Asymptotic Behavior of Solutions of Difference Equations with Continuous Time

Ph. D. dissertation

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'Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.'

'The Hermit Clad in Crane Feathers'
in R.. van Gulik's The Chinese Maze Murders.

Preface

The essence of the qualitative theory of differential and difference equations consists of the examination of the characteristics of the solutions of the considered equations, not knowing them explicitly. Consequently, it serves as a direct source of information about solutions. Some of these characteristics are: oscillation, boundedness, periodicity, asymptotic behavior, stability properties, and others. The significance of this theory is tremendous since it can be applied also in the case when the solution can be expressed in an explicit form but this form is very complicated and not suitable for further investigation.

The present work is a treatment of the field of qualitative analysis of generalized difference equations and describes the asymptotic behavior of solutions. In the first part we define the concept of functional equations from a historical perspective as well as the particular cases such as the discrete difference equations and difference equations with continuous time.

In the second part we mention the origin of the pantograph differential equation and cite some results about the asymptotic behavior of solutions because the research has been motivated by this type of differential equations.

In the third part we sum up the usefulness and applicability of generalized difference equations approaching the problem with the applications from several branches of sciences.

In the fourth, fifth and sixth part the new results of the asymptotic theory of generalized difference equations are given. In the main results we give asymptotic lower and upper bounds of solutions of difference equations specifying the asymptotic behavior of solutions and estimating the rate of the convergence.

The graphics constructed by *MATHEMATICA* show the characteristics of the asymptotic behavior of solutions of discrete difference equations and difference equations with continuous time emphasizing the analogies as well as the main differences between them. Through the presented examples the message of the main results becomes understandable and we also illustrate the comprehensiveness of using computers in mathematical research.

Contents

1. Basic Notions and Historical Review

Mathematical computations are frequently based on equations that allow us to compute the value of a function recursively from a given set of values. Such equations are called *iterative functional equations* and they do not include equations in which infinitesimal operations are performed on the unknown function. So, differential equations, integral equations, integro - differential equations, functional differential equations do not fall under this notion.
Let $\mathbf{R}$ be the set of real numbers and n a given positive integer. For given functions $f_i : \mathbf{R} \to \mathbf{R}$, $i = 1, 2, ..., n$, and $F : \mathbf{R}^{n+2} \to \mathbf{R}$, the *iterative functional equation* or *functional equation* is the equation of general form

$$F(t, x(t), x(f_1(t)), ..., x(f_n(t))) = 0, \tag{1.1}$$

where x is the unknown function. The number n occuring in Equation (1.1) is referred to as the *order* of the equation. Thus the functional equation of order 1 becomes

$$F(t, x(t), x(f_1(t))) = 0.$$

Functional equations of order zero are *implicit functions*

$$F(t, x(t)) = 0.$$

It is useful to consider Equation (1.1) in the form solved with respect to the unknown function x, if there is a function $h : \mathbf{R}^{n+1} \to \mathbf{R}$ with the property $F(t, x, x_1, ..., x_n) = x - h(t, x_1, ..., x_n)$. Under some suitable conditions, Equation (1.1) is equivalent to equation

$$x(t) = h(t, x(f_1(t)), ..., x(f_n(t))). \tag{1.2}$$

The reader interested in details is referred to [61].

The theory of functional equations is fascinating because of its intrinsic mathematical beauty as well as its applications. Nowadays, functional equations have grown to be a large, independent branch of mathematics. In such a large area further subdivisions are indispensable. The main line of division runs between *equations in several variables* (see [1], [4], [62],) and *equations in a single variable* (see [3], [61], [63]). While functional equations in single variable in the form of recurrent sequences go back to antiquity (Archimedes, for instance), Oresme was the first mathematician in the 14th century who used functional equations in several variables. The difference between equations in single or several variables is not whether the unknown function is single or multiplace, but whether there is one variable in the equation, like in equation

$$f(x+1) = F(f(x)),$$

or several variables as in Jensen's functional equation

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_1)+f(x_2)}{2}.$$

To solve a functional equation is one of the oldest range of problems in analysis. Every mathematician is aware of how this field was developed in the work of authors from the second part of the eighteenth century and throughout all the nineteenth century. Names like D'Alambert, Euler, Gauss, Cauchy, Abel, Riemann, Weierstrass, Darboux, Hilbert, to quote only some among the most famous, are all associated with important and deep research on functional equations. The reader interested in the history of functional equations can consult [2], [3], [21], [22], and others.

A *difference equation* is a special case of functional equations in a single variable. In particular, for $f_n(t) = t - 1$ and for given function $g : \mathbf{R}^n \to \mathbf{R}$ such that $h(t, x_1, ..., x_{n-1}, x_n) = x_n + g(t, x_1, ..., x_{n-1})$, the *difference equation with continuous time* or the so-called *generalized difference equation* is the functional equation (1.2) of the form

$$x(t) - x(t-1) = g(t, x(f_1(t)), ..., x(f_{n-1}(t))). \tag{1.3}$$

Let N denote the set of nonnegative integers. Let $m \in \mathbf{N}$, and let $f_i^* : \mathbf{N} \to \mathbf{N}$, $i = 1, 2, ..., n-1$, and $\overline{g} : \mathbf{N} \times \mathbf{R}^{n-1} \to \mathbf{R}$ be given functions. A *discrete difference equation* is the equation of the form

$$x_{m+1} - x_m = \overline{g}(m, x_{f_1^*(m)}, ..., x_{f_{n-1}^*(m)}). \tag{1.4}$$

Discrete difference equations form an independent branch of mathematics and have their special problems and methods, which are not characteristic of more general iterative functional equations. These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For an elementary course see [73].
The following examples have been chosen to illustrate something of the diversity of the uses of difference equations that also have some historical role. For further discrete and continuous models see [27] and [74].

The Rabbit Problem. In 1202 Fibonacci posed and solved the following problem: suppose that every pair of rabbits can reproduce only twice, when they are one and two months old, and that each time they produce exactly one new pair of rabbits. Assume that all rabbits survive. Starting with a single pair in the first generation, how many pairs will there be after n generation? To solve this problem, define f_n as the number of newborn pairs in generation n. Then f_n satisfies the equation

$$f_{n+1} = f_n + f_{n-1}.$$

For $f_0 = f_1 = 1$ we obtain the sequence of the *Fibonacci numbers*, while for $g_n = f_n$, $g_0 = 2$ and $g_1 = 1$ we obtain the sequence of the *Lucas numbers.* The connection between the Fibonacci and Lucas numbers is as follows:

$$g_n = f_{n-1} + f_{n+1}, \quad f_{2n} = f_n g_n.$$

Annual Plant Propagation Problem. Annual plants produce seeds at the end of their growth season (say August), after which they die. A fraction of these seeds survive the winter, and some of these germinate at the beginning of the season (say May), giving rise to the new generation of plants. The fraction that germinates depends on the age of the seeds.
To simplify the problem we suppose that seeds older than two years are no longer viable and can be neglected. Define γ as the number of seeds produced per plant in August, α as the fraction of one-year-old seeds that germinate in May, β as the fraction of two-year-old seeds that germinate in May, σ as the fraction of seeds that survive a given winter and p_n as the number of plants in generation n. The model can be formulated as one second order difference equation of the form

$$p_n = \alpha\sigma\gamma p_{n-1} + \beta\sigma^2(1-\alpha)\gamma p_{n-2}. \tag{1.5}$$

Here γp_{n-2} is the number of seeds produced two years ago, $\sigma\gamma p_{n-2}$ the number of seeds which then survived the first winter, $(1-\alpha)\sigma\gamma p_{n-2}$ the number of seeds which failed to germinate last year, $\sigma(1-\alpha)\sigma\gamma p_{n-2}$ the number of seeds survived last winter and $\beta\sigma(1-\alpha)\sigma\gamma p_{n-2}$ the number of seeds which were among the fraction of two-year-old seeds that germinated.

In the last few decades the qualitative theory of difference equations and inequalities has attracted growing attention. Most research focuses on discrete difference equations. It was Poincaré [87] who initiated the work on asymptotic theory for both difference and differential equations. His work was later improved by Perron [86], Evgrafov [34], Coffman [19] and Máté and Nevai [70]. Numerous papers have recently appeared that mainly address the extension of the mentioned results. For an introductory account of the subject the reader may consult the book by Agarwal [5], Elaydi [28], Kelley and Peterson [58], Sharkovsky, Maistrenko and Romanenko [88]. The reader interested in this topic is referred to papers by Elaydi and Győri [29], Győri and Pituk [42], [43], Elaydi [30], [31], [32], Zhang [91], Elaydi and Zhang [33], Graef and Qian [40], and others.

Although generalized difference equations appear quite frequently in various mathematical models [64]. Recently, the oscillation of solutions of this kind of equations has been extensively studied by many authors. In paper [26], Domshlak developed a new approach to the study of the oscillatory properties of difference equations with continuous time. In particular, his method gives the possibility to derive a series of effective criteria for the oscillation of all solutions of difference equations with continuous time with variable differences. Also, this method allows to estimate the intervals between the adjacent zeros of the solutions. For preliminary results we refer to the references of [26].
The main objective of the paper by Ferreira and Pedro [35] is to investigate when the linear delay difference system has oscillatory properties by use of the Laplace transform as in the work by Arino and Győri [6].
In the papers by Cook and Ladeira [20], Oliveira Filho and Carvalho [76] and Pelyukh [85] the existence of periodic solutions is investigated, whereas in the papers by Carvalho [16], Kato [57] and Zhang [94] stability conditions of solutions are given.
Papers by Pelyukh [83], [84] and Blizorukov [8] investigate the existence and uniqueness of solutions of generalized linear difference systems. Pelyukh in his works unified some fundamental results known for the theory of difference equations with continuous time or results that may be extended from the theory of differential equations (ordinary as well as with delay) to difference equations. This is important, especially now, when the theory of difference systems has great success, there are many papers devoted to this area and many general and partial results concerning equations of this type.

Difference equations with continuous time with integer-valued differences have much in common with the corresponding discrete equations. For example, the discrete difference equation (1.5) with $a = \alpha\sigma\gamma$, $b = \beta\sigma^2(1-\alpha)\gamma$ and $u_n = p_{n-1}$ has the form

$$u_{n+1} - au_n - bu_{n-1} = 0, \quad n \in \mathrm{N} \tag{1.6}$$

and its general solution is

$$u_n = C_1 e^{\lambda_1 n} + C_2 e^{\lambda_2 n},$$

where C_1, C_2 are arbitrary constants and λ_1, λ_2 are the (distinct) roots of the corresponding characteristic equation

$$e^{2\lambda} - ae^{\lambda} - b = 0.$$

The general solution of difference equation with continuous time

$$u(t+1) - au(t) - bu(t-1) = 0, \quad t \geq t_0$$

which represents the continuous analogue of (1.6), has a quite similar form

$$u(t) = C_1(t)e^{\lambda_1 t} + C_2(t)e^{\lambda_2 t},$$

where $C_1(t)$, $C_2(t)$ are arbitrary periodic functions of period one.

Example 1.1. Let $e^{\lambda_1} = 2$, $e^{\lambda_2} = 0.5$, $C_1 = C_1(t) = 1$ and $C_2 = C_2(t) = -1$. Then we have the solutions $x_n = 2^n - 0.5^n$ and $x(t) = 2^t - 0.5^t$. See Figure 1.

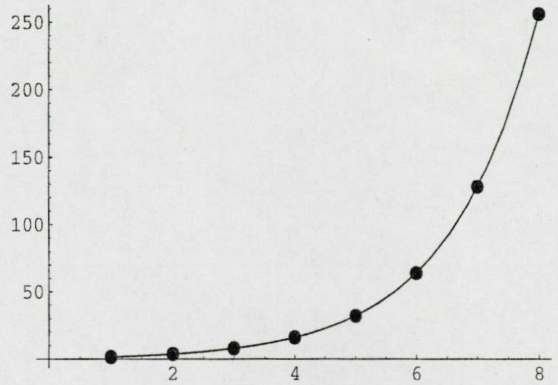

FIGURE 1.

Example 1.2. Let $e^{\lambda_1} = 2$, $e^{\lambda_2} = 0.25$, $C_1 = C_1(t) = 0.01$ and $C_2 = C_2(t) = -10$. Then we have the solutions $x_n = 0.01\ 2^n - 10\ 0.25^n$, $x(t) = 0.01\ 2^t - 10\ 0.25^t$. See Figure 2.

Example 1.3. Let $e^{\lambda_1} = 2$, $e^{\lambda_2} = 0.5$, $C_1(t) = \tan \pi t$ and $C_2(t) = 0$. Then we have the solution $x(t) = 2^t \tan \pi t$. See Figure 3.

Example 1.4. Let $e^{\lambda_1} = 2$, $e^{\lambda_2} = 0.5$, $C_1(t) = 0$ and $C_2(t) = \cos 2\pi t$. Then we have the solution $x(t) = 0.5^t \cos 2\pi t$. See Figure 4.

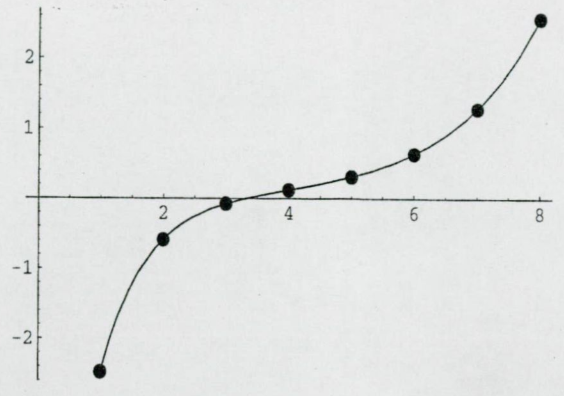

FIGURE 2.

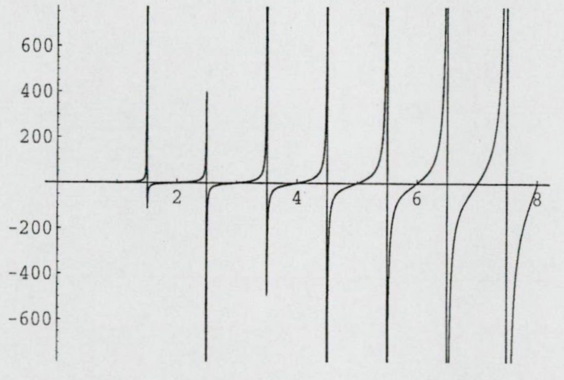

FIGURE 3.

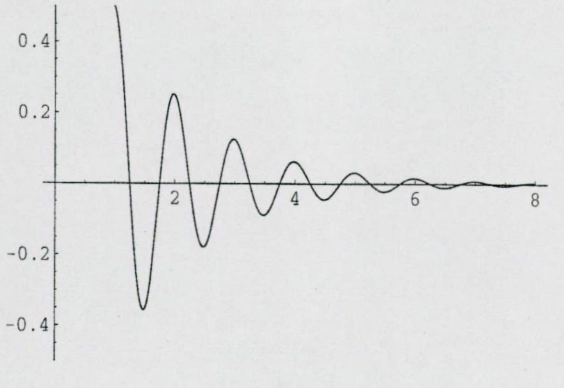

FIGURE 4.

The situation changes drastically if the differences of argument are not integers. For example, the number of linearly independent exponential solutions of the equation

$$x(t+1) - ax(t) - bx(t-\sigma) = 0, \quad t \geq t_0$$

depends on the arithmetic nature of the number σ. The situation becomes even more complicated if the differences (and the coefficients) are not constants.

For given functions $a, b, p : \mathbf{R} \to \mathbf{R}$ we consider the difference equation with continuous argument as a special type of Equation (1.3), which will be investigated in this work:

$$x(t) = a(t)x(t-1) + b(t)x(p(t)). \tag{1.7}$$

For a given real number $p \in (0,1)$, let $p(t) = pt$. Then we obtain the above equation of the form

$$x(t) = a(t)x(t-1) + b(t)x(pt). \tag{1.8}$$

For given functions $a, b_i, p_i : \mathbf{R} \to \mathbf{R}$, $i = 1, 2, ..., m$, we will consider a generalization of the functional equation (1.7), the generalized difference equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)). \tag{1.9}$$

Another generalization of Equation (1.7) is the system a generalized difference equations of the form

$$x(t) = A(t)x(t-1) + B(t)x(p(t)),$$

where $x(t) \in \mathbf{R}^n$, A, B are $n \times n$ matrix functions with real entries.
For given sequences of real numbers $\{a_n\}$, $\{b_n\}$ and for a given sequence of positive integers $\{p_n\}$ we consider the discrete difference equation as a special type of Equation (1.4), which will be also the subject of our investigation:

$$x_{m+1} - x_m = -a_m x_m + b_m x_{p_m}.$$

Difference equation (1.8) is an analogue of a special type of *differential equations*, the so-called *pantograph differential equation*

$$x'(t) = Ax(t) + Bx(pt), \tag{1.10}$$

where $0 < p < 1$ and A, B are constant matrices.

2. Preliminaries

In [75], Ockendon and Taylor studied the motion of a pantograph head on an electric locomotive. The pantograph head is used for collecting current from an overhead supply line. The analysis of the problem leads to Equation (1.10), where A and B are 4×4 matrices with real constant entries and the rank of A is unity.
A similar equation is also obtained by Fox, Mayers, Ockendon and Taylor in [36] for the wave motion of a stretched string under an applied force which moves along the string. Consider a one - dimensional wave motion, wave speed $c > 1$, such as that due to small vertical displacements $Y(x,t)$ of a stretched string under gravity, caused by an applied force which moves along the string with speed unity. Let the force produce a discontinuity in slope $[\partial Y/\partial x]^{+}_{-}$ proportional to its vertical velocity dy/dt where $y(t) = Y(t,t) + Y_s(t)$ and $Y_s(x)$ is the static displacement. Finally let the force move towards a partially reflecting boundary at $x = 0$ where, for example, conditions $[Y]^{+}_{-} = 0$ and $\kappa[\partial Y/\partial x]^{+}_{-} = Y$ are imposed. The solution of the wave equation can be represented by $Y = F_i(ct - x) + G_i(ct + x)$ in the six regions $i = 1, 2, ..., 6$ shown in Figure 5. Boundary conditions that the displacement Y is zero for large values of x along $ct \pm x$ =const. are given and require that $F_4 \equiv 0 \equiv G_6$. Continuity conditions require $G_1 \equiv G_6$ and $F_3 \equiv F_4$, together with $F_1(0) = F_6(0)$ and $G_3(0) = G_4(0)$. For $t > 0$ conditions on $x = t$ give

$$F_1(ct - t) = F_2(ct - t) + G_2(ct + t),$$

and after some manipulation

$$G_2(t) = \alpha y(t/(c+1)),$$

where α is a suitable constant of proportionality. Conditions on $x = 0$ give

$$G_3(t) = F_2(t) + G_2(t) = \kappa[G_2'(t) - F_2'(t) - G_3'(t)],$$

and the displacement $y(t) = F_i(ct - t) + Y_s(t)$. Eliminating F_1, F_2, G_2 and G_3 we obtain

$$\left(1 + \frac{2\kappa}{c-1}\frac{d}{dt}\right) y(t) + \frac{\alpha}{1-\alpha} y(pt) = Y_s + \frac{2\kappa}{c-1}\frac{dY_s}{dt},$$

where $p = (c-1)/(c+1)$. This is a first order linear functional differential equation with constant coefficients. Therefore, the study led to the functional differential equation

$$y'(t) = ay(t) + by(pt), \tag{2.1}$$

where a, b and p are given constants.

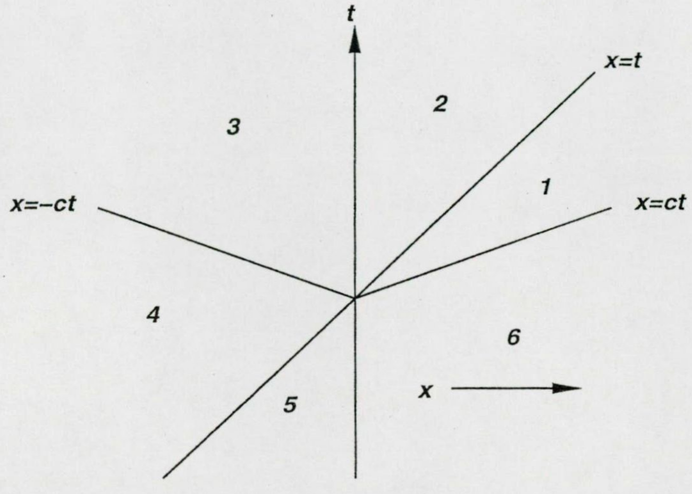

FIGURE 5.

Some basic results about explicit solutions of the scalar differential equation (2.1) will be cited here and can be found in [36].
Trivial or easily-computable solutions of (2.1) satisfying $y(0) = 1$ obviously exist when $b = 0$, $p = 0$ or 1, or when $bp^n + a = 0$ which produces an exact polynomial solution of degree n. If $p > 1$ and $b \neq 0$, the solution is not unique and there exist eigensolutions (nontrivial solutions) satisfying $y(0) = 0$. We therefore expect that any solution will have different properties according to the signs of b, $a + b$ and $1 - p$.
By an obvious method (2.1) can be converted into a difference equation. Though not generally helpful, since exponential terms appear in the coefficients, the case $p < 1$, $a = 0$ has been discussed by De Brujin's method (1953). He obtained a complete set of independent integral representation which satisfies the difference equation. However, only one combination of these representations is finite at $t = 0$, namely

$$y = C\int_{-\infty}^{\infty} \exp\left\{\frac{s^2}{2\log p} + bt\exp\left(-\frac{1}{2}\log p + is\right)\right\} ds,$$

where C is an arbitrary constant. It is interesting to note that this case appears to have been first studied by Mahler (1940) in connection with a partition problem in the theory of numbers [68].

