d$. Moreover, in the paper [**42**] a suitable Riemannian metric was discovered by Hiai and Petz for which the geodesic joining $A, B \in \mathbb{P}_d$ is defined by $\Gamma(t) = M_{\log,t}(A,B)$ and thus with respect to this metric the geodesic midpoint is just the log-Euclidean mean. In this chapter, we discuss three preserver problems related to quasi-arithmetic means.

Problem A. Describe the corresponding homomorphisms with respect to quasi-arithmetic means.

This is a natural question which has been investigated in the case of operator means and has been answered in many subcases [**60, 61, 62**]. As for quasi-arithmetic means, we have the following simple observation. Assume that f is bijective and consider a homomorphism ϕ of $(\mathbb{P}_d, M_{f,1/2})$. Then for the transformation

$$\psi\colon \mathbb{H}_d \to \mathbb{H}_d, \quad A \mapsto f(\phi(f^{-1}(A)))$$

one has

$$\psi\left(\frac{A+B}{2}\right) = \frac{\psi(A)+\psi(B)}{2}, \qquad \text{for } A, B \in \mathbb{H}_d.$$

The above functional equation is usually termed *Jensen equation*. The main result of [**31**] tells us that functions from a nonempty $\mathbb{Q}$-convex of a linear space X over $\mathbb{Q}$ into another linear space Y over $\mathbb{Q}$ satisfying the Jensen equation are of the form

$$x \mapsto A_0 + A(x)$$

with some element $A_0 \in Y$ and additive function $A_1 : X \to Y$. Clearly, additive maps of linear spaces are rational homogeneous and hence any such transformation of $\mathbb{H}_d$ satisfies the latter equality. However, those maps could be totally irregular. To sum up, we see that the homomorphisms $\phi\colon \mathbb{P}_d \to \mathbb{P}_d$ for a quasi-arithmetic mean have no structure in general.

Despite this fact, there are maps preserving such means which possess a general form. We mentioned that for strictly increasing smooth functions f, the curve

$$\Gamma_{A,B} : [0,1] \to \mathbb{P}_d, \quad t \mapsto M_{f,1-t}(A,B) \tag{3.2}$$

is the unique geodesic in $\mathbb{P}_d$ (equipped with a suitable connection) joining A with B. In [**85**] the authors considered certain Riemannian structures on $\mathbb{P}_d$ and investigated maps $\phi \colon \mathbb{P}_d \to \mathbb{P}_d$ with the property

$$\phi(\Gamma_{A,B}(t)) = \Gamma_{\phi(A),\phi(B)}(t), \qquad \text{for } A, B \in \mathbb{P}_d,\ t \in [0,1].$$

Observe that in the case of the geodesics in (3.2), the last equation simply means that ϕ is a homomorphism for any of the operation $M_{f,t}$ for all $t \in [0,1]$. Adopting the arguments given in [**85**], one obtains the following generalization of [**85**, Theorem 2.4].

Proposition 3.1. (Gaál, Nagy [**28**])
Assume that f is bijective and let $\phi \colon \mathbb{P}_d \to \mathbb{P}_d$ be a homomorphism with respect to each of the means $M_{f,t}$ for all $t \in [0,1]$. Then there exists an integer $k \in \mathbb{N}$, further we have matrices $B_1, \ldots, B_k \in M_d(\mathbb{C})$, real numbers $\beta_1, \ldots, \beta_k$ and a matrix $Y \in \mathbb{H}_d$ such that

$$\phi(A) = f^{-1}\left(\sum_{j=1}^{k} \beta_j B_j f(A) B_j^* + Y\right), \qquad \textit{for all } A \in \mathbb{P}_d.$$

We omit the details here.

Problem B. Determine the structure of transformations which preserve norms of quasi-arithmetic means.

This question has not been answered yet for general quasi-arithmetic means, but for the corresponding results on Kubo-Ando means we refer to the papers [**66, 71**]. In this chapter, we solve Problem B under certain quite general conditions. Our corresponding result reads as follows.

Theorem 3.2. (Gaál, Nagy [**28**])
Suppose that $|\lim_{x\to 0} f(x)| = +\infty$ and that $f(]0,+\infty[)$ is any of the sets $\mathbb{R}$, $]0,\infty[$. Moreover, let $t \in]0,1[$ be a fixed real number and $N(.)$ be a unitary invariant norm on $M_d(\mathbb{C})$. In addition, assume that $\phi \colon \mathbb{P}_d \to \mathbb{P}_d$ is a bijection with

$$N(M_{f,t}(\phi(A), \phi(B))) = N(M_{f,t}(A,B)) \tag{3.3}$$

for all $A, B \in \mathbb{P}_d$. Then there is a unitary matrix $U \in M_d(\mathbb{C})$ such that either

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{P}_d$.

Our goal concerning the result is to cover as many of the most fundamental quasi-arithmetic means, the log-Euclidean mean and the λ–power means, as we can. Clearly, the generating functions of the first and, in the case $\lambda < 0$, the second type of means (hence also that of the harmonic mean) satisfy the conditions of Theorem 3.2. However, this is not the case with those functions of λ–power means for $\lambda > 0$. Concerning the latter operations, we have the following somewhat partial result.

THEOREM 3.3. (Gaál, Nagy [**28**])
Suppose that $f(x) = x^\lambda$ $(x > 0)$ with some scalar $\lambda > 0$ and let $t \in]0, 1[$ be a fixed real number. Denote by $\|.\|_\infty$ the spectral norm on $M_d(\mathbb{C})$. In addition, assume that $\phi\colon \mathbb{P}_d \to \mathbb{P}_d$ is a bijection satisfying

$$\|M_{f,t}(\phi(A), \phi(B))\|_\infty = \|M_{f,t}(A, B)\|_\infty \tag{3.4}$$

for all $A, B \in \mathbb{P}_d$. Then there is a unitary matrix $U \in M_d(\mathbb{C})$ such that either

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{P}_d$.

In the remaining part of this section, let $\mathcal{A}$ be a C^*-algebra. Then for any $t \in [0, 1]$ and for every elements $a, b \in \mathcal{A}^{++}$ one can define the mean $M_{f,t}(a, b)$ by the formula (3.1). Now suppose that f is increasing and smooth. Then for all elements $a, b \in \mathcal{A}^{++}$ and $t \in [0, 1]$ the element $\Gamma_{a,b}(t)$ can be defined by the formula in (3.2). Given a norm $|||.|||$ on $\mathcal{A}$, we shall say that a map ϕ of $\mathcal{A}^{++}$ preserve the norm $|||.|||$ of the geodesic correspondence whenever

$$|||\Gamma_{\phi(a),\phi(b)}(t)||| = |||\Gamma_{a,b}(t)||| \tag{3.5}$$

holds for all $t \in [0, 1]$ and for every $a, b \in \mathcal{A}^{++}$. Our third problem can be formulated in the following way.

Problem C. Characterize those maps of $\mathcal{A}^{++}$ which preserve a norm of the geodesic correspondence.

Problem C was partially solved in [**85**, Theorem 4.1] by Szokol et. al for $\mathcal{A} = M_d(\mathbb{C})$ in the case of the Schatten p-norm. Observe that our statement in Theorem 3.2 is much more stronger than their corresponding result in the following respects:

1) we consider quite general quasi-arithmetic means, not just the ones whose generating function is log;
2) we require that (3.5) should hold not for all but *one* fixed $t \in [0, 1]$;
3) we consider an arbitrary unitary invariant norm, not necessarily $\|.\|_p$.

Our further contribution to Problem C is the following.

THEOREM 3.4. (Gaál, Nagy [**28**])
Let $\mathcal{A}$ be a C^*-algebra such that a faithful normalized trace τ on $\mathcal{A}$ can be given. Fix a number $p \geq 1$ and assume that $f = \log$. If $\phi\colon \mathcal{A}^{++} \to \mathcal{A}^{++}$ is a bijective transformation preserving the Schatten p-norm of the geodesic correspondence, then there exist a Jordan *-isomorphism J of $\mathcal{A}$ and a central element $c \in \mathcal{A}^{++}$ such that

$$\phi(a) = cJ(a), \quad \textit{for all } a \in \mathcal{A}^{++}.$$

Moreover, c and J satisfy

$$\tau(cJ(x)J(y)) = \tau(xy), \quad \textit{for every } x, y \in \mathcal{A}. \tag{3.6}$$

3.2. Proofs

This section is devoted to the proofs of our results formulated in the previous section. First we present some assertions which are required for the verification of Theorem 3.2. In the sequel $N(.)$ stands for a unitary invariant norm on $M_d(\mathbb{C})$. Moreover, we denote by $\leq$ the usual order on $\mathbb{H}_d$ and by $\mathbb{H}_d^D$ the collection of the elements of $\mathbb{H}_d$ whose spectra lie in the set $D \subseteq \mathbb{R}$. Since the rank-one projections in $M_d(\mathbb{C})$ are pairwise unitary similar to each other, one has a constant c_N satisfying that $N(P) = c_N$ for any such matrix P. Our first assertion reads as follows.

LEMMA 3.5. *Let $]-\infty, 0[\subseteq D \subseteq \mathbb{R}$ be an open interval and $g\colon D \to [0, +\infty[$ be a continuous increasing function such that $\lim_{\alpha\to-\infty} g(\alpha) = 0$. If $A \in \mathbb{H}_d^D$ is a matrix and $x \in \mathbb{C}^d$ is a unit vector, then*

$$\lim_{\alpha\to\infty} N(g(A + \alpha(xx^* - I))) = c_N \cdot g(x^*Ax).$$

Proof. The crucial step of the proof is to show that there exists a real number $K > 0$ such that for all scalars $t \geq K$, we have

$$\left(x^*Ax - t^{-\frac{1}{2}}\right)xx^* + t(xx^* - I) \leq A \leq \left(x^*Ax + t^{-\frac{1}{2}}\right)xx^* + \frac{t}{2}(I - xx^*). \tag{3.7}$$

We verify only the first inequality, the second can be proved in a similar fashion. To this end, let $t > 0$ be a number and define

$$T(t) := A - \left(\left(x^*Ax - t^{-\frac{1}{2}}\right)xx^* + t(xx^* - I)\right).$$

Fix an orthonormal basis in $\mathbb{C}^d$ containing x. Then the off-diagonal entries of the matrix of $T(t)$ with respect to this basis are constants, the first entry of the diagonal is $t^{-1/2}$ and the i^{th} one is of the form "t+constant" for $i \in \{2, \ldots, n\}$. It is apparent that the first leading principal minor $\Delta_1(t)$ of the matrix of $T(t)$ is positive. Now let $j \in \{2, \ldots, n\}$. It is easy to check that the j^{th} leading principal minor $\Delta_j(t)$ of the matrix of $T(t)$ is a linear combination of powers of t with maximal exponent $j - 3/2$ and the coefficient of $t^{j-3/2}$ is 1. It follows that $\lim_{t\to+\infty} \Delta_j(t) = +\infty$ and thus for large enough t we have $\Delta_j(t) > 0$. We infer that for large enough t this inequality holds for all $j \in \{1, \ldots, n\}$ which, by Sylvester's criterion, implies the existence of a scalar $K > 0$ satisfying that for all $t \geq K$ the matrix of $T(t)$ is positive definite. We easily conclude that the first inequality in (3.7) is valid for each $t \geq K$.

