

TAUBERIAN THEOREMS FOR ORDINARY AND STATISTICAL CONVERGENCE, STATISTICAL LIMIT

Thesis

Árpád Fekete

2006

The importance of Tauberian theorems emanated from Littlewood's theorem (1910) which became the starting point of a new branch of mathematical analysis: the Tauberian theory. A typical Tauberian theorem says if a given sequence is summable by some regular summability method plus an additional so-called Tauberian condition is satisfied, then the sequence in question is convergent to the same limit. Tauberian theorems have a wide range of application, for example in number theory or in probability theory. The aim of this research is to give Tauberian theorems for locally integrable functions, under which convergence follows from summability by weighted mean methods and to give Tauberian conditions for integrals and double sequences under which the statistical limit/ statistical convergence follows from their statistical summability.

In 2003 Ferenc Móricz introduced the notion of statistical limit of measurable functions as the nondiscrete analogue of statistical convergence. In the fourth chapter of dissertation we give necessary and sufficient condition for the existence of statistical limit. The dissertation consists of 6 chapters based on 4 papers: [4], [3], [6] and [5].

1. The sharpening one of Karamata's Tauberian theorems

Jovan Karamata published one of his popular theorems in 1937 [7]. If P is continuous and strictly increasing to ∞ ,

$$\sigma(t) = \frac{1}{P(t)} \int_0^t s(x) dP(x) \rightarrow c, \quad t \rightarrow \infty,$$

and $s(t)$ is slowly decreasing with respect to P , that is

$$\lim_{\lambda \rightarrow 1+} \liminf_{t \rightarrow \infty} \min_{t \leq x \leq P^{-1}(\lambda P(t))} \{s(x) - s(t)\} \geq 0,$$

then

$$s(t) \rightarrow c, \quad t \rightarrow \infty.$$

This chapter give necessary and sufficient conditions for locally integrable functions under which convergence follows from summability by weighted mean methods.

Let P be a function defined on $R_+ := [0, \infty)$ such that

$$(1) \quad P \text{ is nondecreasing on } R_+, \quad P(0) = 0 \text{ and } \lim_{t \rightarrow \infty} P(t) = \infty.$$

P is called a *weight function*, due to the fact that it induces a positive Borel measure on R_+ .

For any complex-valued function $f : R_+ \rightarrow C$ which is integrable in Lebesgue's sense over every finite interval $(0, t)$ for $0 < t < \infty$, in symbol: $f \in L^1_{\text{loc}}(R)$, we set

$$(2) \quad s(x) := \int_0^x f(y)dy \text{ and } \sigma(t) := \frac{1}{P(t)} \int_0^t s(x)dP(x), \quad t > 0,$$

provided that $P(t) > 0$. The integral in the definition of $\sigma(t)$ exists as a Riemann-Stieltjes integral.

Now, σ is called the weighted mean of s and the formal integral

$$(3) \quad \int_0^\infty f(x)dx$$

is called *summable by the weighted mean method* determined by the weight function P , shortly: *summable* (W, P) , if the following finite limit exists:

$$(4) \quad \lim_{t \rightarrow \infty} \sigma(t) = L.$$

Let $\rho : R_+ \rightarrow R_+$ be a strictly increasing, continuous function such that $\rho(t) \rightarrow \infty$ as $t \rightarrow \infty$. We say that ρ is an *upper allowed function* with respect to P if

$$\liminf_{t \rightarrow \infty} \frac{P(\rho(t))}{P(t)} > 1.$$

Similarly, we say that ρ is a *lower allowed function* with respect to P if

$$\liminf_{t \rightarrow \infty} \frac{P(t)}{P(\rho(t))} > 1.$$

We denote by Λ_u and Λ_ℓ the classes of all upper and lower allowed functions, respectively. For real-valued functions f we shall prove the following one-sided Tauberian theorem.

Theorem 1. ([4]) *Assume that P satisfies (1), $f : R_+ \rightarrow R$ and $f \in L^1_{\text{loc}}(R_+)$. The convergence of integral (3) follows from its summability (W, P) to the same limit if and only if both of the following two conditions are satisfied:*

$$(5) \quad \sup_{\rho \in \Lambda_u} \liminf_{t \rightarrow \infty} \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} \{s(x) - s(t)\} dP(x) \geq 0$$

and

$$(6) \quad \sup_{\rho \in \Lambda_\ell} \liminf_{t \rightarrow \infty} \frac{1}{P(t) - P(\rho(t))} \int_{\rho(t)}^t \{s(t) - s(x)\} dP(x) \geq 0.$$

Karamata's theorem is an immediate consequence of this theorem. For complex-valued functions f we shall prove the following two-sided Tauberian theorem.

Theorem 2. ([4]) *Assume that P satisfies (1), $f : R_+ \rightarrow C$ and $f \in L^1_{\text{loc}}(R_+)$. Then the convergence of integral (3) follows from its summability (W, P) to the same limit if and only if one of the following two conditions is satisfied:*

$$(7) \quad \inf_{\rho \in \Lambda_u} \limsup_{t \rightarrow \infty} \left| \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} \{s(x) - s(t)\} dP(x) \right| = 0$$

or

$$(8) \quad \inf_{\rho \in \Lambda_\ell} \limsup_{t \rightarrow \infty} \left| \frac{1}{P(t) - P(\rho(t))} \int_{\rho(t)}^t \{s(t) - s(x)\} dP(x) \right| = 0.$$

In the complex case we can give a Karamata-type corollary, but this time we have to use the condition of slowly oscillating, that is

$$\lim_{\lambda \rightarrow 1+} \limsup_{t \rightarrow \infty} \max_{t \leq x \leq P^{-1}(\lambda P(t))} |s(x) - s(t)| = 0,$$

in place of the condition of slowly decreasing. The main results of this chapter apply to all weighted mean methods and unify the results known in the literature for particular methods.

2. The nondiscrete analogue of Schoenberg's theorem

The notion of statistical convergence was introduced by H. Fast in 1951 [1]. The number sequence x_k is said to be *statistically convergent* to some number ξ , if for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - \xi| \geq \epsilon\}| = 0,$$

where the vertical bars denote the cardinality of the set which they enclose. In 1959 Schoenberg proved [10] that x_k is statistically convergent to some ξ if and only if for every $t \in R$,

$$\lim_{\ell \rightarrow \infty} \frac{1}{\ell} \sum_{k=1}^{\ell} e^{itx_k} = e^{it\xi}$$

holds.

In 2003 Ferenc Móricz introduced the notion of statistical limit of a measurable function. Let f be measurable (in Lebesgue's sense) on the interval $(0, \infty)$. We say that f has a *statistical limit at ∞* if there exists a number ξ such that for every $\epsilon > 0$,

$$(9) \quad \lim_{a \rightarrow \infty} \frac{1}{a} |\{x \in (0, a) : |f(x) - \xi| > \epsilon\}| = 0,$$

where the vertical bars denote the Lebesgue measure of the set which they enclose. We extend Schoenberg's theorem for functions of n -variables. We shall consider a complex-valued function f defined on R_+^n , $R_+ := [0, \infty)$, which is measurable in the sense of the n -dimensional Lebesgue measure, where $n \geq 1$ is a fixed integer. Motivated by (9), we say that the function $f(\mathbf{u}) := f(u_1, u_2, \dots, u_n)$ has a statistical limit at ∞ if there exists a number ξ such that for every $\epsilon > 0$,

$$(10) \quad \lim_{\mathbf{b} \rightarrow \infty} |\mathbf{b}|^{-1} |\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u}) - \xi| \geq \epsilon\}| = 0,$$

where we agree that $\mathbf{b} := (b_1, b_2, \dots, b_n) \rightarrow \infty$ means $\min_{1 \leq j \leq n} b_j \rightarrow \infty$, $|\mathbf{b}| := b_1 b_2 \dots b_n$, and $\mathbf{0} \leq \mathbf{u} \leq \mathbf{b}$ means $0 \leq u_j \leq b_j$ for each $j = 1, 2, \dots, n$. If this is the case, then we shall write

$$\text{st-} \lim_{\mathbf{u} \rightarrow \infty} f(\mathbf{u}) = \xi.$$

Theorem 3. ([3]) *Let $f : R_+^n \rightarrow C$ be a measurable function, where $n \geq 1$ is a fixed integer. For*

$$\text{st-} \lim_{\mathbf{u} \rightarrow \infty} f(\mathbf{u}) = \xi$$

it is necessary and sufficient that for every $t \in R$,

$$\lim_{\mathbf{b} \rightarrow \infty} \frac{1}{|\mathbf{b}|} \int_0^{b_1} \dots \int_0^{b_n} e^{itf(\mathbf{u})} du_1 \dots du_n = e^{it\xi}.$$

Schoenberg's theorem can be extended from single to multiple sequences of complex numbers. To this effect, let N_+^n be the set of n -tuples $\mathbf{k} := (k_1, k_2, \dots, k_n)$ with positive integers for the coordinates k_j , where $n \geq 1$ is a fixed integer. We shall consider an n -multiple sequence $(x_{\mathbf{k}} : \mathbf{k} \in N_+^n)$ of complex numbers. We say that $(x_{\mathbf{k}})$ is statistically convergent if there exists a number ξ such that for every $\epsilon > 0$,

$$\lim_{\mathbf{l} \rightarrow \infty} |\mathbf{l}|^{-1} |\{\mathbf{1} \leq \mathbf{k} \leq \mathbf{l} : |x_{\mathbf{k}} - \xi| \geq \epsilon\}| = 0,$$

where

$$|\mathbf{l}| = |(\ell_1, \ell_2, \dots, \ell_n)| = \ell_1 \ell_2 \dots \ell_n \quad \text{and} \quad \mathbf{1} := (1, 1, \dots, 1).$$

If this is the case, we shall write that

$$\text{st-} \lim_{\mathbf{k} \rightarrow \infty} x_{\mathbf{k}} = \xi.$$

Theorem 4. ([3]) *Let $x_{\mathbf{k}} : N_+^n \rightarrow C$, where $n \geq 1$ is a fixed integer. For*

$$\text{st-} \lim_{\mathbf{k} \rightarrow \infty} x_{\mathbf{k}} = \xi$$

it is necessary and sufficient that for every $t \in R$,

$$\lim_{\mathbf{l} \rightarrow \infty} \frac{1}{|\mathbf{l}|} \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{l}} e^{itx_{\mathbf{k}}} = e^{it\xi}.$$

Let ν be an arbitrary positive measure defined on the Borel measurable subsets of R_+^n (or possibly concentrated only on N_+^n) with the property that $\nu(R_+^n) = \infty$. We introduce the concept of statistical limit (at ∞) of Borel measurable functions on R_+^n , as well as statistical convergence of multiple sequences on N_+^n with respect to the measure ν as follows. A Borel measurable function $f : R_+^n \rightarrow C$ is said to have a statistical limit at ∞ with respect to ν if there exists a number ξ such that for every $\epsilon > 0$,

$$(11) \quad \lim_{\mathbf{b} \rightarrow \infty} \frac{\nu(\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u}) - \xi| \geq \epsilon\})}{\nu(\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b}\})} = 0.$$

Now, definition (10) is the special case of (11) when ν is the ordinary Lebesgue measure on R_+^n . It can be proved that both Theorems 3 and 4 remain valid if we use the more general definition (11).

Theorem 5. ([3]) *Let $f : R_+^n \rightarrow C$ be a Borel measurable function and ν a positive Borel measure on R_+^n such that $\nu(R_+^n) = \infty$. Then the limit relation (11) holds for every $\epsilon > 0$ if and only if for every $t \in R$,*

$$\lim_{\mathbf{b} \rightarrow \infty} \frac{1}{\nu(\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b}\})} \int_0^{b_1} \cdots \int_0^{b_n} e^{itf(\mathbf{u})} d\nu(u_1, \dots, u_n) = e^{it\xi}.$$

3. Tauberian theorems for statistical limit

In view of the notion of statistical limit it was a natural idea to define the statistical summability and to find Tauberian conditions under which statistical limit follows from statistical summability.

Let $0 \neq P : R_+ \rightarrow R_+$ be a nondecreasing function such that $P(0) = 0$ and

$$(12) \quad \text{st-}\liminf_{t \rightarrow \infty} \frac{P(\lambda t)}{P(t)} > 1 \quad \text{for every } \lambda > 1.$$

We introduce the functions $s(x)$ and $\sigma(t)$ as in (2). If the finite limit

$$(13) \quad \text{st-}\lim_{t \rightarrow \infty} \sigma(t) = l \quad \text{exists,}$$

then we say that

$$\int_0^\infty f(x) dx$$

is *statistically summable to l* with respect to the weight function $P(t)$.

First we consider real-valued function f and prove the following theorem under one-sided Tauberian conditions.

Theorem 6. ([6]) *If a real-valued function $f \in L_{loc}^1(R_+)$ is such that (12) and (13) hold, then $\text{st-}\lim_{x \rightarrow \infty} s(x) = l$ holds if and only if for every $\epsilon > 0$, we have*

$$\inf_{\lambda > 1} \limsup_{a \rightarrow \infty} \frac{1}{a} |\{t \in (0, a) :$$

$$(14) \quad \frac{1}{P(\lambda t) - P(t)} \int_t^{\lambda t} [s(x) - s(t)] dP(x) < -\epsilon \} = 0$$

and

$$\inf_{0 < \lambda < 1} \limsup_{a \rightarrow \infty} \frac{1}{a} |\{t \in (0, a) :$$

$$(15) \quad \frac{1}{P(t) - P(\lambda t)} \int_{\lambda t}^t [s(t) - s(x)] dP(x) < -\epsilon \} = 0.$$

Second, we consider the general case where the function f may take on complex values. We shall prove the following theorem under two-sided Tauberian condition.

Theorem 7. *If a complex-valued function $f \in L^1_{loc}(R_+)$ is such that (12) and (13) hold, then $\text{st-}\lim_{x \rightarrow \infty} s(x) = l$ holds if and only if for every $\epsilon > 0$, we have*

$$\inf_{\lambda > 1} \limsup_{a \rightarrow \infty} \frac{1}{a} |\{t \in (0, a) :$$

$$(16) \quad \left| \frac{1}{P(\lambda t) - P(t)} \int_t^{\lambda t} [s(x) - s(t)] dP(x) \right| > \epsilon \} = 0,$$

or

$$\inf_{0 < \lambda < 1} \limsup_{a \rightarrow \infty} \frac{1}{a} |\{t \in (0, a) :$$

$$(17) \quad \left| \frac{1}{P(t) - P(\lambda t)} \int_{\lambda t}^t [s(t) - s(x)] dP(x) \right| > \epsilon \} = 0.$$

4. Tauberian conditions for statistical summability of double sequences

Móricz and Orhan have recently proved necessary and sufficient Tauberian conditions under which statistical convergence follows from statistical summability by weighted means [9]. We extend this result from single to double sequences.

A double sequence $(x_{jk} : j, k = 0, 1, 2, \dots)$ of (real or complex) numbers is said to be *statistically convergent* to some number L , if for each $\epsilon > 0$,

$$\lim_{m, n \rightarrow \infty} \frac{1}{(m+1)(n+1)} |\{j \leq m \text{ and } k \leq n : |x_{jk} - L| \geq \epsilon\}| = 0.$$

Let $p := \{p_j\}_{j=0}^{\infty}$, $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers ($p_0, q_0 > 0$) with the property that

$$P_m := \sum_{j=0}^m p_j \rightarrow \infty \text{ as } m \rightarrow \infty \text{ and } Q_n := \sum_{k=0}^n q_k \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The weighted means of a given double sequence (x_{jk}) are the (\overline{N}, p, q) means t_{mn} , which are defined by

$$(18) \quad t_{mn} = \frac{1}{P_m Q_n} \sum_{j=0}^m \sum_{k=0}^n p_j q_k x_{jk}, \quad m, n = 0, 1, 2, \dots$$

We say that the sequence x_{jk} is *statistically summable* (\overline{N}, p, q) to L if

$$(19) \quad \text{st-}\lim t_{mn} = L.$$

First, we consider sequences (x_{jk}) of real numbers and give one-sided Tauberian conditions. We use the notion of statistical limit inferior and limit superior introduced by Fridy and Orhan [2].

Theorem 8. ([5]) Let $p := \{p_j\}_{j=0}^{\infty}$, and $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers such that $p_0 > 0$, $q_0 > 0$ and

$$(20) \quad \text{st-}\liminf \frac{P_{\lambda_m}}{P_m} > 1 \text{ and } \text{st-}\liminf \frac{Q_{\lambda_n}}{Q_n} > 1 \text{ for all } \lambda > 1,$$

where $\lambda_m := [\lambda m]$, $\lambda_n := [\lambda n]$, and let (x_{jk}) be a sequence of real numbers, which is statistically summable (\bar{N}, p, q) to a finite number L . Then (x_{jk}) is statistically convergent to the same L if and only if the following two conditions hold: for every $\epsilon > 0$,

$$(21) \quad \inf_{\lambda > 1} \limsup_{M, N \rightarrow \infty} \frac{1}{(M+1)(N+1)} |\{m \leq M \text{ and } n \leq N : \\ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \leq -\epsilon\}| = 0$$

and

$$(22) \quad \inf_{0 < \lambda < 1} \limsup_{M, N \rightarrow \infty} \frac{1}{(M+1)(N+1)} |\{m \leq M \text{ and } n \leq N : \\ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^m \sum_{k=\lambda_n+1}^n p_j q_k (x_{mn} - x_{jk}) \leq -\epsilon\}| = 0.$$

Second, we consider sequences (x_{jk}) of complex numbers and give two-sided Tauberian conditions.

Theorem 9. ([5]) Let $p := \{p_j\}_{j=0}^{\infty}$, and $q := \{q_k\}_{k=0}^{\infty}$ be two sequences of nonnegative numbers such that $p_0 > 0$, $q_0 > 0$ and conditions in (20) are satisfied. Assume that (19) holds. Then (x_{jk}) is statistically convergent to the same L if and only if one of the following two conditions holds: for every $\epsilon > 0$, either

$$(23) \quad \inf_{\lambda > 1} \limsup_{M, N \rightarrow \infty} \frac{1}{(M+1)(N+1)} |\{m \leq M \text{ and } n \leq N :$$

$$\left| \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (x_{jk} - x_{mn}) \right| \geq \epsilon \} = 0$$

or

$$(24) \quad \inf_{0 < \lambda < 1} \limsup_{M, N \rightarrow \infty} \frac{1}{(M+1)(N+1)} |\{m \leq M \quad \text{and} \quad n \leq N : \\$$

$$\left| \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^m \sum_{k=\lambda_n+1}^n p_j q_k (x_{mn} - x_{jk}) \right| \geq \epsilon \} = 0.$$

References

- [1] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), 241-244.
- [2] J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.* **125** (1997), 3625-3631
- [3] Á. Fekete and F. Móricz, A characterization of the existence of statistical limit of measurable functions, *Acta Math. Hungar.* (to appear)
- [4] Á. Fekete and F. Móricz, Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over R_+ , *Publ. Math. Debrecen*, **67** (2005), 65-78.
- [5] Á. Fekete, Tauberian conditions for double sequences that are statistically summable by weighted means, *Sarajevo Journal of Mathematics*, Vol.1 (14) (2005), 197-210
- [6] Á. Fekete, Tauberian conditions under which the statistical limit of an integrable function follows from its statistical summability, *Studia Sci. Math. Hungarica*, **43** (2006), 131-145.
- [7] J. Karamata, *Sur les théorèmes inverses des procédés de sommabilité*, Hermann et Cie, Paris, 1937.
- [8] F. Móricz, Statistical limits of measurable functions, *Analysis* **24** (2004), 207-219.
- [9] F. Móricz and C. Orhan, Tauberian conditions under which statistical convergence follows from statistical summability by weighted means, *Studia Sci. Math. Hungar.* **41** (2004), 391-403.
- [10] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959), 361-375.