Several series representations have been obtained. For example, it is easily verified that (2.1) is satisfied by

$$y = \sum_{n=0}^{\infty} \prod_{q=0}^{n-1} (a + bp^q) \frac{t^n}{n!}. \tag{2.2}$$

This series is uniformly convergent for all finite t if $p < 1$, but has a zero radius of convergence for $p > 1$. The alternative representation (Chambers, 1971)

$$y = C \sum_{n=0}^{\infty} \prod_{q=1}^{n} (1 - p^q)^{-1} \left(-\frac{b}{a}\right)^n e^{atp^n} \tag{2.3}$$

is the well known Dirichlet series solution. It is a rearrangement of (2.2) that is uniformly convergent for all t if $p < 1$ and $|b| < |a|$.
Generalized representation results are obtained by Terjéki [90] for the equation

$$x'(t) = A(t)x(t) + B(t)x(p(t)), \tag{2.4}$$

where A, B are given $n \times n$ matrix functions with real entries and p is a given continuous function such that $p : \mathbf{R}_+ \to \mathbf{R}_+$ and $0 \le p(t) \le t$ for all $t \ge 0$. The equation

$$y'(t) = A(t)y(t) \tag{2.5}$$

is called the ordinary part of Equation (2.4).

Theorem 2.1. (Terjéki [90]) *If $y_0(t)$ denote an arbitrary solution of Equation (2.5) and the sequence $\{y_n(t) : n = 1, 2, 3, ...\}$ is defined by*

$$y_n'(t) = A(t)y_n(t) + B(t)y_{n-1}(p(t))$$

$$y_n(0) = 0,$$

then

$$x(t) = \sum_{n=0}^{\infty} y_n(t)$$

is a solution of Equation (2.4). Moreover, this series absolutely and uniformly converges on every finite subinterval of $[0, \infty)$.

Kato and McLeod [55] studied Equation (2.1) for complex constants a, b and proved that when Re $a < 0$ then for every solution y of (2.1) there are constants $K > 0$ and $N > 0$ such that

$$|y(t)| \le K|t^\alpha| \quad \text{for all} \quad t \ge N,$$

where

$$\alpha = \frac{\log(-a/b)}{\log p}.$$

Lim [66] investigated the asymptotic behavior of the solutions of Equation (1.10), when A, B are constant complex matrices and proved the following results.

Theorem 2.2. **(Lim [66])** *Let* $0 < p < 1$ *and* $A = \operatorname{diag}(a_1, a_2, ..., a_n)$ *with* $0 < \operatorname{Re} a_1 \leq \operatorname{Re} a_2 \leq ... \leq \operatorname{Re} a_n$. *If* $p \operatorname{Re} a_n < \operatorname{Re} a_1$, *then for every solution* x *of (1.10) there exist constants* K *and* $t_0 > 0$ *such that*

$$|x(t)| \leq K|e^{ta_n}| \quad \textit{for all} \quad t \geq t_0.$$

Let x be an n-dimensional vector and f be a complex-valued function. Let x and f be defined on $[0, \infty]$. We say that x is $o(f(t))$ if for any $\epsilon > 0$ there is a constant $N > 0$ such that

$$|x(t)| \leq \epsilon|f(t)| \quad \text{for all} \quad t > N.$$

Corollary 2.3. **(Lim [66])** *Let* $0 < p < 1$. *Let* A *be diagonalizable matrix with eigenvalues* a_i, $i = 1, 2, ..., n$, *satisfying* $0 < \operatorname{Re} a_1 \leq \operatorname{Re} a_2 \leq ... \leq \operatorname{Re} a_n$. *If* $p \operatorname{Re} a_n < \operatorname{Re} a_1$, *then for every solution* x *of (1.10) there exist constants* K *and* $t_0 > 0$ *such that*

$$|x(t)| \leq K|e^{ta_n}| \quad \textit{for all} \quad t \geq t_0.$$

Further, there is a nonsingular matrix P *and complex numbers* ℓ_i, $i = 1, 2, ..., n$, *such that*

$$x(t) = P \operatorname{col}(\ell_1 e^{ta_1}, \ell_2 e^{ta_2}, ..., \ell_n e^{ta_n}) + o(e^{ta_n}) \quad \textit{as} \quad t \to \infty.$$

In the next theorem Lim generalizes the result given by Kato and McLeod in [55].

Theorem 2.4. **(Lim [66])** *Let* $0 < p < 1$. *Let* $A = \operatorname{diag}(a_1, a_2, ..., a_n)$ *with* $\operatorname{Re} a_1 \leq \operatorname{Re} a_2 \leq ... \leq \operatorname{Re} a_n < 0$. *Let* $B = (b_{ij})$. *If* α *is defined by*

$$\alpha = \left\{ \log\left(\frac{-\operatorname{Re} a_n}{\max_{1 \leq i \leq n} \sum_{j=1}^{n} |b_{ij}|} \right) \right\} / \log p,$$

then for every solution x *of Equation (1.10) there exist constants* K *and* $t_0 > 0$ *such that*

$$|x(t)| \leq K|t^{\alpha}| \quad \textit{for all} \quad t \geq t_0.$$

In paper [77] Pandolfi generalized the results obtained by Lim in [66] for Equation (1.10) to the case when A, B are $n \times n$ bounded and measurable matrices for $t \geq t_0 \geq 0$.

Theorem 2.5. (Pandolfi [77]) *Let $A = A(t)$, $B = B(t)$ be $n \times n$ bounded and measurable matrix functions for $t \geq t_0 \geq 0$. Let χ_i, $i = 1, 2, ..., r$, $r \leq n$, be the characteristic exponents of equation*

$$x'(t) = A(t)x(t), \tag{2.6}$$

and χ'_i, $i = 1, 2, ..., r$, $r \leq n$, the characteristic exponents of the adjoint equation

$$y'(t) = -A^*(t)y(t)$$

such that $0 < \chi_1 < ... < \chi_r$ and $p\chi_r + \chi'_1 < 0$. Let $G(t, s)$ be the evolution matrix of Equation (2.6). Then for every solution x of Equation (1.10) there exists $v_x \in \mathbf{R}^n$ such that

$$\lim_{t\to\infty} G(t_0, t)x(t) = v_x.$$

Makay and Terjéki in [69] studied the asymptotic behavior of solutions of differential equation

$$x'(t) = -a(t)x(t) + b(t)x(p(t)), \tag{2.7}$$

where a, b and p are continuous functions on $\mathbf{R}_+$, $p(t) \leq t$, $t \geq 0$, and $\lim_{t\to\infty} p(t) = \infty$. Define the function

$$m(t) := \inf\{s : p(s) > t\}$$

on $\mathbf{R}_+$. Then $t \leq m(t)$, $p(m(t)) = t$ and $m(t)$ is increasing. Let be given $t_0 \geq 0$ such that $p(t_0) < t_0$ and introduce the qualities

$$t_{-1} = \inf\{p(s) : s \geq t_0\}, \quad t_n = m(t_{n-1}), \quad n = 1, 2, ..., \quad t_\infty = \lim_{n\to\infty} t_n.$$

For a given function $\rho : \mathbf{R}_+ \to (0, \infty)$ having bounded differential on finite intervals let us introduce the numbers

$$\rho_n := \max_{t_{n-1}\leq t\leq t_n} \rho(t) \int_{t_{n-1}}^{t} \exp\left\{\int_t^s a(z)dz\right\} \rho^{-2}(s)\rho'(s)ds, \quad n = 1, 2, ...$$

Theorem 2.6. (Makay-Terjéki [69]) *Suppose that there exists a differentiable function $\rho : [t_{-1}, t_\infty) \to (0, \infty)$ such that $\dot{\rho}$ is locally bounded,*

$$|b(t)|\rho(t) \leq a(t)\rho(p(t)) \quad \textit{for all} \quad t \in [t_0, t_\infty)$$

and

$$P := \prod_{n=1}^{\infty} (1 + \rho_n) < \infty.$$

If x is a solution of Equation (2.7) and

$$M_0 := \max_{t_{-1} \le t \le t_0} \rho(t)|x(t)|,$$

then

$$|x(t)| \le \frac{M_0 P}{\rho(t)} \quad \textit{for all} \quad t \in [t_0, t_\infty).$$

The asymptotic behavior of solutions of Equation (1.10) has been studied by many authors, but we mention only some of them. Carr and Dyson [15] and Lim [67] investigated the case where A, B are constant complex matrices, Kato [56] and Carr and Dyson [14] the scalar case, Equation (2.1). McLeod [71] indicated some possible results and stated a theorem proved by Dyson. For other results and applications of delay differential equations, we refer the reader to the monographs by Gopalsamy [38], Győri and Ladas [41], Ladde, Lakshmikantham and Zhang [65], Hale [45] and to the papers by Buhmann and Iserles [9], [10], Burton and Makay [11], Busenberg and Cooke [12], Graef and Qian [39], Hatvani [49], and others.

The paper by Zhou and Yu [95] deals with the asymptotic properties of solutions of difference equation with continuous arguments. For given positive constants τ, δ such that $\tau < \delta$ and for given continuous functions $\delta, p, q, f : \mathbf{R}_+ \to \mathbf{R}$ such that $\tau \le \delta(t) \le \delta$ for $t \ge 0$, they considered the difference equation

$$y(t) = p(t)y(t-\tau) + q(t)y(t-\delta(t)) + f(t) \quad \text{for all} \quad t \ge 0. \tag{2.8}$$

A function x is *exponentially decaying* (at infinity) if there exist constants $M > 0$, $T > 0$, $0 < \xi < 1$ such that

$$|x(t)| \le M\xi^t \quad \text{for all} \quad t \ge T.$$

Under the assumption that the forcing function f is exponentially decaying, there are sufficient conditions for all solutions of (2.8) to be exponentially decaying. These conditions are also necessary when p and q are both positive functions and $\lim_{t\to\infty} p(t)$ and $\lim_{t\to\infty} q(t)$ exist.

Theorem 2.7. **(Zhou-Yu [95])** *Assume that f be an exponentially decaying function and that there exists a constant $h \in (0,1)$ such that*

$$\limsup_{t\to\infty} |p(t)| \le h, \quad \limsup_{t\to\infty} |q(t)| < 1 - h.$$

Then every solution of (2.8) is exponentially decaying.

Theorem 2.8. **(Zhou-Yu [95])** *Assume that p and q are nonnegative functions with*

$$p(t) + q(t) > 0 \quad \textit{for all} \quad t \ge 0$$

and that

$$\lim_{t\to\infty} p(t) = p \quad \textit{and} \quad \lim_{t\to\infty} q(t) = q$$

exist. Then every solution of (2.8) is exponentially decaying for any exponentially decaying function f if and only if

$$p + q < 1.$$

There are several works dealing with the oscillatory properties of difference equations with continuous time which are published in Chinese. For example, the reader is referred to Shen [89], Zhang and Yan [92] and Zhang [93] and the references therein. In paper [95] some works dealing with this topic are also mentioned.

3. Usefulness and Applicability

Applications of functional equations, both inside and outside mathematics, also abound nowadays. We mention just applications to iteration, dynamical systems to probabilistic metric spaces, to group theory, mean values, physics and, in particular, to economics, decision and information theory. These and other applications of functional equations and their closely related theory show the usefulness of functional equations.

Another reason why the investigations presented in Sections 4. and 5. may be very useful is that the theory of generalized difference equations is part of the theory of neutral functional differential equations. See [44].

Neutral differential equations are differential equations in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. Suppose $r \geq 0$ is a given real number and set $\mathbf{C} = \mathbf{C}([-r,0],\mathbf{R}^n)$. Let $x_t \in \mathbf{C}$ be defined by $x_t(\theta) = x(t+\theta)$, $t \geq 0$, $-r \leq \theta \leq 0$.

The neutral functional differential equation on $\mathbf{C}$ is of the general form

$$\frac{d}{dt}D(x_t) = f(x_t), \tag{3.1}$$

where $D, f : \mathbf{C} \to \mathbf{R}^n$ are continuous and linear. Suppose that r_j, $j = 1, 2, ..., N$, are given real numbers and $g : \mathbf{R}^N \to \mathbf{R}^n$ is a given continuous function. Let $r = \max\{r_1, ..., r_N\}$. If

$$D(\phi) = \phi(0) - g(\phi(-r_1), ..., \phi(-r_N)),$$

the neutral functional differential equation (3.1) on $\mathbf{C}$ is of the form

$$\frac{d}{dt}[x(t) - g(x(t-r_1), ..., x(t-r_N))] = 0. \tag{3.2}$$

Obviously, the solutions of Equation (3.2) are given by

$$x(t) = g(x(t-r_1), ..., x(t-r_N)) + c,$$

where c is a constant. Consequently, neutral functional differential equations include generalized difference equations. For some basic results in the theory of generalized difference equations the reader is referred to [45], [46], [47], [48], [51], [60], [72], [88] and others.

In [60] Krisztin and Wu considered the periodic scalar functional differential equation

$$\frac{d}{dt}[x(t) - c(t)x(t - \lambda(t))] = g(t, x(t), x(t - \omega), x(t - \sigma(t))), \tag{3.3}$$

where c, λ, σ are continuous ω-periodic functions with $0 \leq c(t) < 1$, $0 < \lambda(t) \leq r$, $0 \leq \sigma(t) \leq r$; and g is continuous ω-periodic in t and locally Lipschitz continuous in all the other variables; moreover $g(t, x, x, x) = 0$ for all $(t, x, x) \in \mathbf{R}^3$. Equation (3.3) appeared in the study of the motion of a classically radiating electron population growth, the spread of epidemics, the dynamics of capital stocks and compartmental systems. The asymptotic behavior of solutions of Equation (3.3) is obtained by investigation of solutions of functional equation

$$y(t) - c(t)y(t - \lambda(t)) = h(t) \quad \text{for} \quad 0 \leq t \leq T,$$

where h is a given function and $T > 0$ is a given constant.
In [54] Karydas studied a neutral delay differential equation with piecewise constant argument

$$\frac{d}{dt}[x(t) + A(t)x(t-1)] = B(t)x\left(\left[t + \frac{k-1}{k}\right]\right),$$

where $t \in (0, \infty)$, A, B are $m \times m$ continuous matrices, $k \in \{2, 3, \ldots\}$. The current strong interest in differential equations with piecewise constant argument was motivated by the fact that they describe hybrid dynamical systems (a combination of continuous and discrete) and therefore combine properties of both differential and difference equations. Differential equations with piecewise constant argument may also have applications in certain biomedical models (see [13]).
In [37] Freedman and Wu considered a single-species population growth model which incorporates a periodic time delay in the birth process. They have shown the existence of a stable periodic solution of a retarded functional differential equation of the form

$$x'(t) = x(t)[a(t) - b(t)x(t) + c(t)x(t - \tau(t))],$$

where the net birth rate a, the self-inhibition rate b, the reproduction rate c and the delay τ are continuously differentiable, ω-periodic functions; $a(t) > 0$, $b(t) > 0$, $c(t) \geq 0$, $\tau(t) \geq 0$ for $t \in \mathbf{R}$, provided that the nondifferential functional equation

$$a(t) - b(t)K(t) + c(t)K(t - \tau(t)) = 0$$

has a positive periodic solution.

When determining stability properties of neutral functional differential equations (hereditary equations), the major difficulties arise due to the associated difference equation. Therefore to understand stability problems for hereditary equations it is first necessary to understand these problems for difference equations. In [46], Hale and Cruz defined the concept of stability for the difference operator

$$D(\phi) = \phi(0) - \sum_{i=1}^{k} A_i \phi(-\tau_i).$$

In the applications the numbers τ_i represent time lags in the system, such as the communication time between two electrical circuits , response time in feedback control systems or the time it takes certain chemicals to be transported through the blood stream. In [72] Melvin generalized Hale's and Cruz's results to the Banach spaces and discussed the stability properties of the generalized difference equation

$$x(t) - \sum_{i=1}^{k} A_i x(t-\tau_i) = \phi(0) - \sum_{i=1}^{k} A_i \phi(-\tau_i),$$

where $x \in \mathbf{R}^n$, A_i are $n \times n$ matrices, $i = 1, 2, ..., k$, $0 < \tau_1 < \tau_2 < ... < \tau_k < r$ and $\phi \in \mathbf{C}([-r, 0], \mathbf{R}^n)$ the space of continuous functions.
Heard's paper [50] is devoted to relating the asymptotic properties of solutions of the delay equation

$$x'(t) = ax(t) + bx(p(t)), \quad t \geq t_0, \tag{3.4}$$

to the behavior of solutions of the linear functional nondifferential equation

$$a\varphi(t) + b\varphi(p(t)) = 0, \quad t \geq t_0. \tag{3.5}$$

The resemblance between the asymptotic behavior of solutions of (3.4) and (3.5) has been shown for $a < 0$ and holds for certain delay equations (3.4) with $p(t)$ unbounded. In thc paper by Čermák [18] this asymptotic resemblance is shown also in the case $a > 0$. We remark that the idea of establishing estimates of solutions of linear functional differential equations by means of solutions of auxiliary functional nondifferential equations has also been used by Diblik [23]. In [17] Čermák generalized this idea also for the system case.

4. The System Case

4.1. Introduction

Assume that $t_0 > 0$, and $A, B : [t_0, \infty) \to \mathbf{R}^{n \times n}$ are given $n \times n$ real matrix valued functions. Let $p : [t_0, \infty) \to \mathbf{R}$ be given such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t \to \infty} p(t) = \infty$.
This section discusses the asymptotic behavior of solutions of the functional equation

$$x(t) = A(t)x(t-1) + B(t)x(p(t)), \tag{4.1}$$

where $x(t) \in \mathbf{R}^n$. We investigate both the case when $A(t)$ is a diagonal matrix and when it is not.
Another aim of this section is to study the special case of system (4.1), that is, the system

$$x(t) = Ax(t-1) + Bx(p(t)) \tag{4.2}$$

with constant $n \times n$ real matrices A and B.

In this section we obtain asymptotic estimates for the rate of convergence of the solutions of Equation (4.1). Section 4.2. contains the notation and definitions. The results in Subsections 4.3. and 4.4. are obtained in the case when matrix $A(t)$ is diagonal. In Subsection 4.3. there is a result for the case when the components of matrix $A(t)$ are between 0 and 1. We illustrate how the rate of convergence of the components of solutions can be estimated in the case when the lag function is $p(t) = pt$, $0 < p < 1$. In Subscction 4.4. are the results when the components of matrix $A(t)$ are greater than 1 and we show how Theorems 4.1 and 4.4 can be generalized to the case, where A, B are constant matrices and A is diagonalizable. In Subsection 4.5. we prove generalizations for the nondiagonal case. Subsection 4.6. describes a representation of solutions of Equation (4.1), in a form of series, using the Cauchy matrix of the linear system

$$x(t) = A(t)x(t-1).$$

4.2. Notation and Definitions

Let $\mathbf{N}$ be the set of nonnegative integers, $\mathbf{R}$ the set of real numbers and $\mathbf{R}_+ = (0, \infty)$.
For given $m \in \mathbf{N}$, $t \in \mathbf{R}_+$ and a function $f : \mathbf{R} \to \mathbf{R}$ we use the standard notation

$$\prod_{\ell=t}^{t-1} f(\ell) = 1, \qquad \prod_{\ell=t-m}^{t} f(\ell) = f(t-m)f(t-m+1)...f(t)$$

and

$$\sum_{\tau=t}^{t-1} f(\tau) = 0, \qquad \sum_{\tau=t-m}^{t} f(\tau) = f(t-m) + f(t-m+1) + ... + f(t).$$

The difference operator Δ is defined by

$$\Delta f(t) = f(t+1) - f(t).$$

For a function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, the difference operator Δ_t is given by

$$\Delta_t g(t,a) = g(t+1,a) - g(t,a).$$

For a real number t and positive integer m, we use the notation

$$t^{(m)} = t(t-1)(t-2)...(t-m+1).$$

Let $p : [t_0, \infty) \to \mathbf{R}$ be given such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$. Set

$$t_{-1} = \min\{\inf\{p(s) : s \ge t_0\}, t_0 - 1\}$$

and

$$t_m = \inf\{s : p(s) > t_{m-1}\} \quad \text{for} \quad \text{all} \quad m = 1, 2, ...$$

Then $\{t_m\}_{m=-1}^{\infty}$ is an increasing sequence such that

$$\lim_{m\to\infty} t_m = \infty, \qquad \bigcup_{m=1}^{\infty} [t_{m-1}, t_m) = [t_0, \infty)$$

$$\text{and} \quad p(t) \in \bigcup_{i=0}^{m} [t_{i-1}, t_i) \quad \text{for} \quad \text{all} \quad t_m \le t < t_{m+1}, \quad m = 0, 1, 2, ...$$

By a *solution* of (4.1) we mean a function $x : [t_{-1}, \infty) \to \mathbf{R}^n$ which satisfies Equation (4.1) for all $t \ge t_0$.