Note that if $S, T \in \mathbb{H}_d$ are operators such that $S \leq T$, then we necessarily have $N(g(S)) \leq N(g(T))$, by [**7**, Corollary III.2.3] and [**84**, Theorem II.3.7]. In virtue of (3.7), for any $t \geq K$, we have

$$\begin{aligned} N\left(g\left(\left(x^*Ax - t^{-\frac{1}{2}}\right)xx^* + 2t(xx^* - I)\right)\right) &\leq N\left(g\left(A + t(xx^* - I)\right)\right) \\ &\leq N\left(g\left(\left(x^*Ax + t^{-\frac{1}{2}}\right)xx^* + \frac{t}{2}(xx^* - I)\right)\right). \end{aligned} \tag{3.8}$$

We compute

$$g\left(\left(x^*Ax - t^{-\frac{1}{2}}\right)xx^* + 2t(xx^* - I)\right) = g\left(x^*Ax - t^{-\frac{1}{2}}\right)xx^* + g(-2t)(I - xx^*)$$

and

$$\begin{aligned} &g\left(\left(x^*Ax + t^{-\frac{1}{2}}\right)xx^* + \frac{t}{2}(xx^* - I)\right) \\ &= g\left(x^*Ax + t^{-\frac{1}{2}}\right)xx^* + g\left(-\frac{t}{2}\right)(I - xx^*). \end{aligned}$$

By the properties of g, it follows that both of the latter expressions tend to $g(x^*Ax)xx^*$, as t tends to infinity. This gives us that the limit of both sides of (3.8) at infinity is

$$N(g(x^*Ax)xx^*) = c_N \cdot g(x^*Ax),$$

whence taking the limit $t \to +\infty$ in (3.8) yields the desired conclusion. □

We proceed with the other assertions mentioned above.

Lemma 3.6. *Let D be any of the sets $]-\infty, 0[$, $\mathbb{R}$ and $g\colon D \to [0, \infty[$ be a continuous strictly increasing function such that $\lim\limits_{\alpha \to -\infty} g(\alpha) = 0$. For any $A, B \in \mathbb{H}_d^D$, we have $A \leq B$ if and only if $N(g(A + X)) \leq N(g(B + X))$ holds for all $X \in \mathbb{H}_d^D$.*

Proof. Let $S, T \in \mathbb{H}_d^D$ such that $S \leq T$. According to [**7**, Corollary III.2.3] and [**84**, Theorem II.3.7] we have $N(g(S)) \leq N(g(T))$, whence the norm-inequality in the statement of Lemma 3.6 is satisfied for all $X \in \mathbb{H}_d^D$.

Now assume that the relation

$$N(g(A + X)) \leq N(g(B + X)) \tag{3.9}$$

is satisfied by any element $X \in \mathbb{H}_d^D$. Then one can check that, by continuity this inequality holds also for each matrix $X \in \mathbb{H}_d^{\overline{D}}$. Now consider a fixed unit vector $x \in \mathbb{C}^d$ and a positive real number α. Then xx^* is a rank-one projection. Plug $X = \alpha(xx^* - I)$ in (3.9). By taking the limit $\alpha \to \infty$ and using Lemma 3.5, we get that $g(x^*Ax) \leq g(x^*Bx)$ which, due to the monotonicity property of g, gives us that $x^*Ax \leq x^*Bx$. Since this inequality holds for all unit vectors $x \in \mathbb{C}^d$, we see that $A \leq B$ and thus we conclude that the desired characterization is valid. □

We proceed with the following structural result which will imply the statement of Theorem 3.2 rather easily.

Lemma 3.7. *Let D be one of the sets $]-\infty, 0[$, $]0, +\infty[$, $\mathbb{R}$ and $g\colon D \to]0, +\infty[$ be a continuous strictly monotone function such that there is an element $\alpha_0 \in D' \cap \{-\infty, +\infty\}$ for which $\lim_{\alpha \to \alpha_0} g(\alpha) = 0$. Moreover, let $t \in]0, 1[$ be an arbitrary but fixed real number and assume that $\phi\colon \mathbb{H}_d^D \to \mathbb{H}_d^D$ is a bijective map with the property that*

$$N(g(t\phi(A) + (1-t)\phi(B))) = N(g(tA + (1-t)B))$$

is satisfied for all $A, B \in \mathbb{H}_d^D$. Then there exists a unitary matrix $U \in M_d(\mathbb{C})$ such that either

$$\phi(A) = UAU^* \quad \text{or} \quad \phi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{P}_d$.

Proof. Assume, as we may, that g is increasing and D coincides with either $]-\infty, 0[$ or $\mathbb{R}$. Now let $A, B \in \mathbb{H}_d^D$. Referring to Lemma 3.6, we conclude that ϕ is an order automorphism of $\mathbb{H}_d^D$. Now define the transformation $\psi\colon -\mathbb{H}_d^D \to -\mathbb{H}_d^D$ by $\psi(A) = -\phi(-A)$. Then ψ is an order automorphism of either $\mathbb{H}_d$ or $\mathbb{P}_d$. The structure of such transformations are described in [**63**, Theorem 2] and in [**59**, Theorem 1], respectively. By those results, we conclude that there is an invertible matrix $T \in M_d(\mathbb{C})$ and a Hermitian matrix $Y \in \mathbb{H}_d$, which is 0 if $D =]-\infty, 0[$, such that either

$$\psi(A) = TAT^* + Y \tag{3.10}$$

or

$$\psi(A) = TA^{tr}T^* + Y$$

holds for all $A \in -\mathbb{H}_d^D$. It follows that ϕ is also of one of these forms. For convenience, assume that (3.10) holds (the other case can be handled in a similar manner). Then referring to (3.10) and the preserver property of ϕ, one has that

$$N(g(TAT^* + Y)) = N(g(A)), \quad \text{for } A \in \mathbb{H}_d^D. \tag{3.11}$$

In what follows, we intend to show that $Y = 0$. Since it holds in the case $D =]-\infty, 0[$, we also suppose that $D = \mathbb{R}$. By putting $T^{-1}A(T^*)^{-1}$ in the place of A in (3.11), for the matrix $S = T^{-1}$ one has

$$N(g(A + Y)) = N(g(SAS^*)), \quad \text{for } A \in \mathbb{H}_d.$$

Now let $x \in \mathbb{C}^d$ be a unit vector and $\alpha \in \mathbb{R}$ be a number. By inserting $A = \alpha(xx^* - I)$ in the last displayed equality and using Lemma 3.5, we get

$$\lim_{\alpha\to\infty} N(g(\alpha S(xx^* - I)S^*)) = c_N \cdot g(x^*Yx).$$

It is apparent that the kernel of the element $S(I - xx^*)S^* \in \mathbb{H}_d^+$ is one-dimensional. It follows that the eigenvalues of $g(\alpha S(xx^* - I)S^*)$ (counted according to their multiplicities) are $g(0), g(\alpha\lambda_1), \ldots, g(\alpha\lambda_{d-1})$ where $\lambda_i < 0$ is a number ($i = 1, \ldots, d-1$). Thus, we have

$$\lim_{\alpha\to\infty} N(g(\alpha S(xx^* - I)S^*)) = c_N \cdot g(0).$$

By the last two displayed equalities $g(x^*Yx) = g(0)$, that is, $x^*Yx = 0$ and since this holds for all unit vectors x, we conclude that $Y = 0$. It implies

$$N(g(TAT^*)) = N(g(A)), \quad \text{for } A \in \mathbb{H}_d^D. \tag{3.12}$$

From now on we do not assume that $D = \mathbb{R}$ and are concerned with verifying that T is a unitary matrix. By polar decomposition, there is a unitary matrix $U \in M_d(\mathbb{C})$ such that $T = UR$ where $R = |T| \in \mathbb{P}_d$. It is an elementary task to check that the quantity $N(g(A))$ is invariant under unitary similarity transformations. Hence applying also (3.12), we deduce

$$N(g(RAR)) = N(g(A)), \quad \text{for } A \in \mathbb{H}_d^D. \tag{3.13}$$

Denote by $\mu_1, \ldots, \mu_d$ the eigenvalues of R and let $\{e_i\}_{i=1}^d$ be an orthonormal basis in $\mathbb{C}^d$ consisting of the corresponding eigenvectors. Choose an index $j \in \{1, \ldots, d\}$ and an arbitrary negative number s. Define

$$P_j := e_j e_j^*, \quad A(s) := -P_j + s(I - P_j).$$

We compute

$$g(RA(s)R) = g\left(-\mu_j^2\right) P_j + \sum_{k \in \{1,\ldots,d\} \setminus \{j\}} g\left(s\mu_k^2\right) e_k e_k^*$$

and

$$g(A(s)) = g(-1)P_j + g(s)(I - P_j)$$

which yield

$$\lim_{s \to -\infty} g(RA(s)R) = g\left(-\mu_j^2\right) P_j, \quad \lim_{s \to -\infty} g(A(s)) = g(-1)P_j.$$

By substituting $A = A(s)$ into the equality (3.13) and taking the limit $s \to -\infty$ in the so obtained relation, the last two displayed formulas imply that $g\left(-\mu_j^2\right) = g(-1)$. This gives us that for the number $\mu_j > 0$ one has $-\mu_j^2 = -1$ and thus $\mu_j = 1$. Since this holds for all $j \in \{1, \ldots, d\}$ we conclude that $R = I$ and thus $T = UR = U$ is a unitary matrix, as wanted. $\square$

We are in a position to verify the second result of chapter.

Proof of Theorem 3.2. Define $D = f(]0, +\infty[)$. Then the bijective map

$$\psi \colon \mathbb{H}_d^D \to \mathbb{H}_d^D, \quad A \mapsto f(\phi(f^{-1}(A))) \tag{3.14}$$

is bijective and possesses the property that

$$N\left(f^{-1}(t\psi(A) + (1-t)\psi(B))\right) = N\left(f^{-1}(tA + (1-t)B)\right)$$

is satisfied for all $A, B \in \mathbb{H}_d^D$. Thus, Lemma 3.7 applies and we deduce that there is a unitary matrix $U \in M_d(\mathbb{C})$ such that either

$$\psi(A) = UAU^* \quad \text{or} \quad \psi(A) = UA^{tr}U^*$$

holds for every $A \in \mathbb{H}_d^D$. Taking coordinates the second case (the first can be handled very similarly), for any $A \in \mathbb{P}_d$ we compute

$$\phi(A) = f^{-1}(Uf(A)^{tr}U^*) = Uf^{-1}(f(A))^{tr}U^* = UA^{tr}U^*$$

which completes the proof. $\square$

Now we present the verification of the third theorem of the chapter.

