Appendix 1.
Experiments for Difference Equations with Continuous Time

■ Acknowledgement

The development was done during the visit of the author at the Department of the Medical Informatics, Albert Szent-Györgyi Medical University.

Introduction

Consider the difference equation with continuous time

$x(t)=a(t)\,x(t-1)+b(t)\,x(p(t)), \quad t\geq t_0,$

where a, b and p are real functions such that $0<a(t)<1$, $0<p(t)<t$, $t\geq t_0$, and $\lim_{t\to\infty} p(t)=\infty$.

Here we do some numerical and graphical experiments on the asymptotic behavior of the solutions of some equation of above form, based on the asymptotic estimates given in Section 5.

Notations

a(t), b(t) :	coefficient functions in the equation
p(t):	lag function
φ (t):	initial function
ρ(t):	function estimating the solutions
x1, x2:	window frame parameters for x(t)
t_0:	initial time
t_{-1}:	minimal time ($[t_{-1},t_0]$ intial interval)
T:	final time for solution
α:	constant such that $\alpha\leq 1-a(t)$, $t\geq t_0$,
M_0:	constant such that $M_0=\sup_{t_{\min}\leq t\leq t_0}\{\rho(t)\,\lvert\varphi(t)\rvert\}$

Example 1: $x(t) = \frac{1}{(t+1)^2}\, x(t-1) + \frac{1}{2}\left(1 - \frac{1}{(t+1)^2}\right) x(\frac{t}{2})$

Consider the particular case $p(t)=\frac{t}{2}$. In virtue of Corollary 5.3, for the solution $x(t)$ holds that $|x(t)| \le \frac{C}{\rho(t)}$ for $t \ge t_0$, where $\rho(t)=t$ and

$$C= \prod_{n=1}^{\infty} \left(1 + \frac{t_0\, 0.5^{n+2}}{\alpha\, 0.5\, (0.5\, t_0 - 0.5^n)^2}\right).$$

■ Definitions

In[9]:=
```
Clear[a, b, φ, x, ρ, p, x1, x2, t0, tmin, T, α, M0, CC]
a[t_] := 1/(1 + t)^2;
b[t_] := (1 - a[t]) 0.5;
p[t_] := 0.5 t;
φ[t_] := j t^0.5;   (* j is a parameter given later *)
ρ[t_] := t;
```

■ Parameters

In[11]:=
```
x1 := -4; x2 := 4;
t0 := 2; tmin := 1; T := 30;
α = 8/9; M0 = 5 2^0.5;
```

■ Recursive definition of the solutions

In[12]:=
```
x[t_] := Which[t ≥ tmin && t < t0, φ[t], t ≥ t0, a[t] x[t - 1] + b[t] x[p[t]]];
```

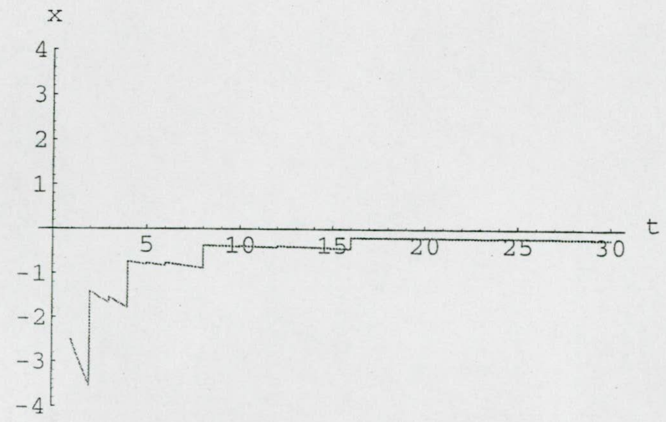
■ Solve the value of the infinite product

In[13]:=
```
CC = M0 * N[Evaluate[Product[1 + t0 0.5^n/(α 0.5 (t0 - 0.5^n)^2), {n, 1, ∞}]]]
```

Out[13]= 25.8848

■ Plot the solutions with several initial functions

In[14]:=
```
plt =
  Table[Plot[x[t], {t, tmin, T}, PlotRange → {-4, 4},
    PlotPoints → 50, PlotStyle → {Hue[(j + 2.5)/10]}, AxesLabel → {t, x}],
   {j, -2.5, 2.5, 1}];
```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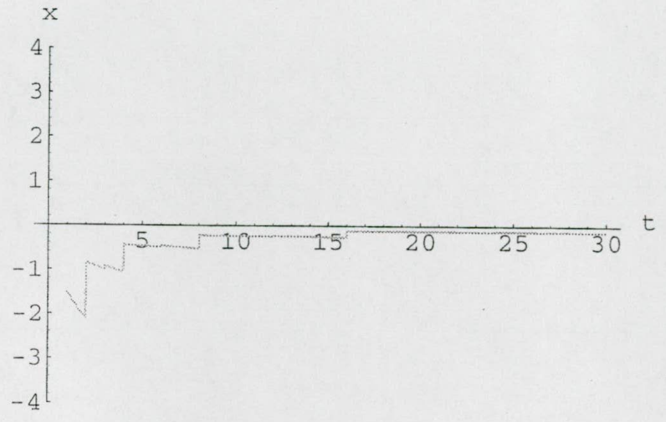
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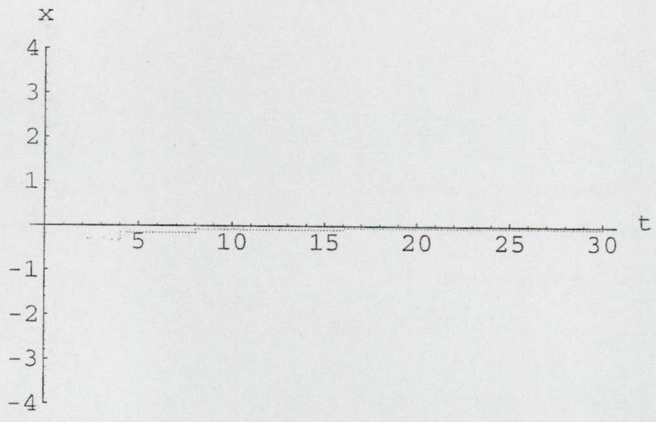
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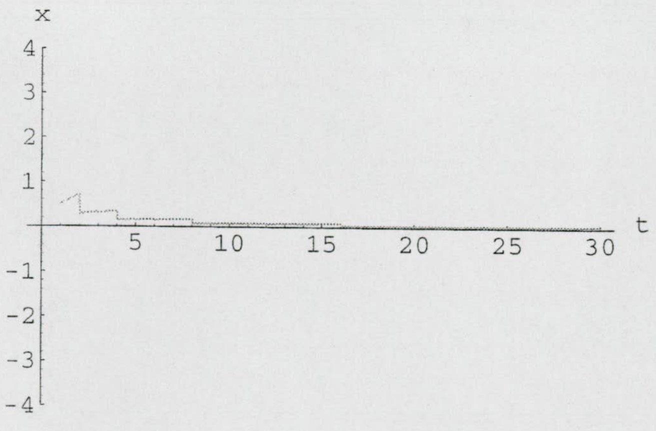
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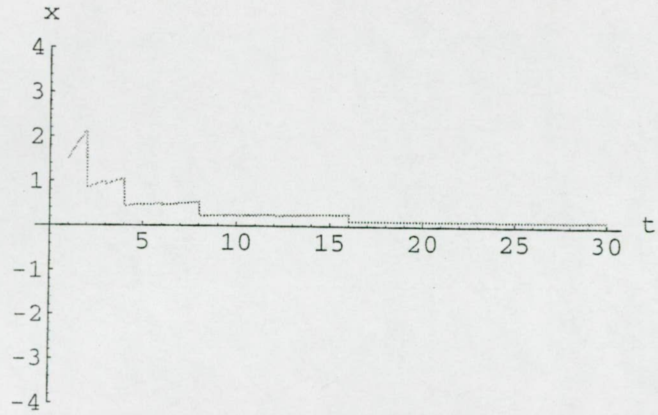
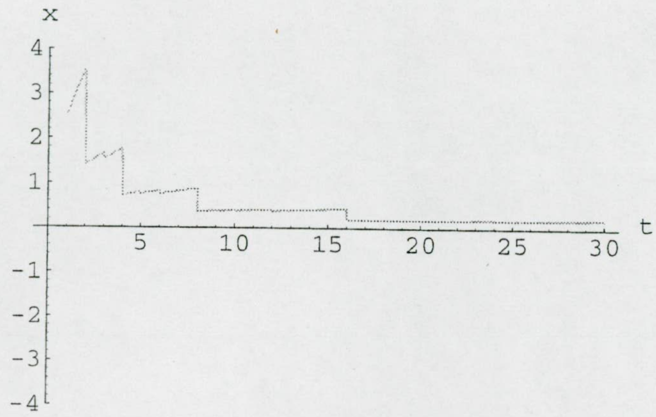
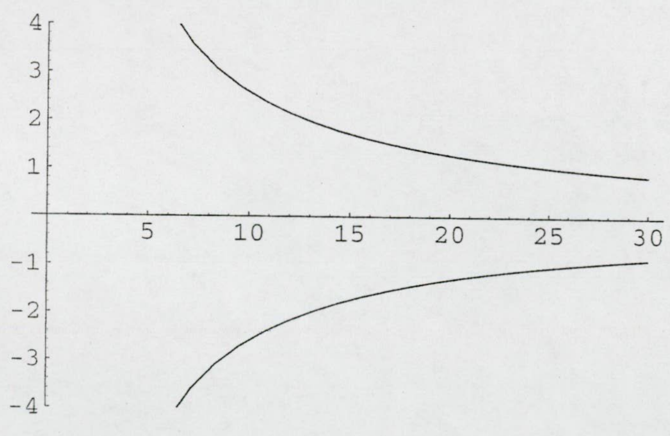




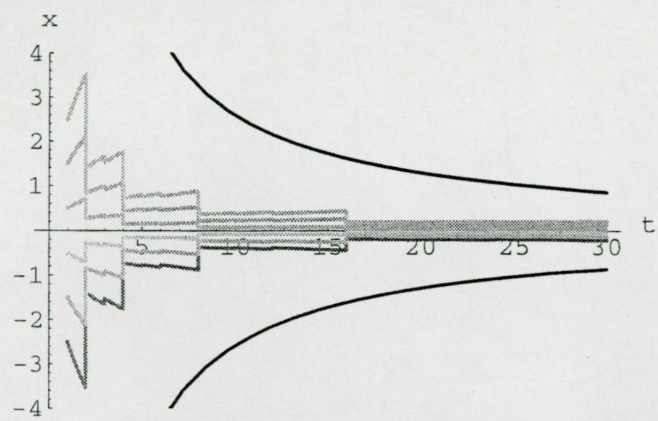
■ Plot the covering curves

In[15]:= $\texttt{gor1 = Plot}\left[\left\{\frac{\texttt{cc}}{\rho[\texttt{t}]}, -\frac{\texttt{cc}}{\rho[\texttt{t}]}\right\}, \{\texttt{t, tmin, T}\}, \texttt{PlotRange} \rightarrow \{\texttt{x1, x2}\}\right];$



■ Plot them together

```
In[16]:= Show[Graphics[Thickness[0.006]], gor1, plt,
           DisplayFunction → $DisplayFunction, AxesLabel → {"t", "x"},
           PlotRange → {x1, x2}, Axes → True];
```



Example 2: $x(t) = \frac{x(t-1)}{(t+1)^2} + \frac{1}{2}\left(1 - \frac{1}{(t+1)^2}\right)x\left(\sqrt{t}\right)$

Consider the particular case $p(t)=\sqrt{t}$. In virtue of Corollary 5.4, for the solution $x(t)$ holds that $|x(t)|\leq C/(\rho(t))$ for $t\geq t_0$, where $\rho(t)=\log t$ and

$C= M_0 \prod_{n=1}^{\infty}$ (1 + (log t 0 2^(n+2)) / (α ((t0)^2^n - 1) log ^2 ((t0)^2^n - 1))).

■ Definitions

```
In[17]:= Clear[a, b, φ, x, ρ, p, x1, x2, t0, tmin, T, α, M0, CC]
                      1
         a[t_] := ────────── ;
                  (1 + t)^2
         b[t_] := (1 - a[t]) 0.5;
         p[t_] := √t ;
         φ[t_] := j t^0.5;   (* j is a parameter given later *)
         ρ[t_] := Log[t];
```

■ Parameters

```
In[19]:= x1 := -4; x2 := 4;
         t0 := 2.; tmin := 1; T := 20;
              8
         α = ─── ;
              9
         M0 := Log[2] 2.5 2^0.5;
```

■ Recursive definition of the solutions

```
In[20]:= x[t_] := Which[t ≥ tmin && t < t0, φ[t], t ≥ t0, a[t] x[t - 1] + b[t] x[p[t]]];
```

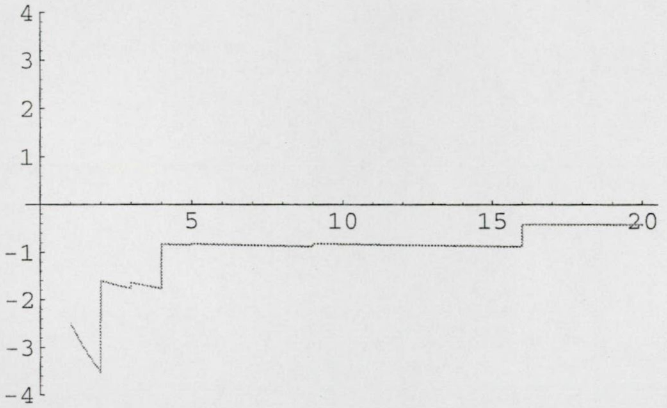
■ Solve the value of the infinite product

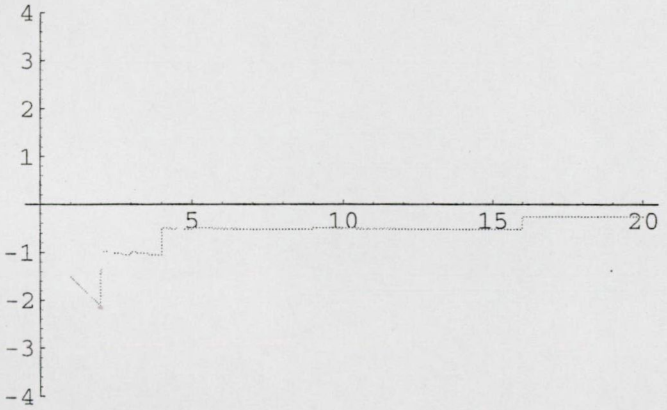
In[21]:=
$$CC = M0\ N\left[\text{Evaluate}\left[\prod_{n=1}^{\infty}\left(1 + \frac{\text{Log}[t0]\ 2.^{n+1}}{\alpha\ (t0^{2.^{n}} - 1)\ \text{Log}[t0^{2.^{n}} - 1]^{2}}\right)\right]\right]$$

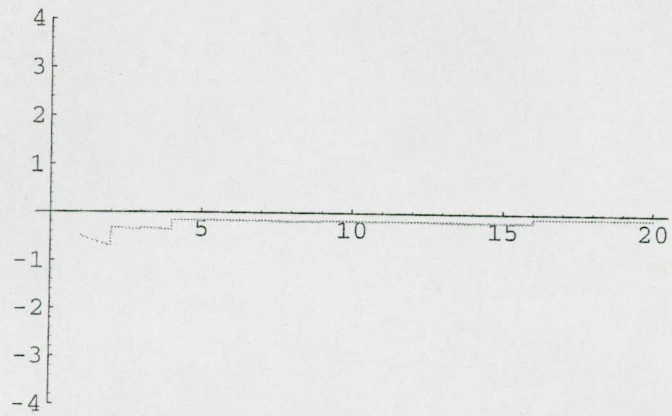
Out[21]= 4.82814

■ Plot the solutions with several initial functions

In[22]:=
```
plt1 = Table[Plot[x[t], {t, tmin, T},
    PlotRange → {x1, x2}, PlotPoints → 50, PlotStyle → {Hue[(j + 3)/10]}],
  {j, -2.5, 2.5, 1}];
```



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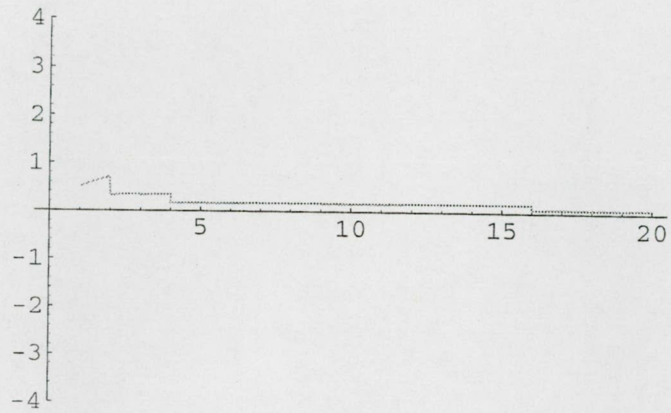
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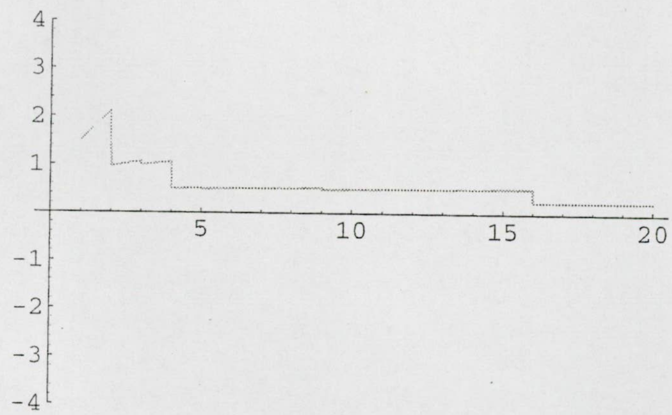
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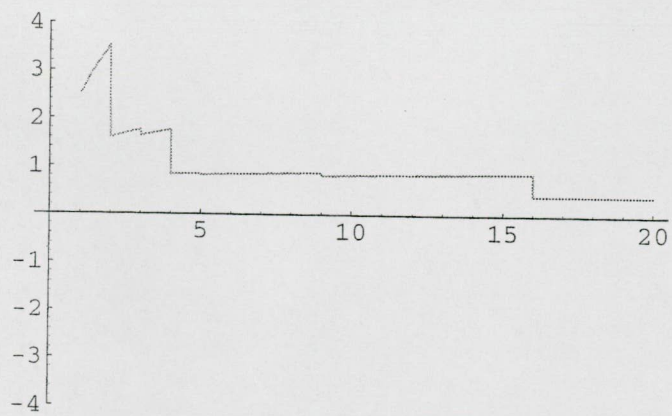
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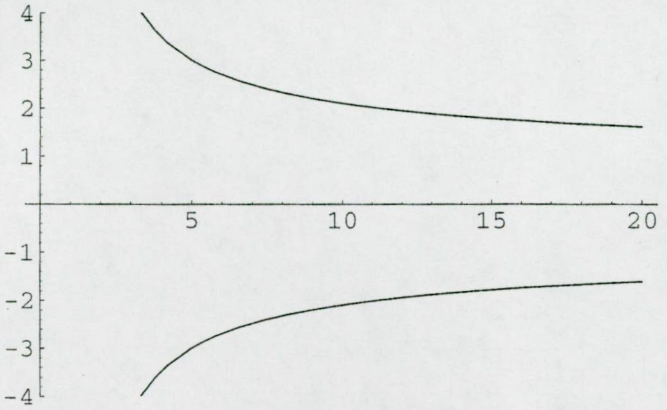
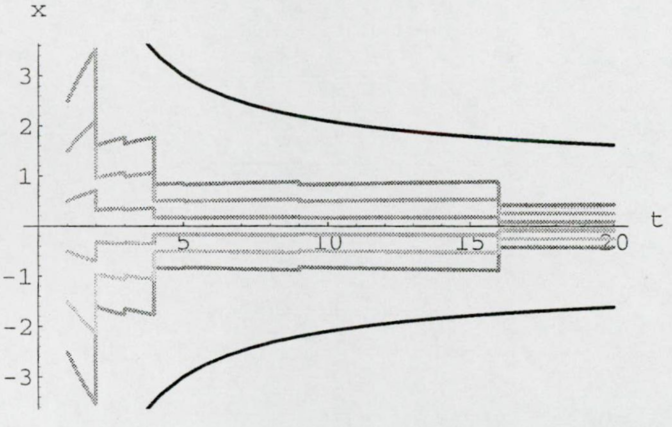
■ Plot the covering curves

In[23]:=
```
gor2 = Plot[{cc/ρ[t], -cc/ρ[t]}, {t, tmin, T}, PlotRange → {x1, x2}];
```



■ Plot them together

In[24]:=
```
Show[Graphics[Thickness[0.006]], gor2, plt1,
  DisplayFunction -> $DisplayFunction, AxesLabel -> {"t", "x"},
  Axes -> True];
```



Example 3: $x(t) = \frac{1}{8} x(t-2) + \frac{1}{4} x(t-1)$

Consider the particular case $p(t)=t-2$. In virtue of Corollary 5.12, for the solution $x(t)$ holds that $|x(t)| \le \frac{M_0}{\rho(t)}$ for $t \ge t_0$, where $\rho(t)=\lambda^t$.

■ Definitions

```
In[25]:= Clear[a, b, φ, x, ρ, p, x1, x2, t0, tmin, T, M0]
         a[t_] := 0.25;
         b[t_] := 1 / 8;
         p[t_] := t - 2;
         φ[t_] := j t^0.5;    (* j is a parameter given later *)
         ρ[t_] := 2^t;
```

■ Parameters

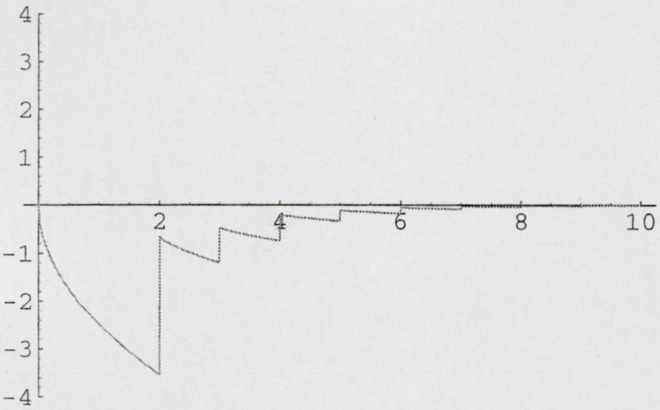
```
In[27]:= x1 := -4; x2 := 4;
         t0 := 2.; tmin := 0; T := 10;
         M0 := 10 2^0.5;
```

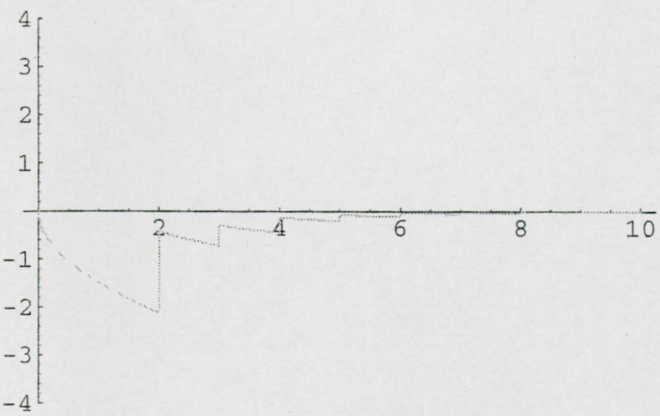
■ Recursive definition of the solutions

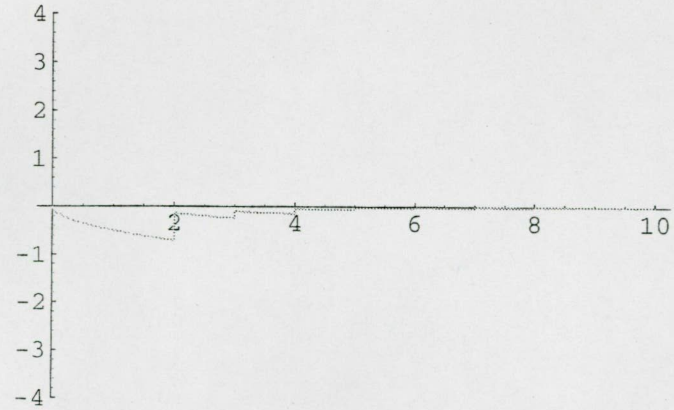
```
In[28]:= x[t_] := Which[t ≥ tmin && t < t0, φ[t], t ≥ t0, a[t] x[t - 1] + b[t] x[p[t]]];
```

■ Plot the solutions with several initial functions

```
In[29]:= plt2 = Table[Plot[x[t], {t, tmin, T},
            PlotRange → {x1, x2}, PlotPoints → 50, PlotStyle → {Hue[(j + 3)/10]}],
           {j, -2.5, 2.5, 1}];
```



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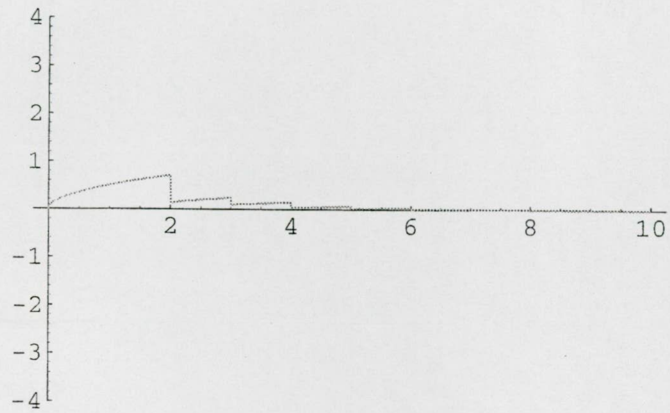
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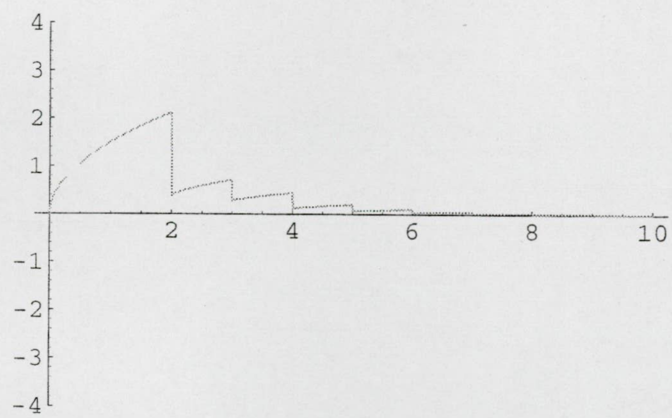
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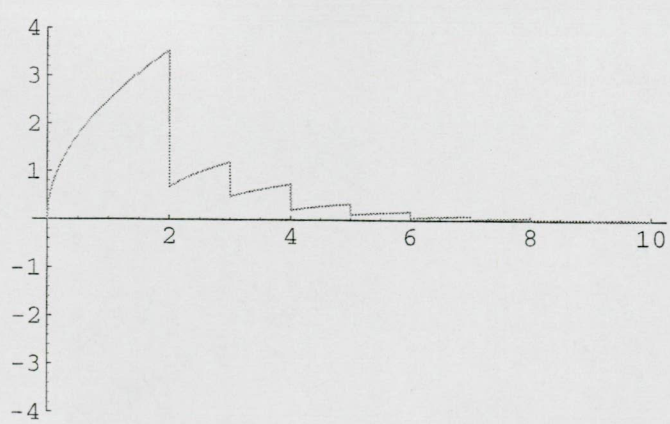
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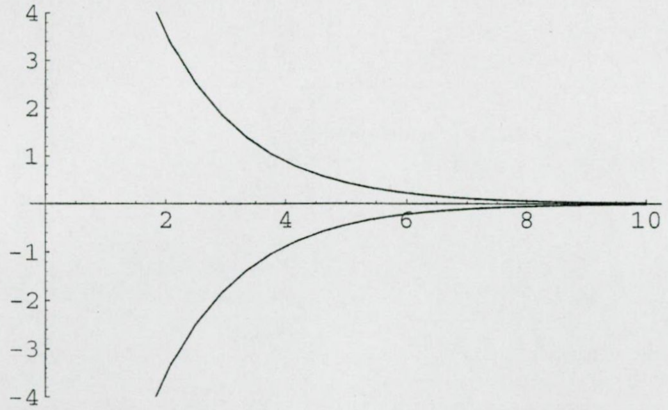
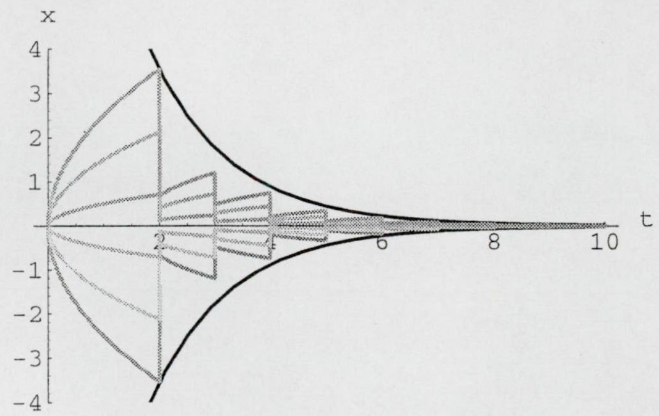
■ Plot the covering curves

In[30]:=
$$\text{gor3 = Plot}\left[\left\{\frac{\text{M0}}{\rho[\text{t}]}, -\frac{\text{M0}}{\rho[\text{t}]}\right\}, \{\text{t, tmin, T}\}, \text{PlotRange} \to \{\text{x1, x2}\}\right];$$



■ Plot them together

In[31]:=
```
Show[Graphics[Thickness[0.006]], gor3, plt2, PlotRange → {x1, x2},
 DisplayFunction → $DisplayFunction, AxesLabel → {"t", "x"}, Axes → True];
```



Appendix 2.
Discrete pantograph-type difference equations

Acknowledgement

The development was done during the visit of the author at the Department of Medical Informatics, Albert Szent-Györgyi Medical University.

Theory

Consider the discrete difference equation

$x_{n+1} = (1 - a_n) x_n + b_n x_{p_n}$, $n \geq n_0$,

where $\{a_n\}$, $\{b_n\}$ are real sequences such that $0 < a_n < 1$, $\{p_n\}$ is *a* sequence of natural numbers such that $p_n \leq n$, $n \geq n_0$, and $\lim_{n \to \infty} p_n = \infty$.

Here we do some numerical and graphical experiments on the asymtotic behavior of the solutions of some equation of above form, based on the asymptotic estimates given in Section 6.

Notations

a_n, b_n : coefficient sequences in the equation

p_n:	lag sequence
φ_n:	initial values ($n=n_{-1}, n_{-1}+1, \ldots, n_0$)
ρ_n:	sequence estimating the solutions
x1, x2:	window frame parameters for x_n
n_0:	initial time
n_{-1}:	minimal time
N1:	final time for solution
α:	constant such that $\alpha \leq a_n$, $t \geq t_0$,
M_0:	constant such that $M_0 = \max_{n_{-1} \leq n \leq n_0} \{\rho_n \, \lvert\varphi_n\rvert\}$

In[1]:= **Example 1 :** $x_{n+1} - x_n = \left(\frac{1}{4} + \frac{1}{(n+1)^3}\right) x_n - \frac{1}{3}\left(\frac{1}{4} + \frac{1}{(n+1)^3}\right) x_{[\frac{n}{2}]}$

Consider the particular case $p_n=[\frac{n}{2}]$. In virtue of Corollary 6.2, for the solution x_n holds that $|x_n| \leq \frac{C}{\rho_n}$ for $n \geq n_0$, where $\rho_n = n$ and

$C = M_0 \prod_{j=1}^{\infty}\left(1+\frac{2\,2^{(j+1)}}{\alpha\,(2\,2^j-1)^2}\right)$.

■ Definitions

```
In[54]:= Clear[an, bn, pn, ρn, xn, C1, x1, x2, N0, N1, MM, NN, α]; NN = 1000;
         an = Table[1/(n + 1)^3 + 1/4, {n, 1, NN}]; bn = Table[-an[[n]]/3, {n, 1, NN}];
         pn = Table[Floor[n/2], {n, 1, NN}];
```

■ Parameters

```
In[55]:= x1 = -5; x2 = 5;
         N1 = 50;
         N0 = 2; α = 0.25;
         MM = 10; M0 = 2;
         soltab = Table[0, {i, 1, MM}, {j, 1, N1 + 1}];
         Table[
          {soltab[[i + 1, 1]] = Cos[i 2 π/MM], soltab[[i + 1, 2]] = Sin[i 2 π/MM]}, {i, 0, MM - 1}];
```

■ Recursive definition of the solutions

```
In[56]:= SetAttributes[Pantograph, HoldFirst]
         Pantograph[x_, n_Integer, an_, bn_, pn_] := (1 - an[[n]]) x[[n]] + bn[[n]] x[[pn[[n]]]];
```

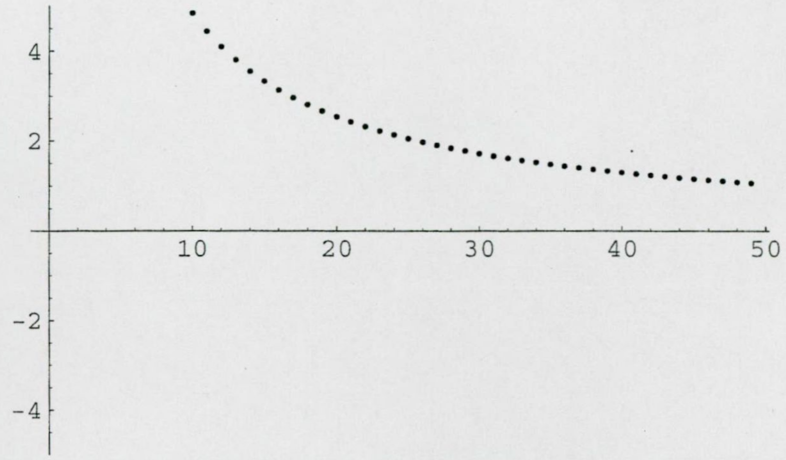
■ Solve the value of the infinite product

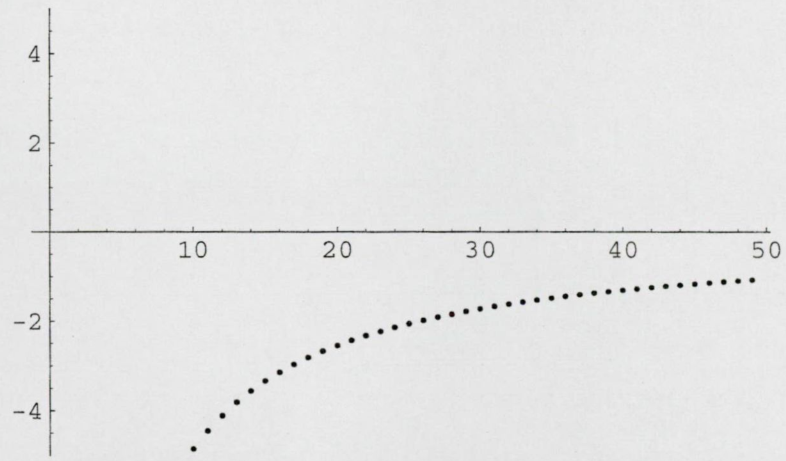
In[58]:= C1 = M0 N[Evaluate[$\prod_{j=1}^{\infty}\left(1+\frac{2\,2^{j+1}}{\alpha\,(2\,2^j-1)^2}\right)$]]

Out[58]= 53.3388

■ Plot the covering curves

In[59]:=
```
plt1 = ListPlot[
    Table[N[C1/n], {n, 2, N1}], Axes → True, PlotRange → {x1, x2}]; plt2 =
  ListPlot[Table[N[-C1/n], {n, 2, N1}], Axes → True, PlotRange → {x1, x2}];
```

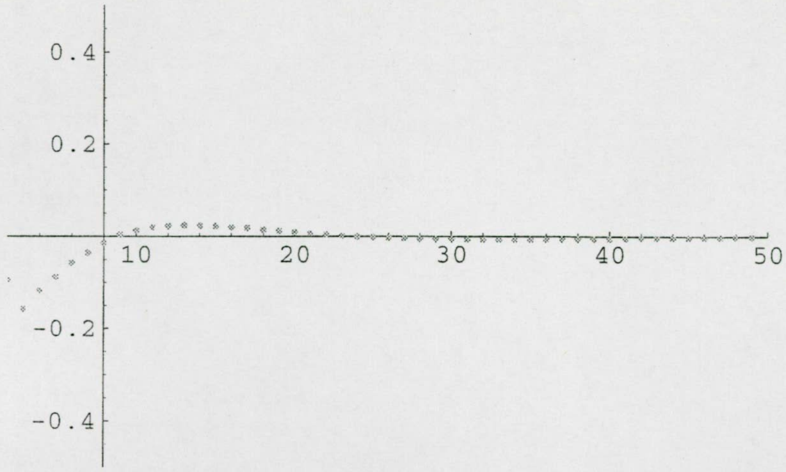




■ Solve the difference equation with several initial condition

In[60]:=
```
Do[soltab[[i, n + 1]] = N[Pantograph[soltab[[i]], n, an, bn, pn]],
  {n, 2, N1}, {i, 1, Length[soltab]}]
```

In[61]:=
```
plt3 = Table[ListPlot[Take[soltab[[i]], {2, N1}], PlotStyle → {Hue[i/10]},
    PlotRange → {{2, N1}, {-1/2, 1/2}}, PlotJoined → False],
   {i, 1, Length[soltab]}];
```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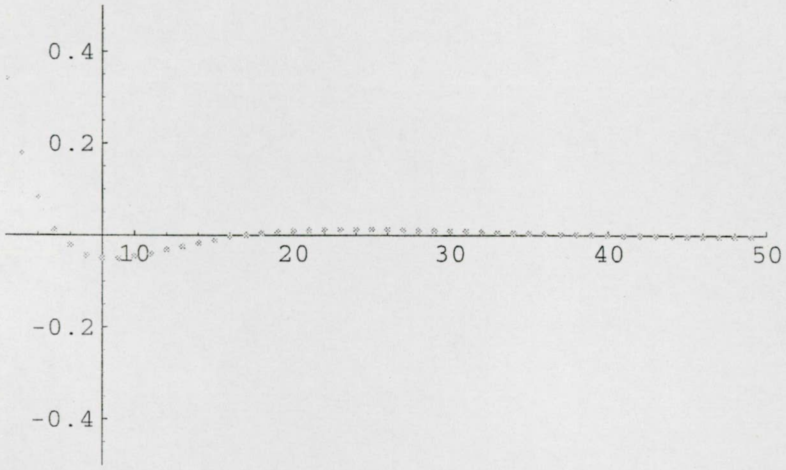
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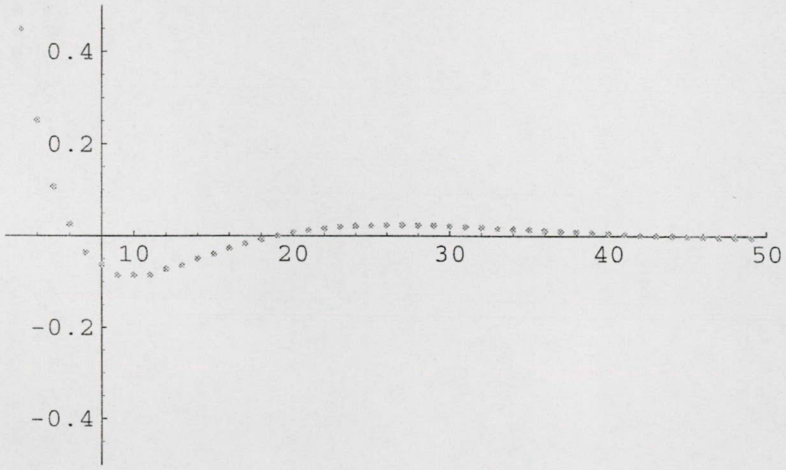
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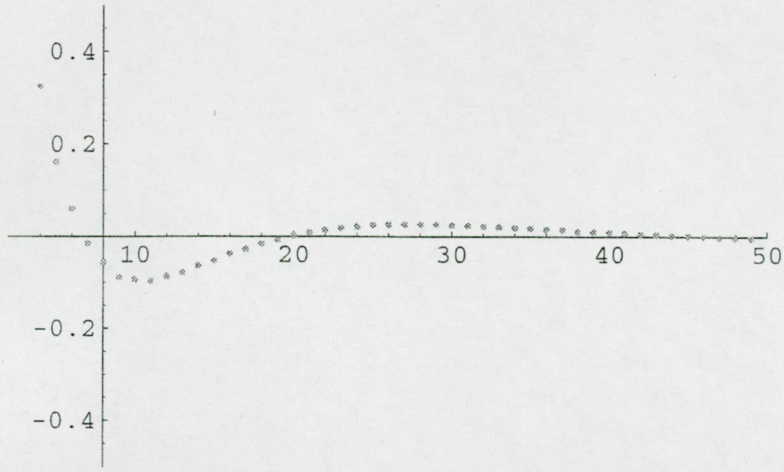
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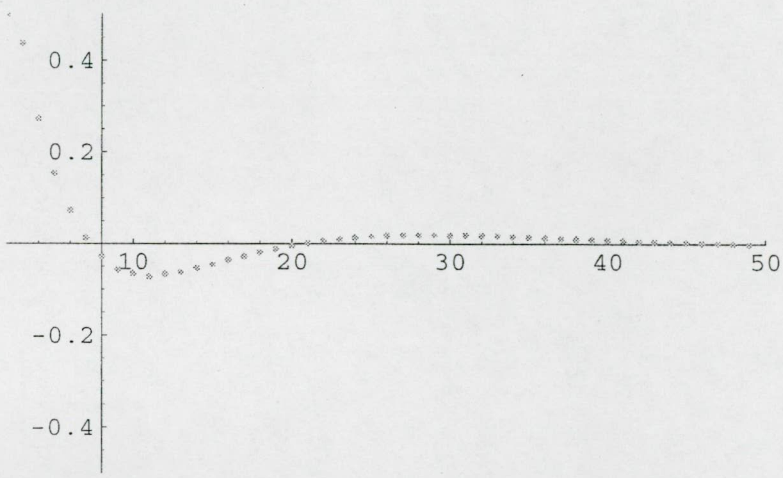
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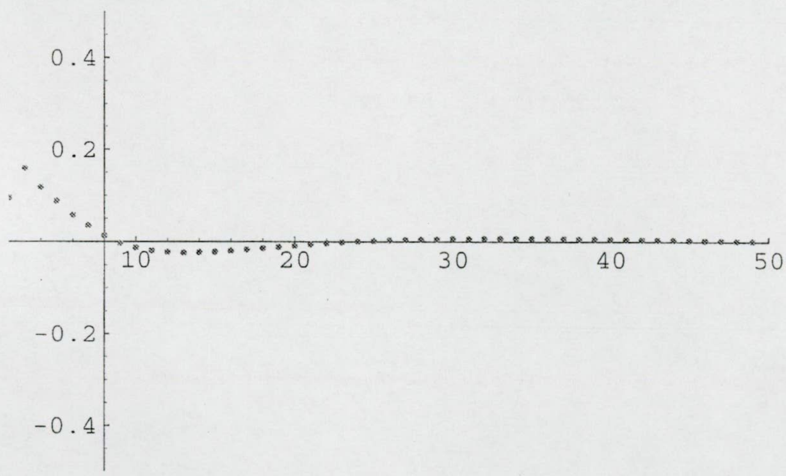
0.4
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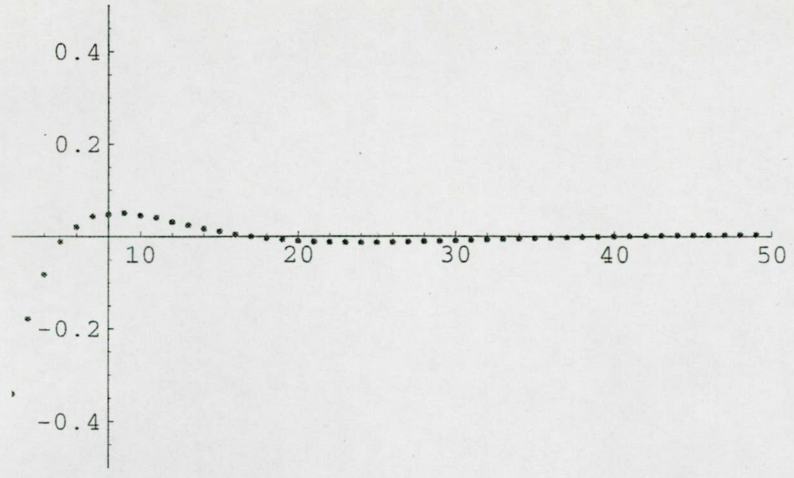
-0.4


0.4
0.2
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-0.4
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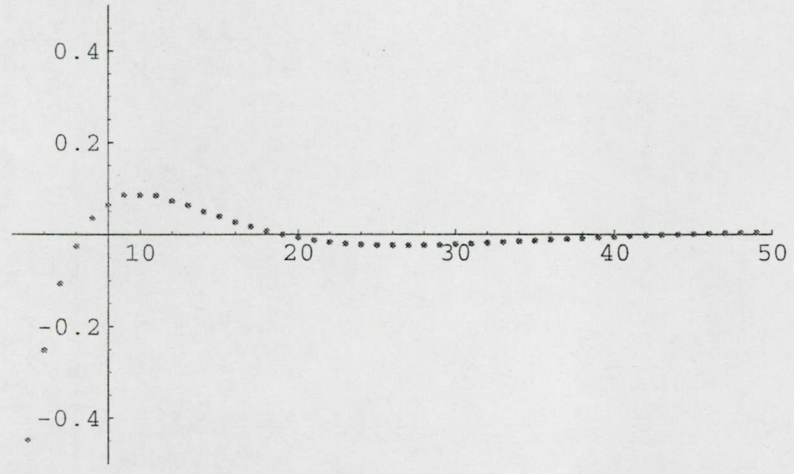
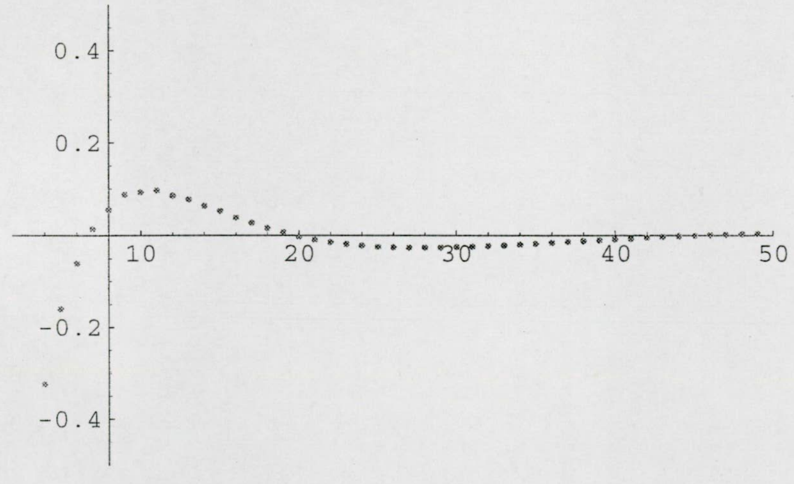
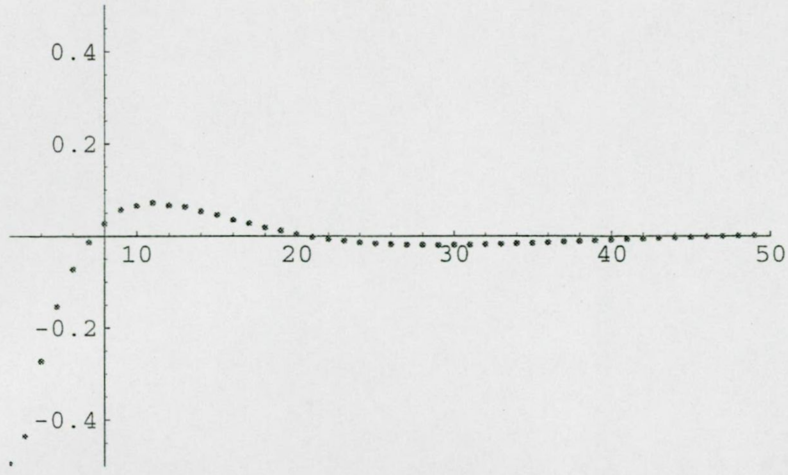
40
50

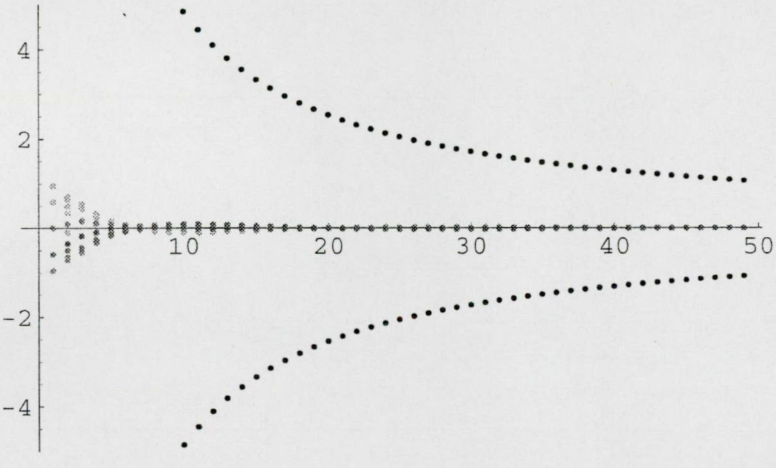

0.4
0.2
-0.2
-0.4
10
20
30
40
50


0.4
0.2
-0.2
-0.4
10
20
30
40
50



Plot them together

In[62]:=
```
Show[plt1, plt2, plt3, DisplayFunction → $DisplayFunction];
```



Example 2: $x_{n+1}-x_n=\left(\frac{1}{4}+\frac{1}{(n+1)^3}\right)x_n-\frac{1}{3}\left(\frac{1}{4}+\frac{1}{(n+1)^3}\right)x_{[\sqrt{n}]}$

Consider the particular case $p_n=[\sqrt{n}\,]$. In virtue of Corollary 6.3, for the solution x_n holds that $|x_n|\le\frac{C}{\rho_n}$ for $n\ge n_0$, where $\rho_n=\log n$ and

$$C=M_0\prod_{j=1}^{\infty}\left(1+\frac{\log n_0\ 2^{(j+1)}}{\alpha\,(n_0 2^j-1)(\log(n_0 2^j-1))^2}\right).$$

Definitions

In[16]:=
```
Clear[an, bn, pn, ρn, xn, C1, x1, x2, N0, N1, α, NN, MM]; NN = 1000;
an = Table[1/(n + 1)^3 + 1/4, {n, 1, NN}]; bn = Table[-an[[n]]/3, {n, 1, NN}];
pn = Table[Floor[√n], {n, 1, NN}];
```

■ Parameters

In[17]:=
```
x1 = -4; x2 = 4;
N1 = 50;
N0 = 2.;  α = 0.25;
MM = 10; M0 = Log[2];
soltab = Table[0, {i, 1, MM}, {j, 1, N1 + 1}];
Table[
  {soltab[[i + 1, 1]] = Cos[i 2 π / MM], soltab[[i + 1, 2]] = Sin[i 2 π / MM]}, {i, 0, MM - 1}];
```

■ Recursive definition of the solutions

In[18]:=
```
SetAttributes[Pantograph, HoldFirst]
Pantograph[x_, n_Integer, an_, bn_, pn_] := (1 - an[[n]]) x[[n]] + bn[[n]] x[[pn[[n]]]];
```

■ Solve the value of the infinite product

In[20]:=

$$C1 = M0\ N\left[\text{Evaluate}\left[\prod_{j=1}^{\infty}\left(1 + \frac{\text{Log}[N0]\ 2.^{j+1}}{\alpha\ (N0^{2.^{j}} - 1)\ \text{Log}[N0^{2.^{j}} - 1]^{2}}\right)\right]\right]$$

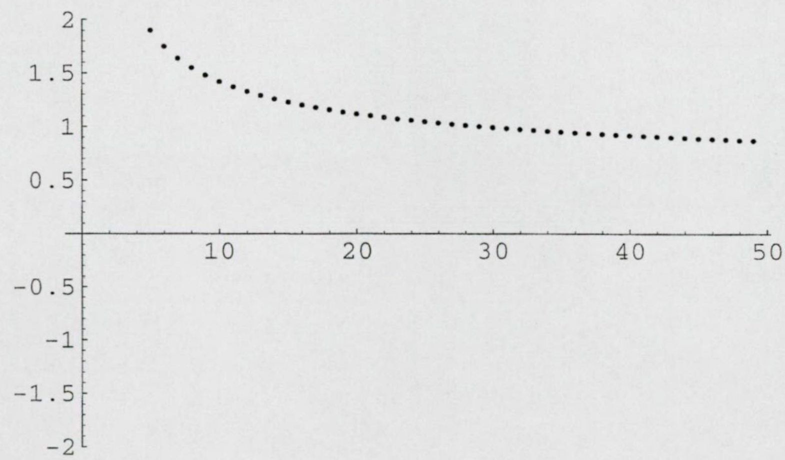
Out[20]= 3.40326

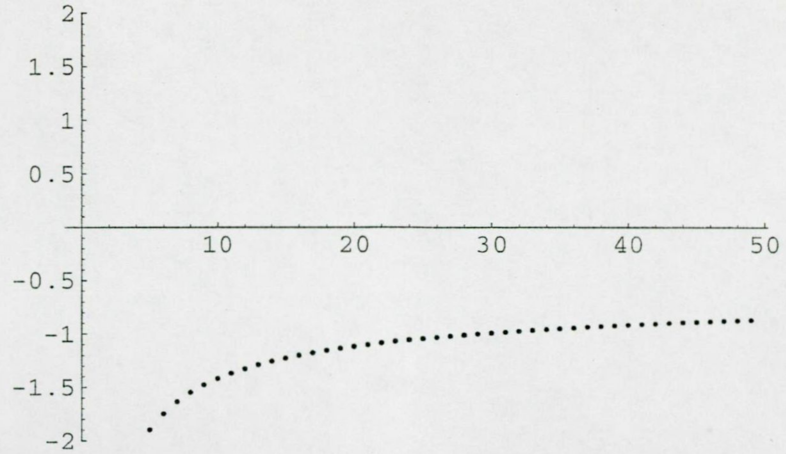
■ Plot the covering curves

In[21]:=

$$\text{plt4} = \text{ListPlot}\left[\text{Table}\left[N\left[\frac{C1}{\text{Log}[n]}\right], \{n, N0, N1\}\right], \text{PlotRange} \to \{-2, 2\}\right];$$

$$\text{plt5} = \text{ListPlot}\left[\text{Table}\left[N\left[-\frac{C1}{\text{Log}[n]}\right], \{n, N0, N1\}\right], \text{PlotRange} \to \{-2, 2\}\right];$$

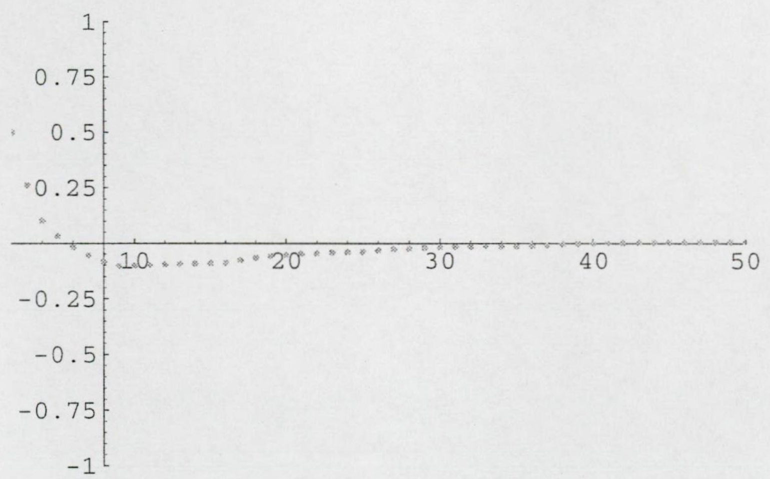


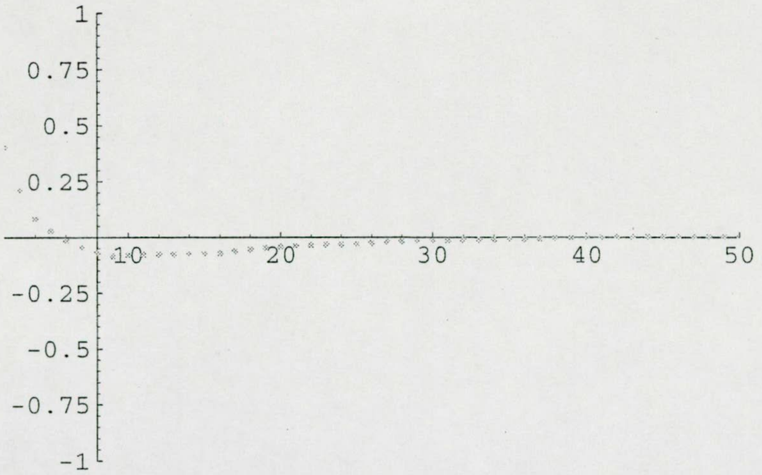


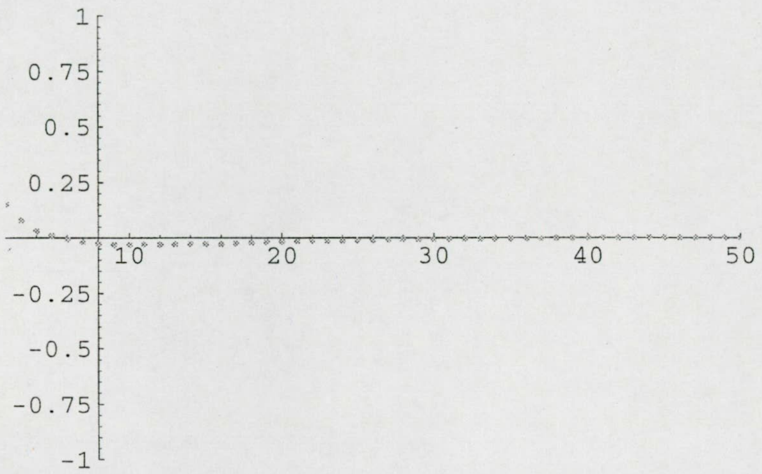
■ Solve the difference equation with several initial condition

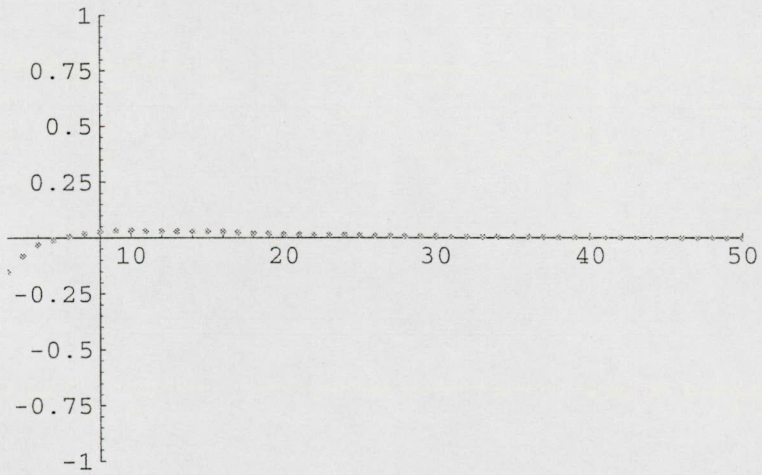
```
In[22]:= Do[soltab[[i, n + 1]] = N[Pantograph[soltab[[i]], n, an, bn, pn]],
          {n, 1, N1}, {i, 1, Length[soltab]}]
```

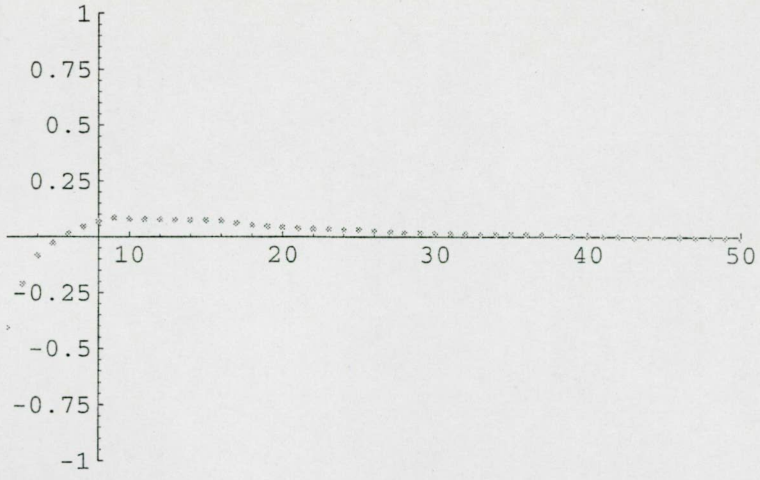
```
In[23]:= plt6 = Table[ListPlot[Take[soltab[[i]], {1, N1}], PlotStyle → {Hue[i/10]},
            PlotRange → {{2, N1}, {-1, 1}}, PlotJoined → False],
           {i, 1, Length[soltab]}];
```

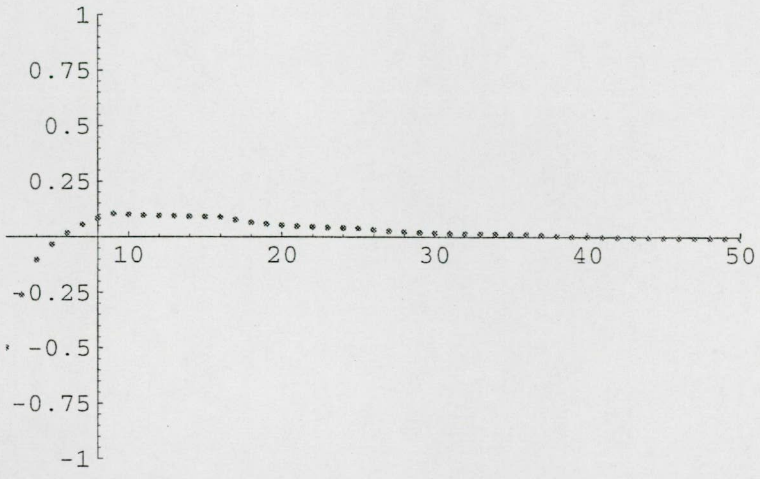


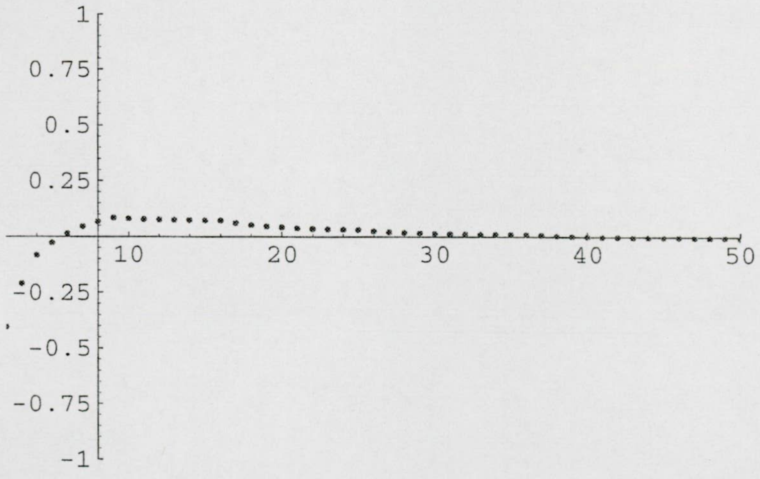

1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1

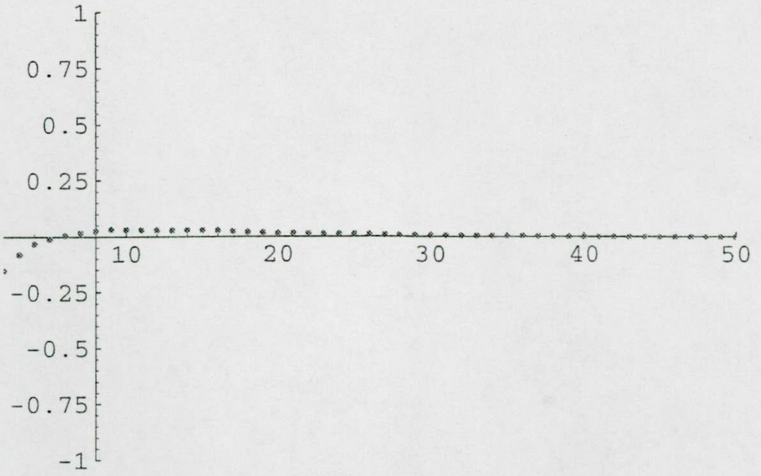

1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1

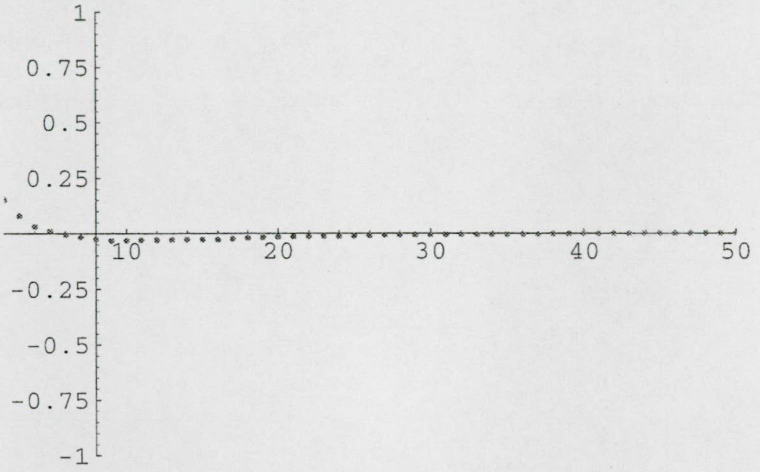

1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1

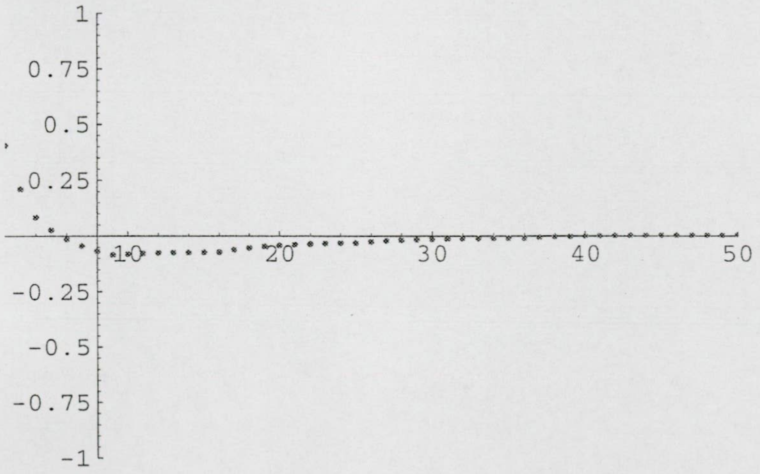

1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
10
20
30
40
50


1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
10
20
30
40
50


1
0.75
0.5
0.25
-0.25
-0.5
-0.75
-1
10
20
30
40
50


1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1


1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1


1
0.75
0.5
0.25
10
20
30
40
50
-0.25
-0.5
-0.75
-1

■ Plot them together

In[25]:=
```
Show[plt4, plt5, plt6, DisplayFunction -> $DisplayFunction,
  AxesOrigin -> {0, 0}];
```