For a given function $\phi : [t_{-1}, t_0) \to \mathbf{R}^n$, Equation (4.1) has the unique solution x^ϕ satisfying the *initial condition*

$$x^\phi(t) = \phi(t) \quad \text{for} \quad t_{-1} \le t < t_0. \tag{4.3}$$

For a given nonnegative integer m, fix a point $t \ge t_0$, and define the natural numbers $k_m(t)$ such that

$$k_m(t) := [t - t_m], \quad m = 0, 1, 2...$$

Then, for $t \in [t_m, t_{m+1})$, we have

$$t - k_m(t) - 1 < t_m \quad \text{and} \quad t - k_m(t) \ge t_m, \quad m = 0, 1, 2, ...$$

and for the arbitrary $t \ge t_0$, $m = 0$ we have

$$t - k_0(t) - 1 < t_0 \quad \text{and} \quad t - k_0(t) \ge t_0.$$

Set

$$T_m(t) := \{t - k_m(t), t - k_m(t) + 1, ..., t - 1, t\}, \quad m = 0, 1, 2, ...$$

and

$$T_0^*(t) := \{t - k_0(t) - 1, t - k_0(t), ...t - 1, t\}.$$

For given functions $g_i : [t_{-1}, \infty) \to (0, \infty)$ and $a_i : [t_0, \infty) \to (0, 1)$, $i = 1, 2, ..., n$, and for a given nonnegative integer m we define the numbers

$$R_{im} := \sup_{t_m \le t < t_{m+1}} \left\{ g_i(t) \sum_{\tau = t - k_m(t)}^{t} \sum_{j=1}^{n} \frac{\Delta_\tau(g_j(\tau - 1))}{g_j(\tau) g_j(\tau - 1)} \prod_{\ell = \tau + 1}^{t} a_i(\ell) \right\}. \tag{4.4}$$

For given functions $g_i : [t_{-1}, \infty) \to (0, \infty)$ and given initial functions ϕ_i, $i = 1, 2, ..., n$, we set

$$M_{i0} = \sup_{t_{-1} \le t < t_0} g_i(t) |\phi_i(t)| \quad \text{for} \quad i = 1, 2, ..., n. \tag{4.5}$$

For given numbers Q_i, $i = 1, 2, ..., n$ and $p \in (0, 1)$ we define

$$k_i = \frac{\log Q_i}{\log p}, \quad k = \min_{1 \le i \le n} k_i, \quad K = \max_{1 \le i \le n} k_i. \tag{4.6}$$

For given numbers a_i, A_i, $i = 1, 2, ..., n$, set

$$\Gamma := \max_{1 \le i \le n} A_i \quad \text{and} \quad \gamma := \min_{1 \le i \le n} a_i \tag{4.7}$$

and

$$\nu := \sup_{t \ge t_0} \left\{ \frac{\Gamma^{\frac{p(t)}{t}}}{\gamma} \right\}. \tag{4.8}$$

4.3. The Case of Diagonal A with $0 < A < 1$

In this subsection we discuss the case when, matrix A is diagonal and their components are between 0 and 1.
Consider the following hypotheses.

- (H_1^1) For every $t \geq t_0$, $A(t) = \text{diag}(a_1(t), ..., a_n(t))$ is an $n \times n$ diagonal matrix with real entries satisfying $0 < a_i(t) < 1$, for all $t \geq t_0$, $i = 1, 2, ..., n$.
- (H_2^1) $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix with real entries for all $t \geq t_0$.
- (H_3^1) There exists a diagonal $n \times n$ matrix $G(t) = \text{diag}(g_1(t), ..., g_n(t))$ for all $t \geq t_{-1}$ so that the diagonal entries $g_i : [t_{-1}, \infty) \to (0, \infty)$ are bounded on the initial interval $[t_{-1}, t_0)$, $i = 1, 2, ..., n$, and such that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(p(t))} \leq (1 - a_i(t)) \sum_{j=1}^{n} \frac{1}{g_j(t-1)} \quad \text{for} \quad t \geq t_0, \quad i = 1, 2, ..., n.$$

- (H_4^1) There are real numbers R_i, $i = 1, 2, ..., n$, such that

$$\prod_{j=0}^{m} (1 + R_{ij}) \leq R_i, \quad \text{for all positive integers } m \quad \text{and} \quad i = 1, 2, ..., n,$$

  where the numbers R_{ij} are defined in (4.4).
- (H_5^1) $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$.

The next theorem gives asymptotic estimates for the rate of convergence of the components of solutions of Equation (4.1). An analogue result was given by Makay and Terjéki for the scalar pantograph differential equations in Theorem 2.6. The construction of the proof of the result given for difference equations is different from the proof given for differential equations. Moreover, the coefficients of Equation (4.1) are not scalar functions as in the result given in Theorem 2.6, but matrix functions.

Theorem 4.1. *Suppose that conditions (H_1^1), (H_2^1), (H_3^1) (H_4^1) and (H_5^1) hold. Let $x = x^{\phi}$ be the solution of the initial value problem (4.1) and (4.3) with bounded components ϕ_i, $i = 1, 2, ..., n$, in (4.3). Then*

$$|x_i(t)| \leq \frac{M_{i0} R_i}{g_i(t)} \quad \textit{for all} \quad t \geq t_0 \quad \textit{and} \quad i = 1, 2, ..., n.$$

Proof. Equation (4.1) can be written as

$$x_i(t) = a_i(t)x_i(t-1) + \sum_{j=1}^{n} b_{ij}(t)x_j(p(t)), \quad i = 1,2,...,n. \tag{4.9}$$

Introduce the functions $y_i(t) := x_i(t)g_i(t)$, $i = 1,2,...,n$. Then the function $y_i(t)$ satisfies the equation

$$\frac{y_i(t)}{g_i(t)} = a_i(t)\frac{y_i(t-1)}{g_i(t-1)} + \sum_{j=1}^{n} b_{ij}(t)\frac{y_j(p(t))}{g_j(p(t))}, \quad i = 1,2,...,n.$$

Let $t \in [t_m, t_{m+1})$ and $\tau \in T_m(t)$. Then system (4.9) is equivalent to

$$\Delta_\tau \left\{ \frac{y_i(\tau-1)}{g_i(\tau-1)} \prod_{\ell=t-k_m(t)}^{\tau-1} \frac{1}{a_i(\ell)} \right\} = \sum_{j=1}^{n} b_{ij}(\tau)\frac{y_j(p(\tau))}{g_j(p(\tau))} \prod_{\ell=t-k_m(t)}^{\tau} \frac{1}{a_i(\ell)},$$

for $i = 1,2,...,n$. Summing up both sides of this equality from $t - k_m(t)$ to t we obtain, for $i = 1,2,...,n$, that

$$\begin{aligned} y_i(t) &= \frac{g_i(t)}{g_i(t-k_m(t)-1)} y_i(t-k_m(t)-1) \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \\ &+ g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} b_{ij}(\tau)\frac{y_j(p(\tau))}{g_j(p(\tau))} \prod_{\ell=\tau+1}^{t} a_i(\ell). \end{aligned} \tag{4.10}$$

Now, the inequality in (H_3^1) implies that, for $i = 1,2,...,n$,

$$\begin{aligned} |y_i(t)| &\le \frac{g_i(t)}{g_i(t-k_m(t)-1)} |y_i(t-k_m(t)-1)| \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \\ &+ g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{1-a_i(\tau)}{g_j(\tau-1)} |y_j(p(\tau))| \prod_{\ell=\tau+1}^{t} a_i(\ell). \end{aligned}$$

Define

$$\mu_{im} := \sup_{t_{m-1}\le t<t_m} |y_i(t)|, \quad M_{im} := \max\{\mu_{i0}, \mu_{i1}, ..., \mu_{im}\} \tag{4.11}$$

for $m = 0,1,2,...$, $i = 1,2,...,n$, and set

$$M_m = \max_{1\le i\le n} M_{im}, \quad m = 0,1,2,... \tag{4.12}$$

Since $|y_i(p(\tau))| \le M_{im}$, for $i = 1,2,...,n$, $\tau \in T_m(t)$ and $t_m \le t < t_{m+1}$, by using the summation by parts formula, it follows that, for $i = 1,2,...,n$,

$$|y_i(t)| \le M_m \left(\frac{g_i(t)}{g_i(t-k_m(t)-1)} \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \right.$$

$$
\begin{aligned}
& + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{1-a_i(\tau)}{g_j(\tau-1)} \prod_{\ell=\tau+1}^{t} a_i(\ell) \Bigg) \\
= \; & M_m \left(\frac{g_i(t)}{g_i(t-k_m(t)-1)} \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \right. \\
& \left. + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{1}{g_j(\tau-1)} \Delta_\tau \left(\prod_{\ell=\tau}^{t} a_i(\ell) \right) \right) \\
= \; & M_m \left(\frac{g_i(t)}{g_i(t-k_m(t)-1)} \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \right. \\
& \left. + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{1}{g_j(\tau-1)} \Delta_\tau \left(\prod_{\ell=\tau}^{t} a_i(\ell) \right) \right) \\
= \; & M_m \left(\frac{g_i(t)}{g_i(t-k_m(t)-1)} \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \right. \\
& + g_i(t) \sum_{j=1}^{n} \frac{1}{g_j(\tau-1)} \prod_{\ell=\tau}^{t} a_i(\ell) \Bigg|_{t-k_m(t)}^{t+1} + \\
& \left. + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{\Delta_\tau g_j(\tau-1)}{g_j(\tau) g_j(\tau-1)} \prod_{\ell=\tau+1}^{t} a_i(\ell) \right) \\
\le \; & M_m \left(1 + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{\Delta_\tau g_j(\tau-1)}{g_j(\tau) g_j(\tau-1)} \prod_{\ell=\tau+1}^{t} a_i(\ell) \right) \\
\le \; & M_m (1 + R_{im}).
\end{aligned}
$$

From the above inequalities follows that

$$1 + R_{im} \ge 0 \quad \text{and} \quad M_{m+1} \le M_m(1 + R_{im})$$

for all $m = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, n$, that implies

$$M_{m+1} \le M_{i0} \prod_{j=0}^{m} (1 + R_{ij}), \quad i = 1, 2, \ldots, n.$$

Therefore

$$|y_i(t)| \le M_{i0} R_i, \quad i = 1, 2, \ldots, n,$$

and the proof is complete.

In the following result we illustrate how the rate of convergence of the components of the solutions can be estimated by a power function in the particular case when the lag function is $p(t) = pt$, $0 < p < 1$.
We shall need the following hypothesis.

- (H_6^1) There exist real numbers Q_i and α_i, $i = 1, 2, ..., n$, such that $0 < Q_i \leq 1$, $0 < \alpha_i < 1$, $i = 1, 2, ..., n$, and

  $$|b_{ij}(t)| \leq Q_j(1 - a_i(t)), \quad \alpha_i \leq 1 - a_i(t) \quad \text{for} \quad t \geq t_0, \quad i, j = 1, 2, ..., n,$$

  where the functions a_i and b_{ij} are given in (H_1^1) and (H_2^1).

Corollary 4.2. *Assume that $p \in (0,1)$, $t_0 \geq p/(1-p)$ and $p(t) = pt$ for all $t \geq t_0$. Suppose that conditions (H_1^1), (H_2^1) and (H_6^1) hold. Let $x = x^\phi$ be a solution of the initial value problem (4.1) and (4.3) with bounded components ϕ_i, $i = 1, 2, ..., n$, and suppose that $k + 1 > K$, where k and K are defined by (4.6). Then*

$$|x_i(t)| \leq \frac{M_{i0} R_i}{(t - p/(1-p))^{k_i}} \quad \textit{for} \quad \textit{all} \quad t \geq t_0, \quad i = 1, 2, ..., n,$$

where

$$M_{i0} = \sup_{t_{-1} \leq t < t_0} \left\{ \left(t - \frac{p}{1-p} \right)^{k_i} |x_i(t)| \right\}$$

and

$$R_i = \prod_{m=0}^{\infty} \left(1 + \frac{t_0^{k_i}}{\alpha_i p^{k_i}} \sum_{j=1}^{n} \frac{k_j (1-p)^{k_j+1}}{(t_0(1-p) - p^m)^{k_j+1}} \left(p^{k-K+1} \right)^m \right)$$

for $i = 1, 2, ..., n$.

Proof. Set

$$t_{-1} = \min\{t_0 - 1, pt_0\}, \quad t_m = \frac{t_0}{p^m} \quad m = 1, 2, ...$$

and

$$g_i(t) = \left(t - \frac{p}{1-p} \right)^{k_i}, \quad i = 1, 2, ..., n.$$

Since $Q_i = p^{k_i}$, $i = 1, 2, ..., n$, it follows that

$$\sum_{j=1}^{n} \frac{|b_{ij}(t)|}{g_j(pt)} \leq \sum_{j=1}^{n} \frac{Q_j(1 - a_i(t))}{\left(pt - \frac{p}{1-p} \right)^{k_j}} \leq \sum_{j=1}^{n} \frac{Q_j(1 - a_i(t))}{p^{k_j} \left(t - \frac{1}{1-p} \right)^{k_j}} =$$

$$= (1 - a_i(t)) \sum_{j=1}^{n} \frac{1}{\left(t - \frac{p}{1-p} \right)^{k_j}} = ((1 - a_i(t)) \sum_{j=1}^{n} \frac{1}{g_j(t-1)}.$$

Therefore, condition (H_3^1) of Theorem 4.1 is valid. Moreover,

$$\begin{aligned}
R_{im} &= \sup_{t_m \le t < t_{m+1}} \left\{ \left(t - \frac{p}{1-p}\right)^{k_i} \times \right. \\
&\quad \left. \times \sum_{\tau=t-k(t)}^{t} \sum_{j=1}^{n} \frac{\left(\tau - \frac{p}{1-p}\right)^{k_j} - \left(\tau - \frac{1}{1-p}\right)^{k_j}}{\left(\tau - \frac{p}{1-p}\right)^{k_j}\left(\tau - \frac{1}{1-p}\right)^{k_j}} \prod_{\ell=\tau+1}^{t} a_i(\ell) \right\} \\
&\le \sup_{t_m \le t < t_{m+1}} \left\{ t^{k_i}(1-\alpha_i)^t \sum_{\tau=t-k(t)}^{t} \sum_{j=1}^{n} \frac{k_j}{\left(\tau - \frac{1}{1-p}\right)^{k_j+1}} \left(\frac{1}{1-\alpha_i}\right)^{\tau} \right\} \\
&\le \sup_{t_m \le t < t_{m+1}} \left\{ \frac{t^{k_i}}{\alpha_i} \sum_{j=1}^{n} \frac{k_j}{(t - k(t) - \frac{1}{1-p})^{k_j+1}} \right\} \\
&\le \frac{t_0^{k_i}}{\alpha_i p^{k_i}} \sum_{j=1}^{n} \frac{k_j(1-p)^{k_j+1}}{(t_0(1-p) - p^m)^{k_j+1}} \left(p^{k_j - k_i + 1}\right)^m \\
&\le \frac{t_0^{k_i}}{\alpha_i p^{k_i}} \sum_{j=1}^{n} \frac{k_j(1-p)^{k_j+1}}{(t_0(1-p) - p^m)^{k_j+1}} \left(p^{k - K + 1}\right)^m .
\end{aligned}$$

Then, Theorem 4.1 implies the assertion.

Remark 4.3. *We can prove similarly the above result by choosing*

$$g_i(t) = t^{k_i} \quad for\ all \quad i = 1, 2, ..., n.$$

Then the statement of Theorem 4.1 holds for $t \ge t_0/p$ that does not disturb the asymptotic behavior of solutions but the comparableness with the functions g_i, $i = 1, 2, ..., m$ is clearer and the rate of the convergence is better.

4.4. The Case of Diagonal A with $A > 1$

In this subsection we suppose that matrix A is diagonal and its components are greater than 1.
Now we introduce the following conditions.

- (H_7^1) For every $t \ge t_0$, $A(t) = \text{diag}(a_1(t), ..., a_n(t))$ is an $n \times n$ diagonal matrix with real entries, and there exist real numbers a_i, A_i, $i = 1, 2, ..., n$, such that $1 < a_i \le a_i(t) \le A_i$, $i = 1, 2, ..., n$.
- (H_8^1) $B(t) = (b_{ij}(t))$ is an $n \times n$ matrix with real bounded entries for $i, j = 1, 2, ..., n$, $t \ge t_0$.
- (H_9^1) $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that for a given real number $q \in (0, 1)$ $p(t) < qt$ holds for all $t \ge t_0$.

In the next result we will show that the solutions tend to infinity and give estimates on the rate of convergence of the components of the solutions of Equation (4.1). A similar result was given in Theorem 2.2 by Lim for the pantograph differential equations with constant coefficients. Here we prove the result for the difference equations with continuous time when the coefficients in Equation (4.1) are not constant matrices as in the Lim's result, but matrix functions.

Theorem 4.4. *Suppose that conditions (H_7^1), (H_8^1) and (H_9^1) hold such that $\Gamma^q < \gamma$, where Γ, γ and ν are defined by (4.7) and (4.8). Let $x = x^\phi$ be a solution of the initial value problem (4.1) and (4.3) with the bounded components ϕ_i, $i = 1,2,...,n$. Then, there are positive constants K_i, $i = 1,2,...,n$, such that*

$$|x_i(t)| \leq K_i \prod_{\ell=t-k_0(t)-1}^{t} a_i(\ell) \quad \text{for all} \quad t \geq t_0, \quad i = 1,2,...,n.$$

Proof. Equation (4.1) can be written in the form (4.9). Introduce the function $y_i(t) := x_i(t)g_i(t)$ with

$$g_i(t) := \prod_{\ell=t-k_0(t)-1}^{t} \frac{1}{a_i(t)}, \quad i = 1,2,...,n.$$

Let $t \in [t_m, t_{m+1})$ and $\tau \in T_m(t)$. Then, the system (4.9) is equivalent to (4.10). Using notations (4.11) and (4.12), we obtain

$$\begin{aligned} |y_i(t)| &\leq \frac{g_i(t)}{g_i(t-k_m(t)-1)}|y_i(t-k_m(t)-1)| \prod_{\ell=t-k_m(t)}^{t} a_i(\ell) + \\ &\quad + g_i(t) \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} \frac{|b_{ij}(\tau)|}{g_j(p(\tau))}|y_j(p(\tau))| \prod_{\ell=\tau+1}^{t} a_i(\ell) \\ &\leq M_{im} + \\ &\quad + \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} |b_{ij}(\tau)| M_{jm} \prod_{\ell=p(\tau)-k_0(p(\tau))-1}^{p(\tau)} a_j(\ell) \prod_{\ell=t-k_0(t)-1}^{\tau} \frac{1}{a_i(\ell)}. \end{aligned}$$

In virtue of the equality $t - k_0(t) = \tau - k_0(\tau)$ for $\tau \in T_0(t)$, it follows that

$$|y_i(t)| \leq M_m \left(1 + \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} |b_{ij}(\tau)| A_j^{k_0(p(\tau))+2} \left(\frac{1}{a_i}\right)^{k_0(\tau)+2}\right).$$

Using notation (4.8), it follows from the hypothesis of the theorem that

$$\nu \leq \frac{\Gamma^q}{\gamma} < 1.$$

Define

$$\beta := \sup_{t\geq t_0} \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}(t)|.$$

Since $\tau - t_0 - 1 \leq k_0(\tau) \leq \tau - t_0$ and $(\gamma/\Gamma)^{t_0} < 1$, we have

$$\begin{aligned} |y_i(t)| &\leq M_m \left(1 + \beta\Gamma^2 \sum_{\tau=t-k(t)}^{t} \left(\frac{\Gamma^{\frac{p(\tau)}{\tau}}}{\gamma}\right)^{\tau}\right) \\ &\leq M_m \left(1 + \beta\Gamma^2 \sum_{\tau=t-k_m(t)}^{t} \nu^{\tau}\right) \\ &\leq M_m \left(1 + \frac{\beta\Gamma^2}{1-\nu}\nu^{t_m}\right), \end{aligned}$$

for $i = 1, 2, ..., n$. Therefore

$$M_m \leq M_0 \prod_{\ell=0}^{m-1} \left(1 + \frac{\beta A^2}{1-\nu}\nu^{t_m}\right), \quad m = 1, 2, ...$$

and the infinite product

$$\prod_{\ell=0}^{m-1} \left(1 + \frac{\beta A^2}{1-\nu}\nu^{t_m}\right)$$

converges for all $m = 1, 2, ...,$ and $\{M_m\}_{m=0}^{\infty}$ is a bounded sequence. This implies that there are constants K_i, $i = 1, 2, ..., n$, such that

$$|x_i(t)| \prod_{\ell=t-k_0(t)-1}^{t} \frac{1}{a_i(\ell)} \leq K_i, \quad i = 1, 2, ..., n,$$

and this completes the proof.

Consider Equation (4.2) with constant coefficients and obtain the following estimates for the components of the solutions of System (4.2) as a special case of the previous theorem. The obtained corollary is analogue to the result given by Lim in Corollary 2.3. It is interesting in its own right because the estimating function is the well known exponential function.

Corollary 4.5. *Let A and B be $n \times n$ constant matrices with real entries, where $B = (b_{ij})$ and $A = \operatorname{diag}(a_1, a_2 ..., a_n)$ is a diagonal matrix such that $a_i > 1$ for $i = 1, 2, ..., n$. Suppose that (H_9^1) holds such that $\Gamma^q < \gamma$, where*

$$\Gamma := \max_{1\leq i\leq n} a_i \quad \text{and} \quad \gamma := \min_{1\leq i\leq n} a_i.$$

Let $x = x^{\phi}$ be a solution of the initial value problem (4.2) and (4.3) with the bounded components ϕ_i, $i = 1, 2, ..., n$. Then, there are positive constants K_i, $i = 1, 2, ..., n$, such that

$$|x_i(t)| \leq K_i a_i^t \quad for \quad t \geq t_0, \quad i = 1, 2, ..., n.$$

Proof. Apply the same technique as in the proof of Theorem 4.4 with $g_i(t) := a_i^{-t}$ for $i = 1, 2, ..., n$. Set

$$\beta := \sum_{i=1}^{n} \sum_{j='}^{n} |b_{ij}|$$

and ν is defined by (4.8). Now, it follows that $\nu < 1$ and

$$\begin{aligned} |y_i(t)| &\leq M_{im} + \sum_{\tau=t-k_m(t)}^{t} \sum_{j=1}^{n} |b_{ij}| M_{jm} \left(\frac{a_j^{\frac{p(\tau)}{\tau}}}{a_i} \right)^{\tau} \\ &\leq M_m \left(1 + \frac{\beta}{1-\nu} \nu^{t_m} \right), \end{aligned}$$

and following the same arguments as in Theorem 4.4, the proof is complete.