Proof of Theorem 3.3. Define the transformation $\psi\colon \mathbb{P}_d \to \mathbb{P}_d$ by the formula (3.14). Let $A, B \in \mathbb{P}_d$. Then we have

$$\|t\psi(A) + (1-t)\psi(B)\|_\infty^{1/\lambda} = \left\|(t\psi(A) + (1-t)\psi(B))^{\frac{1}{\lambda}}\right\|_\infty =$$
$$\left\|(tA + (1-t)B)^{\frac{1}{\lambda}}\right\|_\infty = \|tA + (1-t)B\|_\infty^{1/\lambda}.$$

It follows that

$$\|t\psi(A) + (1-t)\psi(B)\|_\infty = \|tA + (1-t)B\|_\infty. \tag{3.15}$$

Assume now that $A \leq B$. By the monotonicity of the norm $\|.\|_\infty$, one has $\|A + X\|_\infty \leq \|B + X\|_\infty$ for all $X \in \mathbb{P}_d$. Conversely, if the latter inequality holds for all $X \in \mathbb{P}_d$, then we deduce that it is also satisfied by any element $X \in \mathbb{H}_d^+$ and, in particular, by each matrix of the form tQ with some rank-one projection Q and scalar $t \geq 0$. According to [**66**, Lemma] we have $A \leq B$. To sum up, we see that $A \leq B$ if and only if $\|A + X\|_\infty \leq \|B + X\|_\infty$ for all $X \in \mathbb{P}_d$. Referring to (3.15) this implies that the bijection ψ is an order automorphism of $\mathbb{P}_d$ and therefore by [**59**, Theorem 1] we get that there is an invertible matrix $T \in M_d(\mathbb{C})$ such that either

$$\psi(A) = TAT^* \quad \text{or} \quad \psi(A) = TA^{tr}T^*$$

is satisfied for every $A \in \mathbb{P}_d$. Using (3.15) we infer

$$\|TAT^*\|_\infty = \|A\|_\infty \tag{3.16}$$

for each $A \in \mathbb{P}_d$, whence for all $A \in \mathbb{H}_d^+$. Let $x \in \mathbb{C}^d$ be a unit vector and insert $A = xx^*$ in (3.16) to obtain $\|Tx\|^2 = 1$. It follows that T preserves the norm and thus it is a unitary matrix. The proof can be completed in the same way as in the corresponding part of the proof of Theorem 3.2. □

For the proof of the last result of the chapter we recall some elementary properties of Jordan *-isomorphisms listed below. Let $J\colon \mathcal{A} \to \mathcal{A}$ be any such a map. We remark that according to [**77**, 9.9.16 Proposition] J maps central elements to such elements. Furthermore, we note that J preserves the order in both directions on $\mathcal{A}_{sa}$ and $J(e) = e$ where e is the unit of $\mathcal{A}$. The converse statement is a very well-known result of Kadison [**46**, Corollary 5]. Namely, any unital *linear* order automorphism of $\mathcal{A}_{sa}$ can be implemented by a Jordan *-isomorphism of $\mathcal{A}$. From this we obtain rather easily the general form of affine order automorphisms of $\mathcal{A}_{sa}$. These maps are given of the form

$$\phi(a) = b_0 J(a) b_0 + y_0, \quad \text{for } a \in \mathcal{A}_{sa} \tag{3.17}$$

with some elements $b_0 \in \mathcal{A}^{++},\ y_0 \in \mathcal{A}_{sa}$.

Proof of Theorem 3.4. The preserver property of ϕ implies that

$$\begin{aligned} \tau\left((\exp(t\log\phi(a) + (1-t)\log\phi(b)))^p\right) = \\ \tau\left((\exp(t\log a + (1-t)\log b))^p\right) \end{aligned} \tag{3.18}$$

is satisfied for every $a, b \in \mathcal{A}^{++}$ and for all $t \in [0,1]$. For any given $a, b \in \mathcal{A}^{++}$, consider the line segment

$$h :]0,1[\to \mathcal{A}_{sa}, \quad t \mapsto t\log a + (1-t)\log b$$

joining $\log a, \log b \in \mathcal{A}_{sa}$. Differentiate both sides of (3.18) at $t = 0+$ and apply Lemma 1.8. We conclude that

$$\tau\left(\phi(a)^p(\log\phi(b) - \log\phi(a))\right) = \tau\left(a^p(\log b - \log a)\right) \tag{3.19}$$

holds for any $a, b \in \mathcal{A}^{++}$. The remaining part of the proof is an adaptation of that of [**58**, Theorem 1]. However, for the sake of completeness we present a sort of sketch.

One checks easily that for all elements $b, \tilde{b} \in \mathcal{A}^{++}$ the inequality $\log b \leq \log\tilde{b}$ holds if and only if

$$\tau\left(a^p(\log b - \log a)\right) \leq \tau\left(a^p(\log\tilde{b} - \log a)\right), \quad \text{for } a \in \mathcal{A}^{++}.$$

Hence we conclude that the transformation

$$\psi : \mathcal{A}_{sa} \to \mathcal{A}_{sa}, \quad x \mapsto \log\phi(\exp(x))$$

preserves the order in both directions. By literally following the arguments given in [**58**, Theorem 1], we conclude that ψ is affine, too. It follows from (3.17) that

$$\psi(x) = b_0J(x)b_0 + y_0 = J(a_0xa_0 + x_0), \quad \text{for } x \in \mathcal{A}_{sa}$$

with some Jordan *-isomorphism $J\colon \mathcal{A} \to \mathcal{A}$ and elements $a_0, b_0 \in \mathcal{A}^{++}$ and $x_0, y_0 \in \mathcal{A}_{sa}$. By (3.19) we infer that

$$\begin{aligned} \tau\left(\exp(x)^p(y - x)\right) &= \tau\left(\exp(\psi(x))^p(\psi(y) - \psi(x))\right) \\ &= \tau\left(\exp(J(a_0xa_0 + x_0))^p(J(a_0ya_0 + x_0) - J(a_0xa_0 + x_0))\right) \end{aligned}$$

holds for all $x, y \in \mathcal{A}_{sa}$. Rearranging this equality we deduce

$$\begin{aligned} &\tau\left(\exp(J(a_0xa_0 + x_0))^pJ(a_0xa_0 + x_0) - \exp(x)^px\right) \\ &= \tau\left(\exp(J(a_0xa_0 + x_0))^pJ(a_0ya_0 + x_0) - \exp(x)^py\right) \end{aligned}$$

for any $x, y \in \mathcal{A}_{sa}$. As the left-hand side of the last displayed equality does not depend on y, this holds for the other side, too. Hence we have

$$\tau\left(\exp(J(a_0xa_0 + x_0))^p J(a_0ya_0 + x_0) - \exp(x)^p y\right) = \tau\left(\exp(J(a_0xa_0 + x_0))^p J(x_0)\right)$$

or, equivalently,

$$\tau\left(\exp(J(a_0xa_0 + x_0))^p J(a_0ya_0)\right) = \tau\left(\exp(x)^p y\right)$$

for every $x, y \in \mathcal{A}_{sa}$. Now, plug $a_0^{-1}xa_0^{-1}$ in the place of x. Then we have

$$\tau\left(\exp(J(x + x_0))^p J(a_0ya_0)\right) = \tau\left(\exp(a_0^{-1}xa_0^{-1})^p y\right) \tag{3.20}$$

for all $x, y \in \mathcal{A}_{sa}$. Substitute $x = \lambda e$ for $\lambda \in \mathbb{R}$ in the last displayed equality. We obtain that the relation

$$\tau\left(\exp(p\lambda e + J(px_0))J(a_0ya_0)\right) = \tau\left(\exp(p\lambda a_0^{-2})y\right)$$

holds for any $y \in \mathcal{A}_{sa}, \lambda \in \mathbb{R}$. By the uniqueness of the coefficients of power series', we have

$$\tau\left(J(a_0ya_0)\exp(J(px_0))\right) = \tau\left(ya_0^{-2n}\right), \qquad \text{for } n \in \mathbb{N},\ y \in \mathcal{A}_{sa}.$$

This yields $\tau(ya_0^{-2n}) = \tau(y)$ for all $y \in \mathcal{A}_{sa},\ n \in \mathbb{N}$ which, by the faithfulness of τ, entails that $a_0 = e$. Hence $\psi(x) = J(x + x_0)$.

Next we intend to show that x_0 is a central element. To this end, from (3.20) we infer that

$$\tau\left(\exp(J(x + px_0))J(y)\right) = \tau\left(\exp(x)y\right), \qquad \text{for } x, y \in \mathcal{A}_{sa}.$$

Set $x = \log a$ for any $a \in \mathcal{A}^{++}$. We deduce that

$$\tau\left(\exp(J(\log a + px_0))J(y)\right) = \tau\left(ay\right), \qquad \text{for } a \in \mathcal{A}^{++},\ y \in \mathcal{A}_{sa}.$$

The right-hand side of this equality is clearly additive in a, whence so is the other side. This means that the map

$$a \mapsto \exp(\log a + px_0)$$

is additive which can happen only in the case where x_0 is central, by [**58**, Lemma 4]. Further the equality (3.6) can be verified straightforwardly. □

4

Norm preservers of Kubo-Ando means of Hilbert space effects

(The results in this chapter are joint with Gergő Nagy and have been published in [**27**].)

4.1. Introduction and formulation of the result

In what follows, H denotes a complex Hilbert space with $\dim H \geq 2$. Let $B(H)$ denote the algebra of bounded linear operators on H. The cone of positive operators in $B(H)$ is denoted by $B(H)^+$. For a pair A, B of self-adjoint elements in $B(H)$, we write $A \leq B$ whenever $B - A \in B(H)^+$. Our result below concerns the set $E(H)$ of Hilbert space effects on H, by which we mean the operator interval

$$[0, I] = \{X \in B(H) \ : \ X = X^*, \ 0 \leq X \leq I\}$$

where the symbol I stands for the identity operator. Following [**48**], we say that a binary operation

$$\sigma \colon B(H)^+ \times B(H)^+ \to B(H)^+$$

is a mean in the Kubo-Ando sense if it possesses the following properties. For any $A, B, C, D \in B(H)^+$ and sequences $(A_n), (B_n)$ in $B(H)^+$, we have

(i) $I \sigma I = I$;
(ii) if $A \leq C$ and $B \leq D$, then $A \sigma B \leq C \sigma D$;
(iii) $C(A \sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$.

Here, the symbol $\downarrow$ refers for monotone decreasing convergence in the strong operator topology. If σ is a Kubo-Ando mean, then its transpose $\tilde{\sigma}$ is defined as

$$\tilde{\sigma} \colon B(H)^+ \times B(H)^+ \to B(H)^+, \quad A \tilde{\sigma} B := B \sigma A.$$

The mean σ is called symmetric whenever $\tilde{\sigma} = \sigma$.

A function f from a nontrivial interval J into $\mathbb{R}$ is called *n-monotone* (or matrix monotone of order n) if for each pair A, B of self-adjoint operators on an n-dimensional complex Hilbert space whose spectra lie in J, we have $f(A) \leq f(B)$ whenever $A \leq B$ is satisfied. Such a function is obviously increasing, so in the case $J =]0, \infty[$, one can define $f(0) = \lim_{t \to 0} f(t)$. If f is n-monotone for any integer $n \in \mathbb{N}$, then it is called *operator monotone*. We say that f is *n-concave* if for every operator A, B with the aforementioned properties, the inequality

$$f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha) f(B)$$

is valid for all $\alpha \in [0, 1]$.

We learn from the verification of [**48**, Theorem 3.2] that for a Kubo-Ando mean σ and a scalar $t > 0$ the operator $I\sigma(tI)$ is scalar. Thus, one could introduce the function

$$f_\sigma \colon]0, \infty[\to [0, \infty[, \quad f_\sigma(t)I := I\sigma(tI)$$

which is called the generating function of σ. Apparently, the property (i) furnishes $f_\sigma(1) = 1$. Moreover, the mentioned proof also shows that if $d = \dim H < \infty$, then f_σ is d-monotone, otherwise it is operator monotone. Furthermore, σ admits the explicit form

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2} B A^{-1/2}) A^{1/2} \tag{4.1}$$

for all $A, B \in B(H)^+$ with A being invertible, and this together with the property (iv) yields that f_σ uniquely determines σ. Define the function $\widetilde{f_\sigma} := f_{\tilde{\sigma}}$. According to [**48**, Corollary 4.2], we have $\widetilde{f_\sigma}(x) = x f(1/x)$ for $x > 0$.

Note that properties (i)-(ii) ensure that the set $E(H)$ is closed under the binary operation σ defined by the above purely axiomatic way and therefore it is a well-defined operation also on that structure. Thus, Problems A-B in the previous section make sense regarding the effect algebra $E(H)$, too.

As for describing the homomorphisms of $B(H)^+$ and $E(H)$ endowed with any Kubo-Ando mean, we refer to the results of Molnár and Šemrl. Two of the most fundamental means are the geometric and the harmonic mean, their generating functions are given by

$$x \mapsto \sqrt{x} \quad \text{and} \quad x \mapsto \frac{2x}{1+x},$$

respectively. The structure of automorphisms of $B(H)^+$ and $E(H)$ with respect to them was determined in [**61, 62**] and [**80**], respectively. A result on the automorphisms of $B(H)^+$ for a large class of Kubo-Ando means can be found in [**60**], under some mild regularity assumption (a sort of continuity) on the transformations in question.

