Fuzzy reasoning models and fuzzy truth value based inference

Theses of Ph.D. Dissertation

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University of Szeged
Szeged, 2009

1 Introduction

The main contribution of this thesis is that we approach fuzzy reasoning from three different angles. First, by creating a new, hybrid fuzzy rule-learning model with classical inference methods. Second, by introducing a new reasoning method, with intuitive and practical properties. And third, by supervising the so-called fuzzy truth value-based reasoning model, and showing new ways to represent and calculate with fuzzy operations such as conjunctions and implications.

2 The Squashing function

The sigmoid function is defined as

$$\sigma_d^{(\beta)}(x) = \frac{1}{1 + e^{-\beta(x-d)}}.$$

In order to get an approximation of the generalized cut function i.e.

$$[x]_{a,\delta} = \begin{cases} 0, & \text{if} \quad x \leq a - \delta/2 \\ \frac{x-(a-\delta/2)}{\delta}, & \text{if} \quad a - \delta/2 < x < a + \delta/2 \\ 1, & \text{if} \quad a + \delta/2 \leq x \end{cases}$$

we integrated the difference of two sigmoid functions. The result is called the squashing function [1, 2]

$$S_{a,\delta}^{(\beta)}(x) = \frac{1}{2\delta} \ln \left(\frac{\sigma_{a+\delta}^{(-\beta)}(x)}{\sigma_{a-\delta}^{(-\beta)}(x)} \right)^{1/\beta},$$

where $a \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$.

Theorem 2.1. *Let $a \in \mathbb{R}$ and $\delta, \beta \in \mathbb{R}^+$. Then $\lim_{\beta \to \infty} S_{a,\delta}^{(\beta)}(x) = [x]_{a,\delta}$ and $S_{a,\delta}^{(\beta)}(x)$ is continuous in x, a, δ and β.*

The squashing function approximates the cut function. We defined the approximation error of the squashing function as

$$\varepsilon_\beta = S_{0,\delta}^{(\beta)}(-\delta) = \frac{1}{2\delta} \ln \left(\frac{\sigma_{\delta}^{(-\beta)}(-\delta)}{\sigma_{-\delta}^{(-\beta)}(-\delta)} \right)^{1/\beta},$$

where $\beta > 0$. We have the following lemma on the relationship between ε_β and β.

Lemma 2.2. *Let us fix the value of δ. Then $\varepsilon_\beta < c \cdot \frac{1}{\beta}$, where $c = \frac{\ln 2}{2\delta}$ is constant.*

The following derivatives of the squashing function are continuous and can be expressed by itself and sigmoid functions:

$$\begin{aligned}
\frac{\partial S_{a,\delta}^{(\beta)}(x)}{\partial x} &= \frac{1}{2\delta}\left(\sigma_{a-\delta}^{(\beta)}(x) - \sigma_{a+\delta}^{(\beta)}(x)\right), \\
\frac{\partial S_{a,\delta}^{(\beta)}(x)}{\partial a} &= \frac{1}{2\delta}\left(\sigma_{a+\delta}^{(\beta)}(x) - \sigma_{a-\delta}^{(\beta)}(x)\right), \\
\frac{\partial S_{a,\delta}^{(\beta)}(x)}{\partial \delta} &= \frac{1}{2\delta}\left(\sigma_{a+\delta}^{(\beta)}(x) + \sigma_{a-\delta}^{(\beta)}(x)\right) - \frac{1}{\delta} S_{a,\delta}^{(\beta)}(x).
\end{aligned}$$

2.1 Approximation of piecewise linear membership functions

Piecewise linear membership functions can be constructed from generalized cut functions, and thus approximated by using squashing functions with a suitable conjunction operator. We have chosen the Łukasiewicz conjunction. The formula of conjunction also uses the squashing function in place of the cut function. This way, the membership function and the operator are both constructed from the same component.