We can generalize the results given in Theorems 4.1 and 4.4 for equation with constant coefficients when matrix A is not a diagonal matrix.

Remark 4.6. *The method used in the proof of Theorems 4.1 and 4.4 can be developed to estimate the solutions of Equation (4.2). Let the constant matrix A be diagonalizable with eigenvalues a_i, $i = 1, 2, ..., n$. Then, there is a nonsingular matrix P such that $A = PDP^{-1}$, where $D = \operatorname{diag}(a_1, a_2 ..., a_n)$. Let $y(t) = P^{-1}x(t)$. Then (4.2) becomes*

$$y(t) = Dy(t-1) + P^{-1}BPy(p(t)).$$

If the suitable conditions are valid, then we can apply the above theorems.

4.5. Generalizations: Nondiagonal A

Denote by $\mathbf{R}^{n \times n}$ the Banach space of all $n \times n$ real matrices with an arbitrary norm $|| \cdot ||$. We consider the nonautonomous case (4.1), without assuming that A is a diagonal matrix.

Assume that $t_0 > 0$. For given $n \times n$ matrix function A with real entries let us denote the *Cauchy matrix* of the initial value problem

$$x(t) = A(t)x(t-1), \quad x(\theta) = \phi(\theta), \quad t_0 - 1 \leq \theta \leq t_0 \tag{4.13}$$

by $W(\tau;t)$, where

$$W(\tau;t) = A(t)A(t-1)...A(\tau+1),$$

for $t_m \le t < t_{m+1}$, $\tau \in T_0^*(t)$, with $W(t;t) = E$, where E is the n-dimensional unite matrix.
In the particular case, when A is an $n \times n$ matrix with constant entries, the Cauchy matrix of the initial value problem

$$x(t) = Ax(t-1), \quad x(\theta) = \phi(\theta), \quad t_0 - 1 \le \theta \le t_0 \tag{4.14}$$

has the form:

$$W(\tau;t) = A^{t-\tau},$$

for $t_m \le t < t_{m+1}$, $\tau \in T_0^*(t)$ and $A^0 = E$.
For given scalar functions $a, \rho : [t_0, \infty) \to \mathbf{R}$ and nonnegative numbers m we define the numbers

$$R_m := \sup_{t_m \le t < t_{m+1}} \left\{ \rho(t) \sum_{\tau=t-k_m(t)}^{t} \frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)} \prod_{\ell=\tau+1}^{t} a(\ell) \right\} \tag{4.15}$$

and

$$M_0 = \sup_{t_{-1} \le t < t_0} ||M(t)\phi(t)||.$$

We shall assume the following.

- (H_{10}^1) For given $n \times n$ real matrix valued functions $A, B : [t_0, \infty) \to \mathbf{R}^{n \times n}$, where A is nonsingular and B is with bounded entries, there is a nonsingular $n \times n$ matrix $M(t) = (m_{ij}(t))$ with $m_{ij} : [t_{-1}, \infty) \to (0, \infty)$ bounded on the interval $[t_{-1}, t_0)$ for $i, j = 1, 2, ..., n$, and there exist functions $a : [t_0, \infty) \to \mathbf{R}$ and $\rho : [t_{-1}, \infty) \to \mathbf{R}_+$ such that $0 < a(t) < 1$,

$$||M(t)W(\tau;t)M^{-1}(\tau)|| \le \frac{\rho(t)}{\rho(\tau)} \prod_{\ell=\tau+1}^{t} a(\ell), \tag{4.16}$$

  for all $t \ge t_0$, $\tau \in T_0^*(t)$ and

$$||M(t)B(t)M^{-1}(p(t))|| \le \frac{(1-a(t))\rho(t)}{\rho(t-1)} \quad \text{for} \quad t \ge t_0, \tag{4.17}$$

  where W is the Cauchy matrix of the initial value problem (4.13).

- (H_{11}^1) For every $m \in \mathbf{N}$, there exist real numbers R_m such that

$$\prod_{m=0}^{j} (1 + R_m) \le R, \quad \text{for all positive integers } j,$$

  where the numbers R_m are defined in (4.15).

The following result generalizes Theorem 4.1 to the case when matrix A is not diagonal. We suppose only the regularity of matrix A. The new aspect in the proof is using the Cauchy matrix of the linear part of the considered equation.

Theorem 4.7. *Suppose that (H_5^1), (H_{10}^1) and (H_{11}^1) hold. Let $x = x^\Phi$ be a solution of the initial value problem (4.1) and (4.3). Then*

$$||x(t)|| \le M_0 R ||M^{-1}(t)|| \quad for \quad t \ge t_0.$$

Proof. Introduce the transformation $y(t) = M(t)x(t)$. Then, for $t \in [t_m, t_{m+1})$ and $\tau \in T_m(t)$, the function $y(t)$ satisfies the equation

$$\begin{aligned} y(t) &= M(t)W(t-k_m(t)-1;t)M^{-1}(t-k_m(t)-1)y(t-k_m(t)-1) + \\ &+ \sum_{\tau=t-k_m(t)}^{t} W(\tau;t)B(\tau)M^{-1}(p(\tau))y(p(\tau)). \end{aligned} \tag{4.18}$$

Now, using the hypotheses (4.16) and (4.17), the scalar function $u(t) = ||y(t)||$ satisfies the inequality

$$\begin{aligned} u(t) &\le \frac{\rho(t)}{\rho(t-k_m(t)-1)} u(t-k_m(t)-1) \prod_{\ell=t-k_m(t)}^{t} a(\ell) + \\ &+\rho(t) \sum_{\tau=t-k_m(t)}^{t} \frac{1-a(\tau)}{\rho(\tau-1)} u(p(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell). \end{aligned}$$

Define

$$\mu_m := \sup_{t_{m-1} \le t < t_m} u(t) \quad \text{and} \quad M_m := \max\{\mu_0, \mu_1, ..., \mu_m\} \tag{4.19}$$

for $m = 0, 1, 2, \ldots$. Since $u(p(\tau)) \le M_m$ for $\tau \in T_m(t)$ and $t_m \le t < t_{m+1}$, by using the summation by parts formula, it follows that

$$\begin{aligned} u(t) &\le M_m \left(1 + \rho(t) \sum_{\tau=t-k_m(t)}^{t} \frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)} \prod_{\ell=\tau+1}^{t} a(\ell) \right) \\ &\le M_m(1+R_m). \end{aligned}$$

From the above inequality follows that

$$1 + R_m \ge 0 \quad \text{and} \quad M_{m+1} \le M_m(1+R_m) \quad \text{for all} \quad m = 0, 1, 2, \ldots$$

that implies

$$M_{m+1} \le M_0 \prod_{j=0}^{m} (1+R_j)$$

and

$$u(t) \le M_0 R.$$

Therefore

$$\begin{aligned} ||x(t)|| &\le ||M^{-1}(t)||||M(t)x(t)|| \\ &= ||M^{-1}(t)||u(t) \\ &\le M_0 R||M^{-1}(t)||, \end{aligned}$$

and the proof is complete.

Assume that:

- (H_{12}^1) For a real function p given in (H_5^1), for a nonsingular $n\times n$ matrix function $A(t)$ with real entries and for the Cauchy matrix W defined for the initial value problem (4.13), there exist real numbers γ, Γ such that $1 < \gamma \le \Gamma$ with

$$||W^{-1}(t - k_0(t) - 1; t)|| \le \gamma^{-t}, \quad \text{for all} \quad t \ge t_0, \tag{4.20}$$

and

$$||W(p(t) - k_0(p(t)) - 1; p(t))|| \le \Gamma^{p(t)}, \quad \text{for all} \quad t \ge t_0. \tag{4.21}$$

The next result is a generalization of Theorem 4.4. The theorem is formulated applying the concept of the Cauchy matrix W defined for the initial value problem (4.13), similarly as in the result by Pandolfi given in Theorem 2.5 for pantograph differential equations. The arguments in the proof given for difference equations are quite different because of the specific construction of the considered equations.

Theorem 4.8. *Suppose that conditions (H_8^1), (H_9^1) and (H_{12}^1) hold, and assume that $\Gamma^q < \gamma$, where ν is defined by (4.8). Let $x = x^\phi$ be the solution of the initial value problem (4.1) and (4.3). Then, there exists a positive constant R such that*

$$||x(t)|| \le R||W(t - k_0(t) - 1; t)|| \quad for \quad t \ge t_0.$$

Proof. Introduce the transformation $y(t) = M(t)x(t)$ with

$$M(t) := W^{-1}(t - k_0(t) - 1; t) \quad \text{for} \quad t \ge t_0.$$

Then, for $t \in [t_m, t_{m+1})$ and $\tau \in T_m(t)$, the function $y(t)$ satisfies Equation (4.18) which is equivalent to

$$\begin{aligned} y(t) &= y(t - k_m(t) - 1) + \\ &+ \sum_{\tau = t - k_m(t)}^{t} W^{-1}(t - k_0(t) - 1; \tau)B(\tau) \times \\ &\times W(p(\tau) - k_0(p(\tau)); p(\tau))y(p(\tau)). \end{aligned}$$

Using hypotheses (4.20) and (4.21), the scalar function $u(t) := ||y(t)||$, with $\beta := ||B(t)||$, satisfies the inequality

$$u(t) \leq u(t-k_m(t)-1) + \beta \sum_{\tau=t-k_m(t)}^{t} \left(\frac{\Gamma^{\frac{p(\tau)}{\tau}}}{\gamma} \right)^{\tau} u(p(\tau)).$$

Since $u(p(\tau)) \leq M_m$ for $\tau \in T_m(t)$, by using notation (4.19) it follows that

$$\begin{aligned} u(t) &\leq M_m \left(1 + \beta \sum_{\tau=t-k_m(t)}^{t} \nu^{\tau} \right) \\ &\leq M_m \left(1 + \frac{\beta}{1-\nu} \nu^{t_m} \right). \end{aligned}$$

Applying the same arguments as in Theorem 4.4, there exists a positive real number R such that

$$u(t) \leq R \quad \text{for} \quad t \geq t_0.$$

Then,

$$\begin{aligned} ||x(t)|| &= ||W(t-k_0(t)-1;t)W^{-1}(t-k_0(t)-1;t)x(t)|| \\ &\leq ||W(t-k_0(t)-1;t)||u(t) \leq R||W(t-k_0(t)-1;t)||, \end{aligned}$$

for all $t \geq t_0$, and the proof is complete.

4.6. A Representation of Solutions

In this subsection we obtain a series representation of solutions of Equation (4.1), using the Cauchy matrix W of Equation (4.13). We use the series representation of solutions to obtain sufficient conditions for the convergence of solutions.

We shall assume that:

- (H_{13}^1) For every $t \geq t_0$, $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ are $n \times n$ matrices with real entries, for all $t \geq t_0$, $i, j = 1, 2, ..., n$.

First of all we prove a simple but fundamental result. Terjéki gave an analogue result in Theorem 2.1 for the pantograph differential equations. The presented result has the specific characteristics of generalized difference equations.

Theorem 4.9. *Assume that the hypotheses (H_5^1) and (H_{13}^1) hold. Let $y_0(t)$ denote the solution of the initial value problem (4.13) with $\phi(t) \not\equiv 0$ for $t_{-1} \leq t < t_0$, and the sequence $\{y_n(t),\ n = 1, 2, ...\}$ is defined by*

$$y_n(t) = A(t)y_n(t-1) + B(t)y_{n-1}(p(t)), \quad t \geq t_0,$$

$$y_n(t) \equiv 0, \quad t_{-1} \le t < t_0, \quad n = 1, 2, \ldots$$

Then

$$x(t) = \sum_{n=0}^{\infty} y_n(t) \tag{4.22}$$

is a solution of the initial value problem (4.1) and (4.3). Moreover, this series is finite on every finite subinterval of $[t_0, \infty)$.

Proof. First we show that series (4.22) is absolutely convergent on $[t_0, \infty)$. Define

$$M(F,T) := \sup_{t_0 \le t \le T} ||F(t)||$$

for any matrix or vector function F for $T > t_0$ and

$$M(W,T) := \sup_{t_0 \le \tau \le t \le T} ||W(\tau; t)||.$$

Since for $t \ge t_0$

$$y_n(t) = \sum_{\tau = t - k_0(t)}^{t} W(\tau; t) B(\tau) y_{n-1}(p(\tau)), \quad n = 1, 2, \ldots$$

hence, for $T > t_0$ and $t_0 \le t \le T$, the following inequalities hold:

$$\begin{aligned} ||y_n(t)|| &\le \sum_{\tau = t - k_0(t)}^{t} M(W,t) M(B,t) ||y_{n-1}(p(\tau))|| \\ &\le M(W,T) M(B,T) \sum_{\tau = t - k_0(t)}^{t} ||y_{n-1}(p(\tau))||. \end{aligned}$$

By using mathematical induction we will show that

$$y_n(t) = 0 \quad \text{for} \quad t_0 \le t < t_0 + (n-1)\delta. \tag{4.23}$$

For $n = 2$ we have

$$||y_2(t)|| \le M(W,T) M(B,T) \sum_{\tau = t - k_0(t)}^{t} ||y_1(p(\tau))||.$$

For $t_0 \le t < t_0 + \delta$ we have $p(t) < t_0$ and $y_1(p(t)) = 0$. Therefore,

$$y_2(t) = 0 \quad \text{for} \quad t_0 \le t < t_0 + \delta.$$

Suppose that statement (4.23) is valid for $n = k$ and prove it for $n = k + 1$. Then

$$||y_{k+1}(t)|| \le M(W,T) M(B,T) \sum_{\tau = t - k_0(t)}^{t} ||y_k(p(\tau))||.$$

For $t_0 \le t < t_0 + k\delta$ we have $p(t) \le t - \delta < t_0 + (k-1)\delta$ and by the inductional hypothesis $y_k(p(t)) = 0$, and so $y_{k+1}(t) = 0$.
Then, exists a natural number N such that

$$y_m(t) = 0 \quad \text{for all} \quad m \ge N \quad \text{and} \quad t_0 \le t \le T.$$

Therefore,

$$x(t) = \sum_{n=0}^{N-1} y_n(t) \quad \text{for} \quad t_0 \le t \le T$$

and the convergence is clear. Moreover,

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} y_n(t) \\ &= y_0(t) + \sum_{n=1}^{\infty} y_n(t) \\ &= A(t)y_0(t-1) + \sum_{n=1}^{\infty} A(t)y_n(t-1) + \sum_{n=1}^{\infty} B(t)y_{n-1}(p(t)) \\ &= A(t)\sum_{n=0}^{\infty} y_n(t-1) + B(t)\sum_{n=0}^{\infty} y_n(p(t)) \\ &= A(t)x(t-1) + B(t)x(p(t)), \end{aligned}$$

and the proof is complete.

In the space of vector or matrix functions $f(t)$ let the operators S_p and W^* be defined by

$$S_p f(t) = f(p(t)), \quad W^* f(t) = \sum_{\tau = t - k_0(t) - 1}^{t} W(\tau; t) f(\tau),$$

where W is the Cauchy matrix defined for the initial value problem (4.13).
Then

$$y_n = W^*(BS_p y_{n-1}) = (W^* BS_p)^n y_0, \quad n = 1, 2, \ldots$$

Therefore Theorem 4.9 implies the next corollary.

Corollary 4.10. *The unique solution of the initial value problem (4.1) and (4.3) is given by*

$$x(t) = \sum_{n=0}^{\infty} (W^* BS_p)^n W(t - k_0(t) - 1; t)\phi(t - k_0(t) - 1). \tag{4.24}$$

In the next result we give conditions guaranteeing that series (4.24) is absolutely and uniformly convergent.
Consider the following hypotheses.

- (H_{14}^1) There exist positive constants M, b and a, such that $0 < a < 1$, and for $t \geq t_0$, there exists a positive scalar function $f(t)$ such that

$$\sup_{t_{-1} \leq \theta \leq t_0} \sum_{\tau=\theta}^{\infty} f(\tau) = f_0 < \infty,$$

$$||W(\tau;t)|| \leq Ma^t, \quad t_0 - 1 \leq \tau \leq t_0 \leq t, \tag{4.25}$$

$$||W(\tau;t)|| \leq Ma^{t-\tau}, \quad t_0 \leq \tau \leq t, \tag{4.26}$$

$$||B(t)|| \leq b + f(t), \quad t \geq t_0, \tag{4.27}$$

$$M\left(\frac{b}{1-a} + f_0\right) < 1, \quad t_{-1} \leq \theta \leq t_0, \tag{4.28}$$

with the Cauchy matrix W defined for the initial value problem (4.13).

- (H_{15}^1) There exists a positive constant p such that

$$0 < pt \leq p(t) \quad \text{for} \quad \text{all} \quad t \geq t_0. \tag{4.29}$$

Theorem 4.11. *Suppose that conditions (H_5^1) and (H_{14}^1) hold. Then series (4.24) is absolutely and uniformly convergent for $t \geq t_0$. If, in addition, condition (H_{15}^1) also holds, then the solution of the initial value problem (4.1) and (4.3) tends to zero, as $t \to \infty$.*

Proof. Let p_0 be a real number such that

$$0 \leq p_0 < 1 \quad \text{and} \quad p_0 t \leq p(t).$$

Introduce the sequence $\{\gamma_n\}$ as follows.

$$\gamma_0 := 1, \quad \gamma_n := M\gamma_{n-1}\left(\frac{b}{1-a^{1-p_0^n}} + f_0\right), \quad n = 1, 2, \ldots$$

In virtue of (4.28) it is easy to see that the series

$$\sum_{n=0}^{\infty} \gamma_n$$

is finite. Let

$$y_0(t) = W(t - k_0(t) - 1; t)\phi(t - k_0(t) - 1)$$

and $\{y_n(t)\}$ be defined as in Theorem 4.9. We claim that

$$||y_n(t)|| \leq M\gamma_n a^{p_0^n t}||\phi||, \quad n = 0, 1, 2, \ldots \tag{4.30}$$

Inequality (4.25) implies (4.30) for $n = 0$. Suppose that (4.30) holds, for $n-1$. Then

$$\begin{aligned}
||y_n(t)|| &\le \sum_{\tau=t-k_0(t)}^{t} ||W(\tau;t)||||B(\tau)||||y_{n-1}(p(\tau))|| \\
&\le \sum_{\tau=t-k_0(t)}^{t} M^2 a^{t-\tau}(b+f(\tau))\gamma_{n-1} a^{p_0^{n-1}p(\tau)}||\phi|| \\
&\le M^2 a^t \gamma_{n-1}||\phi|| \sum_{\tau=t-k_0(t)}^{t} a^{(p_0^n-1)\tau}(b+f(\tau)) \\
&= M^2 a^t \gamma_{n-1}||\phi|| \left(\sum_{\tau=t-k_0(t)}^{t} b a^{(p_0^n-1)\tau} + \sum_{\tau=t-k_0(t)}^{t} a^{(p_0^n-1)\tau} f(\tau) \right) \\
&\le M^2 a^t \gamma_{n-1}||\phi|| \left(\frac{b}{a^{p_0^n-1}-1} a^{(p_0^n-1)\tau} \Big|_{t-k_0(t)}^{t+1} + a^{(p_0^n-1)t} \sum_{\tau=t-k_0(t)}^{t} f(\tau) \right) \\
&\le M^2 a^t \gamma_{n-1}||\phi|| \left(\frac{b}{a^{p_0^n-1}-1} a^{(p_0^n-1)(t+1)} + a^{(p_0^n-1)t} f_0 \right) \\
&= M^2 \gamma_{n-1}||\phi|| a^{p_0^n t} \left(\frac{b}{1-a^{1-p_0^n}} + f_0 \right) \\
&= M\gamma_n a^{p_0^n t}||\phi||,
\end{aligned}$$

that is, (4.30) holds, for all positive integers n. It means that (4.24) is absolutely and uniformly convergent on $[t_0, \infty)$ and the first part of the theorem is proved.

If (4.29) is satisfied, then we can choose $p_0 = p$ and for all $\epsilon > 0$, we can find an integer N such that

$$2M \sum_{n=N}^{\infty} \gamma_n < \epsilon.$$

Then

$$\begin{aligned}
||x(t)|| &\le \sum_{n=0}^{\infty} ||y_n(t)|| \le \sum_{n=0}^{\infty} M\gamma_n a^{p^n t}||\phi|| \\
&\le \left(M \sum_{n=N}^{\infty} \gamma_n + M \sum_{n=0}^{N-1} a^{p^{N-1}t}\gamma_n \right) ||\phi|| \\
&< \epsilon ||\phi||,
\end{aligned}$$

if t is so large that

$$2Ma^{p^{N-1}t} \sum_{n=0}^{N-1} \gamma_n < \epsilon.$$

This proves the second part of the theorem.

