ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENCE EQUATIONS WITH CONTINUOUS TIME

THESES OF THE Ph.D DISSERTATION

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1. Introduction

The essence of the qualitative theory of differential and difference equations consists of the examination of the characteristics of the solutions of the considered equations, not knowing them explicitly. Consequently, it serves as a direct source of information about solutions. Some of these characteristics are: oscillation, boundedness, periodicity, asymptotic behavior, stability properties, and others. The significance of this theory is tremendous since it can be applied also in the case when the solution can be expressed in explicit form but this form is very complicated not suitable for further investigation.

The present work is a treatment of the field of qualitative analysis of generalized difference equations and describes the asymptotic behavior of solutions. There are only few papers dealing with this topic and hence the given results are new in the qualitative theory of generalized difference equations. The investigation of the solutions of generalized difference equations is very useful because it is a part of the theory of neutral functional differential equations, pantograph differential equations and it can be applied also in the study of some other types of differential equations.

In the first part we define the concept of functional equations from a historical perspective as well as the particular cases such as the discrete difference equations and difference equations with continuous time. We mention the origin of the pantograph differential equation and cite some results about the asymptotic behavior of solutions because the research has been motivated by this type of differential equations. We sum up the usefulness and applicability of generalized difference equations approaching the problem with the applications from several branches of sciences.

In the main results we give asymptotic bounds for the solutions of difference equations specifying the asymptotic behavior of solutions and estimating the rate of the convergence. We obtain some results for systems and for special forms of the lag function. The study of the scalar form of the generalized difference equation is also useful because we can prove sharper results than in system case. We obtain asymptotic lower and upper bounds for the solutions of the scalar difference equation and apply the given results for some special lag functions. The reason why we formulate particular theorems for the discrete difference equations is that they form an independent branch and have their special problems and approximating methods for solving differential equations, which are not characteristic of more general functional equations.

With some typical models given in the *MATHEMATICA* developments we emphasize the connection between difference equations with continuous time and discrete difference equations underlining the principial differences between them. By presented examples the message of the main results becomes understandable and also we illustrate the comprehensiveness of using computers in mathematical research.

3. Preliminaries

We describe the concept of generalized difference equations. Let $\mathbf{R}$ be the set of real numbers and let n be a given positive integer. For given functions $f_i : \mathbf{R} \to \mathbf{R}$, $i = 1, 2, ..., n$, and $g : \mathbf{R}^{n+1} \to \mathbf{R}$, the *difference equation with continuous time* or the so-called *generalized difference equation* is the equation of the form

$$x(t) - x(t-1) = g(t, x(f_1(t)), ..., x(f_n(t))). \tag{1}$$

In the dissertation we study particular cases of generalized difference equations: delay difference equations with continuous time. Consider the following special form of Equation (1). Assume that $t_0 > 0$, and $A, B : [t_0, \infty) \to \mathbf{R}^{n\times n}$ are $n \times n$ real matrix valued functions. Let $p : [t_0, \infty) \to \mathbf{R}$ be given such that for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$. We discuss the asymptotic behavior of the solutions of the generalized difference equation

$$x(t) = A(t)x(t-1) + B(t)x(p(t)), \tag{2}$$

where $x(t) \in \mathbf{R}^n$, and estimate the rate of convergence of the solutions depending on the properties of matrix A. If the components of matrix A are between 0 and 1, then we can estimate the rate of the convergence to zero. If the components of matrix A are greater than 1, we can estimate the rate of the convergence to infinity.