The approximation of a trapezoid membership function (ATR) is the following:

$$ATR(x, a_l, \delta_l, a_r, \delta_r, \beta) = S_{1/2,1/2}^{(\beta)}\left(S_{a_l,\delta_l}^{(\beta)}(x) + S_{a_r,\delta_r}^{(-\beta)}(x) - 1\right).$$

3 Rule based fuzzy classification using squashing functions

Our proposed three-stage fuzzy rule construction algorithm is the following [3].

1. The training data is fuzzified using approximated trapezoidal membership functions for each input dimension.
2. The structures of the logic rules are evolved by a genetic algorithm.
3. A gradient based local optimization is applied to fine-tune the membership functions.

The first step of rule learning is a discretization procedure, the fuzzification of the training data. To avoid complex formulas, we are only concerned with disjunctions of conjunctions i.e. formulas given as disjunctive normal forms. A set of rules is represented by a constrained neural network so that the hidden layer contains only conjunctive neurons and the output layer contains only disjunctive neurons. Every output neuron corresponds to one rule. For multi class problems several networks (with one output node) can be trained, one network per class. The output class is decided by taking the maximum of the activations of the networks' output. In the second step this structure is optimized by a genetic algorithm to give the best possible result. The third step of the rule construction algorithm requires that both the membership functions and the logical connectives have a continuous gradient. The continuous squashing function-based approximation is used.

3.1 Applications of the classification method

The following shorthand notation will be used for the description of membership functions:

$$[a_1 <_{\delta_1} x <_{\delta_2} a_2] \, ,$$

where a_i denote the centers and δ_i denote the widths of the left and right slopes. If one side of the trapezoid is outside of the corresponding input interval then it is omitted.

The problem sets are from the UCI machine learning repository. Our results on the Iris dataset are:

- Iris Setosa: $[x_3 <_{1.7} 3.8]$
- Iris Virginica: $[1.5 <_{0.5} x_4]$
- Iris Versicolor: $[0.35 <_{3.76} x_3 <_{1.55} 6.6]$
  AND $[0.27 <_{1.28} x_4 <_{0.32} 1.9]$

These rules give 96% accuracy with 5 misclassified samples. Only two features are used, and the average certainty factors are [98% 92% 96%] for the classes.

Our results on the Wine dataset are:

- Wine 1: $[435 <_{683} x_{13}]$
- Wine 2: $[x_{10} <_{3.36} 5.9]$
- Wine 3: $[x_7 <_{1.26} 1.74]$

These rules give 95% accuracy with 6 misclassified and 3 undecided samples. Note that only three features are used (x_7, x_{10}, x_{13}) in the rules. The average certainty factors are [88% 85% 85%].

Our results on the Ionosphere dataset are:

$$[0.69 <_{0.5} x_1] \text{ AND } [-0.19 <_{0.013} x_5]$$

With 1/2 threshold, this simple rule gives 88% accuracy.

Our results on the Thyroid dataset are:

- Normal: $[4.92 <_{2.46} x_2 <_{6.95} 14.57]$
- Hypo: $[x_2 <_{3.75} 6.2]$
- Hyper: $[10.95 <_{4.31} x_2 <_{12.77} 36.8]$

These rules give 94.8% accuracy with 11 misclassifications. Note that only x_2 is used. The average certainty factors are [95% 88% 94%].

4 Reasoning using approximated fuzzy intervals

Let $A, A^* \in \mathscr{F}(X)$ and $B \in \mathscr{F}(Y)$ be fuzzy sets on the universes of discourse X and Y, where $\mathscr{F}$ is the set of all fuzzy sets. The compositional rule of inference (CRI) introduced by Zadeh [7] states that knowing A^* and the rule "IF A THEN B", the conclusion $B^* \in \mathscr{F}(Y)$ is calculated by means of the combination/projection principle of the form

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \wedge R(A(x), B(y))\},$$

where $R \subset \mathcal{I}^X \times \mathcal{I}^Y$ is a fuzzy relation.

We established conditions under which the CRI reasoning scheme with a continuous t-norm $\triangle$ and its residual $\triangleright$ as the fuzzy relation is closed under sigmoid-like functions [4].

4.1 Closure properties of the generalized CRI

Theorem 4.1 ($\wedge$-based CRI)**.** *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the original CRI. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity, i.e. both are strictly increasing or decreasing, then

$$B^*(y) = B(y) \vee A^*(A^{-1}(B(y)))$$

where A^{-1} is the (unique) inverse function of A, and B^* is also a sigmoid-like function.

If $A^*(x) = A'(x)$ for a negation $'$ i.e. if the functions have different type of monotonicity, then $B^*(y) = 1$, i.e. the conclusion is interpreted as unknown.

Corollary 4.2. *Three special cases of Theorem 4.1 are:*

- *If $A^*(x) > A(x)$ for all x then $A^*(A^{-1}(x)) > x$ and so $B^*(y) = A^*(A^{-1}(B(y)))$ for all y.*
- *If $A^*(x) \leq A(x)$ for all x then $A^*(A^{-1}(x)) \leq x$ and so $B^*(y) = B(y)$ for all y.*
- *If A^* is ν-sharper than A (or vice versa) then $B^*(y)$ has two parts divided by $A^{-1}(\nu)$ and can be calculated according to the previous two cases.*

Theorem 4.3 (Product-based CRI)**.** *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the generalized CRI with the product t-norm and its residual, the Goguen implication. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity then

$$B^*(y) = B(y) \cdot \bigvee\nolimits_{x:A(x) \geq B(y)} \{A^*(x)/A(x)\},$$

where B^* is also a sigmoid-like function.

If $A^*(x) = A'(x)$ for a negation $'$, then $B^*(y) = 1$.

Theorem 4.4 (Nilpotent CRI)**.** *Let A, A^*, B be sigmoid-like fuzzy sets, and let the reasoning scheme be the generalized CRI with the Łukasiewicz t-norm and its residual. Then B^* can be calculated as follows.*

If A and A^* have the same type of monotonicity then

$$B^*(y) = B(y) + \bigvee_{x:A(x) \geq B(y)} \{A^*(x) - A(x)\}.$$

Here B^* is a sigmoid-like function if and only if $A^* \subseteq A$.

If $A^*(x) = A'(x)$ for a strict negation then $B^*(y) = 1$.

These theorems are also valid regarding respective isomorphic t-norms to the minimum, the product and the Łukasiewicz t-norm. In case of the minimum, it is easy to see, since it is only isomorph to itself. Regarding Archimedean t-norms, a strictly increasing bijection transformation of the generator function does not change the assertions of the proofs. Regarding ordinal sums, the previous Theorems can be applied for each t-norm summand separately.

Theorem 4.5. *Let $\triangle$ be an arbitrary ordinal sum of a family of continuous t-norms, $\triangleright$ its residual. Let A,A^* and B be sigmoid-like functions. Let*

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \triangle (A(x) \triangleright B(y))\}.$$

If A and A^* are both increasing or decreasing, then B^* is sigmoid-like if all summands of $\triangle$ are either the minimum or strict. If additionally $A^* \subseteq A$ then B^* is sigmoid like for all continuous ordinal sums $\triangle$. If A and A^* have different type of monotonicity then $B^* \equiv 1$.

The Łukasiewicz t-norm based generalized CRI is closed under sigmoid-like functions only if $A^* \subseteq A$, since in this case $B^* \equiv B$. In the general case, it introduces a non-zero level of indetermination of the conclusion. Using the product t-norm, the conclusion is sigmoid-like. The min-based CRI is also closed under sigmoid-like functions.

4.2 The Membership Driven Inference

The Membership Driven Inference (MDI) reasoning scheme [4] is

$$B^* = A^* \circ A^{-1} \circ B,$$

where A and B are the antecedent and the consequent of a rule, A^* is the input and B^* is the output of the rule.