5. The Scalar Case

5.1. Introduction

Assume that $t_0 > 0$ and $a, b : [t_0, \infty) \to \mathbf{R}$ are given real functions. Let $p : [t_0, \infty) \to \mathbf{R}$ be given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p(t) = \infty$. Now, we investigate the scalar difference equation with continuous arguments

$$x(t) = a(t)x(t-1) + b(t)x(p(t)), \tag{5.1}$$

where $x(t) \in \mathbf{R}$. Equation (5.1) is a special case of Equation (4.1) for $n = 1$, $a_1(t) = a(t)$ and $b_{11}(t) = b(t)$. We shall need the following definitions.
For a given function $\varphi : [t_{-1}, t_0) \to \mathbf{R}$, Equation (5.1) has the unique solution x^φ satisfying the *initial condition*

$$x^\varphi(t) = \varphi(t) \quad \text{for} \quad t_{-1} \le t < t_0. \tag{5.2}$$

In this section we study the asymptotic behavior of solutions of Equation (5.1) and apply our results to particular cases. In Subsection 5.2. we obtain asymptotic lower and upper bounds for the solutions which can estimate the rate of the convergence of solutions. We apply our main result to particular cases such as $p_1 t \le p(t) \le p_2 t$, for real numbers $0 < p_1 \le p_2 < 1$ and $\sqrt[p_2]{t} \le p(t) \le \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \le p_2$.
Subsection 5.3. presents the characteristic equation associated with the initial value problem (5.1) and (5.2), and using them obtain an asymptotic estimate of solutions of Equation (5.1) which can be applied to the difference equations with constant delay and to the case $t - p_2 \le p(t) \le t - p_1$ for real numbers $1 < p_1 \le p_2$.
Subsection 5.4. generalizes the main results given in Subsections 5.2. and 5.3. to the equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)),$$

where $a, b_i : [t_0, \infty) \to \mathbf{R}$ are given functions for $i = 1, 2, ..., m$, and $p_i : [t_0, \infty) \to \mathbf{R}$ are given such that, for every $T > t_0$ there exists a $\delta > 0$ such

that $p_i(t) \leq t-\delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. We apply the obtained results to particular cases such as $p_i(t) = (1/p_i)t$ for $i = 1, 2, ..., m$, $p_i(t) = \sqrt[p_i]{t}$ for $i = 1, 2, ..., m$, where $1 < p_1 < p_2 < ... < p_m$ are natural numbers and $p_i(t) = t - p_i$ for $i = 1, 2, ..., m$, where $1 \leq p_1 < p_2 < ... < p_m$ are real numbers.
In Subsection 5.5. we give a series representation of solutions of the equation

$$x(t) = ax(t-1) + bx(pt),$$

with constant coefficients a, b, and $p \in (0, 1)$. From the series form of solution we obtain a necessary and sufficient condition for the convergence.
In Subsection 5.6. we give conditions for the existence of bounded solutions of the scalar difference equation

$$x(t) = (1 - c(t))x(t-1) + c(t)x(p(t)),$$

where $c : [t_0, \infty) \to \mathbf{R}$ is a given function such that $0 < c(t) < 1$, and we formulate an open problem of the asymptotic convergence of bounded solutions.

5.2. Asymptotic Estimates

For given scalar functions $a, \rho : [t_0, \infty) \to \mathbf{R}$, for given initial function φ and nonnegative numbers n we define the numbers

$$R_n := \sup_{t_n \leq t < t_{n+1}} \left\{ \rho(t) \sum_{\tau = t - k_n(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau-1)} \prod_{\ell=\tau+1}^{t} a(\ell) \right\} \tag{5.3}$$

and

$$M_0 := \sup_{t_{-1} \leq t < t_0} \rho(t)|\varphi(t)|.$$

We shall need the following hypotheses.

- (H_1^2) Let $a : [t_0, \infty) \to \mathbf{R}$ be a given real function satisfying $0 < a(t) < 1$, for all $t \geq t_0$ and $b : [t_0, \infty) \to \mathbf{R}$ an arbitrary given real function for all $t \geq t_0$.

- (H_2^2) Let $p : [t_0, \infty) \to \mathbf{R}$ be a given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \leq t - \delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p(t) = \infty$.

- (H_3^2) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$, which is bounded on the initial interval $[t_{-1}, t_0)$, and such that

$$|b(t)|\rho(t-1) \leq (1 - a(t))\rho(p(t)) \quad \text{for all} \quad t \geq t_0, \tag{5.4}$$

  where functions a and b are as in (H_1^2).

- (H_4^2) There exists a real number R such that

$$\prod_{n=0}^{j}(1+R_n) \le R, \tag{5.5}$$

for all positive integers j, where the numbers R_n are defined by (5.3).

The next theorem is the main result of this section and gives an asymptotic estimate for the solutions of Equation (5.1). It is without proof because it is a direct consequence of Theorem 4.1 for $n = 1$, $a_1(t) = a(t)$ and $b_{11} = b(t)$. We emphasize it because of its importance in the theory of difference equations with continuous time and in virtue of its interesting consequences which give sharper results than the system case.

Theorem 5.1. *Suppose that conditions (H_1^2), (H_2^2), (H_3^2) and (H_4^2) hold. Let $x = x^\varphi$ be the solution of the initial value problem (5.1) and (5.2) with bounded function φ in (5.2). Then*

$$|x(t)| \le \frac{M_0 R}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

Remark 5.2. *In Theorem 5.1, let the function ρ be monotone increasing. Then the sequence $\{R_n\}$, defined by (5.3), has only positive members and the assumption*

$$\prod_{n=0}^{\infty}(1+R_n) < \infty$$

implies the existence of a real number R that satisfies the condition (5.5). If the function ρ is monotone decreasing, then the condition (5.5) is satisfied with $R = 1$.

Zhou and Yu in Theorems 2.7 and 2.8 obtained for estimating function the exponential function for the case when the lag function is between two constant delays. Applying Theorem 5.1, by finding an appropriate estimating function ρ, we can determine the rate of convergence of the solutions of generalized difference equations for different types of the lag function. We will provide two examples.
If the lag function p is such that $p_1 t \le p(t) \le p_2 t$, for real numbers $0 < p_1 \le p_2 < 1$, then the estimating function is a power function, namely $\rho(t) = t^k$.

Corollary 5.3. *Suppose that condition (H_1^2) holds. For given real numbers p_1, p_2 and t_0 such that $0 < p_1 \le p_2 < 1$ and $t_0 > p_1/(1-p_1)$, let p be given real function such that $p_1 t \le p(t) \le p_2 t$ for all $t \ge t_0$. Suppose that there exist real numbers Q and α such that $0 < Q \le 1$, $0 < \alpha < 1$,*

$$|b(t)| \le Q(1-a(t)), \quad \alpha \le 1-a(t) \quad \text{for} \quad \text{all} \quad t \ge t_0$$

and

$$\log\frac{1}{Q}\left(\log\frac{1}{p_1}-\log\frac{1}{p_2}\right)<\log\frac{1}{p_1}\log\frac{1}{p_2}.$$

Let $x=x^{\varphi}$ be a solution of the initial value problem (5.1) and (5.2), and let

$$k=\frac{\log Q}{\log p_1},\quad \rho(t)=\left(t-\frac{p_1}{1-p_1}\right)^k.$$

Then

$$|x(t)|\le\frac{C}{\rho(t)}\quad \textit{for}\quad \textit{all}\quad t\ge t_0,$$

where

$$C=\sup_{t_{-1}\le t<t_0}\{t^k|\varphi(t)|\}\prod_{n=0}^{\infty}\left(1+\frac{kt_0^k(1-p_1)^{k+1}}{\alpha p_1^k(t_0(1-p_1)-p_2^n)^{k+1}}\left(\frac{p_2^{k+1}}{p_1^k}\right)^n\right).$$

Proof. Set

$$\rho(t)=\left(t-\frac{p_1}{1-p_1}\right)^k.$$

The relations $p(t_{n+1})=t_n$ and $p_1t\le p(t)\le p_2t$ imply that

$$\frac{t_0}{p_2^n}\le t_n\le\frac{t_0}{p_1^n}\quad\text{for}\quad\text{all}\quad n=1,2,...$$

Since $Q=p_1^k$, it follows that

$$|b(t)|\rho(t-1)\le Q(1-a(t))(t-\frac{1}{1-p_1})^k\le$$

$$\le\frac{Q}{p_1^k}(1-a(t))p_1^k(t-\frac{1}{1-p_1})^k=(1-a(t))(p_1t-\frac{p_1}{1-p_1})^k\le$$

$$\le(1-a(t))(p(t)-\frac{p_1}{1-p_1})^k=(1-a(t))\rho(p(t)).$$

Therefore, condition (5.4) is valid. Moreover

$$\begin{aligned}
R_n &= \sup_{t_n\le t<t_{n+1}}\Bigg\{(t-\frac{p_1}{1-p_1})^k\times\\
&\quad\times\sum_{\tau=t-k_n(t)}^{t}\frac{(\tau-\frac{p_1}{1-p_1})^k-(\tau-\frac{1}{1-p_1})^k}{(\tau-\frac{p_1}{1-p_1})^k(\tau-\frac{1}{1-p_1})^k}\prod_{\ell=\tau+1}^{t}a(\ell)\Bigg\}\\
&\le \sup_{t_n\le t<t_{n+1}}\left\{t^k(1-\alpha)^t\sum_{\tau=t-k_n(t)}^{t}\frac{k}{(\tau-\frac{1}{1-p_1})^{k+1}}\left(\frac{1}{1-\alpha}\right)^{\tau}\right\}\\
&\le \sup_{t_n\le t<t_{n+1}}\left\{\frac{kt^k}{\alpha(t-k_n(t)-\frac{1}{1-p_1})^{k+1}}\right\}\\
&\le \frac{kt_0^k(1-p_1)^{k+1}}{\alpha p_1^k(t_0(1-p_1)-p_2^n)^{k+1}}\left(\frac{p_2^{k+1}}{p_1^k}\right)^n.
\end{aligned}$$

Then, Theorem 5.1 and Remark 5.2 imply the assertion.

Remark 5.4. *We can prove similarly the above result by choosing* $\rho(t) = t^k$. *Then the statement of Theorem 5.1 holds for* $t \geq t_0/p_1$ *that does not disturb the asymptotic behavior of solutions but the comparableness with the function* ρ *is clearer and the rate of the convergence is better.*

Example 5.1. Let $A(t) = 1/(1+t)^2$, $B(t) = (1 - 1/(1+t)^2)/2$, $t_0 = 2$, $p_1 = p_2 = 0.5$. Then, conditions of the Corollary 5.3 are satisfied with $k = 1$, $\rho(t) = t$ and $C = 26$. See Figure 6.

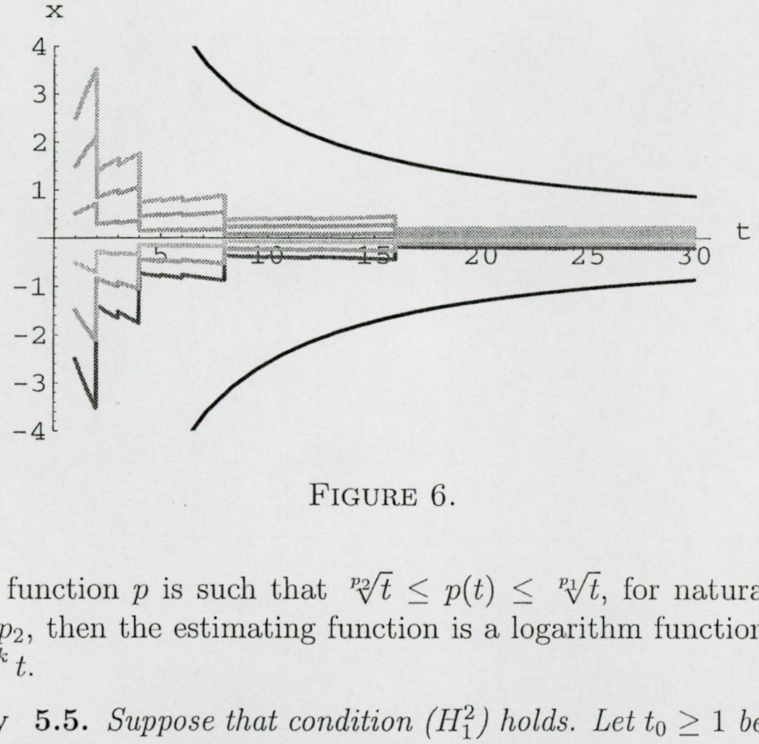

FIGURE 6.

If the lag function p is such that $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \leq p_2$, then the estimating function is a logarithm function. Namely, $\rho(t) = \log^k t$.

Corollary 5.5. *Suppose that condition* (H_1^2) *holds. Let* $t_0 \geq 1$ *be given real number,* p_1, p_2 *be given natural numbers such that* $1 < p_1 \leq p_2$. *Let* p *be given real function such that* $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$ *for all* $t \geq t_0$. *Suppose that there exist real numbers* Q *and* α *such that* $0 < Q \leq 1$, $0 < \alpha < 1$ *and*

$$|b(t)| \leq Q(1 - a(t)), \quad \alpha \leq 1 - a(t) \quad for \quad all \quad t \geq t_0.$$

Let $x = x^\varphi$ *be a solution of the initial value problem (5.1) and (5.2) and let*

$$k = -\frac{\log Q}{\log p_2}.$$

Then

$$|x(t)| \leq \frac{C}{\log^k t} \quad for \quad all \quad t \geq t_0,$$

where

$$C = \sup_{t_{-1} \le t < t_0} \left\{ \log^k t |\varphi(t)| \right\} \prod_{n=0}^{\infty} \left(1 + \frac{k p_2^k p_2^{nk} \log^k t_0}{\alpha(t_0^{p_1^n} - 1) \log^{k+1}(t_0^{p_1^n} - 1)} \right).$$

Proof. The relations

$$p(t_{n+1}) = t_n, \quad \sqrt[p_2]{t} \le p(t) \le \sqrt[p_1]{t}$$

imply that

$$t_0^{p_1^n} \le t_n \le t_0^{p_2^n} \quad \text{for} \quad \text{all} \quad n = 0, 1, 2, \ldots$$

Let $\rho(t) = \log^k t$. Since $Qp_2^k = 1$ and

$$|b(t)|\rho(t-1) \le Q(1 - a(t)) \log^k(t-1) <$$

$$< Q(1 - a(t)) \log^k t = Q(1 - a(t)) \log^k (\sqrt[p_2]{t})^{p_2} =$$

$$= Qp_2^k (1 - a(t)) \log^k \sqrt[p_2]{t} \le (1 - a(t))\rho(p(t)),$$

condition (5.4) is valid. Moreover,

$$\begin{aligned} R_n &= \sup_{t_n \le t < t_{n+1}} \left\{ \log^k t \sum_{\tau = t - k_n(t)}^{t} \frac{\log^k \tau - \log^k(\tau - 1)}{\log^k \tau \log^k(\tau - 1)} \prod_{\ell = \tau + 1}^{t} A(\ell) \right\} \\ &\le \sup_{t_n \le t < t_{n+1}} \left\{ \log^k t (1 - \alpha)^t \sum_{\tau = t - k_n(t)}^{t} \frac{k}{(\tau - 1) \log^{k+1}(\tau - 1)} \left(\frac{1}{1 - \alpha} \right)^{\tau} \right\} \\ &\le \sup_{t_n \le t < t_{n+1}} \left\{ \frac{k \log^k t}{\alpha(t - k_n(t) - 1) \log^{k+1}(t - k_n(t) - 1)} \right\} \\ &\le \frac{k p_2^k p_2^{nk} \log^k t_0}{\alpha(t_0^{p_1^n} - 1) \log^{k+1}(t_0^{p_1^n} - 1)}. \end{aligned}$$

So, Theorem 5.1 and Remark 5.2 imply the assertion.

Remark 5.6. *If the initial function φ defined in (5.2) and the function b are positive, then the solution x^{φ} of the initial value problem (5.1) and (5.2) is nonnegative for all $t \ge t_0$.*

Example 5.2. Let $A(t) = 1/(1+t)^2$, $B(t) = (1 - 1/(1+t)^2)/2$, $t_0 = 2$, $p_1 = p_2 = 2$. Then, conditions of the Corollary 5.5 are satisfied with $k = 1$, $\rho(t) = \log t$ and $C = 5$. See Figure 7.

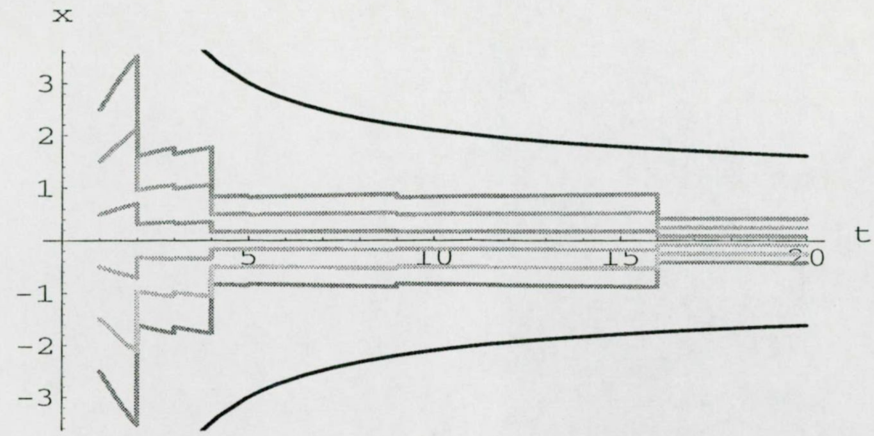

FIGURE 7.

For given scalar functions $a, \rho : [t_0, \infty) \to \mathbf{R}$, for the initial function φ and nonnegative numbers n we define the numbers

$$r_n := \inf_{t_n \le t < t_{n+1}} \left\{ \rho(t) \sum_{\tau = t - k_n(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{\ell = \tau + 1}^{t} a(\ell) \right\} \tag{5.6}$$

and

$$N_0 := \inf_{t_{-1} \le t < t_0} \rho(t)|\varphi(t)|.$$

We assume that:

- (H_5^2) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$ bounded on the initial interval $[t_{-1}, t_0)$ such that

$$b(t)\rho(t-1) \ge (1 - a(t))\rho(p(t)) \quad \text{for all} \quad t \ge t_0. \tag{5.7}$$

- (H_6^2) There exists a positive real number r such that

$$\prod_{n=0}^{j} (1 + r_n) \ge r, \tag{5.8}$$

  for all positive integers j, where the numbers r_n, for $n = 0, 1, \dots$, are defined by (5.6).

The next result gives an asymptotic lower bound estimate for the positive solutions of (5.1). We emphasize that in the system case we can not prove a similar result because of the construction of the available tools.

Theorem 5.7. *Suppose that conditions (H_1^2), (H_2^2), (H_5^2) and (H_6^2) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (5.1) and (5.2) with positive bounded function φ in (5.2). Then*

$$x(t) \geq \frac{N_0 r}{\rho(t)} \quad \textit{for} \quad \textit{all} \quad t \geq t_0.$$

Proof. Introduce the function $y(t) := x(t)\rho(t)$. Then the function $y(t)$ satisfies the equation

$$\frac{y(t)}{\rho(t)} = a(t)\frac{y(t-1)}{\rho(t-1)} + b(t)\frac{y(p(t))}{\rho(p(t))}.$$

Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then Equation (5.1) is equivalent to

$$\Delta_\tau \left\{ \frac{y(\tau-1)}{\rho(\tau-1)} \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{a(\ell)} \right\} = b(\tau)\frac{y(p(\tau))}{\rho(p(\tau))} \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{a(\ell)}.$$

Summing up both sides of this equality from $t - k_n(t)$ to t, we obtain

$$\begin{aligned} y(t) &= \frac{\rho(t)}{\rho(t-k_n(t)-1)} y(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \\ &\quad + \rho(t) \sum_{\tau=t-k_n(t)}^{t} b(\tau)\frac{y(p(\tau))}{\rho(p(\tau))} \prod_{\ell=\tau+1}^{t} a(\ell). \end{aligned} \tag{5.9}$$

Using hypothesis (5.7), it is easy to see that the function $b(t)$ is positive, by virtue of Remark 5.6, and it implies that the function $y(t)$ is also positive. Therefore, the following estimate can be obtained:

$$\begin{aligned} y(t) &\geq \frac{\rho(t)}{\rho(t-k_n(t)-1)} y(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \\ &\quad + \rho(t) \sum_{\tau=t-k_n(t)}^{t} \frac{1-a(\tau)}{\rho(\tau-1)} y(p(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell). \end{aligned}$$

Define

$$\nu_n := \inf_{t_{n-1} \leq t < t_n} y(t) \quad \text{and} \quad N_n := \min\{\nu_0, \nu_1, ..., \nu_n\}$$

for $n = 0, 1, 2, \ldots$. Since $y(p(\tau)) \geq N_n$ for $\tau \in T_n(t)$, $t_n \leq t < t_{n+1}$ and $n = 0, 1, 2, \ldots$, by using the summation by parts formula, it follows that

$$\begin{aligned} |y(t)| &\geq N_n \left(\frac{\rho(t)}{\rho(t-k_n(t)-1)} \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \right. \\ &\quad \left. + \rho(t) \sum_{\tau=t-k_n(t)}^{t} \frac{1-a(\tau)}{\rho(\tau-1)} \prod_{\ell=\tau+1}^{t} a(\ell) \right) \end{aligned}$$

$$
\begin{aligned}
&= N_n\left(\frac{\rho(t)}{\rho(t-k_n(t)-1)}\prod_{\ell=t-k_n(t)}^{t}a(\ell)+\right.\\
&\quad\left.+\rho(t)\sum_{\tau=t-k_n(t)}^{t}\frac{1}{\rho(\tau-1)}\Delta_\tau\left(\prod_{\ell=\tau}^{t}a(\ell)\right)\right)\\
&= N_n\left(\frac{\rho(t)}{\rho(t-k_n(t)-1)}\prod_{\ell=t-k_n(t)}^{t}a(\ell)+\right.\\
&\quad+\rho(t)\frac{1}{\rho(\tau-1)}\prod_{\ell=\tau}^{t}a(\ell)\Big|_{t-k_n(t)}^{t+1}+\\
&\quad\left.+\rho(t)\sum_{\tau=t-k_n(t)}^{t}\frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)}\prod_{\ell=\tau+1}^{t}a(\ell)\right)\\
&= N_n\left(1+\rho(t)\sum_{\tau=t-k_n(t)}^{t}\frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)}\prod_{\ell=\tau+1}^{t}a(\ell)\right)\\
&\geq N_n(1+r_n).
\end{aligned}
$$

From the above inequality follows that

$$N_{n+1}\geq N_n(1+r_n)\geq N_0\prod_{j=0}^{n}(1+r_j).$$

Therefore

$$y(t)\geq N_0 r,$$

and the proof is complete.