Problem B is investigated for Kubo-Ando means and symmetric norms in [**26, 66**] on $B(H)^+$. Recall that a norm $N(.)$ is called symmetric if

$$N(AXB) \leq \|A\| N(X) \|B\|$$

is satisfied by any elements $A, B, X \in B(H)$ where $\|.\|$ denotes the operator norm. Several norms appearing in matrix theory, including the Schatten and the Ky Fan norms are symmetric. Further we note that every symmetric norm $N(.)$ on $B(H)$ is easily seen to be unitarily invariant. Moreover, in the case where $\dim H < \infty$ these latter two properties are equivalent [**7**, Proposition IV.2.4.].

Now the result of the chapter follows, in which we determine the form of the bijective transformations of $E(H)$ preserving any given symmetric norm of a fixed Kubo-Ando mean σ under certain conditions on the generating function f_σ.

THEOREM 4.1. (Gaál, Nagy [**27**])
Let σ be a Kubo-Ando mean, and suppose that f_σ is strictly concave and either $f_\sigma(0) = 0$ or $\tilde{f}_\sigma(0) = 0$ is fulfilled. Furthermore, let N be a symmetric norm. Then the bijection $\phi\colon E(H) \to E(H)$ has the property

$$N(\phi(A)\sigma\phi(B)) = N(A\sigma B), \quad \text{for all } A, B \in E(H) \tag{4.2}$$

if and only if there exists either a unitary or an antiunitary operator U on H such that

$$\phi(A) = UAU^*, \quad \text{for every } A \in E(H).$$

Let us make some remark here. First note that the assumptions $f_\sigma(0) = 0$ and $\tilde{f}_\sigma(0) = 0$ are equivalent to assuming $A\sigma 0 = 0$ and $0\sigma A = 0$ for all $A \in B(H)^+$, respectively. Further we mention that for each number $d > 1$, any d-monotone function on $]0, \infty[$ is concave, since it is concave of order $[d/2]$, see [**54**, Theorem 2.1]. Moreover, assuming that the function f_σ in Theorem was affine such that $f_\sigma(0) = 0$ or $\tilde{f}_\sigma(0) = 0$, it would be the identity or constant and (4.2) would be of the form $\alpha N(\phi(A)) = \alpha N(A)$ for all $A \in E(H))$ with a number $\alpha \geq 0$. Therefore, the transformation ϕ would have no regular form in this case. The latter observation shows that it is reasonable to postulate strict concavity above.

4.2. Proof

In the proof of Theorem 4.1 rank-one orthogonal projections $P_1(H)$ on H will show up frequently. In analogue with the finite dimensional case, the members of $P_1(H)$ are exactly the operators of the form

$$z \otimes z, \qquad \text{for } z \in H,\ \|z\| = 1$$

where the operation $\otimes$ is defined by $(x \otimes y)(w) := \langle w, y\rangle x$ for all $x, y, w \in H$.

Before turning to the verification of Theorem 4.1, we collect some basic facts about the notion of *strength* of an effect along any element of $P_1(H)$. Following [**9**, p. 329], for $A \in E(H),\ P \in P_1(H)$, we define

$$\lambda(A, P) = \sup\{t \geq 0 \ :\ tP \leq A\}.$$

This quantity is called the strength of A along P. According to [**9**, Theorem 4] for any effect $A \in E(H)$ and for the projection $P_z = z \otimes z$, one has

$$\lambda(A, P_z) = \begin{cases} \left\| \left(A^{1/2}|_{\overline{\operatorname{rng} A^{1/2}}} \right)^{-1} z \right\|^{-2}, & \text{if } z \in \operatorname{rng} A^{1/2} \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

According to [**9**, Theorem 1], for any $A, B \in E(H)$, we have $A \leq B$ exactly when $\lambda(A, P) \leq \lambda(B, P)$ is satisfied by each $P \in P_1(H)$.

The proof of Theorem 4.1 owes much to the recent work of Šemrl on order automorphism of effect algebras.

Theorem 4.2. (Šemrl [**81**])
Assume that $\phi : E(H) \to E(H)$ is an order automorphism. Then there exists a number $p < 0$ and an invertible bounded either linear or conjugate-linear operator T on H such that ϕ is of the form

$$\phi(A) = f_p\left(\sqrt{I + (TT^*)^{-1}}(I - (I + TAT^*)^{-1}) \sqrt{I + (TT^*)^{-1}} \right) \tag{4.4}$$

for all effects $A \in E(H)$. Here, the function $f_p \colon [0, 1] \to \mathbb{R}$ is defined by

$$f_p(x) = \frac{x}{px + 1 - p}, \qquad \text{for } x \in [0, 1].$$

Our basic strategy in the proof is to establish that the transformation in question is an order automorphism of $E(H)$, which will follow rather easily using the monotonicity of Kubo-Ando means and an explicit formula for $A\sigma_f P_z$ when P_z is any rank-one projection, just as in [**66**]. The hard part of the proof is to verify that then the form of ϕ in (4.4) reduces to either a unitary or an antiunitary similarity transformation. To reaching this aim, we rest deeply on the following auxiliary lemma.

Lemma 4.3. *Let $C, D, E \in B(H)$ be such that*

$$\langle Cx, x\rangle\langle Dx, x\rangle = \langle Ex, x\rangle\langle x, x\rangle \tag{4.5}$$

holds for all $x \in H$. Then C or D is a scalar multiple of the identity.

Proof. Let $x, y \in H$ be arbitrary elements and substitute $e^{\mathrm{i}t}x + y$ for $t \in [0, 2\pi[$ in place of x in (4.5). After some straightforward calculations, we obtain that both sides of (4.5) are trigonometric polynomials such that the coefficients of $e^{2\mathrm{i}t}$ are $\langle Cx, y\rangle\langle Dx, y\rangle$ and $\langle Ex, y\rangle\langle x, y\rangle$, accordingly. Since the coefficients of Fourier series are unique, the latter two products must be equal. Therefore, we get

$$\langle Cx, y\rangle\langle Dx, y\rangle = \langle Ex, y\rangle\langle x, y\rangle$$

which, in virtue of [**67**, Lemma] entails that C or D is a scalar operator. □

Now we are in a position to present the proof of the main result of the chapter.

Proof of Theorem 4.1. The 'if' part of the statement is in fact quite easy to prove. Indeed, one should only use that the continuous functional calculus is compatible with both the unitary and the antiunitary similarity transformations. Then the formula (4.1), the property (iv) of Kubo-Ando means listed above and the unitary invariance of N yields that if a bijection of $E(H)$ has the form appearing in our result, then it satisfies (4.2).

Let us continue with the highly nontrivial 'only if' part. First observe that since the elements of $P_1(H)$ are unitarily similar to each other, we may and do assume that $N(P) = 1$ holds for any projection $P \in P_1(H)$. Let $f(x) := f_\sigma(x)$. Next we show that $f(0) = 0$ may be supposed, too. By [**75**, Lemma 1.3.2], a function g from an interval J to $\mathbb{R}$ is strictly concave if and only if for any numbers $x_1 < x_2 < x_3$ in J, one has

$$\det\begin{pmatrix} 1 & x_1 & g(x_1) \\ 1 & x_2 & g(x_2) \\ 1 & x_3 & g(x_3) \end{pmatrix} < 0.$$

It can be checked easily that for each scalars $0 < x_1 < x_2 < x_3$, the equality

$$\frac{1}{x_1x_2x_3}\det\begin{pmatrix} 1 & x_1 & \tilde{f}(x_1) \\ 1 & x_2 & \tilde{f}(x_2) \\ 1 & x_3 & \tilde{f}(x_3) \end{pmatrix} = \det\begin{pmatrix} 1 & 1/x_3 & f(1/x_3) \\ 1 & 1/x_2 & f(1/x_2) \\ 1 & 1/x_1 & f(1/x_1) \end{pmatrix}$$

holds. This gives us that $\tilde{f}$ is strictly concave and then it follows that, since $\tilde{f} = f_{\tilde{\sigma}}$, we may and do assume that $f(0) = 0$.

Next let $A \in E(H)$ be an effect and $P \in P_1(H)$ be a projection. We are going to show that

$$N(A\sigma P) = \tilde{f}(\lambda(A, P)). \tag{4.6}$$

To this end, first observe that if $\dim H = \infty$, then by the introduction, f is operator monotone and $f(0) = 0$. Thus, by equation (17) in [**66**] one has $A\sigma P = \tilde{f}(\lambda(A, P))P$. This clearly yields (4.6) in the present case.

Now suppose that $d := \dim H < \infty$. Assume in addition that A is invertible. Using (4.3), we compute

$$A\sigma P = A^{1/2} f\left(A^{-1/2} z \otimes z A^{-1/2}\right) A^{1/2} =$$

$$f\left(\|A^{-1/2}z\|^2\right) A^{1/2}((1/\|A^{-1/2}z\|)A^{-1/2}z) \otimes ((1/\|A^{-1/2}z\|)A^{-1/2}z)A^{1/2}$$

$$= \frac{f\left(\|A^{-1/2}z\|^2\right)}{\|A^{-1/2}z\|^2} P = \lambda(A, P) f\left(\frac{1}{\lambda(A, P)}\right) P = \tilde{f}(\lambda(A, P))P.$$

Turning back to the general case when $A \in E(H)$ is arbitrary, we infer from the property (iv) and the last displayed chain of equalities that

$$A\sigma P = \lim_{n\to\infty} ((A + I/n)\sigma P) = \lim_{n\to\infty} \tilde{f}(\lambda(A + I/n, P))P.$$

Using the formula (4.3), it is not difficult to see that

$$\lim_{n\to\infty} \lambda(A + I/n, P) = \lambda(A, P).$$

Indeed, if $A \in E(H)$ with spectral resolution $A = \sum_{a\in\sigma(A)} aP_a$, then

$$(A + I/n)^{-1} = nP_0 + \sum_{a\in\sigma(A)\setminus 0} (a + 1/n)^{-1} P_a$$

and it follows that

$$\lambda(A + I/n, P) = \frac{1}{n \operatorname{Tr} P_0 P + \sum_{a\in\sigma(A)\setminus 0} (a + 1/n)^{-1} \operatorname{Tr} P_a P}.$$

If $\operatorname{rng} P \subseteq \operatorname{rng} A$, then $\operatorname{Tr} P_0 P = 0$, whence the right-hand side tends to $\left(\operatorname{Tr} A|_{\operatorname{rng} A}^{-1} P\right)^{-1}$ as n tends to infinity. Otherwise, the above limit is zero. Since $\dim H < \infty$, the assertion follows by noting that the ranges of the operators A, $\sqrt{A}$ coincide.

As d-monotone functions are continuous, so is the generating function $\tilde{f}$. Hence it follows that $A\sigma P = \tilde{f}(\lambda(A, P))P$ implying the validity of (4.6) also in the case where $\dim H < \infty$.

We proceed with establishing a characterization of the usual order on $E(H)$. We claim that for any effects $A, B \in E(H)$ one has

(4.7) $\quad A \leq B \iff N(A\sigma X) \leq N(B\sigma X)$ for all $X \in E(H)$.