This reasoning scheme is simple, and it depends only on the sigmoid-like membership functions of A, A^* and B. It does not contain explicitly any conjunctive, implicative or other operation, nor any similarity or distance measure. Although, MDI originates from the min-based generalized CRI it can also be regarded as a modified fuzzy truth value (FTV) reasoning where the mapping M_I responsible for the inference in truth value space is the identity. There is no t-norm for which $M_I \equiv id$, hence the MDI is not a special case of the FTV reasoning scheme. Its properties are summarized as follows.

Theorem 4.6. *The MDI reasoning scheme with sigmoid-like membership functions fulfills the following properties:*

i) If $A^* = A$ then $B^* = B$ (generalized modus ponens)

ii) If $B^* = {}' \circ B$ then $A^* = {}' \circ A$ for any negation $'$ (generalized modus tollens)

iii) If $C^* = B^* \circ B^{-1} \circ C$ then $C^* = A^* \circ A^{-1} \circ C$ (generalized chain rule)

A more general rule is valid, covering the first two cases:

iv) For any unary operator f, $A^* = f \circ A$ if and only if $B^* = f \circ B$. Note that with the proper f function this case involves the ν-sharpening of A, too.

Moreover, let alone sigmoid-like functions, for any A and B, $A^* \equiv 0$ i.e. undefined if and only if $B^* \equiv 0$, and $A^* \equiv 1$ i.e. unknown if and only if $B^* \equiv 1$.

The problem of fuzzy abduction is also fulfilled by the MDI reasoning scheme: in case B^* is given, and A^* is unknown, then it is easy to see that $A^* = B^* \circ B^{-1} \circ A$.

Strictly speaking, this inference mechanism is not equivalent to the generalized CRI reasoning scheme of Zadeh (due to different fulfilled axioms), nor to similarity based reasoning (due to the lack of similarity measure). And since $A^* \circ A^{-1}$ can be treated as an unary operator (a truth-function) or as some kind of (non-commutative) similitude between A and A^*, this reasoning scheme can be positioned in between the generalized CRI and the similarity based reasoning schemes.

4.3 Efficient computation of the MDI reasoning scheme

Theorem 4.7. *If all fuzzy sets are squashing functions, i.e. if*

$$A(x) = \langle a <_{\delta_a} x\rangle_\beta$$
$$A^*(x) = \langle a^* <_{\delta_{a^*}} x\rangle_\beta$$
$$B(x) = \langle b <_{\delta_b} x\rangle_\beta$$

then $B^*(x) = \langle b^* <_{\delta_{b^*}} x\rangle_\beta$*, where*

$$b^* = b + \frac{\delta_b}{\delta_a}(a^* - a) \qquad\qquad \delta_{b^*} = \frac{\delta_b \delta_{a^*}}{\delta_a}$$

Theorem 4.8. *Suppose* $\beta > 0$ *and finite. If all fuzzy sets are trapezoidal fuzzy intervals approximated by squashing functions, i.e. if*

$$A(x) = A\Pi(x; \beta, a_L, \delta_a^L, a_R, \delta_a^R),$$
$$A^*(x) = A\Pi(x; \beta, a_L^*, \delta_{a^*}^L, a_R^*, \delta_{a^*}^R),$$
$$B(x) = A\Pi(x; \beta, b_L, \delta_b^L, b_R, \delta_b^R),$$

then $B^*(x) = A\Pi(x; \beta, b_L^*, \delta_{b^*}^L, b_R^*, \delta_{b^*}^R)$*, where*

$$b_L^* = b_L + \frac{\delta_b^L}{\delta_a^L}(a_L^* - a_L), \qquad\qquad \delta_{b^*}^L = \frac{\delta_b^L \delta_{a^*}^L}{\delta_a^L},$$
$$b_R^* = b_R + \frac{\delta_b^R}{\delta_a^R}(a_R^* - a_R), \qquad\qquad \delta_{b^*}^R = \frac{\delta_b^R \delta_{a^*}^R}{\delta_a^R}.$$

5 Calculations of operations on fuzzy truth values

In general, the convolutions that define the operations on fuzzy truth values are difficult to calculate. If we restrict our investigations to extensions of continuous and Archimedean t-norms and t-conorms, we will get less complex results [5].