Remark 5.8. *In Theorem 5.7, let the function ρ be monotone decreasing. Then the members of the sequence $\{r_n\}$ defined by (5.6) are all negative, and the assumption*

$$\prod_{n=0}^{\infty}(1+r_n)>0$$

implies the existence of a real number $r>0$ which satisfies the condition (5.8). If the function $\rho(t)$ is monotone increasing, then the condition (5.8) is satisfied with $r=1$.

We shall assume the following.

- (H_7^2) There exists a real function $\rho:[t_{-1},\infty)\to(0,\infty)$, which is bounded on the initial interval $[t_{-1},t_0)$, and such that

$$b(t)\rho(t-1)=(1-a(t))\rho(p(t))\quad\text{for}\quad\text{all}\quad t\geq t_0.$$

Combining Theorems 5.1 and 5.7 we can estimate the positive solutions of Equation (5.1) from both sides and obtain the next corollary which is an immediate consequence of the mentioned theorems.

Theorem 5.9. *Suppose that conditions (H_1^2), (H_2^2), (H_4^2), (H_6^2) and (H_7^2) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (5.1) and (5.2) with positive bounded function φ in (5.2). Then*

$$0 < \frac{N_0 r}{\rho(t)} \le x(t) \le \frac{M_0 R}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

Remark 5.10. *In Corollary 5.3, let $p_1 = p_2 = p$ and $b(t) = Q(1 - a(t))$. Then, with a positive initial function in (5.2) and*

$$\rho(t) = (t - p/(1-p))^k, \quad k = \frac{\log Q}{\log p},$$

the conditions of Theorem 5.9 are satisfied. Moreover, the next sharp estimates can be obtained for the solution $x(t)$ of the initial value problem (5.1) and (5.2):

$$0 < \frac{N_0}{\rho(t)} \le x(t) \le \frac{C}{\rho(t)} \quad \text{for} \quad \text{all} \quad t \ge t_0,$$

where

$$N_0 = \inf_{t_{-1} \le t < t_0} \rho(t)\varphi(t)$$

and

$$C = \sup_{t_{-1} \le t < t_0} \{t^k |\varphi(t)|\} \prod_{n=0}^{\infty} \left(1 + \frac{k t_0^k (1-p)^{k+1} p^n}{\alpha p^k (t_0(1-p) - p^n)^{k+1}}\right).$$

Remark 5.11. *In Corollary 5.5 let $p_1 = p_2 = p$ and $b(t) = Q(1 - a(t))$. Let $\rho(t)$ be a solution of the functional equation*

$$\rho(t-1) = Q\rho(\sqrt[p]{t}).$$

Then, with a positive initial function in (5.2), the conditions of Theorem 5.9 are satisfied, and for the solution $x(t)$ of the initial value problem (5.1) and (5.2) similar sharp estimates can be obtained as in Remark 5.10.

5.3. The Characteristic Equation

The characteristic equation is a useful tool in the qualitative analysis of the theory of differential and difference equations. Knowing the solutions or the behavior of solutions of the characteristic equations we can obtain some properties of solutions of the considered differential or difference equations. This method is usually applied in the investigations of the theory of oscillation, asymptotic behavior, stability, etc. Using the solutions of the characteristic equation we can, from a new point of view, describe the asymptotic behavior of solutions of the difference equation. We apply the obtained results for some particular cases.

Let $x = x^{\varphi}$ be a solution of the initial value problem (5.1) and (5.2). Then

$$1 = a(t)\frac{x(t-1)}{x(t)} + b(t)\frac{x(p(t))}{x(t)} \quad \text{for} \quad t \geq t_0.$$

Define the new function

$$\lambda(t) := \frac{x(t-1)}{x(t)} \quad \text{for} \quad t \geq t_0. \tag{5.10}$$

Since, now

$$x(t) = \varphi(t - k_0(t) - 1) \prod_{\ell=t-k_0(t)}^{t} \frac{1}{\lambda(\ell)} \quad \text{for} \quad t \geq t_0,$$

the function λ defined by (5.10) is a solution of the characteristic equation of the form

$$1 - a(t)\lambda(t) =$$

$$= b(t)\frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p(t)-k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)}, \tag{5.11}$$

for $t \geq t_0$. Characteristic equation (5.11) associated with the initial value problem (5.1) and (5.2) is a generalization of the characteristic equation given in [78] and [79] for discrete difference equations.

Assume that:

- (H_8^2) For the function a and b given in (H_1^2) and (H_2^2), there is a real function $\lambda : [t_0, \infty) \to (1, \infty)$ and there is an initial function φ in (5.2) such that

$$|b(t)|\frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p(t)-k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)} \leq$$

$$\leq 1 - a(t)\lambda(t), \quad t \geq t_0. \tag{5.12}$$

In the next result we use the concept of the characteristic equation but it is not necessary to have the solution of characteristic equation (5.11). It is sufficient only to have a solution of Inequality (5.12) that is a much weaker condition.

Theorem 5.12. *Suppose that conditions (H_1^2), (H_2^2) and (H_8^2) hold. Let $x = x^{\varphi}$ be a solution of the initial value problem (5.1) and (5.2). Then*

$$|x(t)| \le \left(\varphi(t-k_0(t)-1) \max_{t_{-1}\le t\le t_0} \lambda(t)\right) \prod_{\ell=t-k_0(t)}^{t} \frac{1}{\lambda(\ell)}, \quad t \ge t_0.$$

Proof. Introduce the transformation

$$y(t) := \frac{x(t)}{\varphi(t-k_0(t)-1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell).$$

Then the function $y(t)$ satisfies the equation

$$\begin{aligned} y(t) &= a(t)\lambda(t)y(t-1) + \\ &+ b(t)\frac{\varphi(p(t)-k_0(p(t))-1)}{\varphi(t-k_0(t)-1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p(t)-k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)}. \end{aligned}$$

Using hypothesis (5.12) we obtain that

$$|y(t)| \le a(t)\lambda(t)|y(t-1)| + (1-a(t)\lambda(t))|y(p(t))|.$$

Let $t \in [t_n, t_{n+1})$, $\tau \in T_n(t)$ and $|y(t)| = u(t)$. Then, the above inequality is equivalent to

$$\begin{aligned} &\Delta_\tau \left(u(\tau-1) \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{a(\ell)\lambda(\ell)}\right) \le \\ &\le (1-a(\tau)\lambda(\tau))u(p(\tau)) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{a(\ell)\lambda(\ell)}. \end{aligned}$$

Summing up both sides of this inequality from $t-k_n(t)$ to t gives that

$$\begin{aligned} u(t) &\le u(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \\ &+ \sum_{\tau=t-k_n(t)}^{t} (1-a(\tau)\lambda(\tau))u(p(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell). \end{aligned}$$

Define

$$\mu_n := \sup_{t_{n-1}\le t<t_n} u(t) \quad \text{and} \quad M_n := \max\{\mu_0, \mu_1, ..., \mu_n\} \tag{5.13}$$

for $n = 0, 1, 2, ...$ The summation by parts formula imply that

$$\begin{aligned} u(t) &\le M_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \sum_{\tau=t-k_n(t)}^{t} (1-a(\tau)\lambda(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell)\right) \\ &= M_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \sum_{\tau=t-k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell=\tau}^{t} a(\ell)\lambda(\ell)\right)\right) \\ &= M_n. \end{aligned}$$

The above inequality implies that

$$M_{n+1} \leq M_n \quad \text{and} \quad u(t) = |y(t)| \leq M_0,$$

and the assertion of the theorem is valid.

The following corollary represents an asymptotic estimate for the solutions of Equation (5.1) and gives information about the rate of the convergence of solutions to particular cases such as $t - p_2 \leq p(t) \leq t - p_1$, for real numbers $1 < p_1 \leq p_2$. We obtain that the solutions are exponentially decaying as in the Theorems 2.7 and 2.7 given by Zhou and Yu.

Corollary 5.13. *Suppose that condition (H_1^2) holds. Let p_1, p_2 and t_0 be given real numbers such that $1 \leq p_1 < p_2$ and $t_0 \geq p_1$. Let $p(t) = t - \delta(t)$, with given real function δ such that $p_1 \leq \delta(t) \leq p_2$ for all $t \geq t_0$. Suppose that there exists a real number $\lambda > 1$ such that*

$$|b(t)| \leq \frac{1 - \lambda a(t)}{\lambda^{p_2}} \quad \textit{for} \quad \textit{all} \quad t \geq t_0. \tag{5.14}$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (5.1) and (5.2), and

$$M_0 = \sup_{t_{-1} \leq t < t_0} \{\lambda^t |\varphi(t)|\}.$$

Then

$$|x(t)| \leq \frac{M_0}{\lambda^t} \quad \textit{for} \quad \textit{all} \quad t \geq t_0.$$

Proof. The relations

$$t_{n+1} - \delta(t_{n+1}) = t_n, \quad t - p_2 \leq t - \delta(t) \leq t - p_1$$

imply that

$$t_0 + np_1 \leq t_n \leq t_0 + np_2 \quad \text{for} \quad n = 1, 2, \ldots$$

Introduce the transformation $y(t) := x(t)\lambda^t$. Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then Equation (5.1) is equivalent to

$$\Delta_\tau \left(y(\tau - 1) \prod_{\ell = t - k_n(t)}^{\tau - 1} \frac{1}{\lambda a(\ell)} \right) = b(\tau) \lambda^{\delta(\tau)} y(\tau - \delta(\tau)) \prod_{\ell = t - k_n(t)}^{\tau} \frac{1}{\lambda a(\ell)}.$$

Summing up both sides of this equality from $t - k_n(t)$ to t gives that

$$\begin{aligned} y(t) &= y(t - k_n(t) - 1) \prod_{\ell = t - k_n(t)}^{t} \lambda a(\ell) + \\ &\quad + \sum_{\tau = t - k_n(t)}^{t} b(\tau) \lambda^{\delta(\tau)} y(\tau - \delta(\tau)) \prod_{\ell = \tau + 1}^{t} \lambda a(\ell). \end{aligned}$$

Define

$$\mu_n := \sup_{t_{n-1}\le t<t_n} |y(t)| \quad \text{and} \quad M_n := \max\{\mu_0, \mu_1, ..., \mu_n\} \tag{5.15}$$

for $n = 0, 1, 2, ...$ Since $|y(p(\tau))| \le M_n$ for $\tau \in T_n(t)$ and $t_n \le t < t_{n+1}$, by using the summation by parts formula, it follows that

$$\begin{aligned} |y(t)| &\le M_n \left(\prod_{\ell=t-k_n(t)}^{t} \lambda a(\ell) + \sum_{\tau=t-k_n(t)}^{t} (1-\lambda a(\tau)) \prod_{\ell=\tau+1}^{t} \lambda a(\ell) \right) \\ &= M_n \left(\prod_{\ell=t-k_n(t)}^{t} \lambda a(\ell) + \sum_{\tau=t-k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell=\tau}^{t} \lambda a(\ell) \right) \right) \\ &= M_n. \end{aligned}$$

The above inequality implies that $M_{n+1} \le M_n$ and $|y(t)| \le M_0$. Therefore

$$|x(t)| \le \frac{M_0}{\lambda^t} \quad \text{for all} \quad t \ge t_0$$

and the proof is complete.

Example 5.3. Let $A(t) = 1/4$, $B(t) = 1/8$, $t_0 = 2$, $p_1 = p_2 = 2$. Then, conditions of the Corollary 5.5 are satisfied with $\lambda = 2$, $\rho(t) = 2^t$ and $M_0 = 102^{0.5}$. See Figure 8.

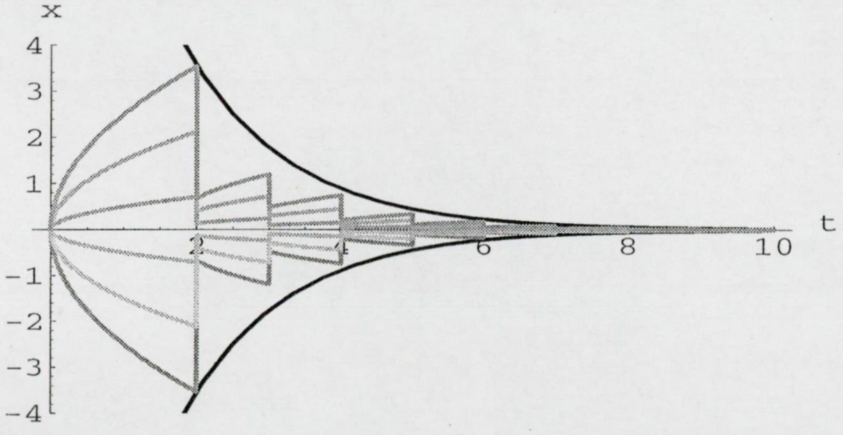

FIGURE 8.

5.4. Generalizations

In this subsection we generalize previous results to the generalized difference equations with several delays and show the usefulness of the new results to the classical particular cases.

Assume that $t_0 > 0$ is a given real number, $a, b_i : [t_0, \infty) \to \mathbf{R}$ are given real functions for $i = 1, 2, ..., m$. Let $p_i : [t_0, \infty) \to \mathbf{R}$ be given functions such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p_i(t) \le t - \delta$ for every $t \in [t_0, T]$, $i = 1, 2, ..., m$, and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. Consider the difference equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)). \tag{5.16}$$

Set

$$t_{-1} = \min\left\{t_0 - 1, \min_{1\le i\le m} \{\inf\{p_i(s), s \ge t_0\}\}\right\},$$

$$t_n = \min_{1\le i\le m} \inf\{s : p_i(s) > t_{n-1}\} \quad \text{for all} \quad n = 1, 2, ...$$

Then

$$p_i(t) \in \bigcup_{j=0}^{n} [t_{j-1}, t_j) \quad \text{for} \quad t_n \le t < t_{n+1},$$

$i = 1, 2, ..., m$ and $n = 0, 1, 2, ...$
For given scalar functions $a, \rho : [t_0, \infty) \to \mathbf{R}$ define the value

$$M_0 := \sup_{t_{-1}\le t<t_0} \rho(t)|\varphi(t)|.$$

We shall need the following hypotheses.

- (H_9^2) $a, b_i : [t_0, \infty) \to \mathbf{R}$ are real functions $i = 1, 2, ..., m$, such that $0 < a(t) < 1$, for all $t \ge t_0$.
- (H_{10}^2) $p_i : [t_0, \infty) \to \mathbf{R}$ are real functions such that for every $T > t_0$ there exists a $\delta > 0$ such that $p_i(t) \le t - \delta$ for every $t \in [t_0, T]$, $i = 1, 2, ..., m$.
- (H_{11}^2) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$, which is bounded on the initial interval $[t_{-1}, t_0)$, and such that

$$\sum_{i=1}^{m} \frac{|b_i(t)|}{\rho(p_i(t))} \le \frac{(1 - a(t))}{\rho(t-1)} \quad \text{for all} \quad t \ge t_0, \tag{5.17}$$

  where functions a and b_i, $i = 1, 2, ..., m$, are as in (H_9^2).

The next result gives an asymptotic estimate for the solutions of the equation (5.16) and generalizes the result obtained in Theorem 5.1.

Theorem 5.14. *Suppose that conditions (H_4^2), (H_9^2), (H_{10}^2) and (H_{11}^2) hold. Let $x = x^\varphi$ be the solution of the initial value problem (5.16) and (5.2) with bounded function φ in (5.2). Then*

$$|x(t)| \leq \frac{M_0 R}{\rho(t)} \quad \textit{for} \quad \textit{all} \quad t \geq t_0.$$

Proof. The function $y(t) = x(t)\rho(t)$ satisfies the equality

$$\begin{aligned} y(t) &= \frac{\rho(t)y(t-k_n(t)-1)}{\rho(t-k_n(t)-1)} \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \\ &+\rho(t) \sum_{\tau=t-k_n(t)}^{t} \sum_{i=1}^{m} b_i(\tau)\frac{y(p_i(\tau))}{\rho(p_i(\tau))} \prod_{\ell=\tau+1}^{t} a(\ell). \end{aligned}$$

Using the notation defined in (5.15) and applying hypothesis (5.17), the following estimate holds for $t_n \leq t < t_{n+1}$:

$$\begin{aligned} |y(t)| &\leq M_n \left(\frac{\rho(t)}{\rho(t-k_n(t)-1)} \prod_{\ell=t-k_n(t)}^{t} a(\ell) + \right. \\ &\left. + \rho(t) \sum_{\tau=t-k_n(t)}^{t} \frac{1-a(\tau)}{\rho(\tau-1))} \prod_{\ell=\tau+1}^{t} a(\ell) \right) \end{aligned}$$

and the same consequences as in the proof of Theorem 5.1 imply the assertion.

In the next two corollaries we apply Theorem 5.14 to the particular cases such as $p_i(t) = (1/p_i)t$ for $i = 1, 2, ..., m$ and $p_i(t) = \sqrt[p_i]{t}$ for $i = 1, 2, ..., m$, where $1 < p_1 < p_2 < ... < p_m$ are natural numbers.

Corollary 5.15. *Suppose that condition (H_9^2) holds. For given real number $t_0 \geq 1$ and for given natural numbers p_1, p_2,...,p_m such that $1 < p_1 < p_2 < ... < p_m$, let $p_i(t) = (1/p_i)t$ for all $t \geq t_0$, $i = 1, 2, ..., m$. Suppose that there exist real numbers Q and α such that $0 < Q \leq 1$, $0 < \alpha < 1$ and*

$$\sum_{i=1}^{m} |b_i(t)| \leq Q(1-a(t)), \quad \alpha \leq 1 - a(t) \quad \textit{for} \quad \textit{all} \quad t \geq t_0.$$

Let $x = x^\varphi$ be a solution of the initial value problem (5.16) and (5.2) and let

$$k = -\frac{\log Q}{\log p_m}.$$

Then

$$|x(t)| \leq \frac{C}{t^k} \quad \textit{for} \quad \textit{all} \quad t \geq t_0,$$

where

$$C = \sup_{t_{-1} \leq t < t_0} \left\{ |\varphi(t)| t^k \right\} \prod_{n=0}^{\infty} \left(1 + \frac{k p_1^k p_1^{nk} t_0^k}{\alpha(p_1^n t_0 - 1)^{k+1}} \right).$$

Proof. Set

$$t_{-1} = \min\{t_0 - 1, \frac{t_0}{p_m}\}, \quad t_n = p_1^n t_0, \quad \text{for all} \quad n = 1, 2, \ldots$$

Let $\rho(t) = t^k$. Since $Qp_m^k = 1$ and

$$\sum_{i=1}^{m} \frac{|b_i(t)|}{\rho(\frac{1}{p_i}t)} = \sum_{i=1}^{m} \frac{p_i^k |b_i(t)|}{t^k} \le \frac{p_m^k}{t^k} \sum_{i=1}^{m} |b_i(t)| \le$$

$$\le \frac{p_m^k Q(1-a(t))}{t^k} \le \frac{1-a(t)}{(t-1)^k} = \frac{1-a(t)}{\rho(t-1)},$$

condition (5.17) is valid. Moreover,

$$\begin{aligned} R_n &\le \sup_{t_n \le t < t_{n+1}} \left\{ \frac{kt^k}{\alpha(t - k_n(t) - 1)^{k+1}} \right\} \\ &\le \frac{k p_1^k p_1^{nk} t_0^k}{\alpha(p_1^n t_0 - 1)^{k+1}}. \end{aligned}$$

So, Theorem 5.14 implies the assertion.

Corollary 5.16. *Suppose that condition (H_9^2) holds. For given real number $t_0 \ge 1$ and for given natural numbers $p_1, p_2, \ldots, p_m$ such that $1 < p_1 < p_2 < \ldots < p_m$, let $p_i(t) = \sqrt[p_i]{t}$ for all $t \ge t_0$, $i = 1, 2, \ldots, m$. Suppose that there exist real numbers Q and α such that $0 < Q \le 1$, $0 < \alpha < 1$ and*

$$\sum_{i=1}^{m} |b_i(t)| \le Q(1 - a(t)), \quad \alpha \le 1 - a(t) \quad \textit{for all} \quad t \ge t_0.$$

Let $x = x^\varphi$ be a solution of the initial value problem (5.16) and (5.2) and let

$$k = -\frac{\log Q}{\log p_m}.$$

Then

$$|x(t)| \le \frac{C}{\log^k t} \quad \textit{for all} \quad t \ge t_0,$$

where

$$C = \sup_{t_{-1} \le t < t_0} \left\{ |\varphi(t)| t^k \right\} \prod_{n=0}^{\infty} \left(1 + \frac{k p_1^k p_1^{nk} \log^k t_0}{\alpha(t_0^{p_1^n} - 1) \log^{k+1}(t_0^{p_1^n} - 1)} \right).$$

Proof. Set

$$t_{-1} = \min\{t_0 - 1, \sqrt[p_m]{t}\}, \quad t_n = t_0^{p_1^n}, \quad \text{for all} \quad n = 1, 2, \ldots$$

Let $\rho(t) = \log^k t$. Since $Qp_m^k = 1$ and

$$\sum_{i=1}^{m} \frac{|b_i(t)|}{\rho(\sqrt[p_i]{t})} = \sum_{i=1}^{m} \frac{p_i^k |b_i(t)|}{\log^k t} \le \frac{p_m^k}{\log^k t} \sum_{i=1}^{m} |b_i(t)| \le$$

$$\le \frac{p_m^k Q(1-a(t))}{\log^k t} = \frac{1-a(t)}{\log^k t} \le \frac{1-a(t)}{\log^k (t-1)} = \frac{1-a(t)}{\rho(t-1)},$$

condition (5.17) is valid. Moreover,

$$\begin{aligned} R_n &\le \sup_{t_n \le t < t_{n+1}} \left\{ \frac{k \log^k t}{\alpha(t-k_n(t)-1)\log^{k+1}(t-k_n(t)-1)} \right\} \\ &\le \frac{k p_1^k p_1^{nk} \log^k t_0}{\alpha(t_0^{p_1^n}-1)\log^{k+1}(t_0^{p_1^n}-1)}. \end{aligned}$$

So, Theorem 5.14 implies the assertion.

