E-solid locally inverse semigroups as extensions

Ph.D. Thesis

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Chapter 1

Introduction

Group extensions play a fundamental role both in the structure theory and in the theory of varieties of groups. In 1950, Kaloujnine and Krasner proved that any extension of a group N by a group H is embeddable in the wreath product of N by H, see [22]. Note that the wreath product of N by H is a special semidirect product of a direct power of N by H.

Semigroups are natural generalisations of groups. There are two important classes of semigroups where the influence of the Kaloujnine–Krasner Theorem is fundamental, namely the classes of finite monoids and of regular semigroups. We say that a monoid S divides a monoid T if S is a homomorphic image of a submonoid of T. The famous Krohn–Rhodes Theorem ([21]) states that every finite monoid (semigroup with identity element) divides a finite iterated wreath product of copies of a specific three element monoid and finite simple groups. Although it is not an embedding theorem its importance in the structure theory of finite monoids (semigroups) and in the theory of automata and languages is undeniable.

Inverse semigroups are one of the most natural generalisations of groups. By Cayley's Theorem we can think of groups (up to isomorphism) as sets of permutations on a given set which are closed under composition and taking inverse. A similar result, the Wagner–Preston Theorem, shows that inverse semigroups are, (also up to isomorphism) sets of partial permutations on a set X (i.e., bijection between subsets of X) which are closed under composition of partial maps and taking inverse. In a group, every congruence is fully determined by the congruence class which is a subgroup. In an inverse semigroup congruence, there might be several idempotent congruence classes, but they play a similar role. More generally, if S is a regular semigroup and

ϱ is a congruence on S such that S/ϱ is a group (more generally, an inverse semigroup) then ϱ is uniquely determined by the single congruence class (by the set of congruence classes) which is a subsemigroup in S (which are subsemigroups in S). What is more, the subsemigroup congruence classes and their union are all regular. This gives us the opportunity to speak about extensions by inverse semigroups in a sense similar to group extensions, and to study whether they can be described by means of simple constructions like semidirect products.

There are a number of such embedding theorems in the structure theory of regular semigroups. Next we recall some of those considered as origins of our research. An inverse semigroup is said to be E-unitary if it is an extension of a semilattice by a group. The classical result of O'Carroll [25] states that every E-unitary inverse semigroup is embeddable in a semidirect product of a semilattice by a group. By a band we mean a semigroup where every element is idempotent and by an E-unitary regular semigroup we mean an extension of a band by a group. O'Carroll's result was extended by Szendrei [27] for extensions of certain bands, called regular, by groups. She proved that every E-unitary regular semigroup whose band of idempotents is in a regular band variety $\mathcal{V}$ is embeddable in a semidirect product of a band from $\mathcal{V}$ by a group. On the other hand, Billhardt [7] showed that there exists an E-unitary regular semigroup which is not embeddable in a semidirect product of a band by a group.

A congruence on an inverse semigroup S is said to be idempotent separating if every congruence class contains at most one idempotent and so, every class that contains an idempotent is a subgroup of S. On the opposite, a congruence is said to be idempotent pure if each congruence class containing an idempotent consists of idempotents. Houghton [16] proved that every idempotent separating extension of an inverse semigroup is embeddable in a kind of wreath product of inverse semigroups, he introduced for the purpose of this proof. Billhardt [5] showed the same with λ-wreath product instead of Houghton's wreath products. Both Houghton's and Billhardt's proof show similarities to the standard proof of the Kaloujnine–Krasner Theorem. Billhardt [6] also proved that an inverse semigroup S with an idempotent pure congruence ϱ is embeddable in a λ-semidirect product of a semilattice by S/ϱ, which generalises O'Carroll's result in another direction.

Billhardt and Szittyai [9] strengthened the former result on idempotent separating extensions by proving that if S is an inverse semigroup and ϱ is an idempotent separating congruence such that every idempotent ϱ-class is from a group variety $\mathcal{V}$ then S is embeddable in a λ-semidirect product of a group from $\mathcal{V}$ by S/ϱ.

Szendrei noticed in [28] that Houghton's wreath product and Billhardt's λ-wreath product are equivalent in the sense that the same extensions of members of a variety-like class by inverse semigroups can be embedded in them.

The main result of the thesis is a generalisation of the result of Billhardt and Szittyai mentioned recently for a class of regular semigroups which is much wider than that of inverse semigroups, and for congruences on them where the idempotent classes are unions of groups.

Locally inverse semigroups form a large and important class of regular semigroups which contains several well-studied subclasses — above all the class of inverse semigroups and that of completely simple semigroups. Locally inverse semigroups were introduced by Nambooripad in [24] (under the name pseudo-inverse semigroups). The research into the structure of locally inverse semigroups was particularly active in the late 1970's and early 1980's, and several nice and deep results were established by McAlister, Nambooripad and Pastijn. For an exhaustive list of references, see [4].

The class of E-solid semigroups appeared even earlier in the structure theory of regular semigroups, see Hall [14]. This wide class also contains the above mentioned prominent classes. Moreover, it is a common generalization of orthodox semigroups and completely regular semigroups. However, the study of the structure of E-solid semigroups has not been as intensive and successful as that of locally inverse semigroups. By Yamada (and Hall) [32], a regular semigroup is E-solid if and only if the idempotent classes of its least inverse semigroup congruence are completely simple subsemigroups.

The study of classes of regular semigroups outside of inverse semigroups and completely regular semigroups from universal algebraic point of view began in the late 1980's. It has turned out that the classes of locally inverse and of E-solid semigroups are precisely those in which a theory showing close analogy to that for usual varieties of algebras can be developed, see Auinger [1], [2], Hall [15], Yeh [33], Kadourek and Szendrei [19], [20], [29]. For surveys, see Auinger [3], Jones [18] and Trotter [31]. This progress revitalized the structure theoretical investigations in these classes, see e.g. Billhardt and Szendrei [8].

The thesis concentrates on E-solid locally inverse semigroups which are extensions by inverse semigroups and the idempotent classes are completely simple. The main problem we will give an answer to is whether such extensions are embeddable in a λ-semidirect product of a completely simple semigroup by an inverse semigroup.

Chapter 3 and [10] deals with the special case where the extensions are by groups. In this case, the extension itself is necessarily completely simple. The motivation for considering this case first was to check whether the general embedding result we intend to prove holds in this special case. In fact, we prove a somewhat stronger result than that following from our main result (Theorem 4.1.1), namely that each extension of a completely simple semigroup U by a group H is embeddable in a semidirect product of a completely simple semigroup V by H where V is close to U, e.g., the maximal subgroups of V are direct powers of those of U (Theorem 3.2.1). Note that the embedding given in the proof mimics the standard proof of the Kaloujnine–Krasner Theorem. Comparing this easy proof to that of the main result, one can see how much more complicated the extensions by inverse semigroups might be than those by groups.

The semidirect product of V by H constructed in the proof of the result mentioned in the previous paragraph is not the wreath product of U by H. Since completely simple semigroups are fairly close to groups — they are disjoint unions of pairwise isomorphic groups —, it is natural to ask whether the Kaloujnine–Krasner Theorem holds for such extensions. In the first section of Chapter 3, we establish that this is not the case in general, that is, an extension of a completely simple semigroup U by a group H is given which is not embeddable in the wreath product of U by H (Theorem 3.1.2). However, we also show that the Kaloujnine–Krasner Theorem is valid within the class of central completely simple semigroups (Proposition 3.1.1).

In Chapter 4 we give affirmative answer to the main problem formulated above (see Theorem 4.1.1):

Main result. *If S is an E-solid locally inverse semigroup and ϱ is an inverse semigroup congruence on S such that the idempotent ϱ-classes, as subsemigroups of S, are completely simple then S is embeddable in a λ-semidirect product of a completely simple semigroup by S/ϱ.*

As a corollary, we obtain that the E-solid locally inverse semigroups are, up to isomorphism, the regular subsemigroups of the λ-semidirect products of completely simple semigroups by inverse semigroups (Corollary 4.1.2). In the proof of the main result we apply the 'canonical embedding technique' developed by Kuřil and Szendrei [23] for handling embeddability of extensions by inverse semigroups in λ-semidirect products. This chapter's results are contained in [12].

Chapter 2

Preliminaries

In this chapter we summarize the notions and results needed in the thesis.

2.1 Basic definitions

This section follows mainly the structure of [17], for more details the reader is referred to that monograph. In several cases, we give alternative definitions for notions in order to reduce the number of concepts introduced.

An element a of a semigroup S is called *regular*, if there exists an element b of S that satisfies $aba = a$. A semigroup S is called *regular* if all of its elements are regular. If b is an element of S, such that $aba = a$ and $bab = b$ then we call it an *inverse* of a. The set of inverses of a is denoted by $V(a)$. If a is regular then $V(a)$ is known to be non-empty.

An element e of a semigroup S is called *idempotent* if $e^2 = e$. The set of idempotent elements is denoted by E_S. Note that every idempotent element is regular and an inverse of itself. If $aba = a$ for some $a, b \in S$ then both ab and ba are idempotents. There is a *natural order of* E_S defined by $e \leq f$ if $ef = fe = e$ ($e, f \in E_S$). In particular, if E_S is a subsemilattice in S then $\leq$ is just the partial order corresponding to the meet semilattice E_S.

If S is a regular semigroup then the natural order of E_S can be extended to S by the rule $s \leq t$ if $s = et = tf$ from some $e, f \in E_S$ ($s, t \in S$), and is called the *natural order of* S.

By a *full regular subsemigroup* of a regular semigroup S we mean a regular subsemigroup T of S such that $E_T \supseteq E_S$ (or equivalently, $E_T = E_S$).

A semigroup with an identity element is called a *monoid*. For any semigroup S let S^1 denote the monoid we get from S by adding a disjoint identity element to S if there is none in S.

If a is an element of S then the smallest left ideal of S containing a is $S^1a = Sa \cup \{a\}$. The Green relation $\mathcal{L}$ is defined by $a \mathrel{\mathcal{L}} b$ if $S^1a = S^1b$. Dually, $a \mathrel{\mathcal{R}} b$ if $aS^1 = bS^1$. Note that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence, and they commute. Let us denote the smallest equivalence relation containing both $\mathcal{R}$ and $\mathcal{L}$ by $\mathcal{D}$. By the previous observation $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$. The intersection $\mathcal{R} \cap \mathcal{L}$ is denoted by $\mathcal{H}$. The equivalence class of $a \in S$ is denoted by L_a, R_a, D_a, H_a, respectively. If H_a contains an idempotent element, then it is a maximal subgroup of S. Conversely, every maximal subgroup of S is of the form H_e for some $e \in E_S$. Note that in a regular semigroup every $\mathcal{R}$- and $\mathcal{L}$-class contains at least one idempotent element. If a $\mathcal{D}$ class contains a regular element then all of its elements are regular.

If S is a semigroup and K is a subsemigroup in S then we distinguish the $\mathcal{R}$ relation on K from that on S by writing $\mathcal{R}^K$. Note that if K is regular then $\mathcal{R}^K = \mathcal{R} \cap (K \times K)$. Moreover, if K is a full regular subsemigroup in the regular semigroup S then the rule $R \mapsto R^K = R \cap K$ determines a bijection from the set of $\mathcal{R}$-classes of S onto the set of $\mathcal{R}$-classes of K. In particular, for any $x \in K$, we have $(R_x)^K = (R^K)_x$, therefore it is not confusing to write simply R_x^K.

Let ϱ be a congruence on a semigroup. The congruence class of $a \in S$ is denoted by $a\varrho$, and the *natural homomorphism* from S onto the factor semigroup S/ϱ is given by $\varrho^\natural \colon S \to S/\varrho,\ a \mapsto a\varrho$. It is easy to see that every factor semigroup of a regular semigroup is also regular. Obviously, a congruence class $a\varrho \in S/\varrho$ forms a subsemigroup in S if and only if $a\varrho$ is idempotent in S/ϱ. In particular, for every idempotent e of S, the congruence class $e\varrho$ is a subsemigroup of S.

The following result is known as Lallement's Lemma.

Lemma 2.1.1. *For every congruence ϱ on a regular semigroup S, if $a\varrho$ is an idempotent in S/ϱ, then there exists an idempotent e in $a\varrho$.*

A congruence ϱ is said to be a *group [semilattice, …] congruence*, if S/ϱ is a group [semilattice, …]. The kernel of a group congruence $\operatorname{Ker}\varrho$ is the inverse image of the identity element of S/ϱ. If ϱ is a semilattice congruence, and $\varphi \colon S \to Y$ is a surjective homomorphism inducing the congruence ϱ on S (and so $Y \cong S/\varrho$), then S is said to be the *semilattice Y of the subsemigroups* S_α $(\alpha \in Y)$ of S where S_α is the inverse image of α. If there are certain kinds of homomorphisms between these classes, called structure homomorphisms, and we can express the multiplication of S with the help of the multiplication of the S_α's and the structure homomorphisms, then S is said to be a *strong semilattice Y of the subsemigroups S_α ($\alpha \in Y$)*. Since the details of this construction are not needed in the thesis, we omit them.

A semigroup where every element is idempotent is called a *band*. Note that semilattices are just commutative bands. A band with only one $\mathcal{D}$-class is called a *rectangular band*. It is known that every band is a semilattice of rectangular bands. A band is *normal* if it is a strong semilattice of rectangular bands.

A regular semigroup S is *completely regular* if it is the union of its maximal subgroups, and is *completely simple* if it is completely regular and it contains only one $\mathcal{D}$-class. Notice that bands and rectangular bands are just the completely regular and completely simple semigroups, respectively, where the maximal subgroups are trivial. Similarly to bands, every completely regular semigroup is a semilattice of completely simple semigroups.

By a *Rees matrix semigroup* we mean a semigroup $S = \mathcal{M}[G; I, \Lambda; P]$ where G is a group, I, Λ are non-empty sets and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with elements from G, called a *sandwich matrix*. The underlying set of S is $I \times G \times \Lambda$, and the multiplication is defined by

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu).$$

Every Rees matrix semigroup is completely simple and conversely, by the Rees–Suschkewitsch Theorem, every completely simple semigroup is isomorphic to a Rees matrix semigroup. We say that P is *normalised* if there exists $i \in I$ and $\lambda \in \Lambda$ such that $p_{\mu i} = p_{\lambda j} = 1_G$ for every $j \in I$ and $\mu \in \Lambda$. Every Rees matrix semigroup is isomorphic to one with normalised sandwich matrix. A completely simple semigroup is in most cases represented as a Rees matrix semigroup with a normalized sandwich matrix throughout Chapter 3.

A completely simple semigroup is called *central* if the product of any two of its idempotents lies in the centre of the containing maximal subgroup. It is well known that a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ with P normalized is central if and only if each entry of P belongs to the centre of G.

The group congruences of a Rees matrix semigroup with a normalized sandwich matrix are characterized as follows.

Proposition 2.1.2. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup where P is normalized. Assume that N is a normal subgroup of G such* *that every entry of P belongs to N. Define a relation ϱ on S such that, for every (i,g,λ), $(j,h,\mu) \in S$, let*

$$(i,g,\lambda)\,\varrho\,(j,h,\mu) \quad \textit{if and only if} \quad gh^{-1} \in N.$$

Then ϱ is a group congruence on S such that S/ϱ is isomorphic to G/N and $\operatorname{Ker}\varrho = \mathcal{M}[N; I, \Lambda; P]$.

Conversely, every group congruence on S is of this form for some normal subgroup N of G where all entries of P belong to N.

This proposition implies that the kernel of any group congruence of a completely simple semigroup is completely simple. Conversely, it is routine to check that if S is a regular semigroup and ϱ is a group congruence on S such that $\operatorname{Ker}\varrho$ is a completely simple subsemigroup of S then S is necessarily completely simple.

A regular semigroup S is called *inverse* if every element a of S has a unique inverse element denoted by a^{-1}. Equivalently, a semigroup is inverse if it is regular and the set of idempotents is a subsemilattice. An inverse semigroup S is a group if and only if $|E_S| = 1$. The natural order of an inverse semigroup S can be handled more easily than the regular case, since $s \leq t$ if and only if there exists $e \in E_S$ such that $s = et$ (or, equivalently, there exists $f \in E_S$ such that $s = tf$), for any $s, t \in S$. The natural order in an inverse semigroup is compatible with the multiplication.

Let S be a semigroup, and $\mathcal{K}$ a class of semigroups. If ϱ is an inverse semigroup congruence on S (i.e., S/ϱ is an inverse semigroup) then ϱ is said to be a *congruence over* $\mathcal{K}$ if each idempotent ϱ-class, as a subsemigroup of S, belongs to $\mathcal{K}$. In this case, the union of the idempotent ϱ-classes, called the *kernel of* ϱ and denoted $\operatorname{Ker}\varrho$, is a semilattice of subsemigroups belonging to $\mathcal{K}$. Recall that if S is a regular semigroup then, by Lallement's Lemma, the idempotent ϱ-classes are precisely the ϱ-classes $e\varrho$ where $e \in E_S$. Moreover, they are always regular subsemigroups in S.

A regular semigroup S is called *orthodox* if its idempotents form a subsemigroup, i.e. a subband. In an orthodox semigroup every inverse of an idempotent is idempotent. The relation γ defined by $a\,\gamma\,b \Leftrightarrow V(a) = V(b)$ is the smallest inverse semigroup congruence on S. An orthodox semigroup is called a *generalised inverse semigroup* if its idempotents form a normal band.

A regular semigroup is called *locally inverse* if each local submonoid eSe ($e \in E_S$) is an inverse subsemigroup. Note that each inverse semigroup and each completely simple semigroup is locally inverse. In fact, completely simple semigroups are just the regular semigroups where local submonoids are groups. Moreover, generalised inverse semigroups are exactly the orthodox locally inverse semigroups. It is well known that a regular semigroup is locally inverse if and only if the natural order $\leq$ is compatible with the multiplication.

For a regular semigroup S the sandwich set $S(e, f)$ of two idempotent elements $e, f \in S$ is defined by

$$S(e, f) = \{g \in V(ef) \cap E_S \colon ge = fg = g\},$$

and it has the property that if $e \mathrel{\mathcal{L}} e_1$ and $f \mathrel{\mathcal{R}} f_1$ then $S(e, f) = S(e_1, f_1)$. It is well known that a regular semigroups is locally inverse if and only if each sandwich set is a singleton. This allows us to introduce another binary operation $\wedge$ on a locally inverse semigroup S, assigning to any pair of elements $(s, t) \in S \times S$ the unique element $s \wedge t$ of the sandwich set $S(t^*t, ss^*)$, where s^* and t^* are arbitrary inverses of s and t, respectively. We call $\wedge$ the *sandwich operation* on S. It is clear by definition that $s \wedge t \in E_S$ and $s \wedge t = ss^* \wedge t^*t$ for every $s, t \in S$ and for any $s^* \in V(s)$ and $t^* \in V(t)$. In particular, in an inverse semigroup, $s \wedge t = ss^{-1}t^{-1}t$ and, in a completely simple semigroup, $s \wedge t$ is the unique idempotent which is $\mathcal{R}$-related to s and $\mathcal{L}$-related to t, that is, $s \wedge t$ is the identity element of the group H_{st}. Let us also mention that every homomorphism and congruence of locally inverse semigroups respects the sandwich operation, see [33].

The following important property of locally inverse semigroups will be also needed later, see [13, Proposition IX.3.2(4)].

Proposition 2.1.3. *Let S be a locally inverse semigroup, and let $s, t \in S$ with $s \leq t$. Then, for every $b \in R_t$, there exists a unique $a \in R_s$ such that $a \leq b$.*

A regular semigroup is called *E-solid* if the core of S, that is, the subsemigroup generated by the idempotents of S is completely regular. In particular, orthodox semigroups and completely regular semigroups are E-solid. It is also known, that a regular semigroup is E-solid if and only if the least inverse semigroup congruence is over the class of all completely simple semigroups, see Yamada (and Hall) [32]. Thus the kernel of the least inverse semigroup congruence of an E-solid locally inverse semigroup is a locally inverse completely regular semigroup, that is, a strong semilattice of completely simple semigroups. For several equivalent characterizations of the class of strong semilattices of completely simple semigroups, see [26].

In the thesis, we denote by $\mathcal{LI}$, $\mathcal{ES}$ and $\mathcal{CS}$ the classes of all locally inverse semigroups, E-solid semigroups and completely simple semigroups, respectively.

Let K be a semigroup and T an inverse semigroup. If S is a semigroup and ϱ is a congruence on S such that S/ϱ is isomorphic to T and $\operatorname{Ker}\varrho$ is isomorphic to K then the pair (S, ϱ) is called an *extension of K by T*. If, moreover, S is regular then (S, ϱ) is termed a *regular extension of K by T*. In this case, K — being isomorphic to the kernel of an inverse semigroup congruence — is necessarily regular. If (S, ϱ) and (T, σ) are extensions by inverse semigroups then an injective homomorphism $\psi\colon S \to T$ is defined to be an *embedding of the extension (S, ϱ) into the extension (T, σ)* if the congruence induced by $\psi\sigma^\natural$ is just ϱ.

In this terminology, the second statement in the paragraph after Proposition 2.1.2 says that each regular extension of a completely simple semigroup by groups is completely simple. Chapter 3 deals with these extensions. Similarly, the statement on E-solid locally inverse semigroups in the paragraph after Proposition 2.1.3 says that each E-solid locally inverse semigroup is a regular extension of a strong semilattice of completely simple semigroups by an inverse semigroup. Note that the converse of this statement is easy to check. The topic of Chapter 4 is an embedding theorem for regular extensions of strong semilattices of completely simple semigroups by inverse semigroups.

2.2 Semidirect and λ-semidirect products

Let K, T be semigroups. We denote the endomorphism monoid of K by $\operatorname{End} K$. We say that T acts on K by endomorphisms on the left, in short, T acts on K if an antihomomorphism $\varepsilon\colon T \to \operatorname{End} K$, $t \mapsto \varepsilon_t$ is given, that is a map, where $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ for any $u, t \in T$. For brevity, we will use the usual notation ${}^t a$ to denote $a\varepsilon_t$ $(a \in K, t \in T)$. The *semidirect product $K \rtimes T$* is defined on the set $K \times T$ by multiplication

$$(a, t)(b, u) = (a \cdot {}^t b, tu).$$

A related construction is the following. For any semigroups K,T, an action of T on the direct power K^T can be defined in the following natural way: for any $f \in K^T$ and $t \in T$, let ${}^t f$ be the element of K^T where $u({}^t f) = (ut)f$ for any $u \in T$. The semidirect product $K^T \rtimes T$ defined by this action is called the *wreath product of* K *by* T, and is denoted by $K \wr T$. In case K and T are groups, these are the usual definitions of a semidirect product $K \rtimes T$ and of the wreath product $K \wr T$ of K and T.

If K is a semigroup and T is a group then $K \rtimes T$ and $K \wr T$ are regular [inverse, completely simple] if and only if K is. However, in general, a semidirect product $K \rtimes T$ is not regular even if both K and T are inverse. This led Bernd Billhardt [6] to adapt these constructions to the inverse case in the following way. Let K be a semigroup and T an inverse semigroup acting on K. The λ-*semidirect product* $K \rtimes_\lambda T$ is defined on the underlying set

$$\{(a,t) \in K \times T \colon {}^{tt^{-1}}a = a\}$$

by multiplication

$$(a,t)(b,u) = ({}^{(tu)(tu)^{-1}}a \cdot {}^t b, tu),$$

for all $a, b \in K$, $t, u \in T$. The wreath product analogue will not be applied in the thesis, therefore we do not define it.

Remark 2.2.1. If T is a group then the above definition of λ-semidirect product construction is simply the usual semidirect product of K by T.

By specializing the respective statements in [5] and [23], we formulate the properties of this construction in the case where K is a completely simple semigroup.