To see this, let $A, B \in E(H)$ be operators. If $A \leq B$, then property (ii) yields that $A\sigma X \leq B\sigma X$ for each $X \in E(H)$. This implies $N(A\sigma X) \leq N(B\sigma X)$ for all $X \in E(H))$, by [**57**, Lemma 12] which asserts that a symmetric norm on a C^*-algebra is monotone on the set of positive elements of that algebra. To verify the reverse implication, suppose that $N(A\sigma X) \leq N(B\sigma X)$ holds for all $X \in E(H)$. Then let $P \in P_1(H)$ be a projection. It follows that

$$N(A\sigma P) \leq N(B\sigma P)$$

which, by virtue of (4.6), means that $\tilde{f}(\lambda(A, P)) \leq \tilde{f}(\lambda(B, P))$. Since f is strictly concave, the first order divided difference

$$\Delta(x, y) := \frac{f(x) - f(y)}{x - y}, \qquad \text{for } x, y \geq 0$$

is strictly decreasing in both of its variables. This implies that $\tilde{f}$ is strictly increasing. Therefore, it follows that $\lambda(A, P) \leq \lambda(B, P)$ for all $P \in P_1(H)$ which is equivalent to saying $A \leq B$, as wanted.

The equivalence (4.7) gives us that ϕ is an order automorphism of $E(H)$ and thus ϕ is of the form as described in (4.4). Via polar decomposition, we can and do assume that in (4.4) the operator T is positive. Then we intend to show that T is a scalar operator. To this end, let $A \in E(H)$ be a non-scalar invertible operator and $x \in H$ be a unit vector. Note that for every projection $P \in P_1(H)$ and any real number $r \in [0, 1]$, the equality $f_p(rP) = f_p(r)P$ holds. Using this fact, it is straightforward to check that $\phi(x \otimes x) = \varphi_x \otimes \varphi_x$ with $\varphi_x = \| \sqrt{T^2 + I}x\|^{-1} \sqrt{T^2 + I}x$. Moreover, we see from (4.4) that ϕ sends invertible effects to invertible ones. Thus, according to (4.6), (4.3) and (4.2) we have

$$\frac{1}{\langle A^{-1}x, x\rangle} = \frac{1}{\langle \phi(A)^{-1}\varphi_x, \varphi_x\rangle}$$

implying that

$$\langle A^{-1}x, x\rangle\langle (T^2 + I)x, x\rangle = \langle \sqrt{T^2 + I}\phi(A)^{-1} \sqrt{T^2 + I}x, x\rangle.$$

Now an application of Lemma 4.3 in the setting $C = A^{-1}$, $D = T^2 + I$ and $E = \sqrt{T^2 + I}\phi(A)^{-1} \sqrt{T^2 + I}$ furnishes that C or D must be a scalar operator. However, since C is not scalar, the operator $D = T^2 + I$ must be so. Apparently, this can happen only when T is scalar, too.

To conclude the proof, let $\mu \geq 0$ be the number for which $T = \mu I$. Using (4.4), for any $A \in E(H)$ one obtains $\phi(A) = h_{\mu,p}(A)$ where $h_{\mu,p} \colon [0,1] \to \mathbb{R}$ is the function defined by

$$h_{\mu,p}(x) = f_p\left(\frac{(\mu^2+1)x}{\mu^2 x + 1}\right), \qquad \text{for } x \in [0,1].$$

Thus, we get

$$\phi(A) = (\mu^2+1)A((\mu^2+p)A + (1-p)I)^{-1}, \qquad \text{for } A \in E(H). \tag{4.8}$$

By the property (i), one has $A\sigma A = A$ which, together with (4.2), yields that $N(\phi(A)) = N(A)$. Now let $a \in [0,1]$ be a number and insert $A = aI$ in the last equality. Then, using also (4.8), it is easy to check that

$$(\mu^2+p)a + 1 - p = \mu^2 + 1.$$

This holds for arbitrary $a \in [0,1]$, so we find that $\mu^2 + p = 0$, that is, $p = -\mu^2$. This gives us that (4.8) reduces to $\phi(A) = A$. Having in mind the reduction $T \in B(H)^+$, we conclude that ϕ is of the desired form. □

5

Isometry groups of self-adjoint traceless and skew-symmetric matrices

(The content of this chapter is more or less the same as of our joint paper [**25**] with Robert Guralnick.)

5.1. Introduction and formulation of the results

In the paper [**70**] titled *"Isometries of the spaces of self-adjoint traceless operators"* Nagy by means of the invariance of domain proved that the isometries on the real vector space of n-by-n self-adjoint traceless matrices H_n^0 are automatically surjective, whence linear up to a translation, by the celebrated Mazur-Ulam theorem. Further the complete description of the structure of linear isometries with respect to some unitary invariant norms was given. More precisely, for any Schatten p-norm $\|.\|_p$ whenever $n \neq 3$ and also for the spectral norm for every n.

Let $PSU(n)$ denote the image of $SU(n)$ in $GL(n^2-1, \mathbb{R})$ acting on H_n^0 via the adjoint representation

$$\mathrm{Ad} : SU(n) \to GL(n^2-1, \mathbb{R}), \quad U \mapsto \mathrm{Int}_U(.)$$

which is isomorphic to $SU(n)/\{\zeta I\}$ where ζ runs through the set of n-th roots of unity and I is the identity matrix. Assume for a moment that $n > 2$. Then the aforementioned result of Nagy could be reformulated as follows.

Theorem 5.1. (Nagy [**70**])
If $\mathcal{K}$ is the isometry group of $\|.\|_p$ on H_n^0, then one of the following happens:

(a) *$p \neq 2$ and $\mathcal{K}$ is generated by $PSU(n)$, $\mathbb{Z}/2$ and the transpose map;*
(b) *$p = 2$ and $\mathcal{K} = O(n^2-1, \mathbb{R})$.*

Here we consider the group $\mathbb{Z}/2$ as operators acting on H_n^0 by scalar multiplications with modulus one. It was also pointed out in [**70**] that the result remains true when any unitary invariant norm is considered in the case where $n = 2$. Note that in such a case the groups described in Theorem 5.1 coincide.

In the first part of this chapter, we determine the isometry group of any *unitary invariant* norm on H_n^0, which was suggested at the end of [**70**] for further research. In addition, it turns out that we need to require only some weaker invariance property: the norm in question must be invariant just under *unitary similarity transformations*. At the same time, we complete the former work of Nagy where the case $n = 3$ was missing.

The clever proof of the main result in [**70**] (that is, the result on the corresponding isometry groups when $n \geq 3$) relies on Wigner's theorem on quantum mechanical symmetry transformations. Furthermore, one key step in the proof of Nagy is the following characterization of orthogonality in terms of the Schatten p-norm: for self-adjoint traceless matrices $A, B \in H_n^0$, one has

$$\|A + B\|_p^p + \|A - B\|_p^p = 2\|A\|_p^p + 2\|B\|_p^p \quad \Longleftrightarrow \quad AB = 0.$$

Clearly, such a characterization cannot be established when general unitary invariant norms are considered. Furthermore, the orthogonality preserving property says in fact nothing in the particular case where $n = 3$. Indeed, for $n = 3$ it implies only that the zero operator is mapped to itself. These are the main reasons why the approach of Nagy cannot be carried out for the remaining case $n = 3$, or for general unitary invariant norms.

Our proof is based on the description of all compact $PSU(n)$ overgroups on self-adjoint traceless matrices. That is, we determine all the compact Lie groups lying between $PSU(n)$ and $GL(n^2 - 1, \mathbb{R})$. Then we select from the list which one preserve any unitary similarity invariant norm. This approach for general algebraic group G was first suggested by Dynkin for solving various linear preserver problems including various G-invariant properties. Later, this scheme was carried out effectively by Guralnick in [**32**] to achieve results on invertible transformations on the full matrix algebra $M_n(\mathbb{F})$ (over any infinite field $\mathbb{F}$) preserving finite union of similarity classes.

Although the aforementioned classification results of Dynkin on the contains of maximal subgroups are originally obtained for complex Lie groups, using some theory of semisimple Lie groups, we are also able to determine compact overgroups. This has been done in [**34**] where among others compact $PSU(n)$ overgroups were determined on $M_n(\mathbb{C})$. For further investigations in this direction, we also mention the series of publications [**12, 13, 32, 33, 79**].

In the second part of the chapter, adopting the machinery of the first part, we determine $PSO(n,\mathbb{R})$ overgroups on real skew-symmetric matrices. As an application, we present a short proof of an old result of Li and Tsing [**51**] concerning isometry groups of real skew-symmetric matrices, and also a minor revision of their result in the case where $n = 8$.

Let us now turn to the formulation of the results of the chapter. If H is a subgroup of G, then we shall write $H \leq G$. Whenever the group G is generated by subgroups $H_1,\dots H_s$ and elements $g_1,\dots g_t$, we write just $\langle H_1,\dots,H_s,g_1,\dots,g_t\rangle$ in order to denote the generated group. Assume now further that G is embedded into the general linear group $GL(V)$ for some vector space V. Then $C(G)$ and $\mathcal{N}(G)$ denote its centralizer and its normalizer, respectively. We warn the reader that all the centralizers and normalizers are taken in $GL(V)$, if not stated otherwise.

The group $PSU(n)$ clearly preserves the trace form $\langle A,B\rangle = \operatorname{Tr} AB$ and this it embeds in the orthogonal group of H_n^0. Let us denote by $O(n^2-1,\mathbb{R})$ this group. The first purpose of this chapter is to prove the following result on compact $PSU(n)$ overgroups.

Theorem 5.2. (Gaál, Guralnick [**25**])
Let $V := H_n^0$ and assume that $n \geq 3$. Let $\mathcal{K}$ be a compact Lie group satisfying $PSU(n) \leq \mathcal{K} \leq GL(V)$. Then one of the following happens:

(a) $\mathcal{K} \leq \mathcal{N}(PSU(n)) = \langle PSU(n), GL(1,\mathbb{R}), (.)^{tr}\rangle$;
(b) $SO(V) \leq \mathcal{K} \leq \mathcal{N}(SO(V)) = \langle O(V), GL(1,\mathbb{R})\rangle$.

Since multiplication by scalars of modulus different from one cannot preserve the norm, we obtain the following answer to the question posed by Nagy.

Corollary 5.3. *Assume that $n \geq 3$. If $\mathcal{K}$ is the isometry group of any unitary similarity invariant norm on H_n^0, then we have the following possibilities:*

(a) $\mathcal{K} = \langle PSU(n), \mathbb{Z}/2, (.)^{tr}\rangle$;
(b) $\mathcal{K} = O(n^2-1,\mathbb{R})$.

We remark that condition (i) of Corollary 5.3 holds if and only if the norm is not a multiple of the Frobenius norm and then the isometries have a simple structure. It is apparent that condition (ii) is satisfied if and only if the norm is induced by an inner product. In such a case the isometries have no structure, there are plenty of them.

Our second result in this chapter concerns the set of skew-symmetric matrices. Let $K_n(\mathbb{F})$ be the set of n-by-n skew-symmetric matrices over any field $\mathbb{F}$. Denote $PSO(n,\mathbb{R})$ the image of $SO(n,\mathbb{R})$ under the adjoint representation

$$\mathrm{Ad} : SO(n,\mathbb{R}) \to GL(K_n(\mathbb{R})), \quad Q \mapsto \mathrm{Int}_Q(.)$$

THEOREM 5.4. (Gaál, Guralnick [**25**])
Let $W := K_n(\mathbb{R})$. Assume that $n \geq 4$. Let $\mathcal{K}$ be a compact Lie group with the property that $PSO(n,\mathbb{R}) \leq \mathcal{K} \leq GL(W)$. Then one of the following occurs:

(a) $n > 4$, $n \neq 8$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(n,\mathbb{R})) = \langle PO(n,\mathbb{R}), GL(1,\mathbb{R})\rangle;$$

(b) $n = 4$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(4,\mathbb{R})) = \langle PO(4,\mathbb{R}), C(PSO(4,\mathbb{R}))\rangle;$$

(c) $n = 8$ *and*

$$\mathcal{K} \leq \mathcal{N}(PSO(8,\mathbb{R})) = \langle PSO(8,\mathbb{R}), GL(1,\mathbb{R}), S_3\rangle;$$

(d) $SO(W) \leq \mathcal{K} \leq \mathcal{N}(SO(W)) = \langle O(W), GL(1,\mathbb{R})\rangle$.

An explicit realization of the *triality automorphism group* S_3 on $K_8(\mathbb{R})$ can be given as follows [**68**]. Let $\mathbb{O}$ be the set of Cayley numbers, that is, the real vector space $\mathbb{R}^8$ equipped with the multiplication (the symbol ε_{ijk} stands for the Levi-Civita unit tensor)

$$e_i \cdot e_j = \begin{cases} e_j, & \text{if } i = 0 \\ e_i, & \text{if } j = 0 \\ -\delta_{ij}e_0 + \varepsilon_{ijk}e_k & \text{otherwise} \end{cases}$$

defined between the standard basis elements $\{e_0, \ldots, e_7\}$, and extended in the natural way by bilinearity to the whole space. This multiplication makes the vector space $\mathbb{R}^8$ a division algebra, which is neither commutative nor associative.

The skew-symmetric matrices $G_{ij} = e_i e_j^{tr} - e_j e_i^{tr}$ for $0 \leq i \neq j \leq 7$ constitute a basis for $K_8(\mathbb{R})$. Define the family F_{ij} of linear operators on $\mathbb{O}$ by

$$\begin{cases} F_{i0}x = \frac{1}{2}e_i x, \quad F_{0i} = -F_{i0} & 1 \leq i \leq 7 \\ F_{ij}x = \frac{1}{2}e_j(e_i x), & 1 \leq i, j \leq 7, \ i \neq j \end{cases}$$

for every $x \in \mathbb{O}$. Then the so-called *swap* automorphism π is the linear map on $K_8(\mathbb{R})$ which is given by

$$\pi : G_{ij} \to F_{ij}, \quad 0 \leq i, j \leq 7, \ i \neq j.$$

The swap is an outer automorphism of $K_8(\mathbb{R})$ of degree 2 which cannot be implemented by an orthogonal similarity on $K_8(\mathbb{R})$. Therefore, $\mathrm{Int}_Q(.)$

with $Q = \mathrm{diag}(-1, 1, \ldots, 1)$ provides another outer automorphism of degree 2. This is usually called the *companion* map and denoted by κ. Then the outer automorphism group S_3 of the Lie algebra $K_8(\mathbb{R})$ is generated by the non-commuting elements π, κ. An element of S_3 of order 3 can be obtained by taking $\lambda = \pi \circ \kappa$ which is called the *triality* map.

Now define the involution $A^* := \psi(A)$ given by

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

If $n \neq 8$, applying our result on the description of $PSO(n, \mathbb{R})$ overgroups, one could recover the following *folklore* result by Li and Tsing.

Theorem 5.5. (Li, Tsing [**51**])
Let $L : K_n(\mathbb{R}) \to K_n(\mathbb{R})$ be a linear map. Then the following conditions are equivalent:

(a) *L is an isometry with respect to any orthogonal congruence invariant norm on $K_n(\mathbb{R})$ which is not a constant multiple of the Frobenius norm;*
(b) *there exist a real number $\eta \in \{-1, 1\}$ and an orthogonal matrix $Q \in O(n, \mathbb{R})$ such that one of the following hold:*
   (i) *$L(X) = \eta Q X Q^{-1}$ for every $X \in K_n(\mathbb{R})$;*
   (ii) *$n = 4$ and $L(X) = \eta Q \psi(X) Q^{-1}$ for every $X \in K_n(\mathbb{R})$.*

In the case where $n = 8$, the corresponding isometry group can be larger.

Theorem 5.6. (Gaál, Guralnick)
Let $\mathcal{K}$ be the isometry group of an orthogonal congruence invariant norm on $K_8(\mathbb{R})$ which is not proportional to the Frobenius norm. Then one of the following holds:

(a) *$\mathcal{K} = \langle PO(8, \mathbb{R}), \mathbb{Z}/2 \rangle$;*
(b) *$\mathcal{K} = \langle PSO(8, \mathbb{R}), \mathbb{Z}/2, S_3 \rangle$.*

Conversely, both of the groups (a) and (b) are isometry groups of certain orthogonal congruence invariant norms on $K_8(\mathbb{R})$.

We remark that beside the spectral norm, the family of Schatten p-norms with $p \neq 2$ provide important examples of orthogonal congruence invariant norms on $K_8(\mathbb{R})$ with isometry group not admitting triality. This follows from the observation that the swap automorphism of S_3 sends

$$G_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bigoplus \mathbf{0}_6$$

into the element

$$F_{10} = \bigoplus_{k=1}^{3} \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$$

and thus

$$\|G_{10}\|_p^p = 2 \neq 2^{3-p} = \|F_{10}\|_p^p$$

whenever $p \neq 2$.

5.2. Proofs

Before diving into the proof, we need some more preliminaries. Namely, there is an overgroup Λ for $PSU(4)$ acting on the set of trace zero matrices $M_4^0(\mathbb{C})$, which we will consider throughout the sequel. We recall here the definition of Λ. To do so, let us define the homomorphism

$$\Theta : SL(4,\mathbb{C}) \to SL(6,\mathbb{C}), \quad A \mapsto AXA^{tr} \text{ for } X \in K_4(\mathbb{C}).$$

Fix a suitable basis in $K_4(\mathbb{C})$, say $E_{ij} := e_i e_j^{tr} - e_j e_i^{tr}$ for $1 \leq i < j \leq 4$. Define the involution $f \in GL(6,\mathbb{C})$ by

$$\begin{aligned} f(E_{12}) &= E_{34}, & f(E_{13}) &= -E_{24}, & f(E_{14}) &= E_{23}, \\ f(E_{23}) &= E_{14}, & f(E_{24}) &= -E_{13}, & f(E_{34}) &= E_{12}. \end{aligned}$$

It is shown in [**79**] that there is an isomorphism $\wedge^2 K_4(\mathbb{C}) \cong M_4^0(\mathbb{C})$ of $SL(4,\mathbb{C})$-modules, and we can transfer the action of $SL(6,\mathbb{C})$ on $\wedge^2 K_4(\mathbb{C})$ (via $A(X \wedge Y) := A(X) \wedge A(Y)$ where the elements of $SL(6,\mathbb{C})$ are considered as linear operators with unit determinant on the six-dimensional space $K_4(\mathbb{C})$) to M_4^0, using the homomorphism $\rho : SL(6,\mathbb{C}) \to SL(15,\mathbb{C})$ which is given by

$$\rho(A)(Xf(Y) - Yf(X)) := A(X)f(A(Y)) - A(Y)f(A(X))$$

for $X, Y \in K_4(\mathbb{C})$. The group Λ is the image of $SU(6)$ under the map ρ, which is isomorphic to $SU(6)/\langle -I \rangle$, and $PSU(4)$ is the image of $SU(4)$ under the transformation $\rho \circ \Theta$. For more details on the construction of Λ, the interested reader can consult with the publications [**13, 33, 79**]. For our objective, we only need to see that the character[1] of Λ is complex valued.

[1]The character of a finite-dimensional representation $\rho : G \to GL(V)$ is the function $g \mapsto \operatorname{Tr} \rho(g)$.

One of our main ingredients is the concept of normalizer. The following facts will be used repeatedly. If G is a compact semisimple Lie group of adjoint type, viewed inside $GL(V)$ with $V := \mathrm{Lie}(G)$, then the quotient of $\mathcal{N}(G)$ by $C(G)$ is embedded into the automorphism group of G, and the latter is contained in $GL(V)$. Moreover, the outer automorphism group of G is isomorphic to the automorphism group $\mathrm{Aut}(\Delta)$ of the Dynkin diagram (cf. Table 1) Δ of G. We note that for simple G the automorphism group is trivial (type A_1, B, C, E_7, E_8, F_4, or G_2), $\mathbb{Z}/2$ (type A_n for $n \geq 2$, D_n for $n \geq 5$, or E_6), or the dihedral group S_3 of order six (type D_4); see for instance [**83**, Section 16.3].

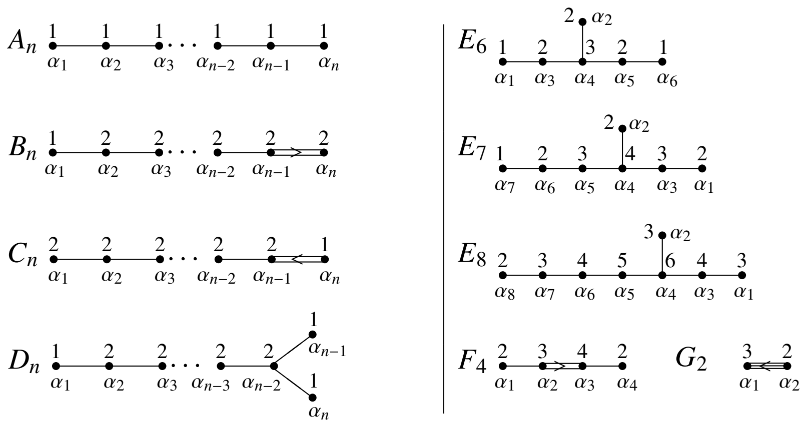

Table 1. Dynkin diagrams of simple Lie groups. The roots are labelled with their coefficients in the highest root [**40**].

In addition, if G is a simple Lie group acting absolutely irreducibly on V, then even more information can be elicited. Namely, the connected one-component of $\mathcal{N} := \mathcal{N}(G)$ is generated by G and the scalars [**30**]. Further let λ be the highest root and denote $\mathrm{Aut}(\Delta, \lambda)$ the subgroup of $\mathrm{Aut}(\Delta)$ fixing λ. Then we have

$$\mathcal{N}/\mathcal{N}_0 \cong \mathrm{Aut}(\Delta, \lambda) = \mathrm{Aut}(\Delta) \tag{5.1}$$

see, for instance, [**6**, Proposition 2.2].

Proof of Theorem 5.2. Let Γ be any compact Lie group lying between $PSU(n)$ and $GL(V)$. By the very well-known structure theorem of compact connected Lie groups (see e.g. [**76**, p. 241]), the connected one-component

Γ_0 is the semidirect product of its commutator subgroup Γ_0' and a subgroup of the center $Z(\Gamma_0)$. In particular, the commutator subgroup is semisimple. Since the adjoint action is absolutely irreducible on V, by Schur's lemma and the compactness of Γ_0, the center $Z(\Gamma_0)$, as a subgroup of the centralizer of Γ_0 in $GL(V)$, lies in both the scalars and $O(V)$. Since $O(V)\cap$ scalars is finite we conclude that Γ_0 is necessarily semisimple.

Next observe that $M_n^0 = V\otimes\mathbb{C}$. The group $PSU(n)$ acts also on M_n^0. The compact semisimple overgroups of $PSU(n)$ in $GL(V\otimes\mathbb{C})$ were determined in [**34**, Corollary 2.3]. By that result, we have the following list of compact semisimple overgroups:

$X(\mathbb{C})$	X	Comments
$PSL(n,\mathbb{C})$	$PSU(n)$	V invariant
$\Lambda(\mathbb{C})$	Λ	$n = 4$
$SL(V\otimes\mathbb{C})$	$SU(V\otimes\mathbb{C})$	
$SO(V\otimes\mathbb{C})$	$SO(V\otimes\mathbb{C})$	V invariant

where Λ is defined above and $X(\mathbb{C})$ stands for the complexification of the Lie group X. The group Λ cannot leave the space V invariant because this representation is not defined over $\mathbb{R}$, as the character of Λ is complex valued. (Isomorphic representations have the same characters.) Clearly, this holds for the representation of $SU(V)$ too. It follows that the compact semisimple overgroups in $GL(V)$ are $\Gamma_0 = PSU(n), SO(V)$.

The inclusion (see, for example, [**2**, p. 959])

$$\Gamma_0 \le \Gamma \le \mathcal{N}(\Gamma) \le \mathcal{N}(\Gamma_0),$$

gives us that any compact overgroup normalizes a semisimple one. (Or, alternatively, we could refer here the folk result that the identity component of a compact Lie group is always a normal subgroup.) Thus, it remains to calculate the normalizers.

As for the normalizer of $PSU(n)$, the identity

$$(UAU^{-1})^{tr} = (U^{tr})^{-1}A^{tr}U^{tr}, \qquad \text{for } A \in H_n^0$$

shows that the transposition map normalizes $PSU(n)$. Since the outer automorphism group of $PSU(n)$ is $\mathbb{Z}/2$, part (a) follows.

The normalizer of $SO(V)$ in $O(V)$ is just $O(V)$, whence according to [**12**, Theorem 2.5] we have $\mathcal{N}(SO(V)) = \langle O(V), C(SO(V))\rangle$ in $GL(V)$, and $C(SO(V))$ consists only of scalar matrices. This completes part (b).

□

In the proof of our second result, we invoke the following classification result of Dynkin on contains of irreducible subgroups of $SL(N, \mathbb{C})$.

THEOREM 5.7. (Theorem 2.3, [**11**])
Table 5 of [**11**] *gives a complete list of all inclusion types*

$$G^* < G < SL(N, \mathbb{C})$$

such that G^*, G are distinct irreducible simple Lie subgroups, and G is not conjugate in $GL(N, \mathbb{C})$ to any of $SL(N, \mathbb{C})$, $SO(N, \mathbb{C})$ or $Sp(N, \mathbb{C})$.

We remark that by the term inclusion type we mean that two pairs of subgroups $A^* < A$ and $B^* < B$ are in the same inclusion type whenever A^* and A are conjugate to B^* and B (in $GL(N, \mathbb{C})$, via the same conjugation).

For temporary use, set $m = n(n-1)/2$, that is, the dimension of $K_n(\mathbb{F})$ and introduce the following groups:

(i) $SO(m, \mathbb{R})$: the special orthogonal group on $K_n(\mathbb{R})$ (with respect to the negative of the Killing form)
(ii) $SO(m, \mathbb{C})$: the special orthogonal group on $K_n(\mathbb{C})$
(iii) $SL(m, \mathbb{C})$: the special linear group on $K_n(\mathbb{C})$
(iv) $SU(m)$: the special unitary group on $K_n(\mathbb{C})$

If n is odd, then $PSO(n, \mathbb{R})$ is isomorphic to $SO(n, \mathbb{R})$. Otherwise, if n is even, $PSO(n, \mathbb{R})$ is isomorphic to $SO(n, \mathbb{R})/\langle -I \rangle$. Moreover, the compact Lie groups $PSO(n, \mathbb{R})$, $SO(m, \mathbb{R})$ and $SU(m)$ have complexifications $PSO(n, \mathbb{C})$, $SO(m, \mathbb{C})$ and $SL(m, \mathbb{C})$, respectively.

PROOF OF THEOREM 5.4. First assume that $n > 4$. Then according to [**23**, Theorem 19.2, 19.14] $K_n(\mathbb{C})$ is an irreducible $PSO(n, \mathbb{C})$-module. If $X(\mathbb{C})$ is a complex semisimple subgroup of $GL(m, \mathbb{C})$ such that $PSO(n, \mathbb{C}) \leq X(\mathbb{C}) \leq GL(m, \mathbb{C})$ holds, then $X(\mathbb{C})$ is simple ($K_n(\mathbb{C})$ is an irreducible and tensor indecomposable module because it is already so under $PSO(n, \mathbb{C})$.) By Theorem 5.7 it follows that if $X(\mathbb{C}) \neq PSO(n, \mathbb{C})$, then either $X(\mathbb{C}) = SL(m, \mathbb{C}), SO(m, \mathbb{C})$ or $X(\mathbb{C})$ is given explicitly in [**11**, Table 5]. Therefore, if $n > 4$, then we have the following list of the corresponding overgroups:

$X(\mathbb{C})$	X	$K_n(\mathbb{R})$ invariant
$PSO(n, \mathbb{C})$	$PSO(n, \mathbb{R})$	yes
$SL(n, \mathbb{C})/\langle -I \rangle$	$SU(n)/\langle -I \rangle$	no
$SL(m, \mathbb{C})$	$SU(m)$	no
$SO(m, \mathbb{C})$	$SO(m, \mathbb{R})$	yes

The case $n = 4$ is a bit more involved because $K_4(\mathbb{C})$ is not a simple module for $PSO(4, \mathbb{C})$. Then either

$$X(\mathbb{C}) = SL(6, \mathbb{C}), SO(6, \mathbb{C}) \cong SL(4, \mathbb{C})/\langle -I \rangle$$

or $X(\mathbb{C})$ is semisimple but not simple. In this latter case, $X(\mathbb{C})$ is a nontrivial product of simple Lie groups. According to part (d) of [**33**, Theorem 2.2] (cf. [**11**, Theorem 1.3]) this can happen only when

$$X(\mathbb{C}) = PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}) \cong PSO(4, \mathbb{C}).$$

In a similar fashion just as in the proof of Theorem 5.2, we conclude that any compact overgroup normalizes a semisimple one.

Let us turn to the computations of the normalizers. If $n > 4$, then $PSO(n, \mathbb{R})$ is simple and so the normalizer $\mathcal{N}(PSO(n, \mathbb{R}))$ modulo scalars is contained in the automorphism group of $PSO(n, \mathbb{R})$. The outer automorphism group of $PSO(n, \mathbb{R})$ is trivial for n odd, $\mathbb{Z}/2$ for $n \neq 8$ even, and the dihedral group S_3 of order six for $n = 8$. This shows that

$$\mathcal{N}(PSO(n, \mathbb{R})) \leq \langle PO(n, \mathbb{R}), GL(1, \mathbb{R}) \rangle$$

when $n \neq 8$, and otherwise

$$\mathcal{N}(PSO(8, \mathbb{R})) \leq \langle PSO(8, \mathbb{R}), GL(1, \mathbb{R}), S_3 \rangle.$$

In virtue of (5.1) the normalizer is the same as we desired.

If $n = 4$, the group $PSO(4, \mathbb{R})$ is not simple but semisimple with Dynkin diagram of type $A_1 \times A_1$. Therefore,

$$\mathcal{N}(PSO(4, \mathbb{R})) \leq \langle PO(4, \mathbb{R}), C(PSO(4, \mathbb{R})) \rangle.$$

Conversely, the group $PSO(4, \mathbb{R})$ is an index two subgroup of $PO(4, \mathbb{R})$, whence normal.

The proof of part (c) is similar to that of the last part of Theorem 5.2. □

Proof of Theorem 5.5. It is plain that (i) is immediate from part (a) of Theorem 5.4 when $n \neq 8$.

As for (ii), we need to the determine the subgroup C of the centralizer of $PSO(4, \mathbb{R})$ which is also a subgroup of the isometry group $\mathcal{K}$. By Schur's lemma, the centralizer is $GL(1, \mathbb{R}) \times GL(1, \mathbb{R})$ with respect to the orthogonal decomposition $K_4(\mathbb{R}) = K_3(\mathbb{R}) \oplus K_3(\mathbb{R})$. Hence $C \leq \mathbb{Z}/2 \times \mathbb{Z}/2$. Since $\det \psi = -1$, a nontrivial element of C can be implemented by ψ (up to an orthogonal similarity) on $K_4(\mathbb{R})$. □

For the proof of our last result, we recall that if V is a normed space and $K \subseteq V$ is a centrally symmetric convex body, then the *Minkowski functional*

$$\|x\|_K := \inf\{t > 0 \ : \ x \in tK\}$$

of K is a norm on V with unit ball K. Consequently, there is a one-to-one correspondence between a norm and its unit ball.

Proof of Theorem 5.6. Clearly, the isometry group of any orthogonal congruence invariant norm $\|.\|$ on $K_8(\mathbb{R})$ must contain $PSO(8,\mathbb{R})$ and $\mathbb{Z}/2$. So $\mathcal{K}$ is either (a) or (b), as $\|.\|$ is not proportional to the Frobenius norm.

For case (a) the spectral norm and the Schatten p-norms with exponent different from two provide easy examples, see the sentences following the formulation of Theorem 5.6.

To construct an orthogonal congruence invariant norm on $K_8(\mathbb{R})$ with isometry group of type (b), we use the one-to-one correspondence between a norm and its unit ball. Let $\mathcal{E}$ be the orbit of

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bigoplus \mathbf{0}_6$$

under the action of $\Gamma = \langle PSO(8,\mathbb{R}), \mathbb{Z}/2, S_3\rangle$. Clearly, $\Gamma(A) \subseteq \mathbb{S}^{27}$ with $\mathbb{S}^{27}$ being the unit sphere of $K_8(\mathbb{R})$ with respect to the Frobenius norm.

Take the unit ball $\mathcal{B}$ of $\|.\|$ to be the convex hull of $\mathcal{E}$. As the action of Γ is non-transitive on $\mathbb{S}^{27}$, the norm $\|.\|$ is not proportional to the Frobenius norm. Since $\mathcal{E}$ is the set of extreme points of $\mathcal{B}$, the linear transformation T is an isometry if and only if it satisfies $T(\mathcal{E}) = \mathcal{E}$. Thus, Γ is contained in $\mathcal{K}$ which cannot be larger. $\square$

Summary

The topic of the dissertation falls into the area of preserver problems. The dissertation consists of five chapters followed by a summary (both in English and in Hungarian) and a bibliography. In Chapters 1-4 we presented the solutions of some preserver problems on the set of positive operators, and Chapter 5 incorporated the study of isometries.

In the forthcoming paragraphs we briefly summarize the results upon which this thesis was built.

Chapter 1. On a finite von Neumann algebra equipped with a faithful tracial state τ, we proved an operator algebraic counterpart of the Minkowski determinant inequality concerning the Fuglede-Kadison determinant

$$\Delta(A) = \exp(\tau(\log A)).$$

The 'traditional' Minkowski determinant inequality for matrices positive definite matrices asserts that

$$\sqrt[d]{\det(A + B)} \geq \sqrt[d]{\det(A)} + \sqrt[d]{\det(B)}$$

with equality if and only if A, B are positive scalar multiples of each other. In terms of the Fuglede-Kadison the inequality can be reformulated as

$$\Delta(A + B) \geq \Delta(A) + \Delta(B).$$

The last displayed inequality was studied earlier by Arveson, and by Hiai and Bourin but here we managed to isolate the condition of equality as well. We showed that for positive invertible operators in a finite von Neumann algebra we have equality, as in case of matrices, exactly when the operators are positive scalar multiples of each other. Then we made crucial use of this result to characterize bijective *unital* determinant additive maps, by which, we mean bijections satisfying

$$\Delta(A + B) = \Delta(\phi(A) + \phi(B))$$

for every A, B, on the cone of positive invertible operators in a finite von Neumann algebra. We obtained the interesting fact that any such map can be implemented by a τ-preserving Jordan *-isomorphism of the underlying algebra. In case of factors, referring to a celebrated result of Herstein, we showed that the corresponding preservers are either algebra isomorphisms or anti-isomorphisms, the trace preserving nature of which is guaranteed automatically.

Chapter 2. The Fuglede-Kadison determinant is a prototype of anti-norms, so the main result of Chapter 1 can be viewed as a characterization of certain anti-norm-additive maps. Chapter 2 is concerned with a preserver problem which is close in spirit to the aforementioned one. Namely, we studied norm-additive maps on the positive definite cone of a C^*-algebra carrying a faithful normalized trace, in the case where the Schatten p-norms are considered. We obtained a similar characterization as that of the main result of Chapter 1.