The next result gives an asymptotic estimate of solutions of Equation (5.16) and generalizes the result given in Theorem 5.12.

Theorem 5.17. *Suppose that conditions (H_9^2) and (H_{10}^2) hold. Suppose that there is a real function $\lambda : [t_0, \infty) \to (1, \infty)$ and there is an initial function φ in (5.2) such that*

$$\sum_{i=1}^{m} |b_i(t)| \frac{\varphi(p_i(t) - k_0(p_i(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p_i(t) - k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)} \le$$

$$\le 1 - a(t)\lambda(t) \quad for \quad all \quad t \ge t_0. \tag{5.18}$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (5.16) and (5.2). Then

$$|x(t)| \le \left(\varphi(t - k_0(t) - 1) \max_{t_{-1} \le t \le t_0} \lambda(t) \right) \prod_{\ell = t - k_0(t)}^{t} \frac{1}{\lambda(\ell)} \quad for \quad all \quad t \ge t_0.$$

Proof. Introduce the transformation

$$y(t) := \frac{x(t)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell).$$

Then the function $y(t)$ satisfies the equation

$$\begin{aligned} y(t) &= a(t)\lambda(t)y(t-1) + \\ &\quad + \sum_{i=1}^{m} b_i(t) y(p_i(t)) \frac{\varphi(p_i(t) - k_0(p_i(t)) - 1)}{\varphi(t - k_0(t) - 1)} \times \\ &\quad \times \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p_i(t) - k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)}. \end{aligned}$$

Let $t \in [t_n, t_{n+1})$, $\tau \in T_n(t)$ and $|y(t)| = u(t)$. Therefore, it follows that

$$\begin{aligned} u(t) &\le u(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \\ &+ \sum_{\tau=t-k_n(t)}^{t} \sum_{i=1}^{m} b_i(t)u(p_i(\tau)) \frac{\varphi(p_i(t)-k_0(p_i(t))-1)}{\varphi(t-k_0(t)-1)} \times \\ &\times \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p_i(t)-k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)} \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell). \end{aligned}$$

Using notation (5.13) and hypothesis (5.18), the same argumentation as in Theorem 5.12 completes the proof.

For the initial function φ and for a given real number λ define the value

$$M_0 := \sup_{t_{-1} \le t < t_0} \{\lambda^t |\varphi(t)|\}.$$

The next result is a special case of the previous theorem and generalizes the result given in Corollary 5.13.

Corollary 5.18. *Suppose that condition (H_9^2) holds. For given real number t_0 and for given natural numbers $p_1, p_2, \dots, p_m$ such that $1 \le p_1 < p_2 < \dots < p_m$ and $t_0 \ge p_1$, let $p_i(t) = t - p_i$ for all $t \ge t_0$, $i = 1, 2, \dots, m$. Suppose that there exists a real number $\lambda > 1$ such that*

$$\sum_{i=1}^{m} |b_i(t)|\lambda^{p_i} \le 1 - \lambda a(t) \quad \textit{for} \quad \textit{all} \quad t \ge t_0. \tag{5.19}$$

Let $x = x^\varphi$ be the solution of the initial value problem (5.16) and (5.2). Then

$$|x(t)| \le \frac{M_0}{\lambda^t} \quad \text{for} \quad \text{all} \quad t \ge t_0.$$

Proof. Introduce the transformation $y(t) := x(t)\lambda^t$. Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then, Equation (5.16) is equivalent to

$$\Delta_\tau \left(y(\tau-1) \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{\lambda a(\ell)} \right) = \sum_{i=1}^{m} b_i(\tau)\lambda^{p_i} y(\tau - p_i) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{\lambda a(\ell)}.$$

Summing up both sides of this equality from $t - k_n(t)$ to t gives that

$$\begin{aligned} y(t) &= y(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} \lambda a(\ell) + \\ &+ \sum_{\tau=t-k_n(t)}^{t} \sum_{i=1}^{m} b_i(\tau)\lambda^{p_i} y(\tau-p_i) \prod_{\ell=\tau+1}^{t} \lambda a(\ell). \end{aligned}$$

Using the notation (5.15) and the summation by parts formula, the same argumentation as in Corollary 5.13 completes the theorem.

Remark 5.19. *Consider the difference equations of the form*

$$x(t) = a(t)x(t-h) + b(t)x(p(t))$$

with $h \in \mathbf{R}_+$. *Then, using the transformation* $y(\frac{t}{h}) = \frac{1}{h}x(t)$ *and* $s = \frac{t}{h}$, *we obtain the equation of the form (5.1) with the unknown function* $y(s)$, *and the above results can be applied.*

5.5. Series Representation of Solutions of Scalar Equations with Constant Coefficients

Consider the scalar difference equation with constant coefficients

$$x(t) = ax(t-1) + bx(pt), \tag{5.20}$$

where a, b, p are real numbers such that $0 < a < 1$ and $0 < p < 1$.
If we apply the results given in Subsection 4.6. to Equation (5.20) the form of the functions $y_n(t)$ will be too complicate, not suitable for further investigation. Therefore, to solve Equation (5.20) by this method, we need a computer. But for the analitical solutions of Equation (5.20) we can obtain a series representation form which is an analogue of the well known Dirichlet series solution (2.3) given in [36] for scalar autonomous pantograph equation (2.1). Of course, it is neccessary for the initial function to be analitical.

Theorem 5.20. *Let* $C_0 \neq 0$ *be a given real number. Let* a, b, p *be given real numbers such that* $0 < a < 1$, $0 < p <$ *and* $|b| < 1 - a$. *Then*

$$x(t) = \sum_{n=0}^{\infty} C_0 b^n \prod_{\ell=1}^{n} (1 - a^{1-p^{\ell}})^{-1} a^{p^n t} \tag{5.21}$$

is a series solution of Equation (5.20) on $[t_0, \infty)$.

Proof. Suppose that a solution of equation (5.20) is a series of the form

$$x(t) = \sum_{n=0}^{\infty} C_n \lambda^{p^n t}.$$

Replacing this form in Equation (5.20) we obtain that

$$\sum_{n=0}^{\infty} C_n \lambda^{p^n t} = a \sum_{n=0}^{\infty} C_n \lambda^{p^n (t-1)} + b \sum_{n=0}^{\infty} C_n \lambda^{p^{n+1} t},$$

and therefore,

$$C_0 \lambda^t + \sum_{n=1}^{\infty} C_n \lambda^{p^n t} = \frac{a}{\lambda} C_0 \lambda^t + \sum_{n=1}^{\infty} \frac{a}{\lambda^{p^n}} C_n \lambda^{p^n t} + \sum_{n=1}^{\infty} b C_{n-1} \lambda^{p^n t}.$$

From the above equality follows that

$$C_0\left(1-\frac{a}{\lambda}\right)=0, \quad \text{so} \quad a=\lambda, \quad C_0\neq 0,$$

$$\begin{aligned} C_1 &= C_1a^{1-p}+bC_0,\\ C_2 &= C_2a^{1-p^2}+bC_1,\\ &\ldots\\ C_n &= C_na^{1-p^n}+bC_{n-1},\\ &\ldots \end{aligned}$$

Using mathematical induction we obtain that

$$C_n=\frac{b^nC_0}{(1-a^{1-p})(1-a^{1-p^2})\ldots(1-a^{1-p^n})} \tag{5.22}$$

for all $n=1,2,\ldots$. Then, the series solution of equation (5.20) is of the form (5.21). Therefore, the necessary and sufficient condition for the convergence is $|b|<1-a$.

5.6. Bounded Solutions and an Open Problem

Huang, Yu and Dai in [52], [53] presented conditions when every solution of discrete difference equation

$$x_n-x_{n-1}=-F(x_n)+F(x_{n-k})$$

is bounded and tends to a constant as $t\to\infty$, where k is a positive integer, F is a continuous, increasing real function. In the following we give conditions for the existence of bounded solutions of generalized difference equations. The obtained results are analogue to the parts of results given by Huang, Yu and Dai for the special case of function F.

Assume that $t_0>0$ and $c:[t_0,\infty)\to\mathbf{R}$ is a given real function. Consider the difference equation with continuous time of the form

$$x(t)=(1-c(t))x(t-1)+c(t)x(p(t)). \tag{5.23}$$

Assume that

- (H_{12}^2) $c:[t_0,\infty)\to\mathbf{R}$ is a given real function satisfying $0<c(t)<1$, for all $t\geq t_0$.

Theorem 5.21. *Suppose that conditions (H_2^2) and (H_{12}^2) hold. Let $x = x^\varphi$ be the solution of the initial value problem (5.23) and (5.2), and let*

$$A_n := \sup_{t_{n-1} \le t < t_n} x(t), \quad n = 0, 1, 2, \ldots$$

Then $A_{n+1} \le A_n$ for all nonnegative integer n.

Proof. For all natural numbers n and for the arbitrary number $\epsilon > 0$ there is a time $t \in [t_n, t_{n+1})$ such that $x(t) \ge A_{n+1} - \epsilon$. Then we obtain

$$\begin{aligned} A_{n+1} - \epsilon &\le x(t) \\ &= x(t - k_n(t) - 1) \prod_{t-k_n(t)}^{t} (1 - c(\ell)) + \\ &\quad + \sum_{t-k_n(t)}^{t} c(\tau) x(p(\tau)) \prod_{\ell=\tau+1}^{t} (1 - c(\ell)) \\ &\le A_n \left(\prod_{t-k_n(t)}^{t} (1 - c(\ell)) + \sum_{t-k_n(t)}^{t} c(\tau) \prod_{\ell=\tau+1}^{t} (1 - c(\ell)) \right) \\ &= A_n. \end{aligned}$$

If $\epsilon \to 0$ we will obtain that $A_{n+1} \le A_n$ for all nonnegative integer n.

Theorem 5.22. *Suppose that conditions (H_2^2) and (H_{12}^2) hold. Let $x = x^\varphi$ be the solution of the initial value problem (5.23) and (5.2), and let*

$$B_n := \inf_{t_{n-1} \le t < t_n} x(t), \quad n = 0, 1, 2, \ldots$$

Then $B_{n+1} \ge B_n$ for all nonnegative integer n.

Proof. For all natural numbers n and for the arbitrary number $\epsilon > 0$ there is a time $t \in [t_n, t_{n+1})$ such that $x(t) \le B_{n+1} + \epsilon$. Then we obtain

$$\begin{aligned} B_{n+1} + \epsilon &\ge x(t) \\ &= x(t - k_n(t) - 1) \prod_{t-k_n(t)}^{t} (1 - c(\ell)) + \\ &\quad + \sum_{t-k_n(t)}^{t} c(\tau) x(p(\tau)) \prod_{\ell=\tau+1}^{t} (1 - c(\ell)) \\ &\ge B_n \left(\prod_{t-k_n(t)}^{t} (1 - c(\ell)) + \sum_{t-k_n(t)}^{t} c(\tau) \prod_{\ell=\tau+1}^{t} (1 - c(\ell)) \right) \\ &= B_n. \end{aligned}$$

If $\epsilon \to 0$ we will obtain that $B_{n+1} \ge B_n$ for all nonnegative integer n.

Corollary 5.23. *Suppose that conditions (H_2^2) and (H_{12}^2) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (5.23) and (5.2). Then the function x is bounded and*

$$\lim_{n\to\infty} A_n = \limsup_{t\to\infty} x(t) \quad \textit{and} \quad \lim_{n\to\infty} B_n = \liminf_{t\to\infty} x(t).$$

Now, we formulate an open problem for the bounded solutions of Equation (5.23).

Open Problem 5.24. *Suppose that conditions (H_2^2) and (H_{12}^2) hold. We know from the previous theorem that all solutions of the initial value problem (5.23) and (5.2) are bounded. The question is whether the bounded solutions tend to a constant as $t \to \infty$ or exist solutions that do not have this behavior.*

Similar problems for differential equations have been studied by many authors. Consider the equation

$$x'(t) = \beta(t)[x(t) - x(t - \tau(t))], \tag{5.24}$$

where τ and β are positive continuous real functions for $t \geq t_0$, $t - \tau(t)$ is an increasing function for $t \geq t_0 - \tau(t_0)$, $\tau(t) \leq \tau$ for $t \geq t_0 - \tau(t_0)$, $0 < \tau$ =const. Differential equation (5.24) was investigated in papers by Atkinson and Haddock [7], by Diblik [24], [25] and in the references therein. The authors developed conditions which ensure that all solutions of Equation (5.24) are asymptotically constant as $t \to \infty$. Makay and Terjéki in [69] proved the existence of a solution which do not tend to a constant, but it is asymptotically logarithmically periodic.

Krisztin in [59] considered the discrete difference equation

$$y(n+1) = y(n) - g(y(n)) + g(y(n-k)),$$

where k is a positive integer, g and $u - g(u)$ are strictly increasing, continuous real functions. The author obtained that under the considered conditions any bounded solution of the above discrete difference equation is asymptotically $(k+1)-$periodic, i.e. exist solutions that do not tend to a constants as $t \to \infty$.

6. The Discrete Case

6.1. Introduction

Assume that n_0 is a positive integer, $\{a_n\}$, $\{b_n\}$ are given sequences of real numbers and $\{p_n\}$ is a given sequence of natural numbers for $n = n_0, n_0+1, \ldots$ Consider the discrete difference equation of the form

$$x_{n+1} = (1 - a_n)x_n + b_n x_{p_n}, \tag{6.1}$$

where x_n is an unknown real sequence for $n = n_0, n_0 + 1, \ldots$
If in Equation (5.1) for all positive integer n, $t_0 = n_0$ and $t \in (n, n+1]$ we put $a(t) = 1 - a_n$, $b(t) = b_n$, $p(t) = p_n$ and $x(t) = x_{n+1}$, then Equation (5.1) will be equivalent with Equation (6.1) and we will formulate the results similar to the results of Section 5.. The reason why we formulate the theorems especially for the discrete difference equations is that they form an independent branch and have their special problems and approximating methods for solving differential equations, which are not characteristic of more general functional equations.
Set

$$n_{-1} = \min\{p_\ell : \ell \geq n_0\}.$$

Define

$$n_j = \min\{\ell : p_\ell > n_{j-1}\} \quad \text{for} \quad j = 1, 2, \ldots$$

Then

$$n_j \geq n_{j-1} + 1, \quad \text{so} \quad \lim_{j\to\infty} n_j = \infty,$$

$$\text{and} \quad n_{-1} \leq p_n < n_j \quad \text{for} \quad n_j \leq n < n_{j+1}.$$

Let

$$x_i^a = a_i, \quad a_i \in \mathbf{R}, \quad i = n_{-1},\ n_{-1} + 1, \ldots, n_0 \tag{6.2}$$

be given *initial values.* The initial value problem (6.1) and (6.2) is well posed with the unique solution x_n^a.
The *difference operator* Δ is defined by

$$\Delta z_i = z_{i+1} - z_i \quad \text{for the sequence } \{z_i\}.$$

If Δ is used for sequences of the form $z_i = \prod_{\ell=i}^{n-1} A_\ell$, we have

$$\Delta\left(\prod_{\ell=i}^{n-1} A_\ell\right) = \prod_{\ell=i+1}^{n-1} A_\ell - \prod_{\ell=i}^{n-1} A_\ell = (1-A_i)\prod_{\ell=i+1}^{n-1} A_\ell.$$

In the following we formulate the main result associated to the discrete difference equations and apply it to particular cases such as the classical pantograph equation with $p_n = [n/p]$ and equations with $p_n = [\sqrt[p]{n}]$. By finding an appropriate sequence $\{\rho_n\}$ we can also estimate the rate of convergence of the solutions.

6.2. Asymptotic Estimates

The following theorem is cited without proof, because it is a special case of Theorem 5.1. We shall need the following hypotheses.

- (H_1^3) Let $a : \mathbf{N} \to \mathbf{R}$ be a given real sequence satisfying $0 < a_n < 1$, for all $n \geq n_0$ and $b : \mathbf{N} \to \mathbf{R}$ an arbitrary given real sequence for all $n \geq n_0$.
- (H_2^3) Let $p : \mathbf{N} \to \mathbf{N}$ be a given sequence such that $p_n \leq n$ holds for all $n \geq n_0$, and $\lim_{n\to\infty} p_n = \infty$.
- (H_3^3) There exists a real sequence $\rho : [n_{-1}, \infty) \cap \mathbf{N} \to (0, \infty)$ such that

  $$|b_n|\rho_n \leq a_n \rho_{p_n} \quad \text{for} \quad n \geq n_0, \tag{6.3}$$

  where the sequences $\{a_n\}$ and $\{b_n\}$ are as in (H_1^3).
- (H_4^3) There exists a real number R such that

  $$\prod_{\ell=0}^{j}(1+R_\ell) \leq R \quad \text{for all positive integers} \quad j, \tag{6.4}$$

  where the numbers R_n are defined by

  $$R_j := \max_{n_j \leq n < n_{j+1}} \left\{\rho_n \sum_{i=n_j-1}^{n-1} \frac{\Delta\rho_i}{\rho_i\rho_{i+1}} \prod_{\ell=i+1}^{n-1}(1-a_\ell)\right\}, \quad j = 0, 1, 2, \ldots$$

Theorem 6.1. *Suppose that conditions (H_1^3), (H_2^3), (H_3^3) and (H_4^3) hold. Let $x_n = x_n^a$ be the solution of the initial value problem (6.1) and (6.2). Then*

$$|x_n| \leq \frac{M_0 R}{\rho_n} \quad \textit{for all} \quad n \geq n_0.$$

In the following we apply Theorem 6.1 to particular cases such as the classical pantograph equation with $p_n = [n/p]$ and equations with $p_n = [\sqrt[p]{n}]$. By finding an appropriate sequence $\{\rho_n\}$ we can also estimate the rate of the convergence of the solutions.

Corollary 6.2. *Suppose that condition (H_1^3) holds. For given natural number $p > 1$ let $n_0 \geq p$ and $p_n = [n/p]$. Suppose that there exist real numbers Q and α such that*

$$0 < Q \leq 1, \quad 0 < \alpha < 1$$

and

$$|b_n| \leq a_n Q, \quad \alpha \leq a_n, \quad \text{for} \quad n \geq pn_0.$$

Let $x_n = x_n^a$ be a solution of the initial value problem (6.1) and (6.2). Then

$$|x_n| \leq \frac{C}{n^k} \quad \text{for all} \quad n \geq pn_0,$$

where

$$C = \max_{n_{-1} \leq n < n_0} \left\{ n^k |a_n| \right\} \prod_{j=0}^{\infty} \left(1 + \frac{k(p^{j+1} n_0)^k}{\alpha (p^j n_0 - 1)^{k+1}} \right)$$

and

$$k = -\frac{\log Q}{\log(p+1)}.$$

Proof. We have

$$n_{-1} = \left[\frac{n_0}{p} \right], \quad n_j = p^j n_0 \quad \text{for} \quad j = 1, 2, \ldots,$$

Since $Q(p+1)^k = 1$ it follows that

$$|b_n| \rho_n \leq a_n Q n^k = a_n Q [n]^k = a_n Q \left[p \frac{n}{p} \right]^k \leq$$

$$\leq a_n Q (p+1)^k \left[\frac{n}{p} \right]^k = a_n \left[\frac{n}{p} \right]^k = a_n (p_n)^k = a_n \rho_{p_n}$$

for $n \geq pn_0$ and condition (6.3) is valid with $\rho_n = n^k$. Therefore

$$R_j = \max_{n_j \leq n < n_{j+1}} \left\{ n^k \sum_{i=n_j-1}^{n-1} \frac{(i+1)^k - i^k}{i^k (i+1)^k} \prod_{\ell=i+1}^{n-1} (1 - a_\ell) \right\}.$$

Using the mean value theorem it follows that

$$\begin{aligned} R_j &\leq \max_{n_j \leq n < n_{j+1}} \left\{ n^k (1-\alpha)^{n-1} \sum_{i=n_j-1}^{n-1} \frac{k}{i^{k+1}} \left(\frac{1}{1-\alpha} \right)^i \right\} \\ &\leq \max_{n_j \leq n < n_{j+1}} \left\{ \frac{n^k (1-\alpha)^{n-1} k}{(p^j n_0 - 1)^{k+1}} \frac{1-\alpha}{\alpha} \left[\left(\frac{1}{1-\alpha} \right)^n - \left(\frac{1}{1-\alpha} \right)^{p^j n_0 - 1} \right] \right\} \\ &\leq \max_{n_j \leq n < n_{j+1}} \left\{ \frac{k n^k}{\alpha (p^j n_0 - 1)^{k+1}} \right\} \leq \frac{k (p^{j+1} n_0)^k}{\alpha (p^j n_0 - 1)^{k+1}}. \end{aligned}$$

Now, Theorem 6.1 implies the assertion.

Example 6.1. Let $a_n = 0.25 + 1/(1+n)^3$, $b_n = -(0.25 + 1/(1+n)^3)/3$, $n_0 = 2$, $p = 2$. Then, conditions of the Corollary 6.2 are satisfied with $k = 1$ and $\rho_n = n$. See Figure 9.