Theorem 5.1. *If $\triangle_1 = \wedge$, $\bigtriangledown = \vee$, and $\triangle_2 = \triangle$ is an arbitrary continuous and Archimedean t-norm, then the following hold for all $f, g \in \mathcal{F}$:*

$$\begin{aligned}
(f \blacktriangle_{\wedge} g)(z) &= \bigvee_{z=x\wedge y} (f(x) \triangle g(y)) &&= \left(\left(f^R \triangle g\right) \vee \left(f \triangle g^R\right)\right)(z), \\
(f \blacktriangledown_{\vee} g)(z) &= \bigvee_{z=x\vee y} (f(x) \triangle g(y)) &&= \left(\left(f^L \triangle g\right) \vee \left(f \triangle g^L\right)\right)(z).
\end{aligned}$$

Theorem 5.2. *If $\triangle_1$ and $\triangle_2$ are t-norms, s.t. $\triangle_1$ is continuous and Archimedean, then the following hold for all $f, g \in \mathcal{F}$. For $z > 0$:*

$$(f \blacktriangle g)(z) = \bigvee_{x \geq z} (f(x) \triangle_2 g(x \rhd_1 z)) = \bigvee_{y \geq z} (f(y \rhd_1 z) \triangle_2 g(y)).$$

If $\triangle_1$ is strict then for $z = 0$:

$$(f \blacktriangle g)(0) = \left(f(0) \triangle_2 g^R(0)\right) \vee \left(f^R(0) \triangle_2 g(0)\right),$$

and if $\triangle_1$ is nilpotent then for $z = 0$:

$$(f \blacktriangle g)(0) = \bigvee_{x} \left(f(x) \triangle_2 g^L(x')\right) = \bigvee_{y} \left(f^L(y') \triangle_2 g(y)\right),$$

where $\rhd_1$ denotes the residual implication of $\triangle_1$, and $x' = (x \rhd_1 0)$ is the strong negation corresponding to $\rhd_1$.

A similar theorem holds for extended Archimedean t-conorms.

Theorem 5.3. *If $\triangle$ is a t-norm, $\bigtriangledown$ is a continuous and Archimedean t-conorm, then the following hold for all $f, g \in \mathcal{F}$. For $z < 1$:*

$$(f \blacktriangledown g)(z) = \bigvee_{x \leq z} (f(x) \triangle g(x \lhd z)) = \bigvee_{y \leq z} (f(y \lhd z) \triangle g(y)).$$

where $\lhd$ denotes the residual coimplication of $\bigtriangledown$. If $\bigtriangledown$ is strict then for $z = 1$:

$$(f \blacktriangledown g)(1) = \left(f^L(1) \triangle g(1)\right) \vee \left(f(1) \triangle g^L(1)\right),$$

and if $\bigtriangledown$ is nilpotent then for $z = 1$:

$$(f \blacktriangledown g)(1) = \bigvee_{x} \left(f(x) \triangle g^R(x')\right) = \bigvee_{y} \left(f^R(y') \triangle g(y)\right),$$

where $\lhd$ denotes the residual coimplication of $\bigtriangledown$, and $x' = (x \lhd 1)$ is a strong negation.

5.1 Left- and right-maximal and monotonic fuzzy truth values

Theorems 5.2 and 5.3 in general do not considerably decrease computational complexity of the extended operations. To this end, we restrict our investigations to special classes of fuzzy truth values: let $\mathcal{F}^+$ and $\mathcal{F}^-$ denote the set of non-decreasing and non-increasing continuous fuzzy truth values, f is end-maximal if $f^L = f^R$, left-maximal if $f^L = f^{LR}$, right-maximal if $f^R = f^{LR}$ and normal if $f^{LR} = 1$, where

$$f^R(x) = \bigvee_{y \geq x} f(y) \quad \text{and} \quad f^L(x) = \bigvee_{y \leq x} f(y).$$

Corollary 5.4. *If f is right-maximal and $g \in \mathcal{F}^-$, then*

$$(f \blacktriangle g)(x) = f^{LR}(x) \bigtriangleup_2 g(x),$$

and $f \blacktriangle g \in \mathcal{F}^-$. Furthermore, if f is also normal, then $f \blacktriangle g = g$, i.e. f acts as a unit element.

Corollary 5.5. *If f is left-maximal and $g \in \mathcal{F}^+$, then*

$$(f \blacktriangledown g)(x) = f^{LR}(x) \bigtriangledown g(x),$$

and $f \blacktriangledown g \in \mathcal{F}^+$. Furthermore, if f is also normal, then $f \blacktriangledown g = g$.

5.2 Continuity of Operations on Fuzzy Truth Values

We gave sufficient conditions for the continuity of the compound fuzzy truth values $f \blacktriangle g$ and $f \blacktriangledown g$. Let $\mathcal{F}_c$ denote the set of continuous fuzzy truth values.

Proposition 5.6. *The strict conjunction $f \blacktriangle g$ of $f, g \in \mathcal{F}_c$ is continuous if f or g is left- or right-maximal.*

Proposition 5.7. *The nilpotent conjunction $f \blacktriangle g$ of $f, g \in \mathcal{F}_c$ is continuous if $f \in \mathcal{F}_c^+$ or $g \in \mathcal{F}_c^+$.*

Proposition 5.8. *The strict disjunction $f \blacktriangledown g$ of $f, g \in \mathcal{F}_c$ is continuous if f or g is left- or right-maximal.*

Proposition 5.9. *The nilpotent disjunction $f \blacktriangledown g$ of $f, g \in \mathcal{F}_c$ is continuous if $f \in \mathcal{F}_c^-$ or $g \in \mathcal{F}_c^-$.*

5.3 Extended Łukasiewicz operations on linear FTVs

The Łukasiewicz conjunction and disjunction of fuzzy truth values are

$$(f \blacktriangle_W g)(z) = \bigvee_{z=(x+y-1)\vee 0} ((f(x) + g(y) - 1) \vee 0)$$

$$(f \blacktriangledown_W g)(z) = \bigvee_{z=(x+y)\wedge 1} ((f(x) + g(y) - 1) \vee 0)$$

Let $\mathcal{L} \subset \mathcal{F}_c$ be the set of linear fuzzy truth values characterized by

$$f_{a,b} \in \mathcal{L} \quad \Longleftrightarrow \quad f_{a,b}(x) = \left\{ \frac{x-a}{b-a} \right\}_0^1,$$

where $a \neq b$, $x \in [0,1]$ and $\{t\}_a^b = a \vee t \wedge b$. Let $\mathcal{L}^+ \subset \mathcal{F}_c^+$ denote the set of non-decreasing, and $\mathcal{L}^- \subset \mathcal{F}_c^-$ the set of non-increasing linear fuzzy truth values. The set of normal, non-decreasing (non-increasing) linear fuzzy truth values is $\mathcal{L}_1^+$ (resp. $\mathcal{L}_1^-$) and characterized by $b \leq 1$ (resp. $b \geq 0$).

Theorem 5.10. *The following hold for all* $f_i = f_{a_i,b_i} \in \mathcal{L}^+$ *(*$i = 1,2$*).*

$$\begin{aligned}(f_1 \blacktriangle_W f_2)(z) = &\ (f_1(1) \vartriangle_W f_2(\{b_1\}_z \rhd_W z)) \\ &\vee (f_2(1) \vartriangle_W f_1(\{b_2\}_z \rhd_W z)),\end{aligned}$$

where $\{x\}_z = z \vee x \wedge 1$*.*

The Łukasiewicz conjunction of non-decreasing linear fuzzy truth values is continuous in any case. Although, it is not always linear, linearity is preserved for normal fuzzy truth values.

Corollary 5.11. *For all* $f_i = f_{a_i,b_i} \in \mathcal{L}_1^+$ *(*$i = 1,2$*)*

$$(f_1 \blacktriangle_W f_2)(z) = f_1(b_2 \rhd_W z) \vee f_2(b_1 \rhd_W z).$$

Furthermore, $f_1 \blacktriangle_W f_2$ *is also linear with parameters*

$$\begin{aligned} a_{\blacktriangle_W} &= (a_1 + b_2 - 1) \wedge (a_2 + b_1 - 1), \\ b_{\blacktriangle_W} &= b_1 + b_2 - 1. \end{aligned}$$

Theorem 5.12. *The following hold for all* $f_i = f_{a_i,b_i} \in \mathcal{L}^-$ *(*$i = 1,2$*). For* $z > 0$*,*

$$\begin{aligned}(f_1 \blacktriangle_W f_2)(z) = &\ (f_1(z) \vartriangle_W f_2(\{b_1\}_z \rhd_W z)) \\ &\vee (f_2(z) \vartriangle_W f_1(\{b_2\}_z \rhd_W z)),\end{aligned}$$

and if $z = 0$*, then*

$$(f_1 \blacktriangle_W f_2)(0) = f_1(0) \vartriangle_W f_2(0).$$

Corollary 5.13. *For* $f_i = f_{a_i,b_i} \in \mathcal{L}^-$*, the Łukasiewicz conjunction* $f_1 \blacktriangle_W f_2$ *is continuous if and only if* $b_1 + b_2 \geq 1$*.*

6 Type-2 implications on fuzzy truth values

Let $\mathbf{A} = (\mathcal{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preccurlyeq)$, where $\mathcal{A} \subseteq \mathcal{F}$. A function $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a type-2 fuzzy implication over $\mathbf{A}$ if and only if it satisfies the boundary conditions

$$\mathbf{0} \bullet \mathbf{0} = \mathbf{0} \bullet \mathbf{1} = \mathbf{1} \bullet \mathbf{1} = \mathbf{1};\ \mathbf{1} \bullet \mathbf{0} = \mathbf{0},$$

and it is antitone in the first and monotone in the second argument w.r.t. at least one of the partial orders $\sqsubseteq$ or $\preccurlyeq$.

6.1 Extended S-implications and S-coimplications

S-implications are formed by a t-conorm $\bigtriangledown$ and a strong negation $'$ according to the formula $x' \bigtriangledown y$. S-coimplications are dual to S-implications, and are defined as $x' \bigtriangleup y$. The extensions to type-2 operations of these operations are $f \blacktriangleright g$ and $f \blacktriangleleft g$.

Proposition 6.1. *The operations $\blacktriangleright$ and $\blacktriangleleft$ are closed on $\mathcal{F}_C$.*

Proposition 6.2. *The operations $\blacktriangleright$ and $\blacktriangleleft$ are closed on $\mathcal{F}_N$. Moreover, $f \blacktriangleright g$ and $f \blacktriangleleft g$ are normal if and only if $f, g \in \mathcal{F}_N$.*

Theorem 6.3 ([6])**.** *The operation $\blacktriangleright$ is a type-2 fuzzy implication over $\mathbf{A} \subseteq \mathbf{F}$ if and only if $\mathbf{A}$ is a subalgebra of the algebra of convex normal functions $\mathbf{F}_{CN}$.*

6.2 The extended residuals of $\wedge$ and $\vee$

As well as the minimum ($\wedge$) and maximum ($\vee$) operators, their type-2 extensions, meet ($\sqcap$) and join ($\sqcup$) are widely used in many applications. The residuals of $\wedge$ and $\vee$ have the well-known formulas

$$x \rhd_{\wedge} y = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{otherwise,} \end{cases} \qquad \text{and} \qquad x \lhd_{\vee} y = \begin{cases} 0 & \text{if } y \le x, \\ y & \text{otherwise.} \end{cases}$$

We considered the extensions of $\rhd_{\wedge}$ and $\rhd_{\vee}$ (resp. $\sqsubset$ and $\sqsupset$). For all $f \in \mathcal{F}$ let

$$f^r(x) = \begin{cases} \bigvee_{y>x} f(y), & \text{if } x < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad f^l(x) = \begin{cases} \bigvee_{y<x} f(y), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is known, that $x \rhd y = 1$ iff $x \le y$ holds for any residual fuzzy implication. The extended counterpart of this equivalence is $f \blacktriangleright g = \mathbf{1}$ iff $f \preceq g$ for a binary relation $\preceq$ over $\mathcal{F}$. The necessary and sufficient conditions in the special case of $\sqsubset$ are summarized as follows.

Theorem 6.4. *For all $f, g \in \mathcal{F}$, $f \sqsubset g = \mathbf{1}$ if and only if*

1. $f, g \in \mathcal{F}_N$, *and*
2. $g^l(x_0) = 0$, *where* $x_0 = \sup\{x \mid f(x) > 0\}$.

6.3 Distributive properties of $\sqsubset$ and $\sqsupset$

Proposition 6.5. *The following distributive laws hold for all $f, g, h \in \mathcal{F}$,*

1. $f \sqsubset (g \vee h) = (f \sqsubset g) \vee (f \sqsubset h)$, $\quad f \sqsupset (g \vee h) = (f \sqsupset g) \vee (f \sqsupset h)$,
2. $(f \vee g) \sqsubset h = (f \sqsubset h) \vee (g \sqsubset h)$, $\quad (f \vee g) \sqsupset h = (f \sqsupset h) \vee (g \sqsupset h)$.

In general, $\sqsubset$ does not distribute over $\sqcap$ and $\sqcup$, only the following inequalities hold.

Theorem 6.6. *For all $f, g, h \in \mathcal{F}$,*

$$f \sqsubset (g \sqcap h) \leq (f \sqsubset g) \sqcap (f \sqsubset h), \quad f \sqsubset (g \sqcup h) \leq (f \sqsubset g) \sqcup (f \sqsubset h).$$

6.4 Algebras of convex and normal fuzzy truth values

An undoubtedly important subalgebra of $\mathbf{F}$ is the algebra of interval fuzzy truth values $\mathbf{F}_I$. It is proved to be isomorphic to the algebra $(I^{[2]}, \wedge, \vee, ', 0, 1)$, where $I^{[2]}$ denotes the set of closed intervals in I. First, we prove two negative results.

Theorem 6.7. *The algebra $\mathbf{F}_I$ of interval fuzzy truth values is not closed w.r.t. $\sqsubset$ and $\sqsupset$.*

Corollary 6.8. *The set $\mathcal{F}_C$ of convex fuzzy truth values is not closed w.r.t. $\sqsubset$ and $\sqsupset$.*

A positive result is proved on normal fuzzy truth values.

Theorem 6.9. *The set $\mathcal{F}_N$ of normal fuzzy truth values is closed w.r.t. $\sqsubset$ and $\sqsupset$. Moreover, $f \sqsubset g \in \mathcal{F}_N$ (resp. $f \sqsupset g \in \mathcal{F}_N$) if and only if $f, g \in \mathcal{F}_N$.*

We also show that the algebra of left- or right-maximal fuzzy truth values is a subalgebra of $\mathcal{F}$ [6].

Theorem 6.10. *The algebra $\mathbf{F}_M = (\mathcal{F}_{LM} \cup \mathcal{F}_{RM}, \sqcap, \sqcup, ^*, \sqsubset, \mathbf{0}, \mathbf{1})$ i.e. the algebra of left- or right-maximal fuzzy truth values is a subalgebra of $(\mathcal{F}, \sqcap, \sqcup, ^*, \sqsubset, \mathbf{0}, \mathbf{1})$.*

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