After Chapter 2 we turned to the investigation of preserver problems related to various operator means. The most fundamental means are the quasi-arithmetic means and the so-called Kubo-Ando means. The quasi-arithmetic mean $M_{f,t}$ generated by a strictly monotone continuous function f on $]0, +\infty[$ with weight $t \in [0, 1]$ is defined by

$$M_{f,t}(A, B) = f^{-1}\left(tf(A) + (1 - t)f(B)\right)$$

between positive definite matrices A, B. Means in the Kubo-Ando sense were introduced via a purely axiomatic way, as binary operations on the set of positive operators satisfying the following properties. For any A, B, C, D and sequences $(A_n), (B_n)$, we have

(i) $I\sigma I = I$;
(ii) if $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$;
(iii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$

where the symbol $\downarrow$ refers for monotone decreasing convergence in the strong operator topology. One of the main result of the beautiful Kubo-Ando theory tells us that there is an n-monotone function f_σ (called the generating function of σ) such that σ admits the explicit form

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all positive operators A, B with A being invertible. Further the function f_σ due to the property (iv) uniquely determines the mean σ.

Chapter 3. The chapter was devoted to the study of preserver problems related to quasi-arithmetic means. Since means can be viewed as binary operations, the natural question arises how one can describe the structure of automorphisms with respect to a given quasi-arithmetic mean $M_{f,t}$. We pointed out that such maps have no general structure, however, those maps that are automorphisms with respect to the mean $M_{f,t}$ for all $t \in [0, 1]$ have a straightforward description. In the main part of Chapter 3, we examined unitary invariant norm preservers of quasi-arithmetic means. We proved that in several cases under certain assumptions on the generating function or on the norm itself the preservers are implemented by either a unitary similarity transformation or a unitary similarity transformation composed by transposition. In addition, we concluded the chapter by presenting a structural result on those transformations of the positive definite cone of a C^*-algebra that preserve the Schatten p-norm of the weighted log-Euclidean mean (the quasi-arithmetic mean associated to the generating function $f =$ log) for every weight $t \in [0, 1]$. As some of the means considered here arise as geodesic points with respect to suitable Riemann metrics, the results in the chapter enjoy meaningful differential geometric connections.

Chapter 4. In this part of the thesis, we dealt with the same problem as in the main part of the previous chapter but we considered Kubo-Ando means on the set of Hilbert space effects. These are positive operators that are majorized by the identity with respect to the usual Löwner order, and they play a significant role in the quantum theory of measurements. Making crucial use of the recent result of Šemrl on order automorphisms of operator intervals, we managed to show that the corresponding preservers are exactly the unitary-antiunitary similarity transformations.

Chapter 5. In the first part of the chapter, we completed and at the same time substantially extended a former result of Nagy on the structure of linear isometries of n-by-n self-adjoint traceless matrices. The main novelty was the consideration of general unitary similarity invariant norms rather than the operator norm and the Schatten norms. Nevertheless, we were also able to handle the case where $n = 3$ which was previously missing in case of Schatten norms. Our proof was based on a group theoretic scheme called 'overgroups'. We considered the group $PSU(n)$ (the image of $SU(n)$ under the adjoint representation) acting on the linear space of self-adjoint traceless matrices, and determined any compact Lie group lying between $PSU(n)$ and the general linear group. In this way we managed to prove that the isometry group of any unitary similarity invariant norm is generated by $PSU(n)$, the transpose map and scalars of modulus one.

Maintaining the techniques of the first part, in the second we recovered an old result of Li and Tsing on the structure of linear isometries of n-by-n skew-symmetric matrices, with some revision in the eight dimensional case.

Összefoglalás

A jelen disszertáció témája a megőrzési problémák témakörébe tartozik. 5 fejezetből áll, melyeket összefoglaló (angol és magyar nyelvű), illetve egy tartalomjegyzék követ. Az 1-4. fejezetekben néhány pozitív operátorokkal kapcsolatos megőrzési problémát oldottunk meg, majd ezt követően az 5. fejezetben izometriákat tanulmányoztunk.

A következőkben áttekintjük a fejezetek fő eredményeit.

1. fejezet. Az első fejezetben egy τ hűséges nyomszerű állapottal rendelkező véges $\mathcal{N}$ Neumann algebrán bizonyítottuk a Minkowski determináns egyenlőtlenség egy operátoralgebrai megfelelőjét, a

$$\Delta(A) = \exp(\tau(\log A))$$

Fuglede és Kadison által bevezetett determináns fogalmat alapul véve. A 'tradícionális' Minkowski determináns egyenlőtlenség szerint pozitív definit mátrixokra

$$\sqrt[d]{\det(A+B)} \geq \sqrt[d]{\det(A)} + \sqrt[d]{\det(B)}$$

teljesül, egyenlőséggel akkor és csak akkor ha az A, B mátrixok egymás pozitív számszorosai. A Fuglede-Kadison determináns bevezetésével a fenti egyenlőtlenség a

$$\Delta(A+B) \geq \Delta(A) + \Delta(B)$$

alakot ölti. A fenti egyenlőtlenséget korábban már Arveson, valamint Hiai és Bourin is igazolták, ugyanakkor a fejezet bizonyítása alkalmas volt az egyenlőség feltételének meghatározására is. Megmutattuk, hogy pozitív definit $A, B \in \mathcal{N}$ operátorok esetén, a mátrix esethez hasonlóan, egyenlőség csak olyan A, B operátorok esetén fordul elő, melyek egymás pozitív skalárszorosai. Az eredményt alkalmazva meghatároztuk egy véges Neumann algebra pozitív definit kúpján értelmezett *egységelemtartó* ún. determináns-additív transzformációk szerkezetét. Ezen leképezések alatt olyan bijekciókat értünk, melyekre

$$\Delta(A + B) = \Delta(\phi(A) + \phi(B))$$

teljesül bármely A, B esetén. Megmutattuk, hogy minden ilyen transzformáció az algebra egy τ-őrző Jordan *-izomorfizmusából származik. Herstein egy ismert eredményének alkalmazásával pedig arra a következtetésre jutottunk, hogy faktor algebrák esetén a kérdéses transzformációk mind algebra *-izomorfizmusok vagy algebra *-antiizomorfizmusok.

2. fejezet. A Fuglede-Kadison determináns az anti-normák prototípusaként szolgáló numerikus mennyiség, így az első fejezet fő eredménye egy fajta anti-norma additív leképezések karakterizációjának tekinthető. A második fejezetben hasonló típusú megőrzési problémát vizsgáltunk. Egy hűséges, normalizált nyommal rendelkező C^*-algebra pozitív definit kúpján, a Schatten p-normára vonatkozóan norma-additív leképezéseket vizsgáltunk. Az elért karakterizáció hasonló az első fejezet fő eredményéhez.

Az ezt követő két fejezetekben különböző operátorközepekkel kapcsolatos megőrzési problémákat vizsgáltunk. A legalapvetőbb operátorközepek a kváziaritmetikai, illetve a Kubo-Ando közepek. Egy szigorúan monoton, folytonos f függvény által generált, $t \in [0, 1]$ súlyhoz tarozó $M_{f,t}$ kváziaritmetikai közepet az

$$M_{f,t}(A, B) = f^{-1}\left(tf(A) + (1 - t)f(B)\right)$$

összefüggéssel definiáljuk pozitív definit A, B mátrixok között. Ezzel szemben a Kubo-Ando közepek tisztán axiomatikusan definiáltak, mint azok a kétváltozós műveletek a pozitív operátorok halmazán, melyek a következő tulajdonságokat teljesítik: tetszőleges A, B, C, D és $(A_n), (B_n)$ sorozat esetén

(i) $I\sigma I = I$;
(ii) ha $A \leq C$ és $B \leq D$, akkor $A\sigma B \leq C\sigma D$;
(iii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) ha $A_n \downarrow A$ és $B_n \downarrow B$, akkor $A_n\sigma B_n \downarrow A\sigma B$

ahol $\downarrow$ a monoton csökkenő konvergenciát jelöli, az erős operátor topológiát tekintve. A Kubo-Ando elmélet egyik fő eredménye szerint létezik egy d-monoton f_σ függvény (a közép generáló függvénye), hogy a közép

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

alakba írható, amennyiben az A operátor invertálható. Továbbá az f_σ függvény a (iv) tulajdonság miatt egyértelműen meghatározza a σ közepet.

3. fejezet. A harmadik fejezetben kvázi-aritmetrikai közepekkel kapcsolatos megőrzési problémákat tanulmányoztunk. Mivel a közepek kétváltozós műveleteket határoznak meg, természetesen adódik a kérdés, hogyan írhatóak le egy adott $M_{f,t}$ kvázi-aritmetikai közép automorfizmusai. Megmutattuk, hogy a kérdéses transzformációknak nincs általános alakjuk, de azon transzformációk melyek automorfizmusok minden $t \in [0, 1]$ esetén az $M_{f,t}$ középre nézve zárt képlettel írhatók le.

A harmadik fejezet fő részében kvázi-aritmetikai közepek unitér invariáns normáit megőrző leképezéseket vizsgáltunk. Beláttuk, hogy a generáló függvényre, illetve a normákra vonatkozó bizonyos feltételek teljesülése mellett minden ilyen transzformáció unitér hasonlósági taranszformáció, vagy egy unitér hasonlósági transzformáció és a transzponálás kompozíciója. A fejezet lezárásaként C^*-algebra pozitív definit kúpján az összes $t \in [0, 1]$ súlyhoz tartozó súlyozott log-Euklideszi közép (amely az $f = \log$ generáló függvényhez tartozó kvázi-aritmetikai közép) Schatten p-normáját megőrző leképezések struktúrájára vonatkozó tételt láttunk be.

4. fejezet. A negyedik fejezetben ugyanazt a kérdéskört vizsgáltuk, mint a harmadik fejezet fő részében, de Kubo-Ando közepeket tekintetve, egy Hilbert tér effekt algebráján. Effektek alatt olyan pozitív operátorokat értünk, melyeket az identiás operátor majorizál a szokásos rendezésre nézve. Megjegyezzük, hogy a Hilbert tér effektek a kvantum méréselméletben játszanak kitüntetett szerepet. Šemrl operátor-intervallumok rendezés automorfizmusaira vonatkozó eredmére támaszkodva megmutattuk, hogy ezen transzformációk éppen az unitér-antiunitér hasonlósági transzformációk.

5. fejezet. Az ötödik fejezet első részében Nagy nulla nyomú önadjungált mátrixok lineáris izometriáira vonatkozó struktúra tételének jelentős általánosítását adtuk meg. Az eredményünk fő újdonsága, hogy az operátor norma és a Schatten normák helyett tetszőleges unitér hasonlósági transzformációkal szemben invariáns normákat tekintettünk. Emellett eredményünk tartalmazta az $n = 3$ esetet is, ami a korábbi tételből hiányzott.

A bizonyításunk során elsősorban Lie csoportok struktúráira vonatkozó eredményeket alkalmaztunk, valamint átalános néhány tétlt a reprezentációelmélet területéről. Meghatároztunk minden olyan kompakt Lie csoportot, amely a $PSU(n)$ (az $SU(n)$ adjungált reprezentáció általi képe) valamint az általános lineáris csoport között helyezkedik el. Így sikerült bebizonyítanunk, hogy egy unitér hasonlóság-invariáns norma izometria csoportját a $PSU(n)$, a transzponálás és az egy abszolút értékű skalárok generálják.

A fejezet végén továbbá egy új bizonyítást adtunk Li és Tsing, a ferdén szimmetrikus mátrixok lineáris izometriáira vonatkozó struktúra tételére is, az $n = 8$ esetben egy kisebb revízióval.

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