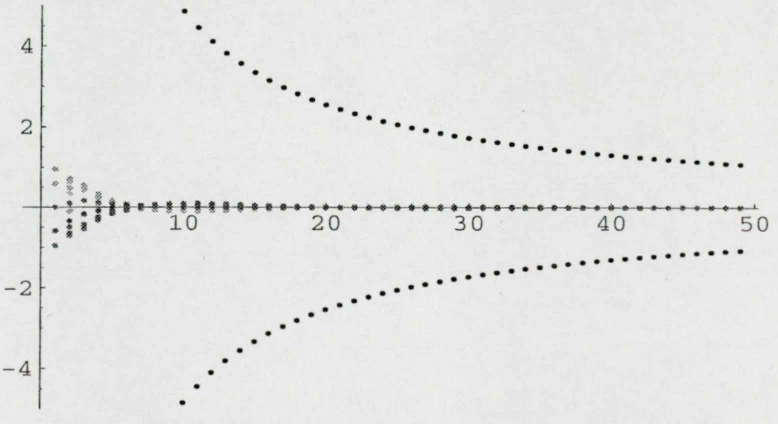

FIGURE 9.

Corollary 6.3. *Suppose that condition (H_1^3) holds. For given natural number $p > 1$ let $n_0 \geq p$ and $p_n = [\sqrt[p]{n}]$. Suppose that there exist real numbers Q and α such that*

$$0 < Q \leq 1, \quad 0 < \alpha < 1$$

and

$$|b_n| \leq a_n Q, \quad \alpha \leq a_n \quad for \quad n \geq n_0^p.$$

Let $x_n = x_n^a$ be a solution of the initial value problem (6.1) and (6.2). Then

$$|x_n| \leq \frac{C}{\log^k n} \quad for \quad all \quad n \geq n_0^p,$$

where

$$C = \max_{n_{-1} \le n < n_0} \left\{ \log^k n |a_n| \right\} \prod_{j=0}^{\infty} \left(1 + \frac{k p^{k(j+1)} \log^k n_0}{\alpha(n_0^{p^j} - 1) \log^{k+1}(n_0^{p^j} - 1)} \right)$$

and $k = -\log Q / \log(p+1)$.

Proof. We have

$$n_{-1} = \left[\sqrt[p]{n_0} \right], \quad n_j = n_0^{p^j} \quad \text{for} \quad j = 1, 2, ...,$$

Since $Q(p+1)^k = 1$ it follows that

$$|b_n| \rho_n \le a_n Q \log^k n \le a_n Q \log^k [\sqrt[p]{n}]^{p+1} =$$

$$= a_n Q (p+1)^k \log^k [\sqrt[p]{n}] = a_n \log^k [\sqrt[p]{n}] = a_n \rho_{p_n}$$

for $n \ge n_0^p$ and condition (6.3) is valid with $\rho_n = \log^k n$. Therefore

$$R_j = \max_{n_j \le n < n_{j+1}} \left\{ \log^k n \sum_{i=n_j-1}^{n-1} \frac{\log^k(i+1) - \log^k i}{\log^k i \log^k(i+1)} \prod_{\ell=i+1}^{n-1} (1 - a_\ell) \right\}.$$

Using the mean value theorem it follows that

$$\begin{aligned} R_j &\le \max_{n_j \le n < n_{j+1}} \left\{ \log^k n \, (1-\alpha)^{n-1} \sum_{i=n_j-1}^{n-1} \frac{k}{i \log^{k+1} i} \left(\frac{1}{1-\alpha} \right)^i \right\} \\ &\le \max_{n_j \le n < n_{j+1}} \left\{ \frac{k \log^k n (1-\alpha)^{n-1}}{(n_j - 1) \log^{k+1}(n_j - 1)} \frac{1-\alpha}{\alpha} \left(\frac{1}{1-\alpha} \right)^n \right\} \\ &\le \max_{n_j \le n < n_{j+1}} \left\{ \frac{k \log^k n}{\alpha(n_j - 1) \log^{k+1}(n_j - 1)} \right\} \\ &\le \frac{k p^{k(j+1)} \log^k n_0}{\alpha(n_0^{p^j} - 1) \log^{k+1}(n_0^{p^j} - 1)}. \end{aligned}$$

Now, Theorem 6.1 implies the assertion.

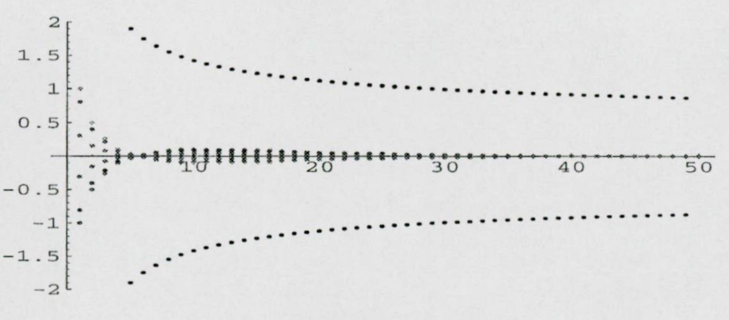

FIGURE 10.

Example 6.2. Let $a_n = 0.25 + 1/(1+n)^3$, $b_n = -(0.25 + 1/(1+n)^3)/2$, $n_0 = 2$, $p = 2$. Then, conditions of the Corollary 6.3 are satisfied with $k = 1$ and $\rho_n = \log n$. See Figure 10.

Introduce the concept of characteristic equation for discrete difference equation (6.1). For this reason divide both sides of Equation (6.1) by x_{n+1} and define the new sequence $\{\lambda_n\}$ with

$$\lambda_n = \frac{x_n}{x_{n+1}}, \quad n = n_0, n_0+1, \ldots$$

Since

$$x_n = x_{n_0} \prod_{i=n_0}^{n-1} \frac{1}{\lambda_i}, \quad n = n_0, n_0+1, \ldots$$

we obtain the characteristic equation associated with Equation (6.1) in the form

$$1 - (1 - a_n)\lambda_n = b_n \prod_{i=p_n}^{n} \lambda_i, \quad n = n_0, n_0+1, \ldots \tag{6.5}$$

Assume that:

- (H_5^3) For the sequences $\{a_n\}$ and $\{b_n\}$ given in (H_1^3), there is a real sequence $\{\rho_n\}$ in the form of product of positive real numbers

  $$\rho_n = \prod_{\ell=n_0}^{n-1} \lambda_\ell \quad \text{for} \quad n \geq n_{-1}$$

  such that $\{\lambda_n\}$ is the solution of the inequality

  $$b_n \prod_{i=p_n}^{n} \lambda_i \leq 1 - (1 - a_n)\lambda_n, \quad n = n_0, n_0+1, \ldots$$

Theorem 6.4. *Suppose that conditions (H_1^3), (H_2^3) and (H_5^3) hold. Let $x_n = x_n^a$ be the solution of the initial value problem (6.1) and (6.2), and let*

$$M_0 = \max_{n_{-1} \leq n < n_0} |a_n|.$$

Then

$$|x_n| \leq M_0 \prod_{\ell=n_0}^{n-1} \frac{1}{\lambda_\ell} \quad for \quad all \quad n \geq n_0.$$

Now, it is possible to apply the last result to the equation with constant delay.

Corollary 6.5. *Suppose that condition (H_1^3) holds. For given natural number $p \geq 1$ let $p_n = n - p$. Assume that there exists real number $\lambda > 1$ such that*

$$|b_n| \leq \frac{1 - \lambda + \lambda a_n}{\lambda^{p+1}} \quad for \quad n \geq n_0. \tag{6.6}$$

Let $x_n = x_n^a$ be the solution of the initial value problem (6.1) and (6.2). Then

$$|x_n| \leq \frac{C}{\lambda^n} \quad for \quad all \quad n \geq n_0,$$

where

$$C = \left\{ \max_{n_{-1} \leq n < n_0} |a_n| \right\} \lambda^{n_0}.$$

Remark 6.6. *In equation (6.1), let $p_n = n - p$ for a natural number $p \geq 1$, $a_n = a$ and $b_n = b$ for real constants a and b. Using the transformation $x_n / x_{n+1} = \xi$ in (6.1), we obtain the characteristic equation of difference equation (6.1) of the form*

$$1 - \xi + a\xi = b\xi^{-(p+1)}. \tag{6.7}$$

If characteristic equation (6.7) has a real solution ξ_0, then condition (6.6) is satisfied with $\lambda = \xi_0$. However, equality is valid in condition (6.6).

It is useful to formulate the next corollary, where the hypotheses are similar to the previous ones and implies the assertion of Corollary 6.5.

Corollary 6.7. *In equation (6.1), let $p_n = n - p$ for a natural number $p \geq 1$. Suppose that there exist real numbers Q and α such that*

$$0 < Q \leq 1, \quad 0 < \alpha < 1$$

and

$$|b_n| \leq a_n Q, \quad \alpha \leq a_n \quad for \quad n \geq n_0.$$

Then, there exists a real number $\lambda > 1$ such that condition (6.6) is satisfied.

Proof. Consider the function

$$F(\lambda) = \frac{\lambda - 1}{\lambda(1 - \lambda^p Q)} \quad \text{for} \quad 1 \leq \lambda < \left(\frac{1}{Q}\right)^{\frac{1}{p}}.$$

Since the first derivative

$$F'(\lambda) = \frac{1 - \lambda^p Q(p + 1 - p\lambda)}{\lambda^2 (1 - \lambda^p Q)^2} > 0 \quad \text{for} \quad 1 \leq \lambda < \left(\frac{1}{Q}\right)^{\frac{1}{p}},$$

and $F(1) = 0$, then function $F(\lambda)$ is positive, continuous and monotone increasing on the interval $[1, \left(\frac{1}{Q}\right)^{\frac{1}{p}})$ and

$$\lim_{\lambda \to \left(\frac{1}{Q}\right)^{\frac{1}{p}} - 0} F(\lambda) = \infty .$$

Therefore, for the given α, there exists a value $\lambda \in [1, \left(\frac{1}{Q}\right)^{\frac{1}{p}})$ such that $F(\lambda) = \alpha$. Now, $1 - \lambda^p Q > 0$ and it follows that

$$a_n \geq \alpha = \frac{\lambda - 1}{\lambda(1 - \lambda^p Q)}$$

and hence

$$Q a_n \leq \frac{\lambda a_n - \lambda + 1}{\lambda^{p+1}}.$$

The last inequality implies assumption (6.6), and this completes the proof.

7. Summary

The subject of this dissertation is the investigation of the asymptotic behavior of solutions of difference equations with continuous time.
In Section 1. we describe the concept of functional equations as well as the particular cases such as the discrete difference equations and generalized difference equations and give a short historical review about the formation and development of these notions. We emphasize the connection between difference equations with continuous time and discrete difference equations underlining the principial differences between them. There are some typical models that illustrate the characteristic behavior of solutions of discrete and continuous difference equations.
In Section 2. we mention the origin of the pantograph differential equation because the research has been motivated by results dealing with the behavior of solutions of this differential equation. We cite some important results connected with the behavior of solutions of the pantograph differential equation and difference equation with continuous time, taking the works by Fox, Mayers, Ockendon and Taylor [36], Kato and McLeod [55], Lim [66], Pandolfi [77], Makay and Terjéki [69], Terjéki [90], Zhou and Yu [95].
In Section 3. we show the diversity of usefulness and applicability of difference equations approaching the problem with some applications.
Since the study of generalized difference equations is a part of the theory of neutral differential equations, we pay special attention to neutral differential equations emphasizing the papers by Freedman and Wu [37], Karydas [54], Krisztin and Wu [60] and Melvin [72].
The theory of functional equations can be successively applied by the investigations of pantograph differential equations. The most important works in this topic are presented by Čermák [17], [18].
Sections 4., 5. and 6. contain the main results obtained for generalized difference equations that are useful in investigations of the asymptotic behavior as well as in stability and asymptotic stability properties of solutions.
Section 4. discusses the asymptotic behavior of solutions of the system of functional equations

$$x(t) = A(t)x(t-1) + B(t)x(p(t)),$$

where $x(t) \in \mathbf{R}^n$, A and B are $n \times n$ real matrix valued functions, p is a

real function such that, for every $T > t_0$ exists a $\delta > 0$ with the property $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$. We investigate both the case when A is a diagonal matrix and when it is not.
Another aim of Section 4. is to study the special case of the previous system, that is, the system

$$x(t) = Ax(t-1) + Bx(p(t)),$$

with constant $n \times n$ real matrices A and B. We obtain asymptotic estimates for the rate of convergence of the solutions of the considered system of functional equations.
The results in Subsections 4.3. and 4.4. are obtained for the case when A is a diagonal matrix with real valued function entries.
In Subsection 4.3., we formulate a result for the case where the components of matrix A are between 0 and 1. We illustrate how the rate of convergence of the solutions can be estimated for the case when the lag function is $p(t) = pt$, $0 < p < 1$.
Subsection 4.4. contains the result when the components of matrix A are greater than 1 and we show how the obtained results can be generalized to the case when A, B are constant matrices and A is diagonalizable.
In Subsection 4.5. are some further generalizations of previous results of the asymptotic behavior of solutions of generalized difference equations to the case when A and B are matrix functions with real valued entries, and A is an arbitrary nonsingular matrix.
Subsection 4.6. describes a representation of solutions of the considered system of nonautonomous functional equations, in the form of series, using the Cauchy matrix of the linear system

$$x(t) = A(t)x(t-1).$$

We give sufficient conditions for the absolute and uniform convergence of the series representation of the solution.
In Section 5. we study the asymptotic behavior of solutions of the scalar difference equation with continuous arguments

$$x(t) = a(t)x(t-1) + b(t)x(p(t)),$$

where a is a real function such that $0 < a(t) < 1$, b is an arbitrary real function and p is a real function such that, for every $T > t_0$ exists a $\delta > 0$ with the property $p(t) \leq t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$, and apply our results to particular cases.
Subsection 5.2. contains asymptotic lower and upper bounds for the solutions. We apply the obtained results to the particular cases such as $p_1 t \leq p(t) \leq p_2 t$, for real numbers $0 < p_1 \leq p_2 < 1$ and $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \leq p_2$. There are some very interesting examples that illustrate how

the rate of convergence of solutions can be estimated by the properly selected auxiliary functions.

In Subsection 5.3. we present the characteristic equation associated with the considered initial value problem and using them obtain asymptotic estimates of solutions which can be applied to the difference equations with constant delay and to the case $t-p_2 \le p(t) \le t-p_1$ for real numbers $1 < p_1 \le p_2$. The example presented here shows that the solution of the generalized difference equations with the lag function between two constant delays is exponentially decaying.

In Subsection 5.4. we generalize the main results obtained in Subsections 5.2. and 5.3. to the equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)),$$

where a is a real function such that $0 < a(t) < 1$, b_i are real functions for $i = 1, 2, ..., m$, and p_i are real functions such that p_i are real functions such that, for every $T > t_0$ exists a $\delta > 0$ with the property $p_i(t) \le t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. We apply obtained results to the particular cases such as $p_i(t) = (1/p_i)t$ for $i = 1, 2, ..., m$, $p_i(t) = \sqrt[p_i]{t}$ for $i = 1, 2, ..., m$, where $1 < p_1 < p_2 < ... < p_m$ are natural numbers and $p_i(t) = t - p_i$ for $i = 1, 2, ..., m$, where $1 \le p_1 < p_2 < ... < p_m$ are real numbers.

In Subsection 5.5. a series representation of solutions of the equation

$$x(t) = ax(t-1) + bx(pt)$$

is given, where a, b, p are real constants, $0 < p < 1$. This series solution gives a necessary and sufficient condition for the convergence.

In Subsection 5.6. we give conditions for the existence of bounded solutions of the scalar difference equation

$$x(t) = (1 - c(t))x(t-1) + c(t)x(p(t)),$$

where c is a positive real function such that $0 < c(t) < 1$ and p is a positive real function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$ and $\lim_{t\to\infty} p(t) = \infty$. We compose an open problem of the asymptotic convergence for the bounded solutions.

In Section 6. we formulate the main result associated to the discrete difference equations

$$x_{n+1} - x_n = -a_n x_n + b_n x_{p_n},$$

where a_n, b_n are given real numbers for all $n = n_0, n_0 + 1, ..$ and $\{p_n\}$ is a given sequence of natural numbers such that $p_n \le n$ for all $n = n_0, n_0 + 1, ..$ and $\lim_{n\to\infty} p_n = \infty$.

We apply the obtained results to particular cases such as the classical pantograph equation with $p_n = [n/p]$ and equations with $p_n = [\sqrt[p]{n}]$, where p is a given positive integer. By finding an appropriate sequence $\{\rho_n\}$ we can also estimate the rate of convergence of the solutions. The reason why we formulate the theorems especially for the discrete difference equations is that they form an independent branch and have their special problems and approximating methods for solving differential equations, which are not characteristic of more general functional equations.
There are two examples that emphasize the basic characteristics of solutions of discrete difference equations making a comparison between the two classical cases. These illustrations also show that under certain conditions the behavior of solutions of discrete difference equations is the same as the behavior of solutions of the associated difference equations with continuous time.
At the end, we enclose the Appendix with *MATHEMATICA* developments and calculations. The graphics of solutions illustrate the main results of the thesis pointing out the characteristic properties of solutions of discrete and continuous difference equations. Our aim is also to refer to the usefulness and diversity of using computers in mathematical research.

8. Összefoglaló

Az értekezés tárgya a folytonos argumentumú differenciaegyenletek megoldásainak aszimptotikus viselkedése.
Az 1. fejezetben leírjuk a függvényegyenletek fogalmát, valamint a jelentősebb speciális eseteket, mint a diszkrét és az általánosított differenciaegyenlet. Rövid történeti áttekintést adunk ezen fogalmak kialakulásáról és fejlődéséről. Kiemeljük a diszkrét és folytonos argumentumú differenciaegyenletek közötti fontosabb összefüggéseket, külön hangsúlyt fektetve a megoldások viselkedése között levő alapvető különbségekre.
Néhány modell segítségével bemutatjuk a diszkrét és folytonos argumentumú differenciaegyenletek megoldásainak jellegzetes tulajdonságait.
A 2. fejezetben szólunk a pantográf differenciálegyenlet keletkezéséről, hiszen ez volt a kutatás fő motivációja.
Idézünk néhány ismert eredményt a pantográf differenciálegyenlet és az általánosított differenciaegyenletek megoldásainak viselkedésével kapcsolatban, kiemelve Fox, Mayers, Ockendon és Taylor [36], Kato és McLeod [55], Lim [66], Pandolfi [77], Makay és Terjéki [69], Terjéki [90], Zhou és Yu [95] munkáit.
A 3. fejezetben egy áttekintést adunk a differenciaegyenletek felhasználhatóságának sokszinűségéről, konkrét alkalmazások bemutatásával közelítve meg ezt a problémát.
Mivel az általánosított differenciaegyenletek tanulmányozása a neutrális differenciálegyenletek elméletének szerves részét képezi, ezért ezzel kapcsolatban leginkább Freedman és Wu [37], Karydas [54], Krisztin és Wu [60] és Melvin [72] munkáit hangsúlyozzuk ki.
A függvényegyenletek elmélete sikeresen alkalmazható a pantográf differenciálegyenletek tanulmányozásánál is, ahogyan ezt Čermák [17], [18] cikkeiben láthatjuk.
A 4., 5. és 6. fejezetek az általánosított differenciaegyenletekkel kapcsolatos fő eredményeket tartalmazzák, melyek igen hasznosak a megoldások aszimptotikus viselkedésével, stabilitásával és aszimptotikus stabilitásával kapcsolatos kutatásokban.
A megoldások aszimptotikus viselkedését először az

$$x(t) = A(t)x(t-1) + B(t)x(p(t))$$

függvényegyenletrendszeren mutatjuk be a 4. fejezetben, ahol $x(t) \in \mathbf{R}^n$, A és B $n \times n$-es valós értékű mátrixfüggvények, p pedig olyan valós függvény,

hogy minden $T > t_0$ esetén van olyan $\delta > 0$, hogy $p(t) < t - \delta$, $t_0 \leq t \leq T$, és $\lim_{t\to\infty} p(t) = \infty$.
Vizsgálatunk tárgyát képezi az az eset is, amikor A diagonálmátrix, és az is, amikor ezt a feltételt nem elégíti ki.
A 4. fejezetben célkitűzéseink közé tartozik még az

$$x(t) = Ax(t-1) + Bx(p(t)),$$

speciális függvényegyenletrendszer tanulmányozása is, ahol A, B $n \times n$-es konstans mátrixok. A tekintett egyenletrendszer megoldásaira olyan aszimptotikus becsléseket kapunk, melyek segítségével felmérhető a megoldások nullához tartásának sebessége.
A 4.3. és 4.4. alfejezetekben olyan eredmények találhatók, ahol az A olyan diagonálmátrix, melynek elemei valós értékű függvények.
A 4.3. alfejezetben az A mátrix komponensei 0 és 1 közé esnek. Külön vizsgáljuk a $p(t) = pt$, $0 < p < 1$ esetet és megmutatjuk, hogy milyen konkrét függvény segítségével tudjuk felülről megbecsülni a nullához tartás sebességét.
A 4.4. alfejezet olyan eredményeket tartalmaz, ahol az A mátrix komponensei 1-nél nagyobbak. Megmutatjuk azt is, hogyan lehet a kapott eredményeket általánosítani arra az esetre, mikor A és B konstans mátrixok, de A nem diagonálmátrix.
A 4.6. alfejezetben megadjuk a tekintett nemautonóm függvényegyenletrendszer megoldásának néhány sor alakú reprezentációját az

$$x(t) = A(t)x(t-1)$$

lineáris rendszerhez tartozó Cauchy mátrix segítségével.
A 5. fejezetben az

$$x(t) = a(t)x(t-1) + b(t)x(p(t))$$

skaláris folytonos argumentumú differenciaegyenlet megoldásainak viselