Thesis of Ph.D. Dissertation

Complete polynomial vector fields and warped product manifolds

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Chapter 1

Overview

My dissertion contains the main results of my research activity carried out during the time of my PhD studies at the University of Szeged. It includes three papers of mine which have been published. These works are related to the following two different fields of mathematics:

1- complete polynomial vector fields

2- geodesics on warped product manifolds.

In accordance with our papers, the dissertation is divided into three main Chapters:

1) In **Chapter 2** we shall describe complete polynomial vector fields on a finite-dimensional simplex $S := (x_1 + x_2 + \cdots + x_n = 1)$ with an application to differential equations in genetical dynamic systems,
2) **Chapter 3** deals with the complete polynomial vector fields on the Euclidean unit ball $B := (x_1^2 + \ldots + x_n^2 < 1)$,
3) **Chapter 4** is devoted to the geometry of the central symmetric warped product structures on $\mathbb{R}_0^N \times \mathbb{R}$.

In **Chapter 2** we are going to describe the complete polynomial vector fields and their fixed points in a finite-dimensional simplex. We apply the results to differential equations of genetical evolution models.

There are several well-known models in literature [4], [5], [2] on the time evolution of a closed population consisting of $\mathbb{N}$ different species - with the whole population at time $t \geq 0$ as the solution of a system of ordinary differential equations $\frac{d}{dt} v_k(t) = F_k(v_1(t), v_2(t), \ldots, r_N(t))$ $(k = 1, 2, \ldots, N)$ where the functions F_k are some polynomials of at most 3-rd. degree. During a seminar on such models one has raised the problem what are the strange consequences of the assumption that the evolution has no starting point in time, in particular what can be stated on non-changing distribution in that case. In this chapter we provide the complete algebraic description of all polynomial vector fields (with arbitrary degrees), $V(x) = (F_1(x), F_2(x), \ldots, F_N(x))$ on $\mathbb{R}^N$ which give rise to solutions for the evolution equation defined for all time parameters $t \in \mathbb{R}$, and satisfying the natural rate conditions $r_1(t),\ r_2(t),\ r_3(t),\ \ldots,\ r_N(t) \geq 0;\ \sum_{k=1}^N r_k(t) = 1$ whenever $r_1(0),\ r_2(0),\ \ldots,\ r_N(0) \geq 0$ and $\sum\limits_{k=1}^{N} r_k(0) = 1$. On the basis of the explicit formulas obtained we describe the structure of the set of zeros for such vector fields which corresponded to the non-changing distribution.

In **Chapter 3** we are going to describe the complete polynomial vector fields in the unit ball $B := (x_1^2 + x_2^2 + \cdots + x_N^2 < 1)$ of $\mathbb{R}^N$. This work originates from a nice parametric formula due to L.L. Stachó [3] for the complete real polynomial vector fields on the unit disc $\mathbb{K}$ of the space $\mathbb{C}$ of complex numbers. He has shown that a real polynomial vector field $p : \mathbb{C} \to \mathbb{C}$ is complete in $\mathbb{K}$ iff p is a finite real linear combination formed by the functions iz, $\gamma\overline{z}^m - \overline{z}z^{m+2}$, $(z \in \mathbb{C},\ m = 0, 1, \ldots)$ and $(1 - |z|^2)Q$ where Q is any real polynomial from $\mathbb{C}$ to $\mathbb{C}$. Our result in this chapter establishes that $p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a complete polynomial vector field in the unit ball B if and only if $p(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$ for some polynomials $Q, R : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. This theorem not only generalizes the result of [3] on $\mathbb{K}$, but it even simplifies it by showing that the complete polynomial vector fields on the unit disc of $\mathbb{C}$ have the form $[ip(z)z + q(z)(1 - |z|^2)]$ where $p, q : \mathbb{C} \to \mathbb{R}$ are any real polynomials.

In **Chapter 4** we shall study the geometry of the central symmetric warped product manifold structures on $\mathbb{R}^N_0 \times \mathbb{R}^1$ where $\mathbb{R}^N_0 = \mathbb{R}^N \setminus \{0\}$, which correspond to the potential functions $a\|x\|$, $a \geq 0$, and equipped with the Riemannian scalar product $\langle \cdot, \cdot \rangle$ defined by the following properties:

i) the projection onto $\mathbb{R}^N$ along $\mathbb{R}^1$ of this Riemannian scalar $\langle \cdot, \cdot \rangle$ is canonical Euclidean,

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}^N$ with respect to $\langle \cdot, \cdot \rangle$,

iii) the projection onto $\mathbb{R}^1$ along $\mathbb{R}^N$ of $\langle \cdot, \cdot \rangle$ at $(a, \alpha) \in \mathbb{R}^N_0 \times \mathbb{R}^1$ is the canonical one multiplied by $U(|a|^2)$, where $U : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth.

Notice that these properties determine uniquely the scalar product of the vectors (X, ξ), $Y(\eta) \in T_{(a,\alpha)}(\mathbb{R}^N_0 \times \mathbb{R}^1)$ and it can be written in the form

$$\langle (X, \xi) \cdot (Y, \eta) \rangle = \langle X, Y \rangle + U(|a|^2) \cdot \xi \cdot \eta.$$

Chapter 2

Complete polynomial vector fields in a simplex

Throughout the whole work $\mathbb{R}^N := \{(\xi_1, \dots, \xi_N) : \xi_1, \dots, \xi_N \in \mathbb{R}\}$ denotes the vector space of all real N-tuples. We reserve the notations $x_1, \dots, x_N$ for the standard coordinate functions $x_k : (\xi_1, \dots, \xi_N) \to \xi_k$ on $\mathbb{R}^N$. Also we reserve the notation S for the unit simplex

$$S := \big(x_1 + \cdots + x_N = 1, \quad x_1, \dots, x_N \geq 0\big) = \\ = \{p \in \mathbb{R}^N : \ x_1(p) + \cdots + x_N(p) = 1, \quad x_1(p), \dots, x_N(p) \geq 0\} \ .$$

Recall [6] that by a vector field on S we simply mean a function $S \to \mathbb{R}^N$. A function $\varphi : S \to \mathbb{R}$ is said to be polynomial if it is the restriction of some polynomial of the linear coordinate functions $x_1, \dots, , x_N$: for some finite system of coefficients $\alpha_{k_1 \dots k_N} \in \mathbb{R}$ with $k_1, \dots, k_N \in \{0, 1, \dots\})$ we can write $\varphi(p) = \sum_{k_1, \dots, k_N} \alpha_{k_1 \dots k_N} x_1^{k_1} \cdots x_N^{k_N}$ $(p \in S)$. In accordance with this terminology, a vector field V on S is a polynomial vector field if its components $V_k := x_k \circ V$ (that is $V(p) = (V_1(p), \dots, V_N(p))$ for $p \in S$) are polynomial functions. It is elementary that given two polynomials $P_m = P_m(x_1, \dots, x_N) : \mathbb{R}^N \to \mathbb{R}$ $(m = 1, 2)$, their restrictions to S coincide if and only if the difference $P_1 - P_2$ vanishes on the affine subspace $A_S := \big(x_1 + \cdots + x_N = 1\big)$ generated by S. We shall see later that a polynomial $P = P(x_1, \dots, x_N)$ vanishes on the affine subspace $M := \big(\gamma_1 x_1 + \cdots + \gamma_N x_N = \delta\big)$ iff $P = (\gamma_1 x_1 + \cdots + \gamma_N x_N - \delta) Q(x_1, \dots, x_N)$ for some polynomial Q. Thus polynomial vector fields on S admit several polynomial extensions to $\mathbb{R}^N$ but any two such extensions differ only by a vector field of the form $(x_1 + \cdots + x_N - 1)W$.

Definition. A locally Lipschitzian (e.g. polynomial) vector field $V : \mathbb{R}^N \to \mathbb{R}^N$ is said to be *complete* in a (non-empty) subset $K \subset \mathbb{R}^N$ if for any point $p \in K$ there is a (necessarily unique) curve $C_p : \mathbb{R} \to K$ such that $C_p(0) = p$ and $\frac{d}{dt} C_p(t) = V(C_p(t))$ $(t \in \mathbb{R})$.

Our purpose will be to describe the complete polynomial vector fields on the simplex S and we apply the results to differential equations of genetical evolutions models.

Our main results are as follows.

2.2. Theorem. *A polynomial vector field $V : S \to \mathbb{R}^N$ is complete in S if and only if with the vector fields*

$$Z_k := x_k \sum_{j=1}^N x_j (e_j - e_k) \qquad (k = 1, \dots, N)$$

where e_j *is the standard unit vector* $e_j := (0,\dots,0,\overbrace{\ 1\ }^{j},0,\dots,0)$*, we have*

$$V = \sum_{k=1}^{N} P_k(x_1,\dots,x_N) Z_k$$

for some polynomial functions $P_1,\dots,P_N : \mathbb{R}^N \to \mathbb{R}$.

2.3. Theorem. *Given a complete polynomial vector field* V *of* S*, there are polynomials* $\delta_1,\dots,\delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ *of degree less than that of* V *such that the vector field*

$$\begin{aligned} \widetilde{V} := \sum_{k=1}^{N-1} x_k \Big[\delta_k(x_1,\dots,x_{N-1}) - \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1,\dots,x_{N-1})\Big] e_k + \\ + (x_1+\dots+x_{N-1}-1) \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1,\dots,x_{N-1}) e_N \end{aligned}$$

coincides with V *on* S*. The points of the zeros of* V *inside the facial subsimplices* $S_K := S \cap (x_1,\dots,x_K > 0 = x_{K+1} = \dots = x_N)$ $(K=1,\dots,N)$ *can be described as*

$$\begin{aligned} S_N \cap (V=0) &= S \cap \bigcup_{k=1}^{N-1} \big(\delta_k(x_1,\dots,x_{N-1}) = 0\big), \\ S_K \cap (V=0) &= S_K \cap \big(\delta_1(x_1,\dots,x_{N-1}) = \dots = \\ &\qquad = \delta_K(x_1,\dots,x_{N-1})\big) \qquad (K<N). \end{aligned} \tag{$*$}$$

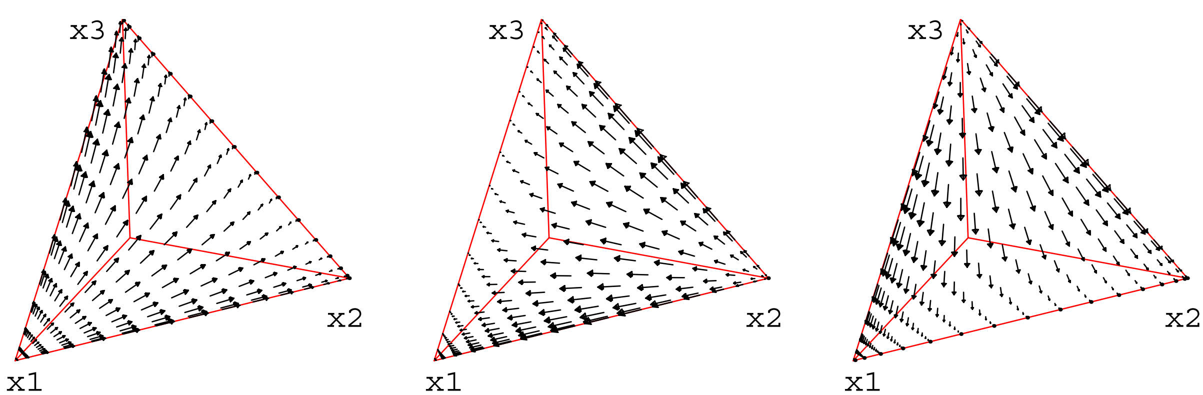

Fig1. The fundamental vector fields Z_1, Z_2, Z_3 in the case $N=3$.

Concerning the genetical time evolution equation for the distribution of species within a closed population, in [7] we have the system

$$(**)\qquad \begin{aligned} \frac{d}{dt}x_k =& \Big(\sum_{i=1}^N g(i)x_i - g(k)\Big)x_k + \\ &+ \sum_{i,j=1}^N w(i,j)x_ix_j\Big[\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,k) - x_k\Big] \end{aligned}$$

for describing the behaviour of the rates $x_1(t),\dots,x_N(t)$ at time t of the N species of the population. Here the terms $g(k)$, $M(i,j,\ell)$ and $\varepsilon(i,j,\ell,k)$ are non-negative constants with $\sum_{\ell=1}^N M(i,j,\ell) = \sum_{k=1}^N \varepsilon(i,j,\ell,k) = 1$. Observe that this can be written as

$$\frac{d}{dt}x = \sum_{k=1}^N g(k)Z_k + W$$

with the vector fields

$$\begin{aligned} Z_k &:= x_k \sum_{j=1}^N x_j(e_j - e_k), \\ W &:= \sum_{i,j,k=1}^N w(i,j)x_ix_j\Big[\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,k) - x_k\Big]e_k, \end{aligned}$$

respectively. As a consequence of Theorems 2.1 and 2.2 we obtain the following.

2.4. Theorem. *Let $N \geq 3$. Then the time evolution of the population can be retrospected up to any time $t \leq 0$ starting with any distribution $(x(0),\dots,x_N(0)) \in S$ if and only if the term W vanishes on S, that is if simply $d/dt\ x = \sum_{k=1}^N g(k)Z_k(x_1,\dots,x_N)$. In this case the set of the stable distributions has the form*

$$\bigcup_{\gamma\in\{g(1),\dots,g(N)\}} S \cap (x_m = 0 \text{ for } m \notin J_\gamma) \qquad \text{where } J_\gamma := \{m:\ g(m) = \gamma\}\ .$$

2.5. Corollary. *If $g(1),\dots,g(N) \geq 0$ and the vector field (**) is complete in S then*

$$\frac{d}{dt}\sum_{k=1}^N g(k)x_k(t) \geq 0$$

for any solution $t \mapsto x(t) \in S$ of the evolution equation $dx/dt = V(x)$.

Chapter 3

Complete polynomial vector
fields of the Euclidean ball

In this chapter we will describe the complete polynomial vector fields in the unit ball of a finite dimensional inner product space which we identify with $\mathbb{R}^N$.

Our work arises from an idea of a nice result of L.L. Stachó [4] in 2001 where he characterized the complete real polynomial vector fields in the (two-dimensional) unit disc $\mathbb{K}$ of the complex plane $\mathbb{C}$. We will show that our result not only generalizes the results of [4] on $\mathbb{K}$, but it even simplifies them.

3.1. Definition. Given any subset K in $\mathbb{R}^N$ the set of real n tuples and a mapping $v : \mathbb{R}^N \to \mathbb{R}^N$, we say that v is a *complete vector field* in K if for every point $k_0 \in K$ there exists a curve $x : \mathbb{R} \to K$ such that $x(0) = k_0$ and $\frac{dx(t)}{dt} = v(x(t))$ for all $t \in \mathbb{R}$.

In Chapter 2 we represented complete polynomial vector fields on a simplex as polynomial combinations of some finite family of complete vector fields of third degree. This idea motivates the formulation of our main result in this section.

First let us reformulate Stachó's theorem [4] in terms of polynomial combinations instead of linear combinations asserting (in complex notations, when identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner) that a polynomial vector field $v : \mathbb{C} \to \mathbb{C}$ is complete in the unit disc $\mathbb{K}$ if and only if it is a finite $\mathbb{R}$-linear combination of the vector fields from the family

$$\mathcal{F} := \big\{ iz,\ \mu\overline{z}^n - \overline{\mu}z^{n+2},\ (1-|z|^2)Q : \\ n = 0,1,\ldots;\ \mu = 1, i;\ Q \in \mathrm{Pol}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \big\} .$$

Actually we have the simpler form for the real linear span (the family of all finite linear combinations) of $\mathcal{F}$ as

$$\mathrm{Span}_{\mathbb{R}}\mathcal{F} = \big\{ P \cdot iz + Q(1-|z|^2) : \\ P \in \mathrm{Pol}(\mathbb{C},\mathbb{R}),\ Q \in \mathrm{Pol}(\mathbb{C},\mathbb{C}) \big\} .$$

3.2. Remark. Recall that a mapping $v : \mathbb{R}^N \to \mathbb{R}^N$ is said to be a polynomial vector field if $v(x) = (p_1(x), \ldots, p_N(x))$; $x \in \mathbb{R}^N$ for some polynomials $p_1, \ldots, p_n : \mathbb{R}^N \to \mathbb{R}$, of N variables (that is each p_i is a finite linear combination of functions of the form $x_1^{m_1} \ldots x_N^{m_N}$ with non-negative integers m_j where $x_j : (\xi_1, \xi_2, \ldots, \xi_N) \mapsto \xi_j$ denotes the j-th canonical coordinate function of $\mathbb{R}^N$).

3.3. Definition. By writing $\langle(\xi_1,\xi_2,\ldots,\xi_N),(\eta_1,\eta_2,\ldots,\eta_N)\rangle := \sum_{i=1}^N \xi_i\eta_i$ for the inner product in $\mathbb{R}^N$, it is easy to see that a polynomial (or even smooth) vector field is complete in the ball $B := (\langle x,x\rangle < 1)$ if and only if it is complete in the sphere $S := (\langle x,x\rangle = 1)$. Furthermore, v is complete in S if and only if it is orthogonal to the radius vector on S, i.e. if $\langle v(x),x\rangle = 0$ for $x \in S$.

We know from Chapter 2 that if $F : \mathbb{R}^N \to \mathbb{R}$ is a polynomial and $p : \mathbb{R}^N \to \mathbb{R}$ be any polynomial such that $p(M) = 0$ and $M \subset R^N$, then there is a polynomial $q : \mathbb{R}^N \to \mathbb{R}$ such that $P = q\cdot F$, when $F(x) = \phi_1(x)\cdot\phi_2(x)\cdots\phi_N(x)$, and the ϕ_i are linearly independent affine functions. Now we will prove the case when $f(x) = 1 - \langle x,x\rangle$ which is important to formulate our main result.

3.4. Lemma. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a polynomial such that $f(x) = 0$ for $x \in S$ where $S := (\langle x,x\rangle = 1)$. Then there exists a polynomial $Q : \mathbb{R}^N \to \mathbb{R}$, such that $f(x) = (1 - \langle x,x\rangle)Q(x)$.*

Lemmas with such a character seem to be very important in the theory of complete polynomial vector fields of domains defined by polynomial inequalities. In the complex case, due to the algebraic closedness of the field $\mathbb{C}$, there are similar results but the proofs cannot be imitated in the real case, even in the case of a ball.

3.5. Theorem. *Let $P : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial mapping. Then P is a complete polynomial vector field in the sphere $S := (\langle x,x\rangle = 1)$ if and only if*

$$[P(x) = R(x) - \langle R(x),x\rangle x + (1 - \langle x,x\rangle)Q(x)]$$

for some polynomial mappings $R, Q : \mathbb{R}^N \to \mathbb{R}^N$.

3.6. Corollary. *Let $V_k : x \to e_k - \langle e_k,x\rangle x$ where $k = 1,2,3,\ldots,N$. Then every complete polynomial vector field on the sphere $S := \big(\sum_{i=1}^N x_i^2 = 1\big)$ coincides with some vector field of the form $V(x) = \sum_{k=1}^N p_k(x)V_k(x)$ when restricted to S where $p_1,\ldots,p_n : \mathbb{R}^n \to \mathbb{R}$ are appropriate polynomials.*

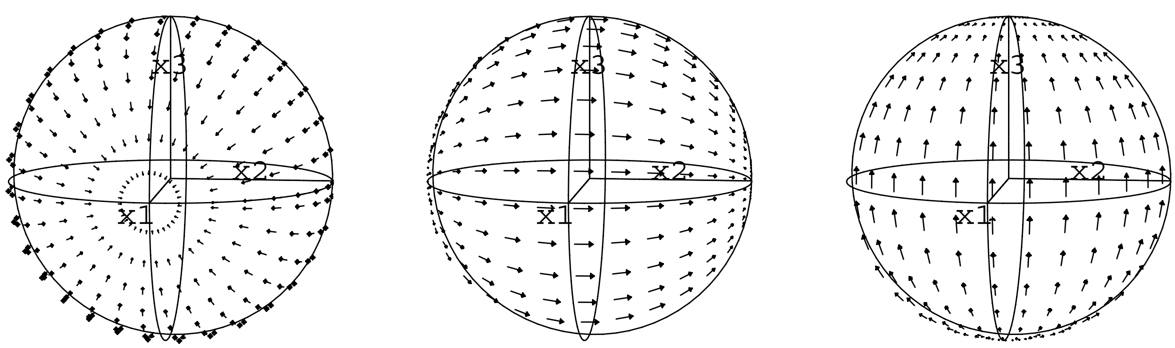

Figure 2. The vector fields $V_k : x \mapsto e_k - \langle e_k,x\rangle x$, $(k{=}1,2,3)$ on $S \subset \mathbb{R}^3$.

3.7. Corollary *The complete polynomial vector fields on S are exactly the resrictrictions of the vector fields of the form*

$$\widetilde{V} : x \mapsto xA(x)$$

where A is any polynomial mapping $\mathbb{R}^N \to \mathrm{Mat}^{(-)}(N, \mathbb{R})$ into the space af all antisymmetric $N \times N$-matrices.

3.8. Remark. There is an interesting link between the complete polynomial vector fields of the unit simplex $P := (x_1 + \cdots + x_N,\ x_1, \ldots, x_N \geq 0)$ and those of the sphere $S := (x_1^2 + \cdots + x_N^2 = 1)$. Namely, the mapping

$$T : (x_1, \ldots, x_N) \mapsto (x_1^2, \ldots, x_N^2)$$

maps the positive part $S_+ := S \cap (x_1, \ldots, x_N \geq 0)$ of the sphere onto P in a one-to-one manner. Given any smooth complete vector field $W : P \to \mathbb{R}^N$ $(W(x) = (w_1(x), \ldots, w_N(x)))$ of the simplex P, its pull-back to S_+ is

$$T^{\#}V : S_+ \ni (x_1, \ldots, x_N) \mapsto \frac{d}{d\tau}\bigg|_{\tau=0} T^{-1}\big(T(x) + \tau W(T(x))\big) =$$

$$= \frac{d}{d\tau}\bigg|_{\tau=0} \big([x_1^2 + \tau w_1(x_1^2, \ldots, x_N^2)]^{1/2}, \ldots, [x_N^2 + \tau w_N(x_1^2, \ldots, x_N^2)]^{1/2}\big) =$$

$$= \frac{1}{2}\big(x_1^{-1} w_1(x_1^2, \ldots, x_N^2), \ldots, x_N^{-1} w_N(x_1^2, \ldots, x_N^2)\big)\ .$$

In particular the operation $T^{\#}$ establishes the following relationship between the fundamental complete polynomial vector fields $Z_k(x) := x_k \sum_{i=1}^N x_i(e_i - e_k)$ of P of P and $V_k(x) := e_k - \langle e_k, x\rangle x$ of S, respectively:

$$T^{\#} Z_k(x) = \tfrac{1}{2} x_k V_k(x) \qquad (k = 1, \ldots, N)\ .$$

Therefore all complete polynomial vector fields of P are pulled back to complete polynomial vector fields of S_+. Namely we have

$$T^{\#}\left(\textstyle\sum_{k=1}^N p_k(x) Z_k(x)\right) = \sum_{k=1}^N \tfrac{1}{2} x_k p_k(x_1^2, \ldots, x_N^2) V_k(x).$$

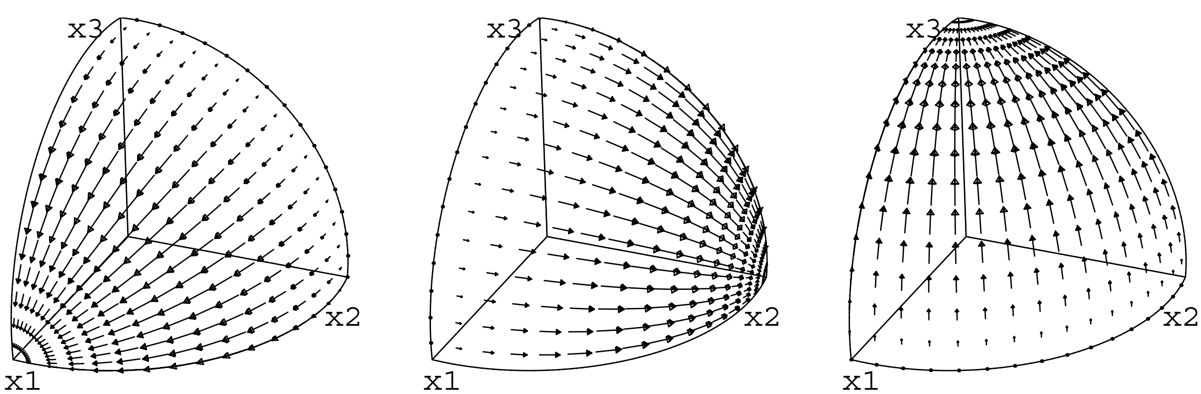

Figure 3. The vector fields $x_k V_k(x)$, $(k{=}1,2,3)$ on $S_+ \subset \mathbb{R}^3$.

Chapter 4

Geodesics on a central symmetric warped product manifold

Let $U : \mathbb{R}_+ \to \mathbb{R}_+$ be a given smooth function. We consider the manifold $\mathbb{R}_0^n \times \mathbb{R}^1$, where $\mathbb{R}_0^n = \mathbb{R}^n \backslash \{0\}$ is equipped with a Riemannian scalar product $\langle \cdot, \cdot \rangle$ satisfying the following conditions:

i) The projection onto $\mathbb{R}_0^n$ along $\mathbb{R}^1$ of the Riemannian scalar product $\langle \cdot, \cdot \rangle$ is the canonical Euclidean one.

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}_0^n$ with respect to $\langle \cdot, \cdot \rangle$.

iii) The projection onto $\mathbb{R}^1$ along $\mathbb{R}_0^n$ of $\langle \cdot, \cdot \rangle$ at $(a, p) \in \mathbb{R}_0^n \times \mathbb{R}^1$ is the canonical one multiplied by the function U.

These properties determine uniquely the scalar product of the tangent vectors (X, ξ), $(Y, \eta) \in T_{(a,\beta)}(\mathbb{R}_0^n \times \mathbb{R}^1)$ and it can written in the form

$$g_{(a,\beta)}((X, \xi), (Y, \eta)) = \langle X, Y \rangle + \xi \cdot \eta \cdot \cup(|a|^2). \tag{1}$$

where $\langle X, Y \rangle = \sum_{i=1}^n X_i \cdot Y_i$. For the sake of simplicity we shall write

$$\langle (X, \xi), (Y, \eta) \rangle_* = g_{(a,\beta)}((X, \xi)(Y, \eta)).$$

This simplification will not lead to any confusion since we know every time which point the tangent vector belongs to. We will regard β in (a, β) like the $(n+1)$-th coordinate.

One of our basic results is formulated in the following theorem.

4.1. Theorem. *The Levi-Civita connection of the Riemannian metric* (1) *introduced above has the following Christoffel symbols*

$$\Gamma_{i,j}^k(a, \beta) = \begin{cases} 0 & \text{if} \quad i, j, k \le n \\ 0 & \text{if} \quad i, j \le n, \ k = n+1 \\ 0 & \text{if} \quad i, k \le n, \ j = n+1 \\ 0 & \text{if} \quad j, k \le n, \ i = n+1 \\ -\partial_k(U(z))/2 & \text{if} \quad k \le n, \ i, j = n+1 \\ \partial_i(U(z))/2U(z) & \text{if} \quad j, k = n+1 \\ \partial_j(U(z))/2U(z) & \text{if} \quad i, k = n+1 \\ 0 & \text{if} \quad i, j, k = n+1 \end{cases},$$

where $1 \le i, j, k \le n+1$, $z = \langle a, \beta \rangle$ and ∂_s is the derivative with respect to the s-th coordinate.

4.2. Corollary. *The system of differential equation of the geodesics is*

$$\dot{\beta} = h/U(Z)$$

$$\ddot{a}_j = a_j h^2 U'(Z)/U^2(Z) \quad 1 \leq j \leq n$$

where h is a suitable constant, $Z = \langle a, \beta \rangle$, and $(a(s), \beta(s))$ is the geodesic whose coordinates are $\{a_j\}_{j=1}^n$ and β.

Hence we get the following description of geodesics.

4.3. Theorem *Let $(\mathbf{x}(s), \xi(s))$ be a geodesic in $\mathbb{R}_0^n \times \mathbb{R}^1$ with respect to the Riemannian metric (1). We denote its initial values at $s = 0$ by $\mathbf{x}(0) = \mathbf{x}_0$, $\xi(0) = \xi_0$, $\dot{\mathbf{x}}(0) = \mathbf{t}_0$, $\dot{\xi}(0) = \tau_0$. Then one has the following possibilities:*

a) *If $\tau_0 = 0$ then the geodesic $(\mathbf{x}(s), \xi(s))$ is contained in the line $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$, $\xi(s) = \xi_0$; this geodesic is complete except in the case if $\xi_0 = 0$ and the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear.*

b) *If $\tau_0 > 0$ then the projection of the geodesic onto $\mathbb{R}_0^n$ is an ellipse with centre $\mathbf{0}$. Its equation has the shape*

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

The corresponding geodesic is complete except in the case if the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear and the projected ellipse is degenerated to a segment with the midpoint $\mathbf{0}$.

c) *If $\tau_0 < 0$ then the projection of the geodesic onto $\mathbb{R}_0^n$ is a hyperbola with center $\mathbf{0}$. Its equation has the shape*

$$\mathbf{x}(s) = \cos h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin h(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

If the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear then the projected hyperbola is degenerated to a half line. The corresponding geodesic is complete.

Now we deal with the second case to give the geometry of Kepler Motions.

In this case the determining function of the metric is $U(z) = c\sqrt{z}$. We have the following description of the geodesics.

4.4. Theorem. *Let $(a(s), \alpha(s))$ be a geodesic in $\mathbb{R}_0^n \times \mathbb{R}^1$ with respect to the Riemannian metric (1). We denote its initial values at $s = 0$ by $a_0 = a(0)$, $\alpha_0 = \alpha(0)$, $T = \dot{a}(0)$, $\tau = \dot{\alpha}(0)$. Let E_1, $E_2 \in \mathbb{R}_0^n$ be orthogonal unit vectors in W which are spanned by a_0 and T. Choose E_1, E_2 satisfying the following*

$$a_0 = a_1 \cdot E_1, \quad T = T_1 \cdot E_1 + T_2 \cdot E_2.$$

If $T_2 \neq 0$ we get the following description of geodesics:

The geodesics do not leave the space spanned by W and $\mathbb{R}^1$. Furthermore, if we denote the projection of T to $\mathbb{R}^1$ along $\mathbb{R}_0^n$ by T_3, there are three possibilities:

i) if $|(T, \tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 < 0$, then the projection of the geodesic onto W is an ellipse, *ii) if* $|T,\tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 = 0$, *then the projection of the geodesic onto* W *is a parabola,*

iii) if $|T,\tau)|_*^2 = T_1^2 + T_2^2 + c \cdot |a_1^0| \cdot T_3^2 > 0$, *then the projection of the geodesic onto* W *is a hyperbola.*

The equation of the projected geodesic in polar-coordinate is

$$P(\gamma) = \frac{2 \cdot |a_1^0|^3 \cdot T_2^2}{-c \cdot T_3^2 \cdot |a_1^0|^3 + v \cdot \cos(\varphi - \omega)},$$

where

$$v = \operatorname{sgn}(c) = \cdot \sqrt{4 \cdot T_1^2 \cdot T_2^2 \cdot |a_1^0|^4 + (2 \cdot T_2^2 \cdot |a_1^0|^2 + c \cdot T_3^2 \cdot |a_1^0|^3)^2},$$

$$\omega = \arcsin\Big(\frac{2 \cdot T_1 \cdot a_1^0 \cdot \operatorname{sgn}(c)}{U}\Big)$$

and $p = |a|$, $\cos\varphi = \langle a, E_1 \rangle / |a|$.

4.5. Corollary. *If* $c > 0$, *then all the projections of geodesics are hyperbolas which have two asymptotic straight lines through the origin with the direction* $\omega - \arccos(1/\varepsilon)$ *and* $\omega + \arccos(1/\varepsilon)$. *The nearest point of these asymptotic lines to the origin is* $(\omega, |a_0| \cdot T_2^2/(u - v))$. *Thus the origin is not contained inside the hyperbola.*

4.6. Corollary. *The projection of a geodesic is a circle if and only if* $c < 0$, T *is perpendicular to* a_0 *and* $|T|^2 + |(T,\tau)|_*^2 = 0$. *The radius of this circle is* $2 \cdot |T|^2/(-c \cdot \tau^2)$. *Its center is the origin.*

4.7. Corollary. *If the projection of a geodesic is an ellipse, and for its eccentricity* $\varepsilon \neq 0$, *then its long axis has direction* ω *and length* $\frac{2 \cdot |a_0| \cdot T_2^2 \cdot u}{u^2 - v^2}$. *It has two focal points: the origin and* $\big(\omega, \frac{2 \cdot |a_0| \cdot T_2^2 \cdot u}{u^2 - v^2}\big)$. *Its short axis has length* $2 \cdot |a_0| \cdot T_2^2$.

4.8. Corollary. *If the projection of the geodesic is a parabola, then it is open in direction* ω. *Its nearest point is* $(\omega + \pi, -T_2^2/(c \cdot \tau^2))$ *and its focal point is the origin.*

4.9. Corollary. *If the projection of the geodesic is a hyperbola and* $c < 0$, *then its focal point is the origin. It has two asymptotic straight lines with direction*

$$\omega + \arccos(1/\varepsilon)(1/\varepsilon) \quad \text{and} \quad \omega - \arccos(1/\varepsilon).$$

4.10. Theorem. *If* $\tau > (<) 0$ *then* α *is strictly increasing (decreasing) and it depends on* $p = |a|$ *according to the following differential equation*

$$\frac{d\alpha}{dp} = \frac{\operatorname{sgn}\sin(\varphi - \omega) \cdot |a_0| \cdot T_2}{\sqrt{p^2(v^2 - u^2) + 2|a_0| T_2^2 \cdot u \cdot p - |a_0|^2 \cdot T_2^4}}, \tag{12}$$

where we have used the notations of our first theorem.

4.11. Corollary. *If the projection of the geodesic is an ellipse, then*

$$p(\alpha) = \frac{c \cdot a_0^2 \cdot \tau^2}{|T, \tau}|_*^2 - \frac{|a_0| \cdot v}{|(T,\tau)|_*^2} \cdot \sin\Big(\frac{\sqrt{|T|^2 + \tau^2}}{\tau \cdot |a_0| \cdot \mathrm{sgn}(\sin(\varphi - \omega))} - \mathrm{const}\Big),$$

where const is such a number, that $p(\alpha) = |a_0|$.

4.12. Corollary. *If the projection of the geodesic is a parabola, then*

$$p(\alpha) = \frac{c \cdot \tau^2}{4} \cdot (\alpha_0 - \alpha) + |a_0|.$$

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List of publications

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[1] M.ben Y. Nuri, *Complete polynomial vector fields in simplexes with application to evolutionary dinamics,* Electronic Journal od Qualitative Theory of Differential Equations Szeged, to appear 2004.

[2] M.ben Y. Nuri, *Complete polynomial vector fields in Euclidean ball,* Publ. Math. Nyíregyháza, to appear 2004.

[3] M.ben Y. Nuri, *Geodesics on a central symmetric warped product manifold,* Publ. Math. Nyíregyháza, to appear 2004.