Another aim of the dissertation is to study the special case of Equation (2), that is, the scalar difference equation with continuous time of the form

$$x(t) = a(t)x(t-1) + b(t)x(p(t)), \tag{3}$$

where $a, b : [t_0, \infty) \to \mathbf{R}$ are given real functions.
Using the asymptotic lower and upper bounds of the solutions we can estimate the rate of convergence of the solutions by the properly selected auxiliary functions. We apply our main result to particular cases such as $p_1 t \leq p(t) \leq p_2 t$, for real numbers $0 < p_1 \leq p_2 < 1$, where the estimating function is a power function, and $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \leq p_2$, where the estimating function is a logarithm function. Using the concept of the characteristic equation of generalized difference equation we obtain a special form of the estimating function for the solutions, and apply it to the classical particular case as $t - p_2 \leq p(t) \leq t - p_1$ for real numbers $1 < p_1 \leq p_2$. We have the estimating function in the form of exponential function.
The pantograph differential equation is of the form

$$x'(t) = Ax(t) + Bx(pt), \tag{4}$$

where $x(t) \in \mathbf{R}^n$, A and B are constant complex matrices and $p \in (0, 1)$ is given real number, the scalar pantograph differential equation is of the form

$$x'(t) = -a(t)x(t) + b(t)x(p(t)), \tag{5}$$

where a, b and p are continuous functions on $\mathbf{R}_+$ with the properties as before.

In the following we cite some results about the asymptotic behavior of solutions of the pantograph differential equations, which have been motivated the investigation presented in the dissertation.
Lim in [1] studied the asymptotic behavior of solutions of Equation (4). He proved the following result.

Theorem A. **(Lim, [1])** *Let $0 < p < 1$ and $A = \text{diag}(a_1, a_2, ..., a_n)$ with $0 < \text{Re } a_1 \leq \text{Re } a_2 \leq ... \leq \text{Re } a_n$. If $p\text{Re } a_n < \text{Re } a_1$, then for every solution x of (4) there exist constants K and $t_0 > 0$ such that*

$$|x(t)| \leq K|e^{ta_n}| \quad \textit{for all} \quad t \geq t_0.$$

Makay and Terjéki in [2] studied the asymptotic behavior of solutions of Equation (5). Set

$$t_{-1} = \inf\{p(s) : s \geq t_0\},$$

and

$$t_m = \inf\{s : p(s) > t_{m-1}\} \quad \text{for} \quad m = 1, 2, ...$$

For a given function $\rho : \mathbf{R}_+ \to (0, \infty)$ having bounded differential on finite intervals let us introduce the numbers

$$\rho_m := \max_{t_{m-1} \le t \le t_m} \rho(t) \int_{t_{m-1}}^{t} \exp\left\{\int_t^s a(z)dz\right\} \rho^{-2}(s)\rho'(s)ds, \quad m = 1, 2, ...$$

and let

$$t_\infty = \lim_{m\to\infty} t_m, \quad M_0 := \sup_{t_{-1} \le t < t_0} \rho(t)|\varphi(t)|.$$

Theorem B. (Makay, Terjéki, [2]) *Suppose that there exists a differentiable function $\rho : [t_{-1}, \infty) \to (0, t_\infty)$ such that ρ' is locally bounded,*

$$|b(t)|\rho(t) \le a(t)\rho(p(t)) \quad \textit{for all} \quad t \in [t_0, t_\infty)$$

and

$$P := \prod_{j=1}^{\infty} (1 + \rho_j) < \infty.$$

If x is a solution of Equation (5), then

$$|x(t)| \le \frac{M_0 P}{\rho(t)} \quad \textit{for all} \quad t \in [t_0, t_\infty).$$

The paper by Zhou and Yu [8] deals with the asymptotic properties of solutions of difference equation with continuous arguments. For given positive constants τ, δ such that $\tau < \delta$ and for given continuous functions $\delta, p, q, f : \mathbf{R}_+ \to \mathbf{R}$ such that $\tau \le \delta(t) \le \delta$ for $t \ge 0$, they considered difference equation

$$y(t) = p(t)y(t-\tau) + q(t)y(t-\delta(t)) + f(t) \quad \text{for all} \quad t \ge 0. \tag{6}$$

Definition. *A function x is exponentially decaying (at infinity) if there exist constants $M > 0$, $T > 0$, $0 < \xi < 1$ such that*

$$|x(t)| \le M\xi^t \quad \textit{for all} \quad t \ge T.$$

Under the assumption that the forcing function f is exponentially decaying, there are sufficient conditions for all solutions of (6) to be exponentially decaying.

Theorem C. (Zhou, Yu, [8]) *Assume that f is an exponentially decaying function and that there exists a constant $h \in (0, 1)$ such that*

$$\lim_{t\to\infty} \sup |p(t)| \le h, \quad \lim_{t\to\infty} \sup |q(t)| < 1 - h.$$

Then every solution of (6) is exponentially decaying.

3. Main Results

The first theorem gives asymptotic estimates for the rate of convergence of the components of the solutions of generalized difference equations. An analogue result was given by Makay and Terjéki in Theorem B for the scalar pantograph differential equations. The construction of the proof of the result given for difference equations is different from the proof given for differential equations. Moreover, the coefficients of Equation (2) are not scalar functions as in the result given in Theorem B, but matrix functions.

Define

$$t_{-1} = \min \left\{ \inf\{p(s) : s \ge t_0\}, t_0 - 1 \right\}.$$

By *a solution* of (2) we mean a function $x : [t_{-1}, \infty) \to \mathbf{R}^n$ which satisfies (2) for all $t \ge t_0$.

For a given function $\phi : [t_{-1}, t_0) \to \mathbf{R}^n$, the equation (2) has the *unique solution* x^ϕ satisfying the *initial condition*

$$x^\phi(t) = \phi(t) \quad \text{for} \quad t_{-1} \le t < t_0 \,. \tag{7}$$

Consider the following hypotheses.

(H_1) For every $t \ge t_0$, $A(t) = \operatorname{diag}(a_1(t), ..., a_n(t))$ is an $n \times n$ diagonal matrix with real entries satisfying $0 < a_i(t) < 1$, for all $t \ge t_0$, $i = 1, 2, ..., n$.

(H_2) $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix with real entries for all $t \ge t_0$.

(H_3) There exists a diagonal $n \times n$ matrix $G(t) = \operatorname{diag}(g_1(t), ..., g_n(t))$ for all $t \ge t_{-1}$ so that the diagonal entries $g_i : [t_{-1}, \infty) \to (0, \infty)$ are bounded on the initial interval $[t_{-1}, t_0)$, $i = 1, 2, ..., n$, and such that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(p(t))} \le (1 - a_i(t)) \sum_{j=1}^{n} \frac{1}{g_j(t-1)} \quad \text{for} \quad t \ge t_0, \quad i = 1, \dots, n.$$

(H_4) There are real numbers R_i, $i = 1, 2, ..., n$, such that

$$\prod_{m=0}^{j}(1 + R_{im}) \leq R_i, \quad \text{for all positive integers } j \quad \text{and} \quad i = 1, 2, ..., n,$$

where the numbers R_{im} are defined by

$$R_{im} := \sup_{t_m \leq t < t_{m+1}} \left\{ g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{\Delta_\tau(g_j(\tau-1))}{g_j(\tau)g_j(\tau-1)} \prod_{\ell=\tau+1}^{t} a_i(\ell) \right\},$$

where

$$k_m(t) := [t - t_m], \quad m = 0, 1, 2, ...$$

and $[x]$ denote the integer part of the number $x \in \mathbf{R}$.

(H_5) $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$.

Theorem 1. *Suppose that conditions (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Let $x = x^\phi$ be the solution of the initial value problem (2) and (7) with bounded components ϕ_i, $i = 1, 2, ..., n$, in (7) and let*

$$M_{i0} = \sup_{t_{-1} \leq t < t_0} g_i(t)|\phi_i(t)| \quad \text{for} \quad i = 1, 2, ..., n.$$

Then

$$|x_i(t)| \leq \frac{M_{i0}R_i}{g_i(t)} \quad \text{for all} \quad t \geq t_0 \quad \text{and} \quad i = 1, 2, ..., n.$$

In the next result we can estimate how fast the components of the solutions of difference equations with continuous time tend to infinity. A similar result was given in Theorem A by Lim for the pantograph differential equations with constant coefficients. Here we prove the result for the difference equations with continuous time when the coefficients in Equation (2) are not constant matrices as in the Lim's result, but matrix functions.

Now we introduce the following conditions.

(H_6) For every $t \geq t_0$, $A(t) = \text{diag}(a_1(t), ..., a_n(t))$ is an $n \times n$ diagonal matrix with real entries, and there exist real numbers a_i, A_i, $i = 1, 2, ..., n$, such that $1 < a_i \leq a_i(t) \leq A_i$, $i = 1, 2, ..., n$.

(H_7) $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix with real bounded entries for $i, j = 1, 2, ..., n$, $t \geq t_0$.

(H_8) $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that for a given real number $q \in (0, 1)$, $p(t) < qt$ holds for all $t \geq t_0$.

Theorem 2. *Suppose that conditions (H_6), (H_7) and (H_8) hold such that $\Gamma^q < \gamma$, where*

$$\Gamma := \max_{1\le i\le n} A_i \quad \text{and} \quad \gamma := \min_{1\le i\le n} a_i.$$

Let $x = x^\phi$ be a solution of the initial value problem (2) and (7) with the bounded components ϕ_i, $i = 1, 2, ..., n$. Then, there are positive constants K_i, $i = 1, 2, ..., n$, such that

$$|x_i(t)| \le K_i \prod_{\ell=t-k_0(t)-1}^{t} a_i(\ell) \quad \text{for} \quad \text{all} \quad t \ge t_0, \quad i = 1, 2, ..., n.$$

Consider equation (2) with constant coefficients. We obtain the following asymptotic estimate as a consequence of the Theorem 2. It is interesting in its own right because the estimating function here is the well known exponential function.

Corollary 1. *Let $A(t) = A$ and $B(t) = B$ be $n \times n$ constant matrices with real entries, where $B = (b_{ij})$ and $A = \mathrm{diag}(a_1, a_2, ..., a_n)$ is a diagonal matrix such that $a_i > 1$ for $i = 1, 2, ..., n$. Suppose that (H_8) holds such that $\Gamma^q < \gamma$, where*

$$\Gamma := \max_{1\le i\le n} a_i \quad \text{and} \quad \gamma := \min_{1\le i\le n} a_i.$$

Let $x = x^\phi$ be a solution of the initial value problem (2) and (7) with the bounded components ϕ_i, $i = 1, 2, ..., n$ in (7). Then, there are positive constants K_i, $i = 1, 2, ..., n$, such that

$$|x_i(t)| \le K_i a_i^t \quad \text{for} \quad t \ge t_0, \quad i = 1, 2, ..., n.$$

The next result is a special case of Theorem 1. We emphasize it because of its importance in the theory of difference equations with continuous time and in virtue of its interesting consequences.

For a given function $\varphi : [t_{-1}, t_0) \to \mathbf{R}$, the equation (3) has the *unique solution* x^φ satisfying the *initial condition*

$$x^\varphi(t) = \varphi(t) \quad \text{for} \quad t_{-1} \le t < t_0 \,. \tag{8}$$

We shall need the following hypotheses.

(H_9) Let $a : [t_0, \infty) \to \mathbf{R}$ be a given real function satisfying $0 < a(t) < 1$, for all $t \ge t_0$.

(H_{10}) Let $b : [t_0, \infty) \to \mathbf{R}$ be an arbitrary real function for all $t \geq t_0$.

(H_{11}) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$, which is bounded on the initial interval $[t_{-1}, t_0)$, and such that

$$|b(t)|\rho(t-1) \leq (1 - a(t))\rho(p(t)) \quad \text{for} \quad \text{all} \quad t \geq t_0,$$

where functions a and b are as in (H_9) and (H_{10}).

(H_{12}) There exists a real number R such that

$$\prod_{m=0}^{j} (1 + R_m) \leq R,$$

for all positive integers j, where the numbers R_m are defined by

$$R_m := \sup_{t_m \leq t < t_{m+1}} \left\{ \rho(t) \sum_{\tau = t - k_m(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{\ell = \tau + 1}^{t} a(\ell) \right\}.$$

Theorem 3. *Suppose that conditions (H_5), (H_9), (H_{10}), (H_{11}) and (H_{12}) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (3) and (8) with bounded function φ in (8). Then*

$$|x(t)| \leq \frac{M_0 R}{\rho(t)} \quad \textit{for} \quad \textit{all} \quad t \geq t_0.$$

In virtue of Theorem 3, by finding an appropriate function $\rho(t)$, we can estimate the rate of convergence of the solutions. If the lag function p is such as $p_1 t \leq p(t) \leq p_2 t$, for real numbers $0 < p_1 \leq p_2 < 1$, then we may have $\rho(t) = t^k$.

Corollary 2. *Suppose that conditions (H_9) and (H_{10}) hold. For given real numbers p_1, p_2 and t_0 such that $0 < p_1 \leq p_2 < 1$ and $t_0 > p_1/(1 - p_1)$, let p be given real function such that $p_1 t \leq p(t) \leq p_2 t$ for all $t \geq t_0$. Suppose that there exist real numbers Q and α such that $0 < Q \leq 1$, $0 < \alpha < 1$,*

$$|b(t)| \leq Q(1 - a(t)), \quad \alpha \leq 1 - a(t) \quad \textit{for} \quad \textit{all} \quad t \geq t_0$$

and

$$\log \frac{1}{Q} \left(\log \frac{1}{p_1} - \log \frac{1}{p_2} \right) < \log \frac{1}{p_1} \log \frac{1}{p_2}.$$

Let $x = x^{\varphi}$ be a solution of the initial value problem (3) and (8), and let

$$k = \frac{\log Q}{\log p_1}, \quad \rho(t) = \left(t - \frac{p_1}{1 - p_1} \right)^k.$$

Then

$$|x(t)| \leq \frac{C}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \geq t_0,$$

where

$$C = \sup_{t_{-1} \leq t < t_0} \{t^k |\varphi(t)|\} \prod_{n=0}^{\infty} \left(1 + \frac{k t_0^k (1-p_1)^{k+1}}{\alpha p_1^k (t_0(1-p_1) - p_2^n)^{k+1}} \left(\frac{p_2^{k+1}}{p_1^k}\right)^n\right).$$

If the lag function p is such as $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \leq p_2$, then $\rho(t) = \log^k t$.

Corollary 3. *Suppose that conditions (H_9) and (H_{10}) hold. Let $t_0 \geq 1$ be given real number, p_1, p_2 be given natural numbers such that $1 < p_1 \leq p_2$. Let p be given real function such that $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$ for all $t \geq t_0$. Suppose that there exist real numbers Q and α such that $0 < Q \leq 1$, $0 < \alpha < 1$ and*

$$|b(t)| \leq Q(1 - a(t)), \quad \alpha \leq 1 - a(t) \quad \text{for} \quad \text{all} \quad t \geq t_0.$$

Let $x = x^\varphi$ be a solution of the initial value problem (3) and (8) and let

$$k = -\frac{\log Q}{\log p_2}.$$

Then

$$|x(t)| \leq \frac{C}{\log^k t} \quad \text{for} \quad \text{all} \quad t \geq t_0,$$

where

$$C = \sup_{t_{-1} \leq t < t_0} \left\{\log^k t |\varphi(t)|\right\} \prod_{n=0}^{\infty} \left(1 + \frac{k p_2^k p_2^{nk} \log^k t_0}{\alpha (t_0^{p_1^n} - 1) \log^{k+1}(t_0^{p_1^n} - 1)}\right).$$

In the following results we assume that:

(H_{13}) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$ bounded on the initial interval $[t_{-1}, t_0)$ such that

$$b(t)\rho(t-1) \geq (1 - a(t))\rho(p(t)) \quad \text{for} \quad \text{all} \quad t \geq t_0.$$

(H_{14}) There exists a positive real number r such that

$$\prod_{n=0}^{j} (1 + r_n) \geq r,$$

for all positive integers j, where the numbers r_n, for $n = 0, 1, \ldots$, are defined by

$$r_n := \inf_{t_n \le t < t_{n+1}} \left\{ \rho(t) \sum_{\tau = t - k_n(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{\ell = \tau + 1}^{t} a(\ell) \right\}.$$

The next result gives an asymptotic lower bound estimate for the positive solutions of (3). We emphasize that in the system case we can not prove a similar result because of the construction of the available tools.

Theorem 4. *Suppose that conditions (H_5), (H_9), (H_{10}), (H_{13}) and (H_{14}) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (3) and (8) with positive bounded function φ in (8). Then*

$$x(t) \ge \frac{N_0 r}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

In the next result we need the following condition.

(H_{15}) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$, which is bounded on the initial interval $[t_{-1}, t_0)$, and such that

$$b(t)\rho(t - 1) = (1 - a(t))\rho(p(t)) \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

Combining Theorems 3 and 4 we can estimate the positive solutions of Equation (3) from both sides and obtain the next corollary which is an immediate consequence of the mentioned theorems.

Theorem 5. *Suppose that conditions (H_5), (H_9), (H_{10}), (H_{12}), (H_{14}) and (H_{15}) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (3) and (8) with positive bounded function φ in (8). Then*

$$0 < \frac{N_0 r}{\rho(t)} \le x(t) \le \frac{M_0 R}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

Zhou and Yu in Theorem C obtained for the estimating function an exponential function for the case when the lag function is between two constant delays. Using the above results we can estimate the behavior of solutions of difference equations with continuous time for some lag functions but not for constant delays. To obtain result that can be applied to lag functions between two constant delays we use the concept of characteristic equation associated with the considered difference

equation. In the conditions of the next result it is not necessary to have the solution of the characteristic equation but we need only a function which satisfies the associated inequality. First we consider the general case.

Assume that:

(H_{16}) For the function a and b given in (H_9) and (H_{10}), there is a real function $\lambda : [t_0, \infty) \to (1, \infty)$ and there is an initial function φ in (8) such that

$$|b(t)| \frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p(t) - k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)} \leq$$

$$\leq 1 - a(t)\lambda(t), \quad t \geq t_0.$$

Theorem 6. *Suppose that conditions (H_5), (H_9), (H_{10}) and (H_{16}) hold. Let $x = x^{\varphi}$ be a solution of the initial value problem (3) and (8). Then*

$$|x(t)| \leq \left(\varphi(t - k_0(t) - 1) \max_{t_{-1} \leq t \leq t_0} \lambda(t) \right) \prod_{\ell = t - k_0(t)}^{t} \frac{1}{\lambda(\ell)}, \quad t \geq t_0.$$

If the lag function is between two constant delays we use the following corollary of Theorem 6 and obtain $\rho(t) = \lambda^t$ for the estimating function.

Corollary 4. *Suppose that conditions (H_9) and (H_{10}) hold. Let p_1, p_2 and t_0 be given real numbers such that $1 \leq p_1 < p_2$ and $t_0 \geq p_1$. Let $p(t) = t - \delta(t)$, with given real function δ such that $p_1 \leq \delta(t) \leq p_2$ for all $t \geq t_0$. Suppose that there exists a real number $\lambda > 1$ such that*

$$|b(t)| \leq \frac{1 - \lambda a(t)}{\lambda^{p_2}} \quad \textit{for all} \quad t \geq t_0.$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (3) and (8), and

$$M_0 = \sup_{t_{-1} \leq t < t_0} \{\lambda^t |\varphi(t)|\}.$$

Then

$$|x(t)| \leq \frac{M_0}{\lambda^t} \quad \textit{for all} \quad t \geq t_0.$$

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