Proposition 2.2.2. *Let K be a completely simple semigroup [rectangular band, group] and T an inverse semigroup acting on K.*

(i) The λ-semidirect product $K \rtimes_\lambda T$ is an E-solid locally inverse semigroup [generalised inverse semigroup, inverse semigroup] with set of idempotents

$$E_{K\rtimes_\lambda T} = \{(e,i) : e \in E_K,\ i \in E_T \text{ and } {}^i e = e\},$$

and for any $(a,t) \in K \rtimes_\lambda T$, we have

$$V_{K\rtimes_\lambda T}\big((a,t)\big) = \{\big(b, t^{-1}\big) : b \in V_K({}^{t^{-1}}a) \text{ and } {}^{t^{-1}t}b = b\}.$$

(ii) The second projection $\pi_2 \colon K \rtimes_\lambda T \to T$, $(a,t) \mapsto t$ is a homomorphism of $K \rtimes_\lambda T$ onto T, and the congruence ϑ_2 induced by π_2 is over completely simple semigroups. The kernel of ϑ_2 is

$$\mathrm{Ker}(\vartheta_2) = \{(a,e) : a \in K,\ e \in E_T \text{ and } {}^{e}a = a\},$$

and it is isomorphic to a strong semilattice of the completely simple subsemigroups [rectangular subbands, subgroups]

$$K_e = \{a \in K : {}^{e}a = a\} \quad (e \in E_T).$$

Statement (ii) of this proposition shows that the extension $(K \rtimes_\lambda T, \vartheta_2)$ is an extension of a strong semilattice of completely simple subsemigroups of K by T. The extension $(K \rtimes_\lambda T, \vartheta_2)$ is referred to as a λ-*semidirect product extension of K by T*.

The following proposition shows how the Green relations $\mathcal{R}$ and $\mathcal{L}$ work in the λ-semidirect product construction.

Proposition 2.2.3. *Let $K \rtimes_\lambda T$ be a λ-semidirect product of a completely simple semigroup K by an inverse semigroup T. Then, for any elements $(a,t), (b,u)$, we have*

(1) $(a,t)\,\mathcal{R}\,(b,u)$ in $K \rtimes_\lambda T$ if and only if $a\,\mathcal{R}\,b$ in T and $t\,\mathcal{R}\,u$ in T,

(2) $(a,t)\,\mathcal{L}\,(b,u)$ in $K \rtimes_\lambda T$ if and only if ${}^{t^{-1}}a\,\mathcal{L}\,{}^{u^{-1}}b$ in K and $t\,\mathcal{L}\,u$ in T.

In particular, if T is a group then $K \rtimes T$ satisfies the same properties, where the second relations are trivial.

2.3 Existence varieties, bifree objects

A class of regular semigroups is termed an *existence variety*, or, for short, an *e-variety* if it is closed under taking direct products, homomorphic images and regular subsemigroups. For example, $\mathcal{LI}$, $\mathcal{ES}$ and $\mathcal{CS}$ form e-varieties. Note also that a class of inverse semigroups or a class of completely simple semigroups constitutes an e-variety if and only if it forms a variety of semigroups with an additional operation $^{-1}$.

If S is a regular semigroup then an *inverse unary operation* is defined to be a mapping $^{\dagger} \colon S \to S$ with the property that $s^{\dagger} \in V(s)$ for every $s \in S$.

In particular, if S is an inverse semigroup then the unique inverse unary operation is denoted in the usual way by $^{-1}$.

By a *binary semigroup* we mean a semigroup having an additional binary operation denoted by $\wedge$. A homomorphism or a congruence of a binary semigroup is always supposed to respect both the multiplication and the $\wedge$ operation. As noticed above, each locally inverse semigroup is also a binary semigroup with respect to the sandwich operation, and the homomorphisms and congruences of locally inverse semigroups, considered as usual semigroups and binary semigroups, respectively, coincide.

Let X be a non-empty set. The free semigroup on X is denoted by X^+. We 'double' X as follows. Consider a set $X' = \{x' : x \in X\}$ disjoint from X together with a bijection $': X \to X',\ x \mapsto x'$, and denote $X \cup X'$ by $\overline{X}$.

Let S be a regular semigroup. A mapping $\nu\colon \overline{X} \to S$ is called *matched* if $x'\nu$ is an inverse of $x\nu$ in S for each $x \in X$. Now let $\mathcal{K}$ be a class of regular semigroups. We say that a semigroup $B \in \mathcal{K}$ together with a matched mapping $\xi\colon \overline{X} \to B$ is a *bifree object in* $\mathcal{K}$ *on* X if, for any $S \in \mathcal{K}$ and any matched mapping $\nu\colon \overline{X} \to S$, there is a unique homomorphism $\varphi\colon B \to S$ extending ν, that is, for which $\xi\varphi = \nu$ holds. We denote the unique homomorphism extending ν by $\overline{\nu}$. It was essentially proved by Yeh [33] that an e-variety admits a bifree object on any alphabet (or, equivalently, on an alphabet of at least two elements) if and only if it is contained either in $\mathcal{LI}$ or in $\mathcal{ES}$. The bifree objects of $\mathcal{LI}$ and $\mathcal{ES}$ are determined by Auinger [1], [2] and by Szendrei [29], respectively. Each of these descriptions fit into a Birkhoff-type theory for the respective class based on an appropriate notion of 'identity', see also [20]. In this thesis we need the model for the bifree objects of $\mathcal{LI}$ published in [2]. For a more complete introduction to the theory of e-varieties, see [3], [18] and [31].

The *free binary semigroup* $F_{\langle 2,2\rangle}(Y)$ *on the alphabet* Y can be interpreted as follows. Its underlying set is the smallest one among the sets W which fulfill the following conditions:

(i) $Y \subseteq W \subseteq (Y \cup \{(,\ \wedge,\ )\})^+$,

(ii) $u, v \in W$ implies $uv \in W$,

(iii) $u, v \in W$ implies $(u \wedge v) \in W$.

The operations $\cdot$ and $\wedge$ are the concatenation and the operation

$$F_{\langle 2,2\rangle}(Y) \times F_{\langle 2,2\rangle}(Y) \to F_{\langle 2,2\rangle}(Y)\,,\ (u, v) \mapsto (u \wedge v)\,,$$

respectively.

One can see that the indecomposable (into a product) terms in $F_{\langle 2,2\rangle}(Y)$ are precisely the elements of Y called *letters* and the terms of the form $(u\wedge v)$. Moreover, each term admits a unique factorization into indecomposable ones.

A *bi-identity* in $\mathcal{LI}$ is a formal equality $u \mathrel{\hat{=}} v$ among terms $u, v \in F_{\langle 2,2\rangle}\big(\overline{X}\big)$. We say that a semigroup $S \in \mathcal{LI}$ *satisfies the bi-identity* $u \mathrel{\hat{=}} v$ if $u\overline{\nu} = v\overline{\nu}$ for each matched mapping $\nu \colon \overline{X} \to S$. The *bi-identity* $u \mathrel{\hat{=}} v$ *holds in the class* $\mathcal{K}$ *of locally inverse semigroups* if it holds in every member of $\mathcal{K}$. For an e-variety $\mathcal{V}$ of locally inverse semigroups, define

$$\begin{aligned}\Theta(\mathcal{V}, X) = \{(u, v) \in F_{\langle 2,2\rangle}\big(\overline{X}\big) \times F_{\langle 2,2\rangle}\big(\overline{X}\big) :&\\ \text{the bi-identity } u \mathrel{\hat{=}} v \text{ holds in } \mathcal{V}\}.&\end{aligned}$$

This relation is obviously a congruence on $F_{\langle 2,2\rangle}\big(\overline{X}\big)$ which is called the *bi-invariant congruence on* $F_{\langle 2,2\rangle}\big(\overline{X}\big)$ *corresponding to* $\mathcal{V}$. The main results of the Birkhoff-type theory for $\mathcal{LI}$ are the following.

Proposition 2.3.1. *A class of locally inverse semigroups forms an e-variety if and only if it is defined by a set of bi-identities.*

Proposition 2.3.2. *Given an e-variety $\mathcal{V}$ of locally inverse semigroups and a non-empty set X, the factor semigroup* $\mathrm{BF}\mathcal{V}(X) = F_{\langle 2,2\rangle}\big(\overline{X}\big)/\Theta(\mathcal{V}, X)$ *together with the matched mapping $\xi \colon \overline{X} \to \mathrm{BF}\mathcal{V}(X)$, $y \mapsto y\Theta(\mathcal{V}, X)$ is a bifree object in $\mathcal{V}$ on X.*

In the sequel we need the description, published in [2], of the bi-invariant congruences corresponding to the variety $\mathcal{CS}$.

For any term $w \in F_{\langle 2,2\rangle}\big(\overline{X}\big)$, denote by ιw $[w\tau]$ the first [last] letter (i.e., element of $\overline{X}$) appearing in w (reading w from the left to the right as a word in the alphabet $\overline{X} \cup \{(, \wedge,)\}$). In the usual way, extend $' \colon X \to X'$ to a mapping $' \colon \overline{X} \to \overline{X}$ by defining $(x')'$ to be x for any $x \in X$.

Let us consider the following *reductions* of the terms in $F_{\langle 2,2\rangle}\big(\overline{X}\big)$ where $u, v \in F_{\langle 2,2\rangle}\big(\overline{X}\big)$ and $x, y, z \in \overline{X}$:

(R0) $(u \wedge v) \rightsquigarrow (\iota u \wedge v\tau)$,

(R1) $x(y \wedge x) \rightsquigarrow x$,

(R2) $(x \wedge y)x \rightsquigarrow x$, (R3) $(x \wedge y)(x \wedge z) \rightsquigarrow (x \wedge z)$,

(R4) $(z \wedge x)(y \wedge x) \rightsquigarrow (z \wedge x)$,

(R5) $x'x \rightsquigarrow (x' \wedge x)$.

A *reduction* $s \rightsquigarrow t$ *is applied* in a term $w \in F_{\langle 2,2\rangle}(\overline{X})$ if a segment s in w is changed for t. A term in $F_{\langle 2,2\rangle}(\overline{X})$ is called *reduced* if no reduction can be applied. Since reductions (R0)–(R5) shorten the terms in $F_{\langle 2,2\rangle}(\overline{X})$, one sees that a reduced form can be obtained for any term by applying finitely many reductions. In [2], each term $w \in F_{\langle 2,2\rangle}(\overline{X})$ is proved to have a uniquely determined reduced form which is denoted by $\mathbf{s}(w)$, and the following result is obtained.

Proposition 2.3.3. *For any non-empty set X, we have*

$$\Theta(\mathcal{CS}, X) = \{(u,v) \in F_{\langle 2,2\rangle}(\overline{X}) \times F_{\langle 2,2\rangle}(\overline{X}) : \mathbf{s}(u) = \mathbf{s}(v)\}.$$

Notice that, applying reduction (R0) for any indecomposable factor of a term in $F_{\langle 2,2\rangle}(\overline{X})$ which is not a letter, we obtain an element of the free semigroup $\widetilde{X}^+$ on the alphabet $\widetilde{X} = \overline{X} \cup (\overline{X} \wedge \overline{X})$ where $(\overline{X} \wedge \overline{X})$ stands for the set $\{(x \wedge y) : x, y \in \overline{X}\}$. In particular, every reduced term belongs to $\widetilde{X}^+$. Thus the model of a bifree object in $\mathcal{CS}$ on X provided by Results 2.3.2 and 2.3.3 can be simplified as follows. Make the free semigroup $\widetilde{X}^+$ to a binary semigroup by defining an additional binary operation $\wedge$ on it by

$$(u \wedge v) = (\iota u \wedge v\tau) \tag{2.3.1}$$

for every $u, v \in \widetilde{X}^+$, and consider the restriction of $\Theta(\mathcal{CS}, X)$ to $\widetilde{X}^+$.

Proposition 2.3.4. *For any non-empty set X, the relation*

$$\widetilde{\Theta}(\mathcal{CS}, X) = \{(u,v) \in \widetilde{X}^+ \times \widetilde{X}^+ : \mathbf{s}(u) = \mathbf{s}(v)\}$$

is a congruence on the binary semigroup $\widetilde{X}^+$ such that $\widetilde{X}^+/\widetilde{\Theta}(\mathcal{CS}, X)$ together with the matched mapping $\xi\colon \overline{X} \to \widetilde{X}^+/\widetilde{\Theta}(\mathcal{CS}, X)$, $y \mapsto y\widetilde{\Theta}(\mathcal{CS}, X)$ is a bifree object in $\mathcal{CS}$ on X.

Later on, we use the latter model for the bifree object of $\mathcal{CS}$ on X, and need an alternative description of $\widetilde{\Theta}(\mathcal{CS}, X)$. In order to distinguish the elements of the two types in the alphabet $\widetilde{X}$, we call the elements of $\overline{X}$, as usual, *letters*, and those of $(\overline{X} \wedge \overline{X})$ $\wedge$-*letters*.

Lemma 2.3.5. *The congruence* $\widetilde{\Theta}(\mathcal{CS}, X)$ *is generated on* $\widetilde{X}^+$*, as a semigroup congruence, by the relation* $\mathrm{I} \cup \Upsilon$ *where*

$$\mathrm{I} = \{(xx'x, x) : x \in \overline{X}\},$$

and Υ *is the union of the following three relations coming from rules* (R3)–(R5)*:*

$$\begin{aligned}\Upsilon_3 &= \{\big((x \wedge y)(x \wedge z), (x \wedge z)\big) : x, y, z \in \overline{X}\},\\ \Upsilon_4 &= \{\big((z \wedge x)(y \wedge x), (z \wedge x)\big) : x, y, z \in \overline{X}\},\\ \Upsilon_5 &= \{\big(x'x, (x' \wedge x)\big) : x \in \overline{X}\}.\end{aligned}$$

Proof. Denote the semigroup congruence described in the lemma by χ. It is obvious by Proposition 2.3.4 that χ is contained in $\widetilde{\Theta}(\mathcal{CS}, X)$, and that, in order to show the reverse inclusion, it suffices to verify that the pairs

$$\big(x(y \wedge x), x\big) \text{ and } \big((x \wedge y)x, x\big) \quad (x, y \in \overline{X}),$$

coming from rules (R1)–(R2), belong to χ. Indeed, applying the relations I, Υ_5, Υ_4, Υ_5, I, we see that

$$x(y \wedge x) \;\chi\; xx'x(y \wedge x) \;\chi\; x(x' \wedge x)(y \wedge x) \;\chi\; x(x' \wedge x) \;\chi\; xx'x \;\chi\; x$$

for every $x, y \in \overline{X}$. The statement for the pairs of the other type is proven dually. □

2.4 Graphs and semigroupoids

Now we summarize the basic notions on graphs and semigroupoids needed in the dissertation.

A *graph* $\mathcal{X}$ consists of a set of *objects* denoted by $\operatorname{Obj} \mathcal{X}$ and, for every pair $g, h \in \operatorname{Obj} \mathcal{X}$, a set of *arrows* from g to h which is denoted by $\mathcal{X}(g, h)$. The sets of arrows corresponding to different pairs of objects are supposed to be disjoint, and the set of all arrows is denoted by $\operatorname{Arr} \mathcal{X}$. If $a \in \mathcal{X}(g, h)$ then we write that $\alpha(a) = g$ and $\omega(a) = h$. By a *loop* we mean an arrow a with $\alpha(a) = \omega(a)$. The arrows $a, b \in \operatorname{Arr} \mathcal{X}$ are called *coterminal* if $\alpha(a) = \alpha(b)$ and $\omega(a) = \omega(b)$, and are termed *consecutive* if $\omega(a) = \alpha(b)$.

A *semigroupoid* is a graph $\mathcal{X}$ equipped with a composition which assigns to every pair of consecutive arrows $a \in \mathcal{X}(g, h)$, $b \in \mathcal{X}(h, i)$ an arrow in

$\mathcal{X}(g,i)$, usually denoted by $a \circ b$, such that the composition is associative, that is, for any arrows $a \in \mathcal{X}(g,h)$, $b \in \mathcal{X}(h,i)$ and $c \in \mathcal{X}(i,j)$, we have $(a \circ b) \circ c = a \circ (b \circ c)$.

The notion of a semigroupoid generalizes that of a semigroup. Indeed, each semigroup can be considered as the set of arrows of a semigroupoid whose set of objects is a singleton. A number of basic notions in semigroup theory can be extended in a natural way for semigroupoids.

Let $\mathcal{X}$ be a semigroupoid. We define *Green's relation* $\mathcal{L}$ *on* $\mathcal{X}$ as follows: for any $a, b \in \operatorname{Arr} \mathcal{X}$, we have $a \,\mathcal{L}\, b$ if and only if either $a = b$, or there exist $u, v \in \operatorname{Arr} \mathcal{X}$ such that $\omega(u) = \alpha(a)$, $\omega(v) = \alpha(b)$ and $u \circ a = b$, $v \circ b = a$. It is routine to check that $\mathcal{L}$ is an equivalence relation on $\operatorname{Arr} \mathcal{X}$, and clearly, for any $a, b \in \operatorname{Arr} \mathcal{X}$ with $a \,\mathcal{L}\, b$, we have $\omega(a) = \omega(b)$. Furthermore, for any $c \in \operatorname{Arr} \mathcal{X}$ with $\alpha(c) = \omega(a) = \omega(b)$, the relation $a \,\mathcal{L}\, b$ implies $a \circ c \,\mathcal{L}\, b \circ c$. Dually, we can introduce Green's relation $\mathcal{R}$ on $\mathcal{X}$ and formulate its basic properties.

By a *regular semigroupoid* we mean a semigroupoid $\mathcal{X}$ in which, for every arrow $a \in \mathcal{X}(g,h)$, there exists an arrow $b \in \mathcal{X}(h,g)$ with $a \circ b \circ a = a$. If the arrows $a \in \mathcal{X}(g,h)$, $b \in \mathcal{X}(h,g)$ have the property that $a \circ b \circ a = a$ and $b \circ a \circ b = b$ then we call b an *inverse* of a, and denote the set of all inverses of a by $V(a)$. Similarly to a regular semigroup, each arrow of $\mathcal{X}$ has an inverse, and each $\mathcal{R}$- and $\mathcal{L}$-class contains an idempotent arrow. Each idempotent arrow is a loop, i.e., belongs to $\mathcal{X}(g,g)$ for some $g \in \operatorname{Obj} \mathcal{X}$, and $\mathcal{X}(g,g)$ is a regular semigroup for every $g \in \operatorname{Obj} \mathcal{X}$. Therefore the notion of the sandwich set $S(e,f)$ is defined for every $g \in \operatorname{Obj} \mathcal{X}$ and $e, f \in E(\mathcal{X}(g,g))$. These sandwich sets are singletons if and only if $E(\mathcal{X}(g,g))$ is a locally inverse semigroup for any $g \in \operatorname{Obj} \mathcal{X}$. If $\mathcal{X}$ is a regular semigroupoid with this property then we call it a *locally inverse semigroupoid*, and we define a sandwich operation on it as follows: if $a, b \in \operatorname{Arr} \mathcal{X}$ such that $\alpha(a) = \omega(b)$ then $(a \curlywedge b)$ is the unique element of $S(b'b, aa')$ where $a' \in V(a)$, $b' \in V(b)$. Note that $(a \curlywedge b)$ is independent of the choice of a', b', and $(a \curlywedge b) \in E\big(\mathcal{X}(\alpha(a), \alpha(a))\big)$. For completeness, let us mention that also the notion of a *natural partial order* can be introduced for regular semigroupoids so that its properties are similar to those well known for regular semigroups. In particular, the natural partial order of $\mathcal{X}$ is compatible with $\circ$ if and only if $\mathcal{X}$ is locally inverse, and in the case, it is compatible also with $\curlywedge$.

Motivated by the notion of a locally inverse semigroupoid which is a semigroupoid with an additional partial binary operation, now we introduce a notion of a *free binary semigroupoid* $F_{\langle 2,2 \rangle}(\mathcal{X})$ *on a graph* $\mathcal{X}$. Consider the smallest set P such that it is the disjoint union of its subsets $P_{g,h}$ ($g, h \in \operatorname{Obj}\mathcal{X}$), and the following conditions are satisfied:

(i) $\mathcal{X}(g,h) \subseteq P_{g,h} \subseteq \big(\operatorname{Arr}\mathcal{X} \cup \{(,\ \wedge,\ )\}\big)^+$,

(ii) $p \in P_{g,h},\ q \in P_{h,k}$ imply $pq \in P_{g,k}$,

(iii) $p \in P_{g,h},\ q \in P_{k,g}$ imply $(p \wedge q) \in P_{g,g}$.

The elements of P are called *paths* (more precisely, *binary paths*), and if $p \in P_{g,h}$ then we define $\alpha(p) = g$ and $\omega(p) = h$. The free binary semigroupoid $F_{\langle 2,2\rangle}(\mathcal{X})$ on $\mathcal{X}$ is defined as follows: its set of objects and arrow sets are $\operatorname{Obj} F_{\langle 2,2\rangle}(\mathcal{X}) = \operatorname{Obj}\mathcal{X}$ and $F_{\langle 2,2\rangle}(\mathcal{X})\,(g,h) = P_{g,h}$ ($g, h \in \operatorname{Obj} F_{\langle 2,2\rangle}(\mathcal{X})$), respectively, and the operations are those in (ii) and (iii) above. Notice that $\operatorname{Arr} F_{\langle 2,2\rangle}(\mathcal{X}) \subseteq F_{\langle 2,2\rangle}(\operatorname{Arr}\mathcal{X})$.

Let us 'double' $\mathcal{X}$ as follows. Consider a graph $\mathcal{X}'$ such that $\operatorname{Obj}\mathcal{X}' = \operatorname{Obj}\mathcal{X}$, the set $\operatorname{Arr}\mathcal{X}'$ is disjoint from $\operatorname{Arr}\mathcal{X}$, and a bijection

$$'\colon \mathcal{X}(g,h) \to \mathcal{X}'(h,g), \quad a \mapsto a'$$

is fixed for every $g, h \in \operatorname{Obj}\mathcal{X}'$. Define the graph $\overline{\mathcal{X}}$ by $\operatorname{Obj}\overline{\mathcal{X}} = \operatorname{Obj}\mathcal{X}$ and $\overline{\mathcal{X}}(g,h) = \mathcal{X}(g,h) \cup \mathcal{X}'(g,h)$ ($g, h \in \operatorname{Obj}\overline{\mathcal{X}}$). Notice that the bijections $'$ from the arrow sets of $\mathcal{X}$ onto those of $\mathcal{C}'$ determine a bijection from $\operatorname{Arr}\mathcal{X}$ onto $\operatorname{Arr}\mathcal{X}'$. Therefore, $\operatorname{Arr}\mathcal{X} \cup \operatorname{Arr}\mathcal{X}'$ is a doubling of the set $\operatorname{Arr}\mathcal{X}$. For brevity, put $A = \operatorname{Arr}\mathcal{X}$, and assume that $A' = \operatorname{Arr}\mathcal{X}'$. Thus $\overline{A} = \operatorname{Arr}\overline{\mathcal{X}}$ also follows, and each (binary) path in $\overline{\mathcal{X}}$ can be also considered as a term in $F_{\langle 2,2\rangle}\big(\overline{A}\big)$. In particular, a $\wedge$-letter $(a \wedge b) \in (\overline{A} \wedge \overline{A})$ is a path in $\overline{\mathcal{X}}$ if and only if $\alpha(a) = \omega(b)$. Such a $\wedge$-letter will be termed a $\wedge$-*loop*. Obviously, a word $a_1 a_2 \cdots a_m \in \widetilde{A}^+$ is a path in $\overline{\mathcal{X}}$ if and only if a_i is either a letter or a $\wedge$-loop for every i ($1 \leq i \leq m$), and $\omega(a_1) = \alpha(a_2),\ \omega(a_2) = \alpha(a_3),\ \ldots,\ \omega(a_{m-1}) = \alpha(a_m)$. It is straightforward that the subgraph of $F_{\langle 2,2\rangle}\big(\overline{\mathcal{X}}\big)$ whose arrows are just the (binary) paths in $\overline{\mathcal{X}}$ belonging to $\widetilde{A}^+$ forms a subsemigroupoid. This subsemigroupid is denoted $\widetilde{\mathcal{X}}^+$. Equality (2.3.1), applied only for $u, v \in \operatorname{Arr}\widetilde{\mathcal{X}}^+$ with $\alpha(u) = \omega(v)$, defines a $\wedge$ operation on $\operatorname{Arr}\widetilde{\mathcal{X}}^+$ so that it can be also considered a binary semigroupoid.

Given a regular semigroupoid $\mathcal{X}$, a transformation ${}^\dagger\colon \operatorname{Arr}\mathcal{X} \to \operatorname{Arr}\mathcal{X}$ is called an *inverse unary operation* on $\mathcal{X}$ if $a^\dagger \in V(a)$ for any arrow a.

Let $\mathcal{X}$ be a semigroupoid and S a semigroup. If $\ell\colon \mathcal{X} \to S$ is a morphism of semigroupoids, i.e., $\ell(a \circ b) = \ell(a) \cdot \ell(b)$ for any pair of consecutive arrows

a, b in $\mathcal{X}$ then ℓ is said to be a *labelling of* $\mathcal{X}$ *by* S. For an arrow $a \in \operatorname{Arr} \mathcal{X}$, the element $\ell(a)$ of S is called the *label of* a. Note that if both $\mathcal{X}$ and S are locally inverse then ℓ is also a binary morphism.

Chapter 3

Extensions of completely simple semigroups by groups

3.1 Embeddability in a wreath product

In this section first we notice that the Kaloujnine–Krasner Theorem can be easily extended to central completely simple semigroups. Moreover, we establish that it fails in general: we present a completely simple semigroup which is an extension of a completely simple semigroup by a group, and is not embeddable in their wreath product.

Now we present an isomorphic copy of the wreath product $T \wr H$ of a Rees matrix semigroup $T = \mathcal{M}[G; I, \Lambda; P]$ by a group H which allows us to make the calculation in the next section easier. First, it is routine to see that the direct power T^H is isomorphic to $\mathcal{M}[G^H; I^H, \Lambda^H; P^H]$ where $P^H = (p^H_{\xi\eta})$ is the following sandwich matrix: for any $\xi \in \Lambda^H$ and $\eta \in I^H$ we have $Ap^H_{\xi\eta} = p_{A\xi, A\eta}$ $(A \in H)$. Moreover, the action in the definition of the wreath product determines the following action when replacing T^H by $\mathcal{M}[G^H; I^H, \Lambda^H; P^H]$: for any $A \in H$ and $(\eta, f, \xi) \in \mathcal{M}[G^H; I^H, \Lambda^H; P^H]$ we have ${}^A(\eta, f, \xi) = ({}^A\eta, {}^Af, {}^A\xi)$, where ${}^A\eta \in I^H$, ${}^Af \in G^H$ and ${}^A\xi \in \Lambda^H$ are the maps defined by $B({}^A\eta) = (BA)\eta$, $B({}^Af) = (BA)f$ and $B({}^A\xi) = (BA)\xi$, respectively, for every $B \in H$.

Notice that, for any $A \in H$, we have

$$A({}^Bp^H_{\xi\eta}) = (AB)p^H_{\xi\eta} = p_{(AB)\xi,(AB)\eta} = p_{A({}^B\xi),A({}^B\eta)} = Ap^H_{{}^B\xi,{}^B\eta},$$

and so

$${}^Bp^H_{\xi\eta} = p^H_{{}^B\xi,{}^B\eta} \tag{3.1.1}$$

for any $B \in H$.

Finally, we sketch a standard proof of the Kaloujnine–Krasner Theorem.

Let G be an extension of N by H. Without loss of generality, we can assume that N is a normal subgroup of G and $H = G/N$. Choose and fix an element r_A from each coset A of N in G such that r_N is the identity element of G. It is straightforward to check that the map

$$\kappa\colon G \to N^H \rtimes H,\ g \mapsto (f_g, gN) \text{ where } f_g\colon H \to N,\ A \mapsto r_A g r_{A\cdot gN}^{-1} \tag{3.1.2}$$

is an embedding. Since κ is a morphism, the equality

$$f_{gh} = f_g \cdot {}^{gN}f_h \tag{3.1.3}$$

holds for every $g, h \in G$.

Let $S = \mathcal{M}[G; I, \Lambda; P]$ be an extension of a completely simple semigroup U by a group H where P is chosen to be normalized. By Proposition 2.1.2, we can assume that there is a normal subgroup N of the group G such that all entries of the sandwich matrix P belong to N, and we have $H = G/N$ and $U = \mathcal{M}[N; I, \Lambda; P] \subseteq S$.

First suppose that S is central, i.e., each entry of P belongs to the centre of the group G. Note that, in this case, U is necessarily also central. In this case, we can mimic the proof of the Kaloujnine–Krasner Theorem sketched in the previous section. For, it is routine to check by applying (3.1.2) and (3.1.3) that the map

$$\nu\colon S \to U \wr H = U^H \rtimes H,\ (i, g, \lambda) \mapsto (f_g^{i\lambda}, gN)$$

where

$$f_g^{i\lambda}\colon H \to U,\ A \mapsto (i, Af_g, \lambda)$$

is an embedding. This verifies the following statement.

Proposition 3.1.1. *Each central completely simple semigroup which is an extension of a (necessarily also central) completely simple semigroup U by a group H is embeddable in the wreath product of U by H.*

Now we turn to investigating the general case where S is an arbitrary completely simple semigroup. Suppose that there exists an embedding $S \to U \wr H$, i.e., an embedding

$$\varphi\colon S \to \mathcal{M}[N^H; I^H, \Lambda^H; P^H] \rtimes H \tag{3.1.4}$$

where $\mathcal{M}[N^H; I^H, \Lambda^H; P^H] \rtimes H$ is the isomorphic copy of $U \wr H$ introduced in this section. Proposition 2.2.3 implies that, in the semidirect product $\mathcal{M}[N^H; I^H, \Lambda^H; P^H] \rtimes H$, we have

$$[(\eta_1, f_1, \xi_1), A] \,\mathcal{R}\, [(\eta_2, f_2, \xi_2), B]$$

if and only if $(\eta_1, f_1, \xi_1) \,\mathcal{R}\, (\eta_2, f_2, \xi_2)$ in $\mathcal{M}[N^H; I^H, \Lambda^H; P^H]$, and this is the case if and only if $\eta_1 = \eta_2$. Moreover,

$$[(\eta_1, f_1, \xi_1), A] \,\mathcal{L}\, [(\eta_2, f_2, \xi_2), B]$$

if and only if ${}^{A^{-1}}(\eta_1, f_1, \xi_1) \,\mathcal{L}\, {}^{B^{-1}}(\eta_2, f_2, \xi_2)$ in $\mathcal{M}[N^H; I^H, \Lambda^H; P^H]$, and this is the case if and only if ${}^{A^{-1}}\xi_1 = {}^{B^{-1}}\xi_2$. Thus we see that the $\mathcal{R}$-class of an element $[(\eta, f, \xi), A]$ depends only on η, while its $\mathcal{L}$-class depends only on ξ and A. Since the morphism φ sends $\mathcal{R}$-equivalent elements to $\mathcal{R}$-equivalent elements, and $\mathcal{L}$-equivalent elements to $\mathcal{L}$-equivalent elements, we obtain that, for each $i \in I$, there exists $\eta_i \in I^H$, and for each $(A, \lambda) \in H \times \Lambda$, there exists $\xi_{A,\lambda} \in \Lambda^H$, such that, for every $g \in G$, we have

$$(i, g, \lambda)\varphi = [(\eta_i, f_g^{i\lambda}, \xi_{gN,\lambda}), gN]$$

for some $f_g^{i\lambda} \in N^H$.

Since φ is a morphism, we have

$$[(\eta_i, f_g^{i\lambda}, \xi_{gN,\lambda}), gN][(\eta_j, f_h^{j\mu}, \xi_{hN,\mu}), hN] = [(\eta_i, f_{gp_{\lambda j}h}^{i\mu}, \xi_{ghN,\mu}), ghN]$$

for any $i, j \in I$, $g, h \in G$ and $\lambda, \mu \in \Lambda$. This equality holds if and only if

$$f_g^{i\lambda} \cdot p^H_{\xi_{gN,\lambda}, {}^{gN}\eta_j} \cdot {}^{gN}f_h^{j\mu} = f_{gp_{\lambda j}h}^{i\mu} \tag{3.1.5}$$

for any $i, j \in I$, $g, h \in G$ and $\lambda, \mu \in \Lambda$, and

$${}^{gN}\xi_{hN,\mu} = \xi_{ghN,\mu} \tag{3.1.6}$$

for any $g, h \in G$ and $\mu \in \Lambda$. Notice that (3.1.6) is equivalent to requiring that

$$\xi_{A,\mu} = {}^{A}\xi_{N,\mu}$$

for every $\mu \in \Lambda$ and $A \in H$. Therefore, later on, we shortly write ξ_μ and ${}^A\xi_\mu$ instead of $\xi_{N,\mu}$ and $\xi_{A,\mu}$, respectively.

By (3.1.1), equality (3.1.5) is equivalent to

$$f^{i\mu}_{gp_{\lambda j}h} = f^{i\lambda}_g \cdot {}^{gN}p^H_{\xi_\lambda \eta_j} \cdot {}^{gN}f^{j\mu}_h. \tag{3.1.7}$$

Substituting $g = h = 1$ and $g = p^{-1}_{\lambda i}$, $h = 1$, $j = i$, respectively, where 1 denotes the identity element of N, we obtain from (3.1.7) that

$$f^{i\mu}_{p_{\lambda j}} = f^{i\lambda}_1 p^H_{\xi_\lambda \eta_j} f^{j\mu}_1, \tag{3.1.8}$$

$$f^{i\mu}_1 = f^{i\mu}_{p^{-1}_{\lambda i} p_{\lambda i} 1} = f^{i\lambda}_{p^{-1}_{\lambda i}} p^H_{\xi_\lambda \eta_i} f^{i\mu}_1,$$

and the latter implies

$$f^{i\lambda}_{p^{-1}_{\lambda i}} = (p^H_{\xi_\lambda \eta_i})^{-1}. \tag{3.1.9}$$

If $p_{\lambda i} = 1$ then the map

$$\iota_{i\lambda}\colon G \to N^H \rtimes H, \quad g \mapsto (p^H_{\xi_\lambda \eta_i} f^{i\lambda}_g, gN) \tag{3.1.10}$$

is an injective group morphism. For, it is injective since φ is injective, and by (3.1.7), we have $p^H_{\xi_\lambda \eta_i} f^{i\lambda}_{gh} = p^H_{\xi_\lambda \eta_i} f^{i\lambda}_{gp_{\lambda i}h} = p^H_{\xi_\lambda \eta_i} f^{i\lambda}_g \cdot {}^{gN}(p_{\xi_\lambda \eta_i} f^{i\lambda}_h)$, and so

$$(p^H_{\xi_\lambda \eta_i} f^{i\lambda}_g, gN)(p^H_{\xi_\lambda \eta_i} f^{i\lambda}_h, hN) = (p^H_{\xi_\lambda \eta_i} f^{i\lambda}_{gh}, ghN).$$

We now give a suitable group G, a normal subgroup N of G and a Rees matrix semigroup $S = \mathcal{M}[G; I, \Lambda; P]$ for which no such injective morphism φ exists.

Let G be the non-commutative group of order 21. To ease our calculations, we present G in the form $G = \mathbb{Z}_7 \rtimes [\overline{2}]$ where $\mathbb{Z}_7$ is the additive group of the ring of residues modulo 7, $[\overline{2}] = \{\overline{1}, \overline{2}, \overline{4}\}$ is the subgroup of the (multiplicative) group of units of the same ring generated by $\overline{2}$, and $[\overline{2}]$ acts on $\mathbb{Z}_7$ by multiplication. The second projection of G is a morphism onto $[\overline{2}]$, its kernel is $N = \{(a, \overline{1}) : a \in \mathbb{Z}_7\}$ isomorphic to $\mathbb{Z}_7$, and the map $H = G/N \to [\overline{2}]$, $(a, k)N \mapsto k$ is an isomorphism. For our later convenience, we identify H with $[\overline{2}]$ via this isomorphism. Let $I = \Lambda = \{1, 2\}$, and denote by P the normalized sandwich matrix of type $\Lambda \times I$ over G consisting of the elements $p_{11} = p_{12} = p_{21} = (\overline{0}, \overline{1})$, the identity element of N, and $p_{22} = (\overline{1}, \overline{1}) \in N$, an element of order 7.

Now we assume that φ is an embedding of the form (3.1.4) from this Rees matrix semigroup S, and apply the general properties deduced so far for this S.

The elements of order 3 in G and $N^H \rtimes H$ play crucial role in our argument. Observe that $(\overline{0}, \overline{2})$ and $(\overline{0}, \overline{4})$ are mutual inverse elements of G of order 3. Moreover all the elements of order 3 in $N^H \rtimes H$ are of the form $(t, \overline{2})$ or $(t, \overline{4})$. Let us mention, although we do not need it explicitly, that $(t, \overline{2})$ and $(t, \overline{4})$ are of order 3 if and only if $(\overline{1}t) \cdot (\overline{2}t) \cdot (\overline{4}t) = (\overline{0}, \overline{1})$.

Applying the injective group morphism $\iota_{11} \colon G \to N^H \rtimes H$ defined in (3.1.10), we see that $p_{22}\iota_{11} = (h, \overline{1})$ with $h = p^H_{\xi_1\eta_1} f^{11}_{p_{22}}$. Since the image of an element of order 3 has order 3, the following two cases occur:

Case 1: $(\overline{0}, \overline{4})\iota_{11} = (t, \overline{4})$. Then we obtain

$$((\overline{0}, \overline{4})^{-1} p_{22} (\overline{0}, \overline{4}))\iota_{11} = ({}^{\overline{2}}(t^{-1}), \overline{2})(h, \overline{1})(t, \overline{4}) = ({}^{\overline{2}}(t^{-1}) \cdot {}^{\overline{2}}h \cdot {}^{\overline{2}}t, \overline{1}) = ({}^{\overline{2}}h, \overline{1}).$$

On the other hand, $(\overline{0}, \overline{4})^{-1} p_{22} (\overline{0}, \overline{4}) = (\overline{0}, \overline{2})(\overline{1}, \overline{1})(\overline{0}, \overline{4}) = (\overline{2}, \overline{1}) = (\overline{1}, \overline{1})^2 = p_{22}^2$, and so $p_{22}^2 \iota_{11} = (h, \overline{1})(h, \overline{1}) = (h^2, \overline{1})$. Thus ${}^{\overline{2}}h = h^2$ which implies, for any $a \in H$, that $a({}^{\overline{2}}h) = a(h^2)$, whence $(\overline{2}a)h = ah \cdot ah = (ah)^2$. Consequently, $\overline{2}h = (\overline{1}h)^2$ and $\overline{4}h = (\overline{2}h)^2 = (\overline{1}h)^4$. Since h is not the identity element of the group N^H, we deduce that $\overline{1}h \neq (\overline{0}, \overline{1})$, the identity element of N. Since N is a cyclic group of order 7, we have $\overline{1}h \neq \overline{2}h$, $\overline{1}h \neq \overline{4}h$ and $\overline{2}h \neq \overline{4}h$. This means that h is injective, and its image does not contain $(\overline{0}, \overline{1})$.

Case 2: $(\overline{0}, \overline{4})\iota_{11} = (t, \overline{2})$. A similar argument shows that $\overline{2}h = (\overline{1}h)^4$ and $\overline{4}h = (\overline{1}h)^2$, and we again deduce that h is injective, and its image does not contain $(\overline{0}, \overline{1})$.

By (3.1.8) and (3.1.9), we have

$$f^{11}_{p_{22}} = f^{12}_{(\overline{0},\overline{1})} p^H_{\xi_2\eta_2} f^{21}_{(\overline{0},\overline{1})} = (p^H_{\xi_2\eta_1})^{-1} p^H_{\xi_2\eta_2} (p^H_{\xi_1\eta_2})^{-1},$$

and so

$$h = p^H_{\xi_1\eta_1} f^{11}_{p_{22}} = p^H_{\xi_1\eta_1} (p^H_{\xi_2\eta_1})^{-1} p^H_{\xi_2\eta_2} (p^H_{\xi_1\eta_2})^{-1}. \tag{3.1.11}$$

This means that we can express h as a product of entries in P^H and their inverses, which sit at the intersections of two rows and two columns. By the definition of P^H, for any $a \in H$ and for any $i, j \in \{1, 2\}$, we have

$$ap^H_{\xi_i\eta_j} = p_{a\xi_i, a\eta_j}.$$

Hence the image of each entry of P^H is contained in $\{(\overline{0}, \overline{1}), p_{22}\}$, and

$$ap^H_{\xi_i\eta_i} = p_{22} \text{ if and only if } a\xi_i = a\eta_j = 2.$$

Consequently, for any $a \in H$,

$$ap^H_{\xi_1\eta_1} = ap^H_{\xi_2\eta_2} = p_{22} \text{ if and only if } ap^H_{\xi_2\eta_1} = ap^H_{\xi_1\eta_2} = p_{22}.$$

Hence we see that it is impossible that two of the four entries $p^H_{\xi_1\eta_1}, p^H_{\xi_2\eta_1}$, $p^H_{\xi_2\eta_2}, p^H_{\xi_1\eta_2}$ sitting neither in the same row nor in the same column assign p_{22} to some $a \in H$. For, in this case, (3.1.11) would imply $ah = (\overline{0}, \overline{1})$, contradicting the property deduced above that the image of h does not contain $(\overline{0}, \overline{1})$. Consequently, for any $a \in H$, at least two of the four entries $p^H_{\xi_1\eta_1}, p^H_{\xi_2\eta_1}, p^H_{\xi_2\eta_2}, p^H_{\xi_1\eta_2}$ assign $(\overline{0}, \overline{1})$ to a, and if precisely two then the respective entries sit either in the same row or in the same column of P^H. So, by (3.1.11), we have $ah \in \{(\overline{0}, \overline{1}), p_{22}, p_{22}^{-1}\}$ for any $a \in H$, contradicting the fact that $ah \neq (\overline{0}, \overline{1})$ and h is injective. This completes the proof that there is no embedding (3.1.4) in the case of S considered, thus proving the following result.

Theorem 3.1.2. *There exists a completely simple semigroup which is an extension of a completely simple semigroup U by a group H and which is not embeddable in the wreath product of U by H.*

3.2 Embeddability in a semidirect product

In the previous section, we established that the Kaloujnine–Krasner Theorem does not generalize for extensions of completely simple semigroups by groups. In this section, we present a modified version of the Kaloujnine–Krasner Theorem which holds for all extensions of completely simple semigroups by groups.

Let S be an extension of a completely simple semigroup U by a group H. Our goal is to give an embedding of S into a semidirect product $V \rtimes H$ of a completely simple semigroup V by H such that, in the special case where S is a group (i.e., I and Λ are singletons), it is just the embedding in (3.1.2). Unlike in the wreath product $U \wr H$, in this semidirect product $V \rtimes H$ the $\mathcal{R}$- and $\mathcal{L}$-classes of V, its sandwich matrix and the action of H on V can be chosen appropriately.

Theorem 3.2.1. *Any extension of a completely simple semigroup U by a group H is embeddable in a semidirect product of a completely simple semigroup V by the group H, where the maximal subgroups of V are direct powers of the maximal subgroups of U.*

Proof. Let S be an extension of U by H. As above, we can assume that $S = \mathcal{M}[G; I, \Lambda; P]$ where the sandwich matrix P is normalized, and by

Proposition 2.1.2, there is a normal subgroup N of G such that every entry of P belongs to N, and $H = G/N$, $U = \mathcal{M}[N; I, \Lambda; P] \subseteq S$. Consider the action of H on N^H defining the wreath product $N \wr H$, and, for any $g \in G$, the map $f_g \in N^H$ defined in (3.1.2).

By means of S, we define a suitable semigroup V, an action of H on V, and an embedding of S into the semidirect product of V by H. Let $V = \mathcal{M}[N^H; I, H \times \Lambda; Q]$, where the entries of Q belong to the direct power N^H: for any $(B, \lambda) \in H \times \Lambda$ and $j \in I$, let

$$q_{(B,\lambda),j} = {}^{B}f_{p_{\lambda j}}.$$

Define an action of H on $H \times \Lambda$ by the rule ${}^{A}(B, \lambda) = (A \cdot B, \lambda)$ $((B, \lambda) \in H \times \Lambda,\ A \in H)$. Now we give an action of H on V as follows: for any $A \in H$ and $(i, f, (B, \lambda)) \in V$, let

$${}^{A}(i, f, (B, \lambda)) = (i, {}^{A}f, {}^{A}(B, \lambda)).$$

For any $A \in H$ and $(i, f, (B, \lambda)), (j, f', (C, \mu)) \in V$, we have

$$\begin{aligned}
{}^{A}(i, f, (B, \lambda)) \cdot {}^{A}(j, f', (C, \mu)) &= (i, {}^{A}f, (A \cdot B, \lambda))(j, {}^{A}f', (A \cdot C, \mu)) \\
&= (i, {}^{A}f \cdot q_{(A \cdot B,\lambda),j} \cdot {}^{A}f', (A \cdot C, \mu)) \\
&= (i, {}^{A}f \cdot {}^{A}q_{(B,\lambda),j} \cdot {}^{A}f', {}^{A}(C, \mu)) \\
&= {}^{A}(i, f \cdot q_{(B,\lambda),j} \cdot f', (C, \mu)) \\
&= {}^{A}\big((i, f, (B, \lambda))(j, f', (C, \mu))\big).
\end{aligned}$$

Hence this is a well-defined action of H on V, and so the semidirect product $V \rtimes H = \mathcal{M}[N^H; I, H \times \Lambda; Q] \rtimes H$ with respect to this action is defined.

Let us consider the mapping

$$\psi\colon \mathcal{M}[G; I, \Lambda; P] \to \mathcal{M}[N^H; I, H \times \Lambda; Q] \rtimes H,$$

where

$$(i, g, \lambda)\psi = ((i, f_g, (gN, \lambda)), gN).$$

We intend to verify that ψ is an embedding. Assume that $(i, g, \lambda)\psi = (j, h, \mu)\psi$, i.e., $(i, f_g, (gN, \lambda)), gN) = (j, f_h, (hN, \mu)), hN)$. Hence $i = j$, $\lambda = \mu$, $gN = hN$ and $f_g = f_h$. Since κ in (3.1.2) is injective, the last two equalities imply $g = h$, and so $(i, g, \lambda) = (j, h, \mu)$ follows.

To prove that ψ is a morphism, we can see for any $(i, g, \lambda), (j, h, \mu) \in \mathcal{M}[G; I, \Lambda; P]$, that

$$\begin{aligned}
(i, g, \lambda)\psi(j, h, \mu)\psi &= ((i, f_g, (gN, \lambda)), gN)((j, f_h, (hN, \mu)), hN) \\
&= ((i, f_g, (gN, \lambda)) \cdot {}^{gN}(j, f_h, (hN, \mu)), ghN) \\
&= ((i, f_g, (gN, \lambda))(j, {}^{gN}f_h, (ghN, \mu)), ghN) \\
&= ((i, f_g \cdot q_{(gN,\lambda),j} \cdot {}^{gN}f_h, (ghN, \mu)), ghN),
\end{aligned}$$

and

$$\big((i, g, \lambda)(j, h, \mu)\big)\psi = (i, gp_{\lambda j}h, \mu)\psi = ((i, f_{gp_{\lambda j}h}, (gp_{\lambda j}hN, \mu)), gp_{\lambda j}hN).$$

We need to prove that the two maps in the middle components are equal. Since $p_{\lambda j} \in N$ and N is the identity element of H, (3.1.3) implies by the definition of Q that

$$\begin{aligned}
f_{gp_{\lambda j}h} &= f_g \cdot {}^{gN}f_{p_{\lambda j}h} \\
&= f_g \cdot {}^{gN}(f_{p_{\lambda j}} \cdot {}^{p_{\lambda j}N}f_h) \\
&= f_g \cdot {}^{gN}(f_{p_{\lambda j}} \cdot f_h) \\
&= f_g \cdot {}^{gN}f_{p_{\lambda j}}{}^{gN}f_h \\
&= f_g \cdot q_{(gN,\lambda),j} \cdot {}^{gN}f_h.
\end{aligned}$$

Thus ψ is, indeed, an embedding, and the proof of the theorem is complete. $\square$

Note that, in the case where S is a group, i.e., where I and Λ are singletons (and so the single entry of P is the identity of G, and S is isomorphic to G), the map ψ coincides with the embedding κ in (3.1.2).

Chapter 4

Extensions of completely simple semigroups by inverse semigroups

4.1 Main result

The aim of this chapter is to prove the main result of the thesis.

Theorem 4.1.1. *Let S be an E-solid locally inverse semigroup and ϱ an inverse semigroup congruence on S such that the idempotent classes of ϱ are completely simple subsemigroups in S. Then the extension (S, ϱ) can be embedded into a λ-semidirect product extension of a completely simple by S/ϱ.*

Recall that, in an E-solid semigroup, the idempotent congruence classes of the least inverse semigroup congruence are completely simple subsemigroups, see [32]. Taking into account Proposition 2.2.2 and that both classes of E-solid and of locally inverse semigroups are closed under taking regular subsemigroups, we immediately deduce the following characterization of E-solid locally inverse semigroups.

Corollary 4.1.2. *A regular semigroup is E-solid and locally inverse if and only if it is embeddable in a λ-semidirect product of a completely simple semigroup by an inverse semigroup.*

In particular, this statement provides a structure theorem that constructs E-solid locally inverse semigroups from completely simple and inverse semigroups by means of two fairly simple constructions: forming λ-semidirect product and taking regular subsemigroup.

Note that, when restricting our attention to inverse semigroups, the extensions considered in Theorem 4.1.1 are just the idempotent separating extensions. Thus the following weaker version of the main result of [9] easily follows from Theorem 4.1.1.

Corollary 4.1.3. *If S is an inverse semigroup and ϱ an idempotent separating congruence on S then the extension (S, ϱ) can be embedded into a λ-semidirect product extension of a group by S/ϱ.*

4.2 Construction

In this section the canonical construction of [23] is adapted to derive an embeddability criterion for the extensions considered in the thesis.

Throughout this section, let (S, ϱ) be an extension by an inverse semigroup where S is an E-solid locally inverse semigroup and ϱ is over the class $\mathcal{CS}$ of all completely simple semigroups. For brevity, denote the factor semigroup S/ϱ by T and its elements by lower case Greek letters. Recall that $\operatorname{Ker}\varrho$ is a strong semilattice of completely simple semigroups.

By making use of Result 2.1.3, it is routine to extend a well-known property of strong semilattices of completely simple semigroups (cf. [26, Lemma II.4.6(ii) and Theorem IV.1.6(iii),(iv)]) to E-solid locally inverse semigroups as follows.

Lemma 4.2.1. *For every $\alpha, \beta \in T$ with $\alpha \geq \beta$, and for every $s \in \alpha$, there exists a unique $t \in \beta$ such that $s \geq t$.*

Proof. Recall that $K = \operatorname{Ker}\varrho$ is a full regular subsemigroup in S, and so the rule $R \mapsto R^K = R \cap K$ determines a bijection from the set of $\mathcal{R}$-classes of S onto the set of $\mathcal{R}$-classes of K. Let $\alpha, \beta \in T$ with $\alpha \geq \beta$, and let $s \in \alpha$, $s' \in V(s)$. Then $(ss')\varrho = s\varrho(s\varrho)^{-1} = \alpha\alpha^{-1} \geq \beta\beta^{-1}$, and $R_s^K = R_{ss'}^K$ is an $\mathcal{R}$-class in the completely simple subsemigroup $\alpha\alpha^{-1}$ of K. Since K is a strong semilattice of the completely simple semigroups ϵ ($\epsilon \in E(T)$), there is a unique $\mathcal{R}$-class R^K of the completely simple semigroup $\beta\beta^{-1}$ such that $R^K \leq R_s^K$. In fact, R^K is the $\mathcal{R}$-class of $\beta\beta^{-1}$ containing the unique idempotent e of K (or, equivalently, of S) such that $e \in \beta\beta^{-1}$ and $e \leq ss'$. The inequality $R^K \leq R_s^K$ implies $R \leq R_s$, R being the $\mathcal{R}$-class of S containing

R^K. Finally, we deduce by Proposition 2.1.3 that there is a unique $t \in R$ with $t \le s$. This proves the existence of t. Uniqueness also follows if we observe that $t \le s$ in S and $t\varrho(t\varrho)^{-1} = \beta\beta^{-1}$ imply that, for any $t' \in V(t)$, we have $R_{tt'} = R_t \le R_s = R_{ss'}$, whence $R^K_{tt'} \le R^K_{ss'}$ follows. Furthermore, $R^K_{tt'}$ is an $\mathcal{R}$-class of the completely simple semigroup $\beta\beta^{-1}$, and so $R^K_{tt'} = R^K$. $\square$

Now we recall the canonical construction of [23] and adapt it to our purposes.

First we define the *derived semigroupoid* $\mathcal{C}$ corresponding to the extension (S,ϱ) as follows. Let $\operatorname{Obj}\mathcal{C} = T$ and, for any $\alpha, \beta \in T$, let

$$\mathcal{C}(\alpha,\beta) = \big\{(\alpha,s,\beta) \in T \times S \times T : \alpha \cdot s\varrho = \beta \text{ and } \beta \cdot (s\varrho)^{-1} = \alpha\big\}.$$

Therefore $\alpha(a) = \alpha$ and $\omega(a) = \beta$ for any arrow $a = (\alpha,s,\beta) \in \operatorname{Arr}\mathcal{C}$. Composition is defined in $\mathcal{C}$ in the following manner: if $(\alpha,s,\beta) \in \mathcal{C}(\alpha,\beta)$ and $(\beta,t,\gamma) \in \mathcal{C}(\beta,\gamma)$ then

$$(\alpha,s,\beta) \circ (\beta,t,\gamma) = (\alpha,st,\gamma).$$

Clearly, this operation is associative, and so $\mathcal{C}$ forms a semigroupoid. Furthermore, by putting $\ell(a) = s$ for every arrow $a = (\alpha,s,\beta) \in \operatorname{Arr}\mathcal{C}$, we define a labelling of $\mathcal{C}$ by S. Since S is a regular semigroup, $\mathcal{C}$ is a regular semigroupoid, in which $V\big((\alpha,s,\beta)\big) = \{(\beta,s^*,\alpha) : s^* \in V(s)\}$. Hence, for every $\alpha \in T$, the semigroup $\mathcal{C}(\alpha,\alpha)$ is easily seen to be regular and isomorphic to a subsemigroup of the locally inverse semigroup S. Therefore $\mathcal{C}$ is a locally inverse semigroupoid, and so the sandwich operation $\curlywedge$ is also defined, and the natural partial order of $\mathcal{C}$, where $a \le b$ for any $a,b \in \operatorname{Arr}\mathcal{C}$ if and only if $\alpha(a) = \alpha(b)$, $\omega(a) = \omega(b)$ and $\ell(a) \le \ell(b)$, is compatible both with $\circ$ and $\curlywedge$.

Consider the graphs $\mathcal{C}'$ and $\overline{\mathcal{C}}$ corresponding to $\mathcal{C}$, and put $A = \operatorname{Arr}\mathcal{C}$, $A' = \operatorname{Arr}\mathcal{C}'$. Then we have $\overline{A} = A \cup A' = \operatorname{Arr}\overline{\mathcal{C}}$.

Let us choose and fix an inverse unary operation † on S. This determines an inverse unary operation, also denoted by †, on $\mathcal{C}$ by letting $(\alpha,s,\beta)^\dagger = (\beta,s^\dagger,\alpha)$ for every $(\alpha,s,\beta) \in \operatorname{Arr}\mathcal{C}$. Consider the congruence θ on the free binary semigroup $F_{\langle 2,2\rangle}\big(\overline{A}\big)$ generated by

$$\Theta(\mathcal{CS},A) \cup \Xi_1 \cup \Xi_2$$

where $\Theta(\mathcal{CS},A)$ is the bi-invariant congruence on $F_{\langle 2,2\rangle}\big(\overline{A}\big)$ corresponding to $\mathcal{CS}$ (see Result 2.3.3), and

$$\begin{aligned} \Xi_1 &= \big\{(a',a^\dagger) : a \in A\big\}, \\ \Xi_2 &= \big\{(ab,c) : a,b,c \in A \text{ and } a \circ b = c \ \text{ in } \mathcal{C}\big\}. \end{aligned}$$

The factor semigroup $F_{\langle 2,2\rangle}\left(\overline{A}\right)/\theta$ is clearly isomorphic to the factor semigroup $\mathrm{BF}\mathcal{CS}(A)/\nu$ where ν is the congruence on $\mathrm{BF}\mathcal{CS}(A)$ generated by $\Xi_1 \cup \Xi_2$. In [23], this is the first factor of the λ-semidirect product constructed to embed the extension (S, ϱ) into. Moreover, it is proved that, up to isomorphism, $\mathrm{BF}\mathcal{CS}(A)/\nu$ is independent of the choice of the inverse unary operation of S involved in the construction.

Now we apply the idea of replacing each term of $F_{\langle 2,2\rangle}\left(\overline{A}\right)$ by the word belonging to $\widetilde{A}^+$ which is obtained from it by applications of (R0). Proposition 2.3.3, Proposition 2.3.4 and Lemma 2.3.5 imply that the completely simple semigroup $F_{\langle 2,2\rangle}\left(\overline{A}\right)/\theta$ is isomorphic to $\widetilde{A}^+/\widetilde{\theta}$ where $\widetilde{\theta}$ is the semigroup congruence generated by $\mathrm{I} \cup \Upsilon \cup \widetilde{\Xi}_1 \cup \widetilde{\Xi}_{21} \cup \widetilde{\Xi}_{22}$ where

$$\begin{aligned}
\widetilde{\Xi}_1 &= \Xi_1,\\
\widetilde{\Xi}_{21} &= \{(ab, c) : a, b, c \in A, \text{ either } a \text{ and } b \text{ or } c \text{ are letter factors,}\\
&\qquad \text{and } a \circ b = c \ \text{ in } \mathcal{C}\},\\
\widetilde{\Xi}_{22} &= \left\{\left((a \wedge y), (c \wedge y)\right) : a, c \in A,\ y \in \overline{A},\ \text{and } a \circ b = c \ \text{ in } \mathcal{C}\right.\\
&\qquad \left.\text{for some } b \in A\right\}\\
&\quad \cup \left\{\left((y \wedge b), (y \wedge c)\right) : b, c \in A,\ y \in \overline{A},\ \text{and } a \circ b = c \ \text{ in } \mathcal{C}\right.\\
&\qquad \left.\text{for some } a \in A\right\}.
\end{aligned}$$

By the well-known description of a semigroup congruence generated by a relation, we easily deduce the following lemma.

Lemma 4.2.2. *Let u, v be words in $\widetilde{A}^+$. Then $(u, v) \in \widetilde{\theta}$ if and only if there exists a finite sequence of words $u = w_0, w_1, \ldots, w_n = v$ in $\widetilde{A}^+$ such that, for any i ($i = 0, 1, \ldots, n-1$), the word w_{i+1} is obtained from w_i by one of the following steps:*

(Sja) *replacing a section s in w_i by t,*

(Sjb) *replacing a section t in w_i by s*

where $j = 1, 21, 22$, $(s, t) \in \widetilde{\Xi}_j$, and

(Tja) *replacing a section s in w_i by t,*

(Tjb) *replacing a section t in w_i by s,*

where $j = 3, 4, 5$, $(s, t) \in \Upsilon_j$.

Put $K = \widetilde{A}^+/\widetilde{\theta}$. Since $\widetilde{\Theta}(\mathcal{CS}, A) \subseteq \widetilde{\theta}$, we have $K \in \mathcal{CS}$. The equality

$$^{\pi}(\alpha, s, \beta) = (\pi\alpha, s, \pi\beta) \qquad (\pi \in T,\ (\alpha, s, \beta) \in A)$$

defines an action of T on the semigroupoid $\mathcal{C}$ in the sense that the following properties hold: $^{\pi}(a \circ b) = {}^{\pi}a \circ {}^{\pi}b$ for any $\pi \in T$ and any arrows $a, b \in A$ with $\omega(a) = \alpha(b)$, and also $^{\pi}({}^{\nu}a) = {}^{\pi\nu}a$ for every $\pi, \nu \in T$ and $a \in A$. Note that $^{\pi}(a \curlywedge b) = ({}^{\pi}a \curlywedge {}^{\pi}b)$ also holds for any $\pi \in T$ and any arrows $a, b \in A$ with $\alpha(a) = \omega(b)$ since morphisms of locally inverse semigroupoids respect $\curlywedge$. Moreover, it is also clear that $^{\pi}(a^{\dagger}) = ({}^{\pi}a)^{\dagger}$ for every $a \in A$ and $\pi \in T$. This ensures that, for every $\pi \in T$, the mapping $A \to A,\ a \mapsto {}^{\pi}a$ can be naturally extended to an endomorphism ε_{π} of K such that $\varepsilon_{\varsigma}\varepsilon_{\pi} = \varepsilon_{\pi\varsigma}$ holds for every $\pi, \varsigma \in T$. Therefore $\pi \mapsto \varepsilon_{\pi}$ defines an action ε of T on K by the rule $^{\pi}(u\widetilde{\theta}) = ({}^{\pi}u)\widetilde{\theta}\quad (\pi \in T,\ u \in \widetilde{A}^+)$ where $^{\pi}u$ is the word obtained from u by replacing each letter a of u by $^{\pi}a$.

Consider the λ-semidirect product $K \rtimes_{\lambda} T$ determined by this action, and define a mapping

$$\kappa\colon S \to K \rtimes_{\lambda} T \quad \text{by} \quad s \mapsto \big((s\varrho(s\varrho)^{-1}, s, s\varrho)\widetilde{\theta}, s\varrho\big).$$

It is easily seen that κ is a homomorphism, and the congruence induced by $\kappa\pi_2$ is just ϱ. Furthermore, the following important property of κ is implied by the main result of [23]:

Result 4.2.3. *Let S be an E-solid locally inverse semigroup, and let ϱ be an inverse semigroup congruence on S over $\mathcal{CS}$. Then the extension (S, ϱ) is embeddable in a λ-semidirect product extension of a completely simple semigroup by an inverse semigroup if and only if κ is an embedding, or, equivalently, if and only if the relations $s\,\varrho\,t$ in S and $(s\varrho(s\varrho)^{-1}, s, s\varrho)\,\widetilde{\theta}\,(t\varrho(t\varrho)^{-1}, t, t\varrho)$ in $\widetilde{A}^+$ imply $s = t$ for every $s, t \in S$.*

In the next section we apply this result to prove our main result Theorem 4.1.1. By Lemma 4.2.2, this is based on the study of the a finite sequences $u = w_0, w_1, \ldots, w_n = v$ of words in $\widetilde{A}^+$ where, for every i ($i = 0, 1, \ldots, n-1$), the word w_{i+1} is obtained from w_i by one of the steps (Sja), (Sjb) with $j = 1, 21, 22$ and (Tja), (Tjb) with $j = 3, 4, 5$. Later on, such a sequence $u = w_0, w_1, \ldots, w_n = v$ is called a *$\mathcal{CS}$-derivation from u to v in $\widetilde{A}^+$.*

4.3 Proof of Theorem 4.1.1

This section is devoted to proving that the homomorphism κ introduced in the previous section is, indeed, an embedding.

Let (S, ϱ) be an extension where S is an E-solid locally inverse semigroup and ϱ is an inverse semigroup congruence on S over $\mathcal{CS}$. Consider the construction — in particular, T, $\mathcal{C}$, $\overline{\mathcal{C}}$, A, $\overline{A}$, $\widetilde{A}^+$, †, $\widetilde{\theta}$, K and κ — corresponding to (S, ϱ) as introduced in the previous section.

Now we observe that adjacency in the semigroupoid $\mathcal{C}$ is closely related to Green's relation $\mathcal{R}$ in T, and that there is a crucial connection between the endpoints and the label of an arrow.

Lemma 4.3.1. *Let $(\alpha, s, \beta) \in T \times S \times T$.*

1. *We have $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$ if and only if $\alpha \,\mathcal{R}\, \beta$ in T and $s\varrho \geq \alpha^{-1}\beta$.*

2. *If $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$ then $s\varrho(s\varrho)^{-1} \geq \alpha^{-1}\alpha$ and $(s\varrho)^{-1}s\varrho \geq \beta^{-1}\beta$ in T.*

3. *If $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$ then the following properties are equivalent:*

   (a) $s\varrho = \alpha^{-1}\beta$,

   (b) $s\varrho(s\varrho)^{-1} = \alpha^{-1}\alpha$,

   (c) $(s\varrho)^{-1}s\varrho = \beta^{-1}\beta$.

Proof. (1) Let $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$; then $\alpha \cdot s\varrho = \beta$ and $\beta \cdot (s\varrho)^{-1} = \alpha$ in T. Hence we deduce $\alpha \mathcal{R} \beta$ and $\alpha^{-1}\alpha \cdot s\varrho = \alpha^{-1}\beta$, and so $s\varrho \geq \alpha^{-1}\beta$ is implied in the inverse semigroup T. Now let α and β be $\mathcal{R}$-related elements in T, that is, let $\alpha\alpha^{-1} = \beta\beta^{-1}$, and let $s \in S$ with $s\varrho \geq \alpha^{-1}\beta$. We clearly have $\alpha \cdot s\varrho \geq \alpha\alpha^{-1}\beta = \beta\beta^{-1}\beta = \beta$ and $\beta \cdot (s\varrho)^{-1} \geq \beta(\alpha^{-1}\beta)^{-1} = \beta\beta^{-1}\alpha = \alpha\alpha^{-1}\alpha = \alpha$, whence $\alpha \geq \alpha \cdot s\varrho(s\varrho)^{-1} \geq \beta \cdot (s\varrho)^{-1}$ and $\beta \geq \beta \cdot (s\varrho)^{-1}s\varrho \geq \alpha \cdot s\varrho$ follow, respectively. So, by the definition of $\mathcal{C}$, we see that $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$.

(2) If $(\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$ then (1) implies $s\varrho(s\varrho)^{-1} \geq \alpha^{-1}\beta(\alpha^{-1}\beta)^{-1} = \alpha^{-1}\beta\beta^{-1}\alpha = \alpha^{-1}\alpha\alpha^{-1}\alpha = \alpha^{-1}\alpha$, and similarly, $(s\varrho)^{-1}s\varrho \geq \beta^{-1}\beta$.

(3) Straightforward. □

Lemma 4.3.1(2),(3) indicate that some arrows in $\mathcal{C}$ are special in the sense that their labels are 'as low as possible'. An arrow (α, s, β) in $\mathcal{C}$ having the property that $s\varrho = \alpha^{-1}\beta$ (cf. (3)) is termed *stable*. Consider the subgraph $\widehat{\mathcal{C}}$ of $\mathcal{C}$ where $\operatorname{Obj}\widehat{\mathcal{C}} = \operatorname{Obj}\mathcal{C}$ and $\operatorname{Arr}\widehat{\mathcal{C}}$ consists of all stable arrows of $\mathcal{C}$.

Lemma 4.3.2. *Let a, b be consecutive arrows in $\mathcal{C}$.*

1. *If $a \in \operatorname{Arr}\widehat{\mathcal{C}}$ then each inverse of a is in $\operatorname{Arr}\widehat{\mathcal{C}}$.*

2. *If $a \in \operatorname{Arr}\widehat{\mathcal{C}}$ then $a \circ b \in \operatorname{Arr}\widehat{\mathcal{C}}$ and $a \,\mathcal{R}\, a \circ b$.*

3. *If $a \in \operatorname{Arr}\widehat{\mathcal{C}}$ then $(b \curlywedge a) \in \operatorname{Arr}\widehat{\mathcal{C}}$ and $a \,\mathcal{L}\, (b \curlywedge a)$.*

Proof. (1) Clear by definition.

(2) Suppose that $a = (\alpha, s, \beta) \in \operatorname{Arr}\widehat{\mathcal{C}}$ and $b = (\beta, t, \gamma) \in \operatorname{Arr}\mathcal{C}$. Then $(s\varrho)^{-1}s\varrho = \beta^{-1}\beta$ by Lemma 4.3.1(3), and $\beta \cdot t\varrho(t\varrho)^{-1} = \beta$. Thus

$$\begin{aligned}
&(st)\varrho \cdot ((st)\varrho)^{-1} = \\
&\quad = s\varrho \cdot t\varrho \cdot (t\varrho)^{-1} \cdot (s\varrho)^{-1} = s\varrho \cdot \beta^{-1}\beta \cdot t\varrho(t\varrho)^{-1} \cdot (s\varrho)^{-1} \\
&\quad = s\varrho \cdot \beta^{-1}\beta \cdot (s\varrho)^{-1} = \alpha^{-1}\alpha.
\end{aligned}$$

Hence $a \circ b = (\alpha, st, \gamma)$ is stable. Moreover, if $s' \in V(s)$, $t' \in V(t)$ in S then Lemma 4.3.1(2),(3) imply $(s's)\varrho = \beta^{-1}\beta \leq (tt')\varrho$. Since $\beta^{-1}\beta$ is an idempotent ϱ-class which is a completely simple subsemigroup of S by assumption, we obtain that $(s'stt')\varrho = \beta^{-1}\beta$ and $s's \,\mathcal{R}\, s'stt'$. Since $\mathcal{R}$ is a left congruence, this implies $s \,\mathcal{R}\, stt' \,\mathcal{R}\, st$ whence $a \,\mathcal{R}\, a \circ b$ follows.

(3) The proof is similar to that of (2). □

An immediate consequence of this lemma is that $\widehat{\mathcal{C}}$ is a locally inverse subsemigroupoid in $\mathcal{C}$. Furthermore, we have the following important property of stable arrows:

Lemma 4.3.3. *For every arrow $a \in \operatorname{Arr}\mathcal{C}$, there is a unique stable arrow $b \in \operatorname{Arr}\widehat{\mathcal{C}}$ such that $b \leq a$.*

Proof. Let $a = (\alpha, s, \beta) \in \operatorname{Arr}\mathcal{C}$. If $b = (\alpha, t, \beta) \in \operatorname{Arr}\widehat{\mathcal{C}}$ with $b \leq a$ then, by definition, we have $t \leq s$ and $t\varrho = \alpha^{-1}\beta$. On the other hand, we see by Lemma 4.2.1 that there exists a unique $t \in S$ such that $t \leq s$ and $t\varrho = \alpha^{-1}\beta$. Lemma 4.3.1 ensures that, in this case, we have $b = (\alpha, t, \beta) \in \operatorname{Arr}\widehat{\mathcal{C}}$, and the proof is complete. □

For any arrow a, denote by $\widehat{a}$ the unique stable arrow b with $b \leq a$, and consider the graph morphism $\widehat{\ }\colon \mathcal{C} \to \widehat{\mathcal{C}}$ whose object mapping is identical and which assigns $\widehat{a}$ to a for every $a \in \operatorname{Arr}\mathcal{C}$. From now on, we put $\widehat{A} = \operatorname{Arr}\widehat{\mathcal{C}}$.

Observe, that the graph morphism $\widehat{\ }$ respects the operations of $\mathcal{C}$, that is, it constitutes a binary semigroupid morphism from $\mathcal{C}$ onto $\widehat{\mathcal{C}}$:

Lemma 4.3.4. *For any $a, b \in A$ with $\omega(a) = \alpha(b)$, we have*

1. $\widehat{\widehat{a}} = \widehat{a}$,
2. $\widehat{a \circ b} = \widehat{a} \circ \widehat{b}$,
3. $\widehat{(b \curlywedge a)} = (\widehat{b} \curlywedge \widehat{a})$.

Proof. (1) Straightforward by definition.

(2) By definition, we have $\widehat{a} \leq a$, $\widehat{b} \leq b$, and so $\widehat{a} \circ \widehat{b} \leq a \circ b$. Moreover, $\widehat{a} \circ \widehat{b}$ is stable by Lemma 4.3.2. Therefore $\widehat{a} \circ \widehat{b} = \widehat{a \circ b}$ follows by Lemma 4.3.3.

(3) The proof is similar to that of (2). $\square$

By making use of the inverse unary operation † on $\mathcal{C}$, we extend the graph morphism $\widehat{\ }\colon \mathcal{C} \to \widehat{\mathcal{C}}$ to a binary semigroupoid morphism from $F_{\langle 2,2 \rangle}(\overline{\mathcal{C}})$ in the way that we consider the graph morphism

$$\delta \colon \overline{\mathcal{C}} \to \widehat{\mathcal{C}},\ a\delta = \widehat{a} \text{ and } a'\delta = (\widehat{a})^\dagger \quad (a \in A),$$

and we define $\widehat{\ }\colon F_{\langle 2,2 \rangle}(\overline{\mathcal{C}}) \to \widehat{\mathcal{C}}$ to be the unique extension of δ to $F_{\langle 2,2 \rangle}(\overline{\mathcal{C}})$. The restriction of $\widehat{\ }$ to $\widetilde{\mathcal{C}}^+$, also denoted by $\widehat{\ }$, is obviously a semigroupoid morphism. In fact, it is also a binary semigroupoid morphism since each semigroup $\widehat{\mathcal{C}}(\alpha, \alpha)$ ($\alpha \in T$) is completely simple.

Now we turn to proving that κ is injective, that is, for every $s, t \in S$, the following implication holds:

$$s \mathrel{\varrho} t \quad \text{and} \quad \big(s\varrho(s\varrho)^{-1}, s, s\varrho\big) \mathrel{\widetilde{\theta}} \big(t\varrho(t\varrho)^{-1}, t, t\varrho\big) \quad \text{imply} \quad s = t. \qquad (4.3.1)$$

Let $s, t \in S$ with $s \mathrel{\varrho} t$ and, for brevity, put $a = (s\varrho(s\varrho)^{-1}, s, s\varrho)$ and $b = (t\varrho(t\varrho)^{-1}, t, t\varrho)$. Notice that a, b are coterminal arrows in $\mathcal{C}$, and, simultaneously, one-letter words in $\widetilde{A}^+$. Suppose that $a \mathrel{\widetilde{\theta}} b$. By Lemma 4.2.2, there exists a $\mathcal{CS}$-derivation

$$a = w_0, w_1, \ldots, w_n = b \qquad (4.3.2)$$

from a to b. We intend to prove that $a = b$ which clearly implies the equality $s = t$. The crucial point in the proof is to describe the special features of the words of $\widetilde{A}^+$ appearing in such derivations. Notice that derivation steps (T3b), (T4b) might introduce $\wedge$-letters which are not $\wedge$-loops. Consequently, w_i ($0 < i < n$) need not be a path in $\overline{\mathcal{C}}$. The idea of our description of the words in such derivations is to indicate the breaking points of these kinds and their ranges by pairs of brackets $\lceil . \rrbracket$ and $\llbracket . \rfloor$, respectively. For example, in the derivation

$$(a \wedge b), \quad (a \wedge c)(a \wedge b), \quad (a \wedge c)(d \wedge c)(a \wedge b),$$

where we apply rules (T3b) and (T4b), and $(a \wedge b)$ is a $\wedge$-loop but $(a \wedge c)$, $(d \wedge c)$ are not, we indicate the breaking points as follows:

$$(a \wedge b), \quad \lceil (a \wedge c) \rrbracket (a \wedge b), \quad \lceil (a \wedge c) \llbracket (d \wedge c) \rfloor \rrbracket (a \wedge b).$$

Now we introduce the set of words with brackets needed in our description. Consider the free monoid $(\widetilde{A} \cup \{\llbracket, \rfloor, \lceil, \rrbracket\})^*$ where the empty word is denoted ε, and let $\widetilde{W}$ be its smallest subset which has the following four properties:

(i) $\varepsilon \in \widetilde{W}$;

(ii) $a \in \widetilde{W}$ for all $a \in \widetilde{A}$;

(iii) $w_1 w_2 \in \widetilde{W}$ for all $w_1, w_2 \in \widetilde{W}$;

(iv) $\llbracket w \rfloor, \lceil w \rrbracket \in \widetilde{W}$ for all $w \in \widetilde{W}$, where $w \neq \varepsilon$.

Notice that $\widetilde{A}^+ \subseteq \widetilde{W}$. In order to distinguish the elements of $\widetilde{A}^+$, called words, from those of $\widetilde{W}$, the latter will be called *bracketed words*. Moreover, the elements of $\widetilde{A}$ will be called $\widetilde{A}$*-letters*. Recall that an $\widetilde{A}$-letter is either a letter of a $\wedge$-letter. If $w \in \widetilde{W}$ then $\mathrm{I}w$ $[w\mathrm{T}]$ denotes the first [last] element of $\widetilde{A} \cup \{\llbracket, \rfloor, \lceil, \rrbracket\}$ appearing in w (reading w from the left to the right as a word in this alphabet).

For our later convenience, we introduce the notation $w{\downarrow}$ for the subword of $w \in \widetilde{W}$ obtained from w by deleting all brackets. Clearly, $\varepsilon{\downarrow} = \varepsilon$ and if $w \neq \varepsilon$ then $w{\downarrow} \in \widetilde{A}^+$.

Now we define three subsets W_n, W_n^{right} and W_n^{left} of $\widetilde{W}$ for every $n \in \mathbb{N}_0$. Simultaneously, we attach a (binary) path $\wp(w) \in \operatorname{Arr} \widetilde{\mathcal{C}}^+$ to each element w of these subsets. If $\wp(w)$ is defined then we use $\widehat{\wp}(w)$ to denote $\widehat{\wp(w)}$. For technical reasons, we put $\wp(\varepsilon) = \varepsilon$ but let $\widehat{\wp}(\varepsilon)$ undefined.

Let $W_0 = \operatorname{Arr} \widetilde{\mathcal{C}}^+$, $W_0^{\varepsilon} = W_0 \cup \{\varepsilon\}$, and for any $w \in W_0$, define $\wp(w) = w$. Moreover, define

$$W_0^{\text{right}} = \{p(y \wedge x) : p \in W_0^{\varepsilon},\ \alpha(y) \neq \omega(x),\ \text{and } \omega(p) = \alpha(y) \text{ if } p \neq \varepsilon\},$$

and for any $w = p(y \wedge x) \in W_0^{\text{right}}$, let $\wp(w) = p(y \wedge y')$. By assumptions, this, indeed, belongs to $\operatorname{Arr}\widetilde{\mathcal{C}}^+$. Similarly, let

$$W_0^{\text{left}} = \{(x \wedge y)p : p \in W_0^{\varepsilon},\ \alpha(x) \neq \omega(y),\ \text{and } \omega(y) = \alpha(p) \text{ if } p \neq \varepsilon\},$$

and for any $w = (x \wedge y)p \in W_0^{\text{left}}$, let $\wp(w) = (y' \wedge y)p$. Notice that $W_0 \cup W_0^{\text{right}} \cup W_0^{\text{left}} \subseteq \widetilde{A}^+$.

Assume that W_n [W_n^{right}, W_n^{left}] is defined for some $n \in \mathbb{N}_0$, and a path $\wp(w) \in \operatorname{Arr}\widetilde{\mathcal{C}}^+$ is assigned to each of its elements w. For brevity, denote the set of all idempotent arrows of $\mathcal{C}$ by E. Define the set W_{n+1} [W_{n+1}^{right}, W_{n+1}^{left}] to consist of the bracketed words in W_n [W_n^{right}, W_n^{left}] and, additionally, of all bracketed words $w \in \widetilde{W}$ of the form

$$w = p_0 B_1 C_1 p_1 B_2 C_2 \cdots B_k C_k p_k \quad (k \in \mathbb{N}), \tag{4.3.3}$$

where the following conditions are satisfied:

(E0)

- (E0a) $p_1, \ldots, p_{k-1} \in W_0$, $p_0 \in W_0^{\varepsilon}$ [W_0^{ε}, W_0^{left}], $p_k \in W_0^{\varepsilon}$ [W_0^{right}, W_0^{ε}], and $\omega(p_{i-1}) = \alpha(p_i)$ for every i $(1 \leq i \leq k)$,
- (E0b) $B_1C_1, \ldots, B_kC_k \neq \varepsilon$;

(E1) for any i $(1 \leq i \leq k)$, we have

- (E1a) $B_i = \llbracket w_1 \rfloor \llbracket w_2 \rfloor \cdots \llbracket w_s \rfloor$, where $s \in \mathbb{N}_0$ and $w_j \in W_n^{\text{right}}$ $(1 \leq j \leq s)$, and
- (E1b) for any j $(1 \leq j \leq s)$, if $w_j\mathrm{T} = (y_j \wedge x_j)$ then
  - (E1bi) $\widehat{\wp}(w_j) \in E$ and $\widehat{y_j}\ \mathcal{R}\ \widehat{\wp}(w_j)$, and
  - (E1bii) $\widehat{x_j}\ \mathcal{L}\ \widehat{\wp}(p_{i-1})$ (in particular, $p_0 \neq \varepsilon$ if $B_1 \neq \varepsilon$);

(E2) for any i $(1 \leq i \leq k)$, we have

- (E2a) $C_i = \lceil w_1 \rrbracket \lceil w_2 \rrbracket \cdots \lceil w_s \rrbracket$, where $s \in \mathbb{N}_0$ and $w_j \in W_n^{\text{left}}$ $(1 \leq j \leq s)$, and
- (E2b) for any j $(1 \leq j \leq s)$, if $\mathrm{I}w_j = (x_j \wedge y_j)$ then
  - (E2bi) $\widehat{\wp}(w_j) \in E$ and $\widehat{y_j}\ \mathcal{L}\ \widehat{\wp}(w_j)$, and
  - (E2bii) $\widehat{x_j}\ \mathcal{R}\ \widehat{\wp}(p_i)$ (in particular, $p_k \neq \varepsilon$ if $C_k \neq \varepsilon$).

For every $w \in W_{n+1} \setminus W_n$ $[W_{n+1}^{\mathrm{right}} \setminus W_n^{\mathrm{right}},\ W_{n+1}^{\mathrm{left}} \setminus W_n^{\mathrm{left}}]$ of the form (4.3.3), define $\wp(w) = \wp(p_0)p_1 \cdots p_{k-1}\wp(p_k)$. Again, $\wp(w)$ is easily seen to belong to $\operatorname{Arr}\widetilde{\mathcal{C}}^+$ by (E0a) and by the definition of $\wp(w)$ for $w \in (W_{n+1}^{\mathrm{right}} \setminus W_n^{\mathrm{right}}) \cup (W_{n+1}^{\mathrm{left}} \setminus W_n^{\mathrm{left}})$. The less trivial part to check is that $\wp(w)$ is non-empty if $w = p_0B_1C_1p_1 \in W_{n+1} \setminus W_n$. However, since either B_1 or C_1 is non-empty by (E0b), we get by (E1b) or (E2b) that $p_0 \neq \varepsilon$ or $p_k \neq \varepsilon$, respectively, whence $\wp(w) \neq \varepsilon$ follows.

Finally, we define

$$W = \bigcup_{n=0}^{\infty} W_n, \quad W^{\mathrm{right}} = \bigcup_{n=0}^{\infty} W_n^{\mathrm{right}} \quad \text{and} \quad W^{\mathrm{left}} = \bigcup_{n=0}^{\infty} W_n^{\mathrm{left}}.$$

Alternatively, the bracketed words in $W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$ can be characterized as follows.

Lemma 4.3.5. *1. A bracketed word $w \in \widetilde{W}$ belongs to W $[W^{\mathrm{right}},\ W^{\mathrm{left}}]$ if and only if it is of the form*

$$w = p_0B_1C_1p_1B_2C_2 \cdots B_kC_kp_k \quad (k \in \mathbb{N}_0), \tag{4.3.4}$$

where either $k = 0$ and $p_0 \neq \varepsilon$, or the slightly modified versions of (E0)–(E2) are satisfied where n is deleted from 'W_n^{right}' and 'W_n^{left}' in (E1a) and (E2a), respectively. Moreover, this form of w is uniquely determined.

2. For any bracketed word $w \in W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$ of the form (4.3.4), we have

$$\wp(w) = \wp(p_0)p_1 \cdots p_{k-1}\wp(p_k).$$

Remark 4.3.6. Notice that the description of the bracketed words belonging to $W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$ which is formulated in Lemma 4.3.5(1) can be modified by deleting (E0b) from the properties required, but then the form obtained is no more uniquely determined.

Later on, when considering a bracketed word from $W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$, we always consider it in its form described in Lemma 4.3.5(1), but when checking whether a bracketed word belongs to $W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$, we disregard checking property (E0b).

Notice that the set W is self-dual in the sense that the reverse of each bracketed word of W belongs to W. E.g., the reverse of the bracketed word

$a\lceil(b\wedge c)\rrbracket(a'\wedge a)$ is $(a\wedge a')\llbracket(c\wedge b)\rfloor a$ and vice versa. Similarly, the sets W^{right} and W^{left} are dual to each other.

Besides bracketed words from $W\cup W^{\text{right}}\cup W^{\text{left}}$, we need also certain prefixes and suffixes of them which, due to properties (E1bii) and (E2bii), fail to belong to this set. Define $W^{\emptyset|}$ $[W^{\emptyset|\text{right}}]$ to consist of all non-empty bracketed words w of the form (4.3.4) where $p_0=\varepsilon$, $B_1\neq\varepsilon$, and w satisfies all conditions (E0)–(E2) for W $[W^{\text{right}}]$ but (E1bii) in case $i=1$. Notice that $\wp(w)$ can be also defined for any $w\in W^{\emptyset|}$ $[W^{\emptyset|\text{right}}]$ in the same way as it was done for bracketed words in W $[W^{\text{right}}]$, but this time $\wp(w)$ might be empty. Clearly, we have $\wp(w)=\varepsilon$ if and only if $k=1$ and $p_1=C_1=\varepsilon$. Dually, we define the set of bracketed words $W^{|\emptyset}$ $[W^{\text{left}|\emptyset}]$.

Given a bracketed word $w\in W\cup W^{\text{right}}\cup W^{\text{left}}$ of the form (4.3.4), the following non-empty sections of w are called *$\widetilde{W}$-suffixes of w of type (a), (b) and (c)*, respectively:

(a) $p_{i2}B_{i+1}C_{i+1}\cdots p_{k-1}B_kC_kp_k$ $(0\le i\le k)$ where p_{i2} is a non-empty suffix of p_i,

(b) $C_{i2}p_i\cdots B_kC_kp_k$ $(1\le i\le k)$ where $C_{i2}=\lceil w_t\rrbracket\lceil w_{t+1}\rrbracket\cdots\lceil w_s\rrbracket$ $(1\le t\le s)$ provided C_i is of the form (E2a) with $s\neq 0$,

(c) $B_{i2}C_i\cdots B_kC_kp_k$ $(1\le i\le k)$ where $B_{i2}=\llbracket w_t\rfloor\llbracket w_{t+1}\rfloor\cdots\llbracket w_s\rfloor$ $(1\le t\le s)$ provided B_i is of the form (E1a) with $s\neq 0$.

It is obvious that a $\widetilde{W}$-suffix of w is of the form (a), (b) and (c) if and only if its first $\widetilde{A}$-letter belongs to p_i, C_i and B_i, respectively. The $\widetilde{W}$-prefixes of w of type (a), (b) and (c) are defined dually. The following statement is straightforward to check by definition.

Lemma 4.3.7. *Let $w\in W$ $[W^{\text{right}},\ W^{\text{left}}]$, and let v be a proper $\widetilde{W}$-suffix of w.*

1. *If v is of type (a) or (b) then $v\in W$ $[W^{\text{right}},\ W]$.*

2. *If v is of type (c) then $v\in W^{\emptyset|}$ $[W^{\emptyset|\text{right}},\ W^{\emptyset|}]$.*

Moreover, $\wp(v)=\varepsilon$ if and only if v is of the form $\llbracket w_1\rfloor\llbracket w_2\rfloor\cdots\llbracket w_s\rfloor$ for some $s\in\mathbb{N}$ and $w_1,w_2,\ldots,w_s\in W^{\text{right}}$.

The first two statements of the next lemma directly follow from the previous lemma.

Lemma 4.3.8. *(1) If $w \in W^{\text{left}}$ then $\mathrm{I}w$ is a $\wedge$-letter which is not a $\wedge$-loop. Moreover, if $w \neq \mathrm{I}w$ then w has a proper $\widetilde{W}$-suffix which is either a path, or of the form $\llbracket u \rfloor$ for some $u \in W^{\text{right}}$. The latter case occurs if and only if the last $\widetilde{A}$-letter of w is a $\wedge$-letter which is not a $\wedge$-loop.*

(2) If $w \in W$ then w has a $\widetilde{W}$-sufffix [$\widetilde{W}$-prefix] which is either a path, or of the form $\llbracket u \rfloor$ [$\lceil u \rrbracket$] for some $u \in W^{\text{right}}$ [$u \in W^{\text{left}}$]. The latter case occurs if and only if the last [first] $\widetilde{A}$-letter of w is a $\wedge$-letter which is not a $\wedge$-loop.

(3) Let $\lceil w \rrbracket$ be a factor of a bracketed word in $W \cup W^{\text{right}} \cup W^{\text{left}}$ where $w \in W^{\text{left}}$ such that $w \neq \mathrm{I}w = (x \wedge y)$ $(x, y \in \overline{A})$, and the last $\widetilde{A}$-letter of w is $(b \wedge a)$ $(a, b \in \overline{A})$. Then, independently of whether $\alpha(b) = \omega(a)$ or not, we have $\widehat{a} \mathrel{\mathcal{L}} \widehat{y} \mathrel{\mathcal{L}} \widehat{\wp}(w) \in E$ and $\omega(a) = \omega(y) = \alpha(\wp(w)) = \omega(\wp(w))$. Moreover, if v is the $\widetilde{W}$-suffix of w obtained from w by deleting $\mathrm{I}w$, then we have $v \in W \cup W^{\emptyset|}$, and if $\wp(v) \neq \varepsilon$ then $\widehat{y} \mathrel{\mathcal{L}} \widehat{\wp}(v) \in E$ and $\omega(y) = \alpha(\wp(v)) = \omega(\wp(v))$.

Proof. (3) Assume that w is of the form (4.3.4). If $\alpha(b) = \omega(a)$ then $(b \wedge a) = p_k\mathrm{T}$ and $p_k \neq \varepsilon$. Hence $\widehat{\wp}(w) \mathrel{\mathcal{L}} \widehat{(b \wedge a)} \mathrel{\mathcal{L}} \widehat{a}$ follows. Applying property (E2bi), we see that $\widehat{\wp}(w) \in E$ and $\widehat{\wp}(w) \mathrel{\mathcal{L}} \widehat{y}$ whence $\widehat{a} \mathrel{\mathcal{L}} \widehat{y} \mathrel{\mathcal{L}} \widehat{\wp}(w) \in E$ follows. If $\alpha(b) \neq \omega(a)$ then the last factor in the form (E1a) of B_k is $\llbracket u \rfloor$ for some $u \in W^{\text{right}}$ with $u\mathrm{T} = (b \wedge a)$. This implies by (E0a) that $p_{k-1} \neq \varepsilon$ and $\wp(w) = \wp(p_0) \cdots \wp(p_{k-1})$, and so $\widehat{\wp}(p_{k-1}) \mathrel{\mathcal{L}} \widehat{\wp}(w)$ follows. By property (E1bii) of u we deduce that $\widehat{a} \mathrel{\mathcal{L}} \widehat{\wp}(p_{k-1})$, and by property (E2bi) of w that $\widehat{\wp}(w) \mathrel{\mathcal{L}} \widehat{y}$. Thus we again obtain that $\widehat{a} \mathrel{\mathcal{L}} \widehat{y} \mathrel{\mathcal{L}} \widehat{\wp}(w) \in E$. In both subcases, this relation implies $\omega(a) = \omega(y) = \alpha(\wp(w)) = \omega(\wp(w))$.

Turning to the second statement, first notice that Lemma 4.3.7 implies $v \in W \cup W^{\emptyset|}$. By definition, we have $\widehat{\wp}(w) = \widehat{(y' \wedge y)}\widehat{\wp}(v)$ where all three elements belong to a completely simple subsemigroup of S. This implies $\widehat{\wp}(w) \mathrel{\mathcal{L}} \widehat{\wp}(v)$. Furthermore, we have seen in the first part of the proof that $\widehat{\wp}(w) \in E$ and $\widehat{\wp}(w) \mathrel{\mathcal{L}} \widehat{y}$. Since $\widehat{(y' \wedge y)} \in E$ and $\widehat{(y' \wedge y)} \mathrel{\mathcal{L}} \widehat{y}$ also holds, we deduce that $\widehat{\wp}(v) \in E$ and $\widehat{\wp}(v) \mathrel{\mathcal{L}} \widehat{y}$. These relations imply $\omega(y) = \alpha(\wp(v)) = \omega(\wp(v))$. $\square$

An easy consequence of this lemma is that the subsets W, W^{right} and W^{left} of $\widetilde{W}$ are almost pairwise disjoint.

Corollary 4.3.9. *For the subsets W, W^{right}, and W^{left} of $\widetilde{W}$, we have $W \cap (W^{\mathrm{right}} \cup W^{\mathrm{left}}) = \emptyset$, and $W^{\mathrm{right}} \cap W^{\mathrm{left}}$ is the set of all $\wedge$-letters which are not $\wedge$-loops.*

Let $w \in W \cup W^{\mathrm{right}} \cup W^{\mathrm{left}}$. We see by definition that if $\tilde{u}$ is any non-empty bracketed subword of w then two possibilities occur: either $\tilde{u}$ is inside a pair of brackets $\llbracket, \rfloor$ or $\lceil, \rrbracket$, or it is not. In the first case, there exists a shortest section v of w such that v contains $\tilde{u}$, and v is either of the form $\llbracket u \rfloor$ for some $u \in W^{\mathrm{right}}$ or of the form $\lceil u \rrbracket$ for some $u \in W^{\mathrm{left}}$. We denote u and v by $\mathrm{sb}_w(\tilde{u})$ and $\mathrm{sbbr}_w(\tilde{u})$, respectively. In the second case, $\mathrm{sb}_w(\tilde{u})$ is defined to be w and $\mathrm{sbbr}_w(\tilde{u})$ is undefined.

Now we are ready to return to proving the equality $a = b$ provided a $\mathcal{CS}$-derivation (4.3.2) is given from a to b where a, b are coterminal arrows in $\mathcal{C}$. It suffices to show that, whenever $w, w^{\wr} \in \widetilde{A}^{+}$ such $w^{\wr}$ is obtained from w by one of the derivation steps, and $\underline{w} \in W$ such that $w = \underline{w}{\downarrow}$, then there exists a bracketed word $\underline{w^{\wr}} \in W$ such that $w^{\wr} = \underline{w^{\wr}}{\downarrow}$ and $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^{\wr}})$. For, if this holds, then we can choose $\underline{w_0}$ to be a, and we obtain $\underline{w_{i+1}}$ for $i = 0, 1, \ldots, n-1$ by induction such that $\widehat{\wp}(\underline{w_i}) = \widehat{\wp}(\underline{w_{i+1}})$. This implies $a = \wp(a) = \widehat{\wp}(\underline{w_0}) = \widehat{\wp}(\underline{w_1}) = \cdots = \widehat{\wp}(\underline{w_{n-1}}) = \widehat{\wp}(\underline{w_n}) = \wp(b) = b$, since $\underline{w_n} = \underline{w_n}{\downarrow} = w_n = b$.

In the rest of the section we verify the above statement for any derivation step. In each subcase considered, the general scheme of the argument is as follows. We consider $u = \mathrm{sb}_{\underline{w}}(\tilde{u})$ and $v = \mathrm{sbbr}_{\underline{w}}(\tilde{u})$ for a bracketed subword $\tilde{u}$ of $\underline{w}$ such that $\underline{u}{\downarrow}$ contains the section of w involved in the derivation step, and define $u^{\wr} \in \widetilde{W}$ such that the following conditions are satisfied:

(Q1) $u^{\wr}{\downarrow}$ is just the term obtained from $u{\downarrow}$ by the derivation step considered,

(Q2) $u^{\wr}$ is of the form (4.3.4), and if $\mathrm{I}u = (x \wedge y)$ $[u\mathrm{T} = (x \wedge y)]$ such that $\alpha(x) \neq \omega(y)$, then $\mathrm{I}u^{\wr} = (x^{\wr} \wedge y^{\wr})$ $[u^{\wr}\mathrm{T} = (x^{\wr} \wedge y^{\wr})]$ such that $\widehat{x^{\wr}}\, \mathcal{R}\, \widehat{x}$ and $\widehat{y^{\wr}}\, \mathcal{L}\, \widehat{y}$,

(Q3) $u^{\wr}$ has property (E0a), and we have $\widehat{\wp}(u^{\wr}) = \widehat{\wp}(u)$ $[\widehat{\wp}(u^{\wr})\mathcal{L}\widehat{\wp}(u),\ \widehat{\wp}(u^{\wr})\mathcal{R}\widehat{\wp}(u)]$ provided $\underline{w} = u$ $[v = \lceil u \rrbracket,\ v = \llbracket u \rfloor]$.

(Q4) $u^{\wr}$ has properties (E1)–(E2).

Notice that relations $\widehat{x^{\wr}}\, \mathcal{R}\, \widehat{x}$ and $\widehat{y^{\wr}}\, \mathcal{L}\, \widehat{y}$ imply $\alpha(x^{\wr}) = \alpha(x)$ and $\omega(y^{\wr}) = \omega(y)$. Thus, by Corollary 4.3.9, (Q2)–(Q4) imply that $u^{\wr} \in W$, $u^{\wr} \in W^{\mathrm{right}}$ and

$u^{\wr} \in W^{\text{left}}$ if and only if $\underline{w} = u$, $v = \lfloor\!\lfloor u\rfloor$ and $v = \lceil u\rceil\!\rceil$, respectively. Define $\underline{w^{\wr}}$ to be the bracketed word obtained from $\underline{w}$ by replacing the section u by $u^{\wr}$. To justify our approach, we have to verify that properties (Q1)–(Q4) imply $\underline{w^{\wr}} \in W$, $w^{\wr} = \underline{w^{\wr}}\!\downarrow$ and $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^{\wr}})$.

Clearly, we have $\underline{w^{\wr}} = u^{\wr}$ if and only if $\underline{w} = u$, and we have $\mathrm{sbbr}_{\underline{w^{\wr}}}(u^{\wr}) = \lfloor\!\lfloor u^{\wr}\rfloor$ $[\lceil u^{\wr}\rceil\!\rceil]$ if and only if $v = \lfloor\!\lfloor u\rfloor$ $[\lceil u\rceil\!\rceil]$. Moreover, property (Q2) implies that the factor $\lfloor\!\lfloor u^{\wr}\rfloor$ $[\lceil u^{\wr}\rceil\!\rceil]$ of $\mathrm{sb}_{\underline{w^{\wr}}}(\lfloor\!\lfloor u^{\wr}\rfloor)$ $[\mathrm{sb}_{\underline{w^{\wr}}}(\lceil u^{\wr}\rceil\!\rceil)]$ satisfies condition (E1bii) [(E2bii)], since $\underline{w} \in W$, and so the factor $\lfloor\!\lfloor u\rfloor$ $[\lceil u\rceil\!\rceil]$ of $\mathrm{sb}_{\underline{w}}(\lfloor\!\lfloor u\rfloor)$ $[\mathrm{sb}_{\underline{w}}(\lceil u\rceil\!\rceil)]$ has property (E1bii) [(E2bii)]. All the details of properties (E0)–(E2) of $\underline{w^{\wr}}$ not checked in (Q2)–(Q4) are obviously inherited from those of $\underline{w}$. This shows that $\underline{w^{\wr}} \in W$. The equality $w^{\wr} = \underline{w^{\wr}}\!\downarrow$ is clear by (Q1) and by the definition of $\underline{w^{\wr}}$. The equality $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^{\wr}})$ is implied. For, if $\underline{w} \neq u$ then $\wp(\underline{w})$ is not affected by the changes done in u to obtain $u^{\wr}$, and so $\wp(\underline{w^{\wr}}) = \wp(\underline{w})$. If $\underline{w} = u$ then we also have $u^{\wr} = \underline{w^{\wr}}$, and the equality follows from (Q3).

Note that, throughout the next proofs, (Q1) and (Q2) will be clear from the definition of $u^{\wr}$, and in a number of cases, the same holds for (Q3). Furthermore, most of the properties to be checked in (Q4) are inherited from the respective properties of u and $\underline{w}$, or they are obvious by definition. For example, (Q3) is clear if $\wp(u^{\wr}) = \wp(u)$, or condition (E1bi) is trivially satisfied in case $w_j = w_j\mathrm{T}$. It is also obvious that if u of the form (4.3.4) and $u^{\wr}$ is obtained from u by deleting a factor $\lfloor\!\lfloor w_j\rfloor$ $[\lceil w_j\rceil\!\rceil]$ (see (E1a) [(E2a)]) then (Q2)–(Q4) are valid. In the proofs of the following propositions, we concentrate on the properties where this is not the case.

Proposition 4.3.10. *Suppose that $w, w^{\wr} \in \widetilde{A}^{+}$ and we get $w^{\wr}$ from w by a derivation step of one of the types* (Sja), (Sjb) *for $j = 1, 21, 22$ and* (T5a), (T5b). *If $\underline{w} \in W$ such that $w = \underline{w}\!\downarrow$ then there exists $\underline{w^{\wr}} \in W$ such that $w^{\wr} = \underline{w^{\wr}}\!\downarrow$ and $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^{\wr}})$.*

Proof. First we consider the case of derivation steps (S22a) and (S22b). By symmetry, we can assume that $w^{\wr}$ is obtained from w by replacing either an occurrence of a $\wedge$-letter $(a \wedge y)$ by $(c \wedge y)$, or an occurrence of a $\wedge$-letter $(c \wedge y)$ by $(a \wedge y)$, where $y \in \overline{A}$ and $a, c \in A$ such that $a \circ b = c$ for some $b \in A$. This equality implies $\widehat{a}\,\mathcal{R}\,\widehat{c}$, and so $\alpha(a) = \alpha(c)$ follows. Hence $(a \wedge y)$ is a $\wedge$-loop if and only if $(c \wedge y)$ is, and in this case, $\widehat{(a \wedge y)} = \widehat{(c \wedge y)}$. If $(a \wedge y)$ is replaced by $(c \wedge y)$ then put $u = \mathrm{sb}_{\underline{w}}((a \wedge y))$, and consider its form (4.3.4). Define $u^{\wr}$ to be the bracketed word obtained from u by replacing $(a \wedge y)$ by $(c \wedge y)$. We see that $(a \wedge y)$ belongs to a section of p_i for some i $(0 \leq i \leq k)$, and $(a \wedge y)$ is not a $\wedge$-loop if and only if either $i = 0$, $(a \wedge y) = \mathrm{I}p_1 = \mathrm{I}u$ and $\mathrm{sbbr}_{\underline{w}}((a \wedge y)) = \lceil u \rrbracket$, or $i = k$, $(a \wedge y) = p_k\mathrm{T} = u\mathrm{T}$ and $\mathrm{sbbr}_{\underline{w}}((a \wedge y)) = \llbracket u \rfloor$. In these subcases, denote by $p_1^{\wr}$ and $p_k^{\wr}$ the words obtained from $p_1 \in W_0^{\mathrm{left}}$ and $p_k \in W_0^{\mathrm{right}}$, respectively, by replacing $(a \wedge y)$ by $(c \wedge y)$. By definition, we have $\wp(p_1) = \wp(p_1^{\wr})$ in the first subcase, and since $\widehat{(a \wedge a')}\, \mathcal{R}\, \widehat{(c \wedge c')}$, we have $\widehat{\wp}(p_k)\, \mathcal{R}\, \widehat{\wp}(p_k^{\wr})$ in the second subcase. These observations imply properties (Q2)–(Q4). The same argument applies if $(c \wedge y)$ is replaced by $(a \wedge y)$.

Turning to the rest of the derivations steps, denote by p the section of w modified by the derivation step, and by q the word p is replaced by in order to obtain $w^{\wr}$. (Using the notation of Lemma 4.2.2, $p = s$, $q = t$ or $p = t$, $q = s$.) With each derivation step considered, p and q are coterminal paths in $\mathrm{Arr}\,\widetilde{\mathcal{C}}^{+}$ such that $\widehat{p} = \widehat{q}$. Let $u = \mathrm{sb}_{\underline{w}}(p)$ be of the form (4.3.4). Then p is a section of p_i for some i $(0 \leq i \leq k)$, and p is not a prefix of p_0 [suffix of p_k] if $u \in W^{\mathrm{left}}$ [W^{right}]. Define $u^{\wr}$ to be the bracketed word obtained from u by replacing the section p of u by q. Thus $u^{\wr}$ is obtained from u by replacing a path section of p_i by q. Properties (Q3)–(Q4) are now easier to check than in case (S22a). □

The respective propositions for derivation steps (T3a), (T3b), (T4a), (T4b) are more complicated to prove. However, (T3a) and (T3b) are duals of (T4a) and (T4b), respectively, therefore we can restrict ourselves to proving the latter ones.

Proposition 4.3.11. *Suppose that $w, w^{\wr} \in \widetilde{A}^{+}$ and we get $w^{\wr}$ from w by a derivation step of type* (T4b)*. If $\underline{w} \in W$ such that $w = \underline{w}{\downarrow}$ then there exists $\underline{w^{\wr}} \in W$ such that $w^{\wr} = \underline{w^{\wr}}{\downarrow}$ and $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^{\wr}})$.*

Proof. Assume that an occurrence of a $\wedge$-letter $(y \wedge x)$ in w is replaced by the word $(y \wedge x)(z \wedge x)$ where $x, y, z \in \overline{A}$. Put $u = \mathrm{sb}_{\underline{w}}((y \wedge x))$ and $v = \mathrm{sbbr}_{\underline{w}}(u)$. If $(y \wedge x)$ is not a $\wedge$-loop then we have either $v = \lceil u \rrbracket$ $(u \in W^{\mathrm{left}})$, or $v = \llbracket u \rfloor$ $(u \in W^{\mathrm{right}})$.

First we suppose that $(y \wedge x)$ is a $\wedge$-loop, or $(y \wedge x)$ is not a $\wedge$-loop, and $v = \lceil u \rrbracket$, $u \in W^{\mathrm{left}}$, $\mathrm{I}u = (y \wedge x)$. If u is of form (4.3.4) then in these cases, $(y \wedge x)$ is in p_i for some i $(1 \leq i \leq k)$, and if $(y \wedge x)$ is not a $\wedge$-loop then necessarily $i = 0$ and $(y \wedge x) = \mathrm{I}p_0$. Thus we have $p_i = p_{i1}(y \wedge x)p_{i2}$ for some i where p_{i1} and p_{i2} are (possibly empty) paths, p_{01} being necessarily empty if $(y \wedge x)$ is not a $\wedge$-loop. Define $u^{\wr}$ to be the bracketed word obtained from u by replacing the $\wedge$-letter $(y \wedge x)$ by the bracketed word $(y \wedge x)(z \wedge x)$ or $(y \wedge x)\llbracket (z \wedge x) \rfloor$ according to whether $(z \wedge x)$ is a $\wedge$-loop or not.

If $\alpha(z) = \omega(x)$ then $u^\wr$ is obtained from u such that p_i is replaced by $p_i^\wr = p_{i1}(y \wedge x)(z \wedge x)p_{i2}$, and section $p_i^\wr$ of $u^\wr$ belongs to W_0 if $p_i \in W_0$, and belongs to W_0^{left} if $i = 0$ and $p_0 \in W_0^{\text{left}}$. Moreover, $\widehat{(z \wedge x)}$ is an idempotent $\mathcal{L}$-related to $\widehat{x}$, therefore $\widehat{\wp}(p_i) = \widehat{\wp}(p_i^\wr)$. Similarly to the end of the proof of Proposition 4.3.10, this equality implies properties (Q3)–(Q4). If $(z \wedge x)$ is not a $\wedge$-loop then we have

$$u^\wr = p_0 \cdots p_{i-1} B_i C_i p_{i1}(y \wedge x) \llbracket (z \wedge x) \rfloor p_{i2} B_{i+1} C_{i+1} p_{i+1} \cdots p_k. \tag{4.3.5}$$

To verify (Q4), it suffices to show that the factor $\llbracket (z \wedge x) \rfloor$ and those of B_{i+1} satisfy condition (E1bii). The former holds since $\widehat{\wp}(p_{i1}(y \wedge x)) \,\mathcal{L}\, \widehat{x}$. To see the latter, we recall the respective relation between p_i and B_{i+1} in u and the facts that if $p_{i2} \neq \varepsilon$ then $\widehat{\wp}(p_i) \,\mathcal{L}\, \widehat{p_{i2}}$, and if $p_{i2} = \varepsilon$ then $\widehat{\wp}(p_i) = \widehat{\wp}(p_{i-1}(y \wedge x)) \,\mathcal{L}\, \widehat{x}$.

Now suppose that $v = \llbracket u \rfloor$ ($u \in W^{\text{right}}$), and so $u\mathrm{T} = (y \wedge x)$. Consider the section $u_+ = \mathrm{sb}_{\underline{w}}(v)$ of $\underline{w}$, and suppose that it is of the form (4.3.4). Then v is a factor of B_i for some i ($1 \leq i \leq k$), therefore B_i is of the form $\llbracket u_{-m} \rfloor \cdots \llbracket u_{-1} \rfloor \llbracket u \rfloor \llbracket u_1 \rfloor \cdots \llbracket u_n \rfloor$ ($m, n \in \mathbb{N}_0$) for some bracketed words $u_j \in W^{\text{right}}$ ($-m \leq j \leq n,\ j \neq 0$). Define $u_+^\wr$ to be the bracketed word obtained from u_+ by replacing $\llbracket u \rfloor$ by $\llbracket u \rfloor (z \wedge x)$ if $(z \wedge x)$ is a $\wedge$-loop, and by $\llbracket u \rfloor \llbracket (z \wedge x) \rfloor$ otherwise. Thus

$$u_+^\wr = p_0 \cdots p_{i-1} \llbracket u_{-m} \rfloor \cdots \llbracket u_{-1} \rfloor \llbracket u \rfloor (z \wedge x) \llbracket u_1 \rfloor \cdots \llbracket u_n \rfloor C_i p_i \cdots p_k$$

and

$$u_+^\wr = p_0 \cdots p_{i-1} \llbracket u_{-m} \rfloor \cdots \llbracket u_{-1} \rfloor \llbracket u \rfloor \llbracket (z \wedge x) \rfloor \llbracket u_1 \rfloor \cdots \llbracket u_n \rfloor C_i p_i \cdots p_k,$$

respectively, in the two subcases. In the second subcase, (Q3) is clear. Since $u\mathrm{T} = (y \wedge x)$ implies by property (E1bii) of u_+ that $\widehat{x} \,\mathcal{L}\, \widehat{\wp}(p_{i-1})$, we immediately obtain that the new factor $\llbracket (z \wedge x) \rfloor$ satisfies condition (E1bii), and so (Q4) also follows. In the first subcase, where $(z \wedge x)$ is a new path factor, the relation $\widehat{x} \,\mathcal{L}\, \widehat{\wp}(p_{i-1})$, seen above, implies $\widehat{p_{i-1}}\widehat{(z \wedge x)} = \widehat{p_{i-1}}$ and $\omega(p_{i-1}) = \alpha(z) = \omega(x)$, since $\widehat{(z \wedge x)}$ is idempotent. This verifies (Q3). Moreover, we obtain that the factors $\llbracket u_j \rfloor$ ($1 \leq j \leq n$) satisfy condition (E1bii) whence (Q4) follows. $\square$

Proposition 4.3.12. *Suppose that $w, w^\wr \in \widetilde{A}^+$ and we get $w^\wr$ from w by a derivation step of type* (T4a)*. If $\underline{w} \in W$ such that $w = \underline{w}{\downarrow}$ then there exists $\underline{w^\wr} \in W$ such that $w^\wr = \underline{w^\wr}{\downarrow}$ and $\widehat{\wp}(\underline{w}) = \widehat{\wp}(\underline{w^\wr})$.*

Proof. Assume that an occurrence of a section $(y \wedge x)(z \wedge x)$ of w is replaced by $(y \wedge x)$ where $x, y, z \in \overline{A}$. Denote $\mathrm{sb}_{\underline{w}}((y \wedge x))$, $\mathrm{sb}_{\underline{w}}((z \wedge x))$ and $\mathrm{sb}_{\underline{w}}((y \wedge x)(z \wedge x))$ by u_1, u_2 and u, respectively. Clearly, u_1 and u_2 are sections of u, and each can be equal to u or can be a proper subsection of u. We proceed by distinguishing the four cases obtained in this way.

Case $u = u_1 = u_2$. If u is of the form (4.3.4) then $(y \wedge x)(z \wedge x)$ is a section of p_i for some i $(0 \leq i \leq k)$. This implies that $\omega(x) = \alpha(z)$, and so $(z \wedge x)$ is a $\wedge$-loop. If $(y \wedge x)$ is not a $\wedge$-loop then (E0a) implies $i = 0$, $p_0 \in W_0^{\mathrm{left}}$ and $\mathrm{I}p_0 = (y \wedge x)$. Define $p_i^{\wr}$ and $u^{\wr}$ to be the bracketed words obtained from p_i and u, respectively, by deleting $(z \wedge x)$. Property (Q3) follows from the fact that, if $(y \wedge x)$ is a $\wedge$-loop then $\widehat{(y \wedge x)}$ and $\widehat{(z \wedge x)}$, if $(y \wedge x)$ is not a $\wedge$-loop then $\widehat{(x' \wedge x)}$ and $\widehat{(z \wedge x)}$ are $\mathcal{L}$-related idempotents, and so we have $\widehat{(y \wedge x)}\widehat{(z \wedge x)} = \widehat{(y \wedge x)}$ and $\widehat{(x' \wedge x)}\widehat{(z \wedge x)} = \widehat{(x' \wedge x)}$, respectively. To check (Q4), it suffices to observe that $\widehat{\wp}(p_i) \,\mathcal{L}\, \widehat{\wp}(p_i^{\wr})$ by the former equalities if $(y \wedge x)(z \wedge x)$ is a suffix of p_i, and by the equality $p_i\mathrm{T} = p_i^{\wr}\mathrm{T}$ otherwise.

Case $u = u_1 \neq u_2$. Put $v_2 = \mathrm{sbbr}_{\underline{w}}(u_2)$ where we have $v_2 = \llbracket u_2 \rfloor$ and $u_2 \in W^{\mathrm{right}}$, or $v_2 = \lceil u_2 \rrbracket$ and $u_2 \in W^{\mathrm{left}}$. Assume that u is of the form (4.3.4). Then $(y \wedge x) = p_{i-1}\mathrm{T}$ and $(z \wedge x)$ is the first $\widetilde{A}$-letter of the bracketed word B_iC_i for some i $(1 \leq i \leq k)$, and if $(y \wedge x)$ is not a $\wedge$-loop then $i = 1$ and $p_0 = (y \wedge x)$.

If $B_i = \varepsilon$ then $(z \wedge x)$ is contained in the first factor of C_i of the form (E2a). Therefore the first factor of C_i is $v_2 = \lceil u_2 \rrbracket$ where $u_2 \in W^{\mathrm{left}}$ and $\mathrm{I}u_2 = (z \wedge x)$, and so $\alpha(z) \neq \omega(x)$. Notice that $\omega(x) = \omega(p_{i-1}) = \alpha(p_i)$ and $\widehat{z}\,\mathcal{R}\,\widehat{\wp}(p_i)$ by properties (E0a) and (E2bii) of u, and the latter relation implies $\alpha(z) = \alpha(p_i)$. Hence we obtain that $\alpha(z) = \omega(x)$, a contradiction.

If $B_i \neq \varepsilon$ then $(z \wedge x)$ is the first $\widetilde{A}$-letter of the first factor $\llbracket w_1 \rfloor$ of B_i of the form (E1a) where $w_1 \in W^{\mathrm{right}}$. By the dual of Lemma 4.3.8(1) we see that either $\alpha(z) \neq \omega(x)$ and $u_2 = w_1 = (z \wedge x)$, or $(z \wedge x)$ is a $\wedge$-loop and $u_2 = w_1$, or else $\alpha(z) \neq \omega(x)$, $u_2 \in W^{\mathrm{left}}$ with $\mathrm{I}u_2 = (z \wedge x)$, and $v_2 = \lceil u_2 \rrbracket$ is a $\widetilde{W}$-prefix w_1. Now we consider these subcases separately.

If $\alpha(z) \neq \omega(x)$ and $u_2 = w_1 = (z \wedge x)$ then define $u^{\wr}$ to be the bracketed word obtained from u by deleting the factor $\llbracket w_1 \rfloor$ of B_i. This obviously fulfils all the requirements.

Now consider the subcase where $(z \wedge x)$ is a $\wedge$-loop, i.e., $\alpha(z) = \omega(x)$, and $u_2 = w_1$. Then the dual of Lemma 4.3.8(3) implies $\alpha(z) = \alpha(a)$, and (E1bii) ensures $\widehat{b}\,\mathcal{L}\,\widehat{\wp}(p_{i-1})\,\mathcal{L}\,\widehat{x}$ since $\wp(p_{i-1})\mathrm{T}$ is either $(y \wedge x)$ or $(x' \wedge x)$, depending on whether $(y \wedge x)$ is a $\wedge$-loop or not. This implies $\omega(b) = \omega(x)$ whence we obtain $\alpha(a) = \omega(b)$, a contradiction.

Finally, let $\alpha(z) \neq \omega(x)$, $u_2 \in W^{\text{left}}$ such that $\mathrm{I}u_2 = (z \wedge x)$ and $v_2 = \lceil u_2 \rrbracket$ is a $\widetilde{W}$-prefix of w_1. Therefore we have $u_2 = (z \wedge x)u_{22}$ and $w_1 = \lceil u_2 \rrbracket w_{12}$ for some $\widetilde{W}$-suffix u_{22} and w_{12} of u_2 and w_1, respectively, whence $u_{22} \in W \cup W^{\emptyset|}$ and $w_{12} \in W^{\text{right}}$ by Lemma 4.3.7. This allows us to define $u^{\wr}$ so that the section $(y \wedge x)\llbracket w_1 \rfloor = (y \wedge x)\llbracket \lceil (z \wedge x)u_{22} \rrbracket w_{12} \rfloor$ of u, where $(y \wedge x) = p_{i-1}\mathrm{T}$, is replaced by $(y \wedge x)u_{22}\llbracket w_{12} \rfloor$. Since $u_2 = (z \wedge x)u_{22} \in W^{\text{left}}$, it is easy to see by definition that $(y \wedge x)u_{22} \in W$ or W^{left} depending on whether $(y \wedge x)$ is a $\wedge$-loop or not. This implies that $u^{\wr}$ is of the form (4.3.4). Applying Lemma 4.3.8(3) for u_2, we obtain that if $\wp(u_{22}) \neq \varepsilon$ then $\widehat{x}\, \mathcal{L}\, \widehat{\wp}(u_{22}) \in E$. Hence $\widehat{\wp}((y \wedge x)u_{22}) = \widehat{\wp}((y \wedge x))\widehat{\wp}(u_{22}) = \widehat{\wp}((y \wedge x))$ follows, and this implies $\widehat{\wp}(p_{i-1}u_{22}) = \widehat{\wp}(p_{i-1})$, and so (Q3) holds for $u^{\wr}$. In order to check (Q4) for $u^{\wr}$, it suffices to verify that the factor $\llbracket w_{12} \rfloor$ satisfies (E1b). Since $w_{12}\mathrm{T} = w_1\mathrm{T}$ and $\wp(w_{12}) = \wp(w_1)$, it is straightforward from property (E1bi) of $\llbracket w_1 \rfloor$ in u that the same property is valid for $\llbracket w_{12} \rfloor$ in $u^{\wr}$. Similarly, these equalities combined with $\widehat{\wp}(p_{i-1}u_{22}) = \widehat{\wp}(p_{i-1})$ allow us to see that property (E1bii) of $\llbracket w_1 \rfloor$ in u implies the same property of $\llbracket w_{12} \rfloor$ in $u^{\wr}$.

Case $u = u_2 \neq u_1$. Assume that u is of the form (4.3.4). Then $(z \wedge x) = \mathrm{I}p_i$ and $(y \wedge x)$ is the last $\widetilde{A}$-letter in the bracketed word B_iC_i for some i $(1 \leq i \leq k)$. Furthermore, if $(z \wedge x)$ is not a $\wedge$-loop then $i = k$ and $(z \wedge x) = p_k$.

First we examine the subcase, where $(z \wedge x)$ is a $\wedge$-loop. If $C_i \neq \varepsilon$ then $s \in \mathbb{N}$ in (E2a), and $(y \wedge x)$ is the last $\widetilde{A}$-letter of w_s. Assume that $\mathrm{I}w_s = (a \wedge b)$ where $\alpha(a) \neq \omega(b)$. Property (E2bii) of w_s implies that $\widehat{a}\ \mathcal{R}\ \widehat{\wp}(p_i)\ \mathcal{R}\ \widehat{(z \wedge x)}\ \mathcal{R}\ \widehat{z}$ whence $\alpha(a) = \alpha(z)$ follows. If $w_s = \mathrm{I}w_s$ then $b = x$ and $\omega(b) = \omega(x)$ are obviuos. If $w_s \neq \mathrm{I}w_s$ then we see by Lemma 4.3.8(3) that $\omega(b) = \omega(x)$. Combining these equalities we obtain $\alpha(z) = \alpha(a) \neq \omega(b) = \omega(x)$, a contradiction.

Let us assume now that $C_i = \varepsilon$, and so $(y \wedge x)$ is the last $\widetilde{A}$-letter of B_i. In the form (E1a) of B_i, we have $s \in \mathbb{N}$ and $(y \wedge x) = w_s\mathrm{T}$. Since $(z \wedge x) = \mathrm{I}p_i$, we have $p_i = (z \wedge x)p_{i2}$ where $p_{i2} \in W_0^{\varepsilon}$ or $i = k$ and $p_{i2} \in W_0^{\text{right}}$. Define

$$u^{\wr} = p_0 \cdots p_{i-1}B_ip_{i2}B_{i+1}C_{i+1}p_{i+1} \cdots p_k.$$

Since $(z \wedge x)$ is a $\wedge$-loop and we have $\widehat{\wp}(p_{i-1})\, \mathcal{L}\, \widehat{x}$ in u by the property (E1bii) of w_s, we get $\widehat{\wp}(p_{i-1})\widehat{(z \wedge x)} = \widehat{\wp}(p_{i-1})$. Hence $\widehat{\wp}(p_{i-1}p_i) = \widehat{\wp}(p_{i-1}(z \wedge x)p_{i2}) = \widehat{\wp}(p_{i-1})\widehat{(z \wedge x)}\widehat{\wp}(p_{i2}) = \widehat{\wp}(p_{i-1})\widehat{\wp}(p_{i2}) = \widehat{\wp}(p_{i-1}p_{i2})$ also if $p_{i2} \neq \varepsilon$, and (Q3) follows. If $p_{i2} \neq \varepsilon$ then the relation $\widehat{p_i} \,\mathcal{L}\, \widehat{p_{i2}}$ implies that the factors of B_{i+1} fulfil condition (E1bii) in $u^{\wr}$ since they do in u. If $p_{i2} = \varepsilon$ then the same follows by observing that $\widehat{p_i} = \widehat{(z \wedge x)} \,\mathcal{L}\, \widehat{x}$ in u, and so $\widehat{p_i} \,\mathcal{L}\, \widehat{p_{i-1}}$.

Secondly, consider the subcase where $(z \wedge x)$ is not a $\wedge$-loop. As we have seen above, u is necessarily in W^{right}, and $p_k = (z \wedge x)$ in its form (4.3.4). If $C_k = \varepsilon$ then $(y \wedge x)$ is the last $\widetilde{A}$-letter in B_k, and so in its form (E1a) we have $s \in \mathbb{N}$ and $(y \wedge x) = w_s\mathrm{T}$. By (E0a) it follows that $\omega(p_{k-1}) = \alpha(p_k) = \alpha(z)$. Also, by applying (E1bii) for w_s, we obtain that $\widehat{x} \,\mathcal{L}\, \widehat{\wp}(p_{k-1}) = \widehat{p_{k-1}}$, which immediately implies $\omega(p_{k-1}) = \omega(x)$. Hence we conclude $\alpha(z) = \omega(x)$, which contradicts the assumption that $(z \wedge x)$ is not a $\wedge$-loop.

If $C_k \neq \varepsilon$ then $(y \wedge x)$ is the last $\widetilde{A}$-letter in C_k where it is of the form (E2a) with $s \in \mathbb{N}$. Assume that $\mathrm{I}w_s = (a \wedge b)$ where, by definition, $\alpha(a) \neq \omega(b)$. By property (E2bii) of w_s in u we see that $\widehat{a} \,\mathcal{R}\, \widehat{\wp}(p_k) \,\mathcal{R}\, \widehat{z}$. If $\mathrm{I}w_s = w_s$ then we have $w_s = (a \wedge b) = (y \wedge x)$, and so $\widehat{y} = \widehat{a} \,\mathcal{R}\, \widehat{z}$ follows. In this case, let us define $u^{\wr} = p_0 B_1 C_1 p_1 \cdots p_{k-1} B_k \lceil w_1 \rrbracket \cdots \lceil w_{s-1} \rrbracket (y \wedge x)$. Obviously, the relation $\widehat{y} \,\mathcal{R}\, \widehat{z}$ implies properties (Q2) and (Q4), the rest being even more straightforward.

Now consider the subcase $\mathrm{I}w_s \neq w_s$. Then $w_s = (a \wedge b) w_{s2}$ such that w_{s2} is the $\widetilde{W}$-suffix of $w_s \in W^{\text{left}}$ obtained by deleting $\mathrm{I}w_s = (a \wedge b)$, and so the last $\widetilde{A}$-letter of w_{s2} is $(y \wedge x)$ and $w_{s2} \in W \cup W^{\emptyset|}$ by Lemma 4.3.7. Recall the relation $\widehat{a} \,\mathcal{R}\, \widehat{z}$ from the previous paragraph, and notice that $\widehat{b} \,\mathcal{L}\, \widehat{x}$ follows by applying Lemma 4.3.8(3) for w_s in u. Consider the section $v = \mathrm{sb}_{\overline{w}}(\llbracket u \rfloor)$ of $\overline{w}$, and let its form (4.3.4) be

$$v = \breve{p}_0 \breve{B}_1 \breve{C}_1 \breve{p}_1 \breve{B}_2 \breve{C}_2 \breve{p}_2 \cdots \breve{p}_{l-1} \breve{B}_l \breve{C}_l \breve{p}_l \quad (l \in \mathbb{N}).$$

Then $\llbracket u \rfloor$ is a factor of $\breve{B}_i$ for some i $(1 \leq i \leq l)$, more precisely, we have

$$\breve{B}_i = \llbracket \breve{w}_1 \rfloor \cdots \llbracket \breve{w}_{j-1} \rfloor \llbracket u \rfloor \llbracket \breve{w}_{j+1} \rfloor \cdots \llbracket \breve{w}_t \rfloor \quad (t \in \mathbb{N}),$$

where $\breve{w}_m \in W^{\text{right}}$ $(1 \leq m \leq t,\ m \neq j)$. For brevity, put

$$\breve{B}_{i1} = \llbracket \breve{w}_1 \rfloor \cdots \llbracket \breve{w}_{j-1} \rfloor \quad \text{and} \quad \breve{B}_{i2} = \llbracket \breve{w}_{j+1} \rfloor \cdots \llbracket \breve{w}_t \rfloor,$$

and so we have $\breve{B}_i = \breve{B}_{i1} \llbracket u \rfloor \breve{B}_{i2}$. Define

$$u_0^{\wr} = p_0 B_1 C_1 p_1 \cdots p_{k-1} B_k \lceil w_1 \rrbracket \cdots \lceil w_{s-1} \rrbracket (a \wedge b)$$

and

$$v^{\wr} = \breve{p}_1 \breve{B}_1 \breve{C}_1 \breve{p}_1 \cdots \breve{p}_{i-1} \breve{B}_{i1} \llbracket u_0^{\wr} \rfloor w_{s2} \breve{B}_{i2} \breve{C}_i \breve{p}_i \cdots \breve{B}_l \breve{C}_l \breve{p}_l.$$

Notice that $u_0^{\wr} \in W^{\mathrm{right}}$ which directly follows from the facts that $u \in W^{\mathrm{right}}$ and $\widehat{a}\,\mathcal{R}\,\widehat{z}$, $\widehat{b}\,\mathcal{L}\,\widehat{x}$. Since $w_{s2} \in W \cup W^{\emptyset|}$ the bracketed word $v^{\wr}$ is of the form (4.3.4), and conditions (Q1) and (Q2) are clearly satisfied by v and $v^{\wr}$. If $\wp(w_{s2}) = \varepsilon$ then (Q3) is also obvious. If $\wp(w_{s2}) \neq \varepsilon$ then, applying Lemma 4.3.8(3) for w_s in u, we see that $\widehat{x}\,\mathcal{L}\,\widehat{b}\,\mathcal{L}\,\wp(w_{s2}) \in E$, and, by using (E1bii) for the factor $\llbracket u \rfloor$ of $\breve{B}_i$, we obtain that $\widehat{\wp}(\breve{p}_{i-1})\,\mathcal{L}\,\widehat{x}$. Hence we conclude that $\widehat{\wp}(\breve{p}_{i-1})\widehat{\wp}(w_{s2}) = \widehat{\wp}(\breve{p}_{i-1})$, and (Q3) holds also if $\wp(w_{s2}) \neq \varepsilon$. Moreover, these observations combined with the respective properties of v imply most items of property (Q4). It remains to observe that if w_{s2} has a non-empty $\widetilde{W}$-prefix of the form $\mathring{B}_1 = \llbracket \mathring{w}_1 \rfloor \llbracket \mathring{w}_2 \rfloor \cdots \llbracket \mathring{w}_n \rfloor$ where $\mathring{w}_m \in W^{\mathrm{right}}$ and $\mathring{w}_m\mathrm{T} = (y_m \wedge x_m)$ $(1 \leq m \leq n)$, in particular, if $\wp(w_{s2}) = \varepsilon$, then $\widehat{x_m}\,\mathcal{L}\,\widehat{\wp}(\breve{p}_{i-1})$. For, $\widehat{x_m}\,\mathcal{L}\,\widehat{b}$ follows from the property (E2bi) of w_s in u.

Case $u \neq u_1, u_2$. Observe that in this case $\mathrm{sbbr}(u_1)$ and $\mathrm{sbbr}(u_2)$ are disjoint. Therefore, considering u in the form (4.3.4), each of u_1 and u_2 is in a factor $\llbracket w_j \rfloor$ of some B_i (see (E1a)) or in a factor $\lceil w_j \rrbracket$ of some C_i (see (E2a)). First assume that $(y \wedge x)$ is the last $\widetilde{A}$-letter of B_i and $(z \wedge x)$ is the first $\widetilde{A}$-letter of C_i for some i $(1 \leq i \leq k)$, and so $(y \wedge x) = w_s\mathrm{T}$ where $\llbracket w_s \rfloor$ is the last factor of B_i of the form (E1a), and $(z \wedge x) = \mathrm{I}\breve{w}_1$ where $\lceil \breve{w}_1 \rrbracket$ is the first factor of C_i of the form (E2a). This implies that $\alpha(y), \alpha(z) \neq \omega(x)$. By (E0a), we have $\omega(p_{i-1}) = \alpha(p_i)$, and by (E1bii), we have $\widehat{\wp}(p_{i-1})\,\mathcal{L}\,\widehat{x}$, whence $\omega(p_{i-1}) = \omega(x)$ follows. Similarly, $\widehat{z}\,\mathcal{R}\,\widehat{\wp}(p_i)$ by (E2bii), and so $\alpha(z) = \alpha(p_i)$. Hence we obtain $\alpha(z) = \omega(x)$, a contradiction.

Now assume that both u_1 and u_2 are in B_i for some i $(1 \leq i \leq k)$. Then, considering B_i in the form (E1a), there exists j $(1 < j \leq s)$ such that $(y \wedge x)$ is the last $\widetilde{A}$-letter of w_{j-1} and $(z \wedge x)$ is the first $\widetilde{A}$-letter of w_j. Thus $(y \wedge x) = w_{j-1}\mathrm{T}$ and $\alpha(y) \neq \omega(x)$ follow, and we have either $w_j = (z \wedge x)$ and $\alpha(z) \neq \omega(x)$, or $w_j \neq w_j\mathrm{T}$. If $w_j = (z \wedge x)$ then define $u^{\wr}$ to be the bracketed word obtained from u by replacing B_i by

$$B_i^{\wr} = \llbracket w_1 \rfloor \cdots \llbracket w_{j-1} \rfloor \llbracket w_{j+1} \rfloor \cdots \llbracket w_s \rfloor.$$

It is straightforward that (Q1)–(Q4) hold.

Now we turn to the subcase $w_j \neq w_j\mathrm{T}$. Then $(z \wedge x)$ is the first $\widetilde{A}$-letter of w_j and $w_j\mathrm{T} = (a \wedge b)$ with $\alpha(a) \neq \omega(b)$. We obtain by the dual of Lemma 4.3.8(1) that either $(z \wedge x)$ is a $\wedge$-loop, and so $u_2 = w_j$, or else $\alpha(z) \neq \omega(x)$, and so $u_2 \in W^{\mathrm{left}}$, $\mathrm{I}u_2 = (z \wedge x)$ and $\lceil u_2 \rrbracket$ is a prefix of w_j. If $(z \wedge x)$ is a $\wedge$-loop then property (E1bi) of w_j implies that $\alpha(z) = \alpha(\wp(w_j)) = \omega(\wp(w_j)) = \omega(a') = \alpha(a)$. Furthermore, applying (E1bii) for w_{j-1} and w_j, we see that

$\widehat{x}\,\mathcal{L}\,\widehat{\wp}(p_{i-1})\,\mathcal{L}\,\widehat{b}$, and so $\omega(x)=\omega(b)$. Thus $\alpha(a)=\alpha(z)=\omega(x)=\omega(b)$, a contradiction.

If $(z\wedge x)$ is not a $\wedge$-loop then $w_j=\lceil u_2\rrbracket w_{j2}$ and $u_2=(z\wedge x)u_{22}$ where w_{j2} and u_{22}, if $u_{22}\neq\varepsilon$, are $\widetilde{W}$-suffixes of w_j and u_2, respectively, and we have $w_{j2}\mathrm{T}=w_j\mathrm{T}$ and $\wp(w_{j2})=\wp(w_j)$. Since w_{j2} is of type (a) or (b), we see by Lemma 4.3.7 that $w_{j2}\in W^{\mathrm{right}}$ and $u_{22}\in W\cup W^{\emptyset|}$. Let us define $u^{\wr}$ to be the bracketed word obtained from u by replacing B_i by

$$B_i^{\wr}=\llbracket w_1\rfloor\cdots\llbracket w_{j-1}\rfloor u_{22}\llbracket w_{j2}\rfloor\llbracket w_{j+1}\rfloor\cdots\llbracket w_s\rfloor.$$

Since $u_{22}\in W\cup W^{\emptyset|}$, $u^{\wr}$ is of the form (4.3.4) and (Q2) holds. If $\wp(u_{22})=\varepsilon$ then (Q3) is obvious. In the opposite case, we apply Lemma 4.3.8(3) for u_2 to see that $\widehat{x}\,\mathcal{L}\,\widehat{\wp}(u_{22})\in E$. By property (E1bii) of the factor $\llbracket w_{j-1}\rfloor$ of B_i we have $\widehat{x}\,\mathcal{L}\,\widehat{\wp}(p_{i-1})$, and so $\widehat{\wp}(p_{i-1})\widehat{\wp}(u_{22})=\widehat{\wp}(p_{i-1})$. Hence (Q3) follows also in case $\wp(u_{22})\neq\varepsilon$. In order to check (Q4), assume that u_{22}, if non-empty, is of the form $u_{22}=\llbracket\mathring{w}_1\rfloor\llbracket\mathring{w}_2\rfloor\cdots\llbracket\mathring{w}_n\rfloor u_{22}^{\mathrm{ab}}$ where $n\in\mathbb{N}_0$, $\mathring{w}_m\in W^{\mathrm{right}}$ $(1\le m\le n)$, and u_{22}^{ab} is empty or is the longest $\widetilde{W}$-suffix of u_{22} of type (a) or (b). By Lemma 4.3.7, either $u_{22}^{\mathrm{ab}}=\wp(u_{22}^{\mathrm{ab}})=\varepsilon$, or $\wp(u_{22})=\wp(u_{22}^{\mathrm{ab}})\neq\varepsilon$. Therefore it suffices to show that (E1bii) is satisfied by the following factors of $B_i^{\wr}$: $\llbracket\mathring{w}_m\rfloor$ $(1\le m\le n)$, provided $n\neq 0$, and $\llbracket w_{j2}\rfloor,\llbracket w_{j+1}\rfloor,\ldots,\llbracket w_s\rfloor$, provided $\wp(u_{22})\neq\varepsilon$. If $\mathring{w}_m\mathrm{T}=(y_m\wedge x_m)$ $(1\le m\le n)$ then property (E1bii) of the former factors in u_2 implies $\widehat{x_m}\,\mathcal{L}\,\widehat{\wp}((z\wedge x))\,\mathcal{L}\,\widehat{x}$. Similarly, the same property of the factors of B_i in u ensures that $\widehat{x}\,\mathcal{L}\,\widehat{b}\,\mathcal{L}\,\widehat{b_r}$ if $w_r\mathrm{T}=(a_r\wedge b_r)$ $(j<r\le s)$, since $w_{j-1}\mathrm{T}=(y\wedge x)$ and $w_j\mathrm{T}=(a\wedge b)=w_{j2}\mathrm{T}$. This verifies property (Q4) because it is seen above that $\widehat{x}\,\mathcal{L}\,\widehat{\wp}(p_{i-1})$, and if $u_{22}\neq\varepsilon$ then also $\widehat{x}\,\mathcal{L}\,\widehat{\wp}(u_{22})$.

Finally, assume that both u_1 and u_2 are in C_i for some $1\le i\le k$. Then, considering C_i in the form (E2a), there exists j $(1<j\le s)$ such that $(y\wedge x)$ is the last $\widetilde{A}$-letter of w_{j-1} and $(z\wedge x)$ is the first $\widetilde{A}$-letter of w_j. Hence we obtain that $(z\wedge x)=\mathrm{I}\underset{\sim}{w_j}$ with $\alpha(z)\neq\omega(x)$, and so $w_j=(z\wedge x)w_{j2}$ where w_{j2}, if non-empty, is a $\widetilde{W}$-suffix of w_j. Define $u^{\wr}$ to be the bracketed word obtained from u by replacing C_i by

$$C_i^{\wr}=\lceil w_1\rrbracket\cdots\lceil w_{j-2}\rrbracket\lceil w_{j-1}w_{j2}\rrbracket\lceil w_{j+1}\rrbracket\cdots\llbracket w_s\rfloor.$$

All we have to show is that $w_{j-1}w_{j2}\in W^{\mathrm{left}}$ and the factor $\lceil w_{j-1}w_{j2}\rrbracket$ of $C_i^{\wr}$ has property (E2bi). For, (E2bii) follows from the same property of C_i due to the equality $\mathrm{I}(w_{j-1}w_{j2})=\mathrm{I}w_{j-1}$. By the same argument applied in the previous paragraph for u_2 and u_{22}, we can deduce that if $w_{j2} \neq \varepsilon$ then $w_{j2} \in W^{\emptyset|}$, and if $\wp(w_{j2}) \neq \varepsilon$ then $\widehat{x} \,\mathcal{L}\, \widehat{\wp}(w_{j2}) \in E$ and $\omega(x) = \alpha(\wp(w_{j2}))$. Put $\mathrm{I}w_{j-1} = (a \wedge b)$. If $w_{j-1} = \mathrm{I}w_{j-1}$ then $a = y$, $b = x$, and if $w_{j-1} \neq \mathrm{I}w_{j-1}$ then Lemma 4.3.8(3) implies that $\widehat{x} \,\mathcal{L}\, \widehat{b} \,\mathcal{L}\, \widehat{\wp}(w_{j-1}) \in E$ and $\omega(x) = \omega(b) = \omega(\wp(w_{j-1}))$. Therefore, whether $w_{j-1} = \mathrm{I}w_{j-1}$ or not, $\omega(\wp(w_{j-1})) = \alpha(\wp(w_{j2}))$ follows if $\wp(w_{j2}) \neq \varepsilon$, and we can deduce that $w_{j-1}w_{j2}$ is of the form (4.3.4) where (E0a) holds with $p_0 \in W_0^{\mathrm{left}}$, and so (Q2), (Q3) are satisfied. To verify property (E1) of $w_{j-1}w_{j2}$, we again refer to the argument on u_2 and u_{22} in the previous paragraph which shows in our present case that if $w_{j2} = \llbracket \mathring{w}_1 \rfloor \llbracket \mathring{w}_2 \rfloor \cdots \llbracket \mathring{w}_n \rfloor w_{j2}^{\mathrm{ab}}$ where $n \in \mathbb{N}_0$, $\mathring{w}_m \in W^{\mathrm{right}}$ with $\mathring{w}_m\mathrm{T} = (y_m \wedge x_m)$ $(1 \leq m \leq n)$, and w_{j2}^{ab} is empty or is the longest $\widetilde{W}$-suffix of w_{j2} of type (a) or (b), then $\widehat{x_m} \,\mathcal{L}\, \widehat{\wp}((z \wedge x)) \,\mathcal{L}\, \widehat{x}$. Combining this with the previous relations $\widehat{x} \,\mathcal{L}\, \widehat{\wp}(w_{j2}) \in E$ if $\wp(w_{j2}) \neq \varepsilon$ and $\widehat{x} \,\mathcal{L}\, \widehat{b} \,\mathcal{L}\, \widehat{\wp}(w_{j-1}) \in E$, we obtain that $\widehat{\wp}(w_{j-1}w_{j2}) = \widehat{\wp}(w_{j-1})\widehat{\wp}(w_{j2}) = \widehat{\wp}(w_{j-1})$, and so (E1) holds in $w_{j-1}w_{j2}$. Since $w_{j-1} \in W^{\mathrm{left}}$ and $w_{j2} \in W^{\emptyset|}$, (E2) is clearly fulfilled in $w_{j-1}w_{j2}$, thus we have shown that $w_{j-1}w_{j2} \in W^{\mathrm{left}}$ where, obviously, $\mathrm{I}(w_{j-1}w_{j2}) = \mathrm{I}w_{j-1}$. $\square$

4.4 Concluding remarks

The main result of [9] proves that, given a group variety $\mathcal{U}$, if S is an inverse semigroup and ϱ a congruence on S such that the idempotent classes of ϱ belong to $\mathcal{U}$ then the extension (S, ϱ) is embeddable in a λ-semidirect product extension of a member of $\mathcal{U}$ by S/ϱ. Moreover, Theorem 3.2.1 establishes that if S is a completely simple semigroup and ϱ a group congruence on S such that the idempotent class of ϱ belongs to the variety $\mathcal{CS}(\mathcal{U})$ of all completely simple semigroups whose subgroups are from $\mathcal{U}$ then (S, ϱ) is embeddable in a semidirect product extension of a member of $\mathcal{CS}(\mathcal{U})$ by S/ϱ.

The question naturally arises whether Theorem 4.1.1 can be strengthened so that the variety of all completely simple semigroups be replaced by any variety of completely simple semigroups.

Problem 4.4.1. For which varieties $\mathcal{V}$ of completely simple semigroups is it true that if S is an E-solid locally inverse semigroup and ϱ an inverse semigroup congruence on S such that the idempotent classes of ϱ belong to $\mathcal{V}$ then the extension (S, ϱ) is embeddable in a λ-semidirect product extension of a member of $\mathcal{V}$ by S/ϱ?

Note that in the special case where $\mathcal{V}$ is the variety of rectangular bands, the answer is affirmative. The approach applied in the proof of Theorem 4.1.1 works, and the technical details are significantly simpler (no $\wedge$ operation is needed, the invariant congruence corresponding to the variety of rectangular bands is easy to handle). Thus the following result yields.

Proposition 4.4.2. *A regular semigroup is a generalized inverse semigroup if and only if it is embeddable in a λ-semidirect product of a rectangular band by an inverse semigroup.*

Summary

Group extensions play a fundamental role both in the structure theory and in the theory of varieties of groups. In 1950, Kaloujnine and Krasner proved that any extension of a group N by a group H is embeddable in the wreath product of N by H, see [22]. Note that the wreath product of N by H is a special semidirect product of a direct power of N by H.

Semigroups are natural generalisations of groups. One of the important classes of semigroups where the influence of the Kaloujnine–Krasner Theorem is fundamental is the class of regular semigroups.

Inverse semigroups are one of the most natural generalisations of groups. By Cayley's Theorem we can think of groups (up to isomorphism) as sets of permutations on a given set which are closed under composition and taking inverse. A similar result, the Wagner–Preston Theorem, shows that inverse semigroups are, (also up to isomorphism) sets of partial permutations on a set X (i.e., bijection between subsets of X) which are closed under composition of partial maps and taking inverse. In a group, every congruence is fully determined by the congruence class which is a subgroup. In an inverse semigroup congruence, there might be several semigroup congruence classes, but they play a similar role. More generally, if S is a regular semigroup and ϱ is a congruence on S such that S/ϱ is a group (more generally, an inverse semigroup) then ϱ is uniquely determined by the single congruence class (by the set of congruence classes) which is a subsemigroup in S (which are subsemigroups in S). What is more, the subsemigroup congruence classes are all regular and their union, called the kernel of ϱ and denoted by $\operatorname{Ker}\varrho$, is also a regular subsemigroup in S.

A regular semigroup is *completely simple* if it is a union of its maximal subgroups and it contains only one $\mathcal{D}$-class. Note that in a completely simple semigroup all maximal subgroups are isomorphic to each other. Completely simple semigroups are also natural generalisations of groups.

Let K be a semigroup and T an inverse semigroup. If S is a semigroup and ϱ is a congruence on S such that S/ϱ is isomorphic to T and $\operatorname{Ker}\varrho$ is isomorphic to K then S is called an *extension of K by T*.

Let K, T be semigroups. We denote the endomorphism monoid of K by $\operatorname{End} K$. We say that T acts on K by endomorphisms on the left, in short, T acts on K if an antihomomorphism $\varepsilon\colon T \to \operatorname{End} K$, $t \mapsto \varepsilon_t$ is given, that is a map, where $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ for any $u, t \in T$. For brevity, we will use the usual notation ${}^t a$ to denote $a\varepsilon_t$ ($a \in K, t \in T$). The *semidirect product* $K \rtimes T$ is defined on the set $K \times T$ by multiplication

$$(a, t)(b, u) = (a \cdot {}^t b, tu).$$

A related construction is the following. For any semigroups K,T, an action of T on the direct power K^T can be defined in the following natural way: for any $f \in K^T$ and $t \in T$, let ${}^t f$ be the element of K^T where $u({}^t f) = (ut)f$ for any $u \in T$. The semidirect product $K^T \rtimes T$ defined by this action is called the *wreath product of K by T*, and is denoted by $K \wr T$. In case K and T are groups, these are the usual definitions of a semidirect product $K \rtimes T$ and of the wreath product $K \wr T$ of K and T.

If K is a semigroup and T is a group then $K \rtimes T$ and $K \wr T$ are regular [inverse, completely simple] if and only if K is. However, in general, a semidirect product $K \rtimes T$ is not regular even if both K and T are inverse. This led Billhardt [6] to adapt these constructions to the inverse case in the following way. Let K be a semigroup and T an inverse semigroup acting on K. The *λ-semidirect product* $K \rtimes_\lambda T$ is defined on the underlying set

$$\{(a, t) \in K \times T\colon {}^{tt^{-1}}a = a\}$$

by multiplication

$$(a, t)(b, u) = ({}^{(tu)(tu)^{-1}}a \cdot {}^t b, tu),$$

for all $a, b \in K$, $t, u \in T$.

There are a number of embedding theorems in the structure theory of regular semigroups. Next we recall some of those considered as origins of our research. An inverse semigroup is said to be E-unitary if it is an extension of a semilattice by a group. The classical result of O'Carroll [25] states that every E-unitary inverse semigroup is embeddable in a semidirect product of a semilattice by a group. By a band we mean a semigroup where every element is idempotent and by an E-unitary regular semigroup we mean an extension of a band by a group. O'Carroll's result was extended by Szendrei [27] for extensions of certain bands, called regular, by groups. She proved that every E-unitary regular semigroup whose band of idempotents is in a regular band variety $\mathcal{V}$ is embeddable in a semidirect product of a band from $\mathcal{V}$ by a group. On the other hand, Billhardt [7] showed that there exists an E-unitary regular semigroup which is not embeddable in a semidirect product of a band by a group.

A congruence on an inverse semigroup S is said to be idempotent separating if every congruence class contains at most one idempotent and so, every subsemigroup congruence class is a subgroup of S. On the opposite, a congruence is said to be idempotent pure if each congruence class containing an idempotent consists of idempotents. Houghton [16] proved that every idempotent separating extension of an inverse semigroup is embeddable in a kind of wreath product of inverse semigroups, he introduced for the purpose of this proof. Billhardt [5] showed the same with λ-wreath product instead of Houghton's wreath products. Both Houghton's and Billhardt's proof show similarities to the standard proof of the Kaloujnine–Krasner Theorem. Billhardt [6] also proved that an inverse semigroup S with an idempotent pure congruence ϱ is embeddable in a λ-semidirect product of a semilattice by S/ϱ, which generalises O'Carroll's result in another direction.

Billhardt and Szittyai [9] strengthened the former result on idempotent separating extensions by proving that if S is an inverse semigroup and ϱ is an idempotent separating congruence such that every idempotent ϱ-class is from a group variety $\mathcal{V}$ then S is embeddable in a λ-semidirect product of a group from $\mathcal{V}$ by S/ϱ.

The thesis concentrates on E-solid locally inverse semigroups which are extensions by inverse semigroups and the idempotent classes are completely simple. The main problem we will give an answer to is whether such extensions are embeddable in a λ-semidirect product of a completely simple semigroup by an inverse semigroup.

Chapter 3 which contains the results of [10] deals with the special case where the extensions are by groups. In this case, the extension itself is necessarily completely simple. The motivation for considering this case first was to check whether the general embedding result we intend to prove holds in this special case. In fact, we prove a somewhat stronger result than that following from our main result, see Theorem 3.2.1.

Theorem. *Any extension of a completely simple semigroup U by a group*

H is embeddable in a semidirect product of a completely simple semigroup V by the group H, where the maximal subgroups of V are direct powers of the maximal subgroups of U.

Note that the embedding given in the proof mimics the standard proof of the Kaloujnine–Krasner Theorem. Comparing this easy proof to that of the main result, one can see how much more complicated the extensions by inverse semigroups might be than those by groups.

The semidirect product of V by H constructed in the proof of the result mentioned in the previous paragraph is not the wreath product of U by H. Since completely simple semigroups are fairly close to groups — they are disjoint unions of pairwise isomorphic groups —, it is natural to ask whether the Kaloujnine–Krasner Theorem holds for such extensions. In the first section of Chapter 3, we establish that this is not the case in general (Theorem 3.1.2).

Theorem. *There exists a completely simple semigroup which is an extension of a completely simple semigroup U by a group H and which is not embeddable in the wreath product of U by H.*

However, we also show that the Kaloujnine–Krasner Theorem is valid within the class of central completely simple semigroups (Proposition 3.1.1). They are defined by the property that the product of any two idempotents lies in the centre of the containing maximal subgroup.

Proposition. *Each central completely simple semigroup which is an extension of a (necessarily also central) completely simple semigroup U by a group H is embeddable in the wreath product of U by H.*

A regular semigroup S is called *locally inverse* if each local submonoid eSe ($e \in E_S$) is an inverse subsemigroup. Note that each inverse semigroup and each completely simple semigroup is locally inverse.

A regular semigroup S is called *E-solid* if the subsemigroup of S generated by the idempotents is the union of subgroups of S. In particular, inverse semigroups, completely simple semigroups, and members ofsevereal other well-studied classes (e.g., orthodox, completely regular) are E-solid. It is also known, that a regular semigroup is E-solid if and only if the semigroup classes of the least inverse semigroup congruence are completely simple semigroups, see Yamada (and Hall) [32]. Thus the kernel of the least inverse semigroup congruence of an E-solid locally inverse semigroup is a semilattice of completely simple semigroups which is also locally inverse.

In Chapter 4 we give affirmative answer to the main question of the thesis formulated above (see Theorem 4.1.1):

Main result. *If S is an E-solid locally inverse semigroup and ϱ is an inverse semigroup congruence on S such that the subsemigroup ϱ-classes are completely simple then S is embeddable in a λ-semidirect product of a completely simple semigroup by S/ϱ.*

As a corollary, we obtain that the E-solid locally inverse semigroups are, up to isomorphism, the regular subsemigroups of the λ-semidirect products of completely simple semigroups by inverse semigroups (Corollary 4.1.2).

Kuřil and Szendrei [23] developed a method, called the 'canonical embedding technique' to prove or disprove whether an extension S of a member K of a given "nice" class $\mathcal{C}$ of regular semigroups by an inverse semigroup T is embeddable in a λ-semidirect product of a member of $\mathcal{C}$ by T. They construct an appropriate K together with a λ-semidirect product $K \rtimes_\lambda T$ of K by T, and a homomorphism from S to $K \rtimes_\lambda T$ such that this homomorphism is injective if and only if S is embeddable in a λ-semidirect product of a member of $\mathcal{C}$ by T. In our case K is a factor of some word algebra. To prove injectivty, we have to show that every congruence class contains at most one special one-letter word. For this, we show combinatorial properties of the words of the congruence classes of these special one-letter words. We proved the main result using this technique. In comparison to other similar results where one had to prove properties fulfilled by applying generator relations of the congruence, we had to describe words of the algebra. For our very general class of semigroup we needed to give an almost full description of words.

Összefoglaló

A csoportbővítések alapvető szerepet játszanak mind a csoportok struktúraelméletében, mind a csoportok varietásainak elméletében. Kaloujnine és Krasner ([22]) 1950-ben bebizonyította, hogy egy N csoport H-val vett bővítése beágyazható N-nek H-val vett koszorúszorzatába. Megjegyezzük, hogy a koszorú szorzat egy speciális szemidirekt szorzat.

A félcsoportok a csoportok természetes általánosításai. Az egyik olyan félcsoport osztály, ahol a Kaloujnine–Krasner-tételnek jelentős hatása volt, a reguláris félcsoportok osztálya.

Az inverz félcsoportok az egyik legtermészetesebb általánosításai a csoportoknak. A Cayley-tétel alapján a csoportokra (izomorfiától eltekintve) úgy tekinthetünk, mint adott halmaz permutációinak olyan halmazaira, melyek zártak a kompozícióra és az inverzképzésre. Egy hasonló eredmény, a Wagner–Preston-tétel, azt állítja, hogy az inverz félcsoportok (izomorfizmus erejéig) éppen egy adott X halmaz parciális permutációinak (azaz X részhalmazai közötti permutációinak) halmazai, melyek zártak a parciális leképezések szorzására és az inverzképzésre. Egy csoportkongruenciát teljesen meghatároz az az osztálya, amely részcsoportot alkot. Egy inverz félcsoportban számos részfélcsoport kongruenciaosztály lehet, de azok hasonló szerepet játszanak. Pontosabban, ha S reguláris félcsoport és ϱ olyan kongruencia S-en, amelyre S/ϱ csoport [inverz félcsoport], akkor ϱ-t egyértelműen meghatározza az egyetlen olyan kongruenciaosztály [azon kongruenciaosztályok halmaza], mely részfélcsoport S-ben [melyek részfélcsoportok S-ben]. Továbbá a részfélcsoport kongruenciaosztályok regulárisak és azok egyesítése, melyet ϱ magjának hívunk és $\operatorname{Ker}\varrho$-val jelölünk, szintén reguláris részfélcsoport S-ben.

Egy félcsoportot *teljesen egyszerűnek* nevezünk, ha előáll, mint maximális részcsoportjainak egyesítése, és egyetlen $\mathcal{D}$-osztályból áll. Egy teljesen egyszerű félcsoportban a maximális részcsoportok izomorfak egymással. A teljesen egyszerű félcsoportok a csoportok is a csoportok általánosításai, csak más irányban.

Legyen K tetszőleges, T pedig inverz félcsoport. Ha S olyan félcsoport és ϱ olyan kongruencia S-en, melyre S/ϱ izomorf T-vel és $\operatorname{Ker}\varrho$ izomorf K-val, akkor azt mondjuk, hogy az S félcsoport *K-nak T-vel vett bővítése*.

Legyenek K,T tetszőleges félcsoportok, és jelöljük K endomorfizmus-monoidját $\operatorname{End} K$-val. Azt mondjuk, hogy T hat K-n, ha adott egy $\varepsilon\colon T \to \operatorname{End} K$, $t \mapsto \varepsilon_t$ antihomomorfizmus, azaz olyan leképezés, melyre $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ bármely $t, u \in T$-re. Az egyszerűség kedvéért a szokásos ${}^t a$ jelölést fogjuk használni $a\varepsilon_t$ $(a \in K, t \in T)$ helyett. A $K \times T$ halmazt az

$$(a, t)(b, u) = (a \cdot {}^t b, tu)$$

egyenlőséggel definiált szorzással így nevezzük: *K szemidirekt szorzata T-vel*, jelölése: $K \rtimes T$.

Ehhez kapcsolódó konstrukció a következő. Tetszőleges K,T félcsoportok esetén T hat a K^T direkt hatványon a következő módon: minden $f \in K^T$ és $t \in T$ esetén ${}^t f$ az az elem K^T-ben, melyre $u({}^t f) = (ut)f$ minden $u \in T$ esetén. Az ennek segítségével definiált szemidirekt szorzatot *K-nak T-vel vett koszorú szorzatának* nevezzük, és $K \wr T$-vel jelöljük.

Ha K félcsoport, T pedig csoport, akkor $K \rtimes T$ és $K \wr T$ pontosan akkor reguláris [inverz, teljesen egyszerű] félcsoport, ha K is az. Általában $K \rtimes T$ viszont nem inverz félcsoport még akkor sem, ha K és T is inverz félcsoport. Ez vezette el Billhardtot [6] a konstrukció inverz félcsoportokra történő következő adaptálásához. Legyen K félcsoport és T olyan inverz félcsoport, amely hat K-n. A *K-nak T-vel vett λ-szemidirekt szorzatán* azt a félcsoportot értjük, melynek alaphalmaza

$$\{(a, t) \in K \times T \colon {}^{tt^{-1}} a = a\}$$

és amelyen a szorzás

$$(a, t)(b, u) = ({}^{(tu)(tu)^{-1}} a \cdot {}^t b, tu)$$

minden $a, b \in K$, $t, u \in T$ esetén.

A reguláris félcsoportok elméletében sok beágyazási tétel született. A következőkben ezek közül említünk meg néhányat, melyek a kutatásunk előzményét képezték. Egy inverz félcsoportot E-unitérnek nevezünk, ha félhálónak csoporttal vett bővítése. O'Carroll klasszikus eredménye [25] azt mondja ki, hogy minden E-unitér inverz félcsoport beágyazható félhálónak csoporttal vett szemidirekt szorzatába. Kötegnek olyan félcsoportot nevezzük, melyben minden elem idempotens, E-unitér reguláris félcsoportnak pedig olyat, amely kötegnek csoporttal vett bővítése. O'Carroll eredményét az úgynevezett reguláris kötegekre Szendrei általánosította [27]-ben. Bebizonyította, hogy minden olyan E-unitér reguláris félcsoport, melyben az idempotensek kötege egy $\mathcal{V}$ reguláris kötegvarietáshoz tartozik, beágyazható egy $\mathcal{V}$-beli köteg csoporttal vett szemidirekt szorzatába. Másfelől Billhardt [7] példát adott olyan E-unitér reguláris félcsoportra, amely nem ágyazható be kötegnek csoporttal vett szemidirekt szorzatába.

Egy S inverz félcsoporton kongruenciát idempotens szétválasztónak nevezünk, ha minden kongruenciaosztály legfeljebb egy idempotenst tartalmaz, azaz minden részfélcsoport kongruenciaosztály részcsoport S-ben. Egy kongruenciát idempotenstisztának nevezünk, ha minden idempotenst tartalmazó kongruenciaosztály csak idempotens elemekből áll. Houghton [16] bebizonyította, hogy inverz félcsoport minden idempotens-szétválasztó bővítése beágyazható inverz félcsoportok valamilyen koszorú szorzatába, melyet a bizonyításhoz konstruált. Billhardt [5] ugyanezt bizonyította be λ-koszorú szorzatokat használva. Mindkét bizonyítás hasonlóságokat mutat a Kaloujnine–Krasner-tétel standard bizonyításával. Billhardt [6]-ban azt is igazolta, hogy ha $|rho$ egy idempotenstiszta kongruencia az S inverz félcsoporton, akkor S beágyazható félhálónak S/ϱ-val vett λ-szemidirekt szorzatába, mely O'Carroll tételének egy másik irányú általánosítása.

Billhardt és Szittyai [9]-ben erősebbé tette az előző eredményt idempotens szétválasztó kongruenciákra. Bebizonyították, hogy ha S inverz félcsoport és ϱ idempotens szétválasztó kongruencia S-en melyre minden idempotens ϱ-osztály egy $\mathcal{V}$ csoportvarietásból származik, akkor S beágyazható egy $\mathcal{V}$-beli csoport S/ϱ-val vett szemidirekt szorzatába.

A disszertáció E-tömör lokálisan inverz félcsoportokkal foglalkozik, melyek olyan inverz félcsoporttal vett bővítések, ahol a részfélcsoport osztályok teljesen egyszerű félcsoportok. A fő probléma, amelyre választ adunk, az, hogy beágyazható-e minden ilyen bővítés teljesen egyszerű félcsoportnak inverz félcsoporttal vett szemidirekt szorzatába.

A 3. Fejezet, mely a [10] cikk eredményeit tartalmazza, azzal a speciális esettel foglalkozik, ahol a bővítések csoporttal történnek. Ilyen esetben maga a bővítés is szükségszerűen teljesen egyszerű félcsoport. A motivációja ennek a kérdésnek az volt, hogy ellenőrizzük, igaz-e az általánosan megfogalmazott kérdés ebben a speciális esetben. Valójában valamivel erősebb állítást látunk be ebben az esetben, mint ami a főtételből következik, lásd 3.2.1. Tétel.

Tétel. *Egy U teljesen egyszerű félcsoportnak H csoporttal vett tetszőleges bővítése beágyazható egy V teljesen egyszerű félcsoportnak H-val vett szemidirekt szorzatába, ahol V maximális részcsoportjai U maximális részcsoportjainak direkt hatványai.*

Megjegyezzük, hogy a fenti tétel bizonyítása során használt beágyazás a Kaloujnine–Krasner tétel standard bizonyításában használt beágyazást általánosítja. Összehasonlítva ezt az egyszerű bizonyítást a disszertáció főtételének bizonyításával, könnyen látható, hogy a csoporttal vett bővítések mennyivel egyszerűbben viselkednek, mint az inverz félcsoporttal vett bővítések.

Bár az előző tételben említett V-nek H-val vett szemidirekt szorzata hasonlít a koszorúszorzatra, valójában nem az. Mivel a teljesen egyszerű félcsoportok nagyon közel állnak a csoportokhoz, így természetesen merül fel a kérdés, hogy a Kaloujnine–Krasner-tétel igaz-e rájuk. A 3. Fejezetben bebizonyítjuk, hogy általában nem ez a helyzet (3.1.2. Tétel)

Tétel. *Van olyan teljesen egyszerű félcsoport, amely egy U teljesen egyszerű félcsoport H csoporttal vett bővítése, de nem ágyazható be U-nak H-val vett koszorúszorzatába.*

Ugyanakkor belátjuk, hogy a Kaloujnine–Krasner-tétel igaz ún. centrális teljesen egyszerű félcsoportokban (3.1.1 Állítás), azaz ahol két idempotens szorzata mindig az őt tartalmazó legbővebb részcsoport centrumában van.

Állítás. *Minden olyan centrális teljesen egyszerű félcsoport, mely egy U (szükségképpen centrális) teljesen egyszerű félcsoport H csoporttal vett bővítése, beágyazható U-nak H-val vett koszorúszorzatába.*

Egy S reguláris félcsoportot *lokálisan inverz félcsoportnak* nevezünk, ha minden eSe ($e \in E_S$) "lokális" részmonoidja inverz részfélcsoport. Megjegyezzük, hogy minden inverz és minden teljesen egyszerű félcsoport lokálisan inverz.

Egy S reguláris félcsoportot *E-tömör félcsoportnak* hívunk, ha az S idempotensei által generált részfélcsoport, S részcsoportjainak egyesítése. Speciálisan az inverz és a teljesen egyszerű félcsoportok E-tömör félcsoportok, és ebbe az osztályba tartoznak további sokat vizsgált félcsoportosztályok (pl. ortodox, teljesen reguláris) tagjai is. Ismert, hogy egy reguláris félcsoport pontosan akkor E-tömör, ha a legkisebb inverz félcsoport kongruenciájának részfélcsoport osztályai teljesen egyszerűek, lásd Yamada (és Hall) [32]. Ebből következik, hogy egy E-tömör lokálisan inverz félcsoport legkisebb inverz félcsoport kongruenciájának magja teljesen egyszerű félcsoportok lokálisan inverz félhálója.

A 4. Fejezetben igenlő választ adunk a disszertáció fő kérdésére, amit korábban megfogalmaztunk (lásd: 4.1.1. Tétel):

Főtétel. *Ha S E-tömör lokálisan inverz félcsoport és ϱ olyan inverz félcsoport kongruencia S-en, melyben minden részfélcsoport osztály teljesen egyszerű, akkor S beágyazható teljesen egyszerű félcsoport S/ϱ-val vett λ-szemidirekt szorzatába.*

A tétel következményeként kimondható, hogy az E-tömör lokálisan inverz félcsoportok izomorfizmus erejéig a teljesen egyszerű félcsoportok inverz félcsoporttal vett λ-szemidirekt szorzatainak reguláris részfélcsoportjai (4.1.2. Következmény).

Kuřil és Szendrei [23]-ben kifejlesztettek egy "kanonikus beágyazási módszert", amelynek segítségével bizonyítható vagy cáfolható, hogy egy S félcsoport, mely valamely $\mathcal{C}$ "szép" osztályból származó félcsoport bővítése egy T inverz félcsoporttal, beágyazható-e egy, a $\mathcal{C}$ osztályból származó K félcsoport T-vel vett λ-szemidirekt szorzatába. A bizonyításhoz leírtak egy megfelelő K félcsoportot és egy $K \rtimes_\lambda T$ λ-szemdirekt szorzatát, valamint adtak egy homomorfizmust S-ből $K \rtimes_\lambda T$-be, amely pontosan akkor injektív, ha létezik beágyazás S-ből valamely $\mathcal{C}$-beli félcsoport T-vel vett λ-szemidirekt szorzatába. Itt K egy megfelelő szóalgebra faktora. Az injektivitás bizonyításához pedig azt kell megmutatnunk, hogy minden kongruenciaosztály legfeljebb egy speciális, egy betűből álló szót tartalmaz. Ehhez az egybetűs szavak kongruenciaosztályaiba tartozó szavak kombinatorikus tulajdonságait kellett leírnunk. Ezt a módszert használva bizonyítottuk be a főtételt. A korábbi hasonló bizonyításokban a szavak leírása helyett elegendő volt észrevenni, hogy a kongruencia elemi lépései (generátorelemeinek alkalmazása) megőriznek bizonyos tulajdonságokat. Az általunk vizsgált nagyon általános félcsoport osztályban lényegében teljes leírásra volt szükség.

Statement

The author's publications used in this thesis are [10] and [12]. The author's further publication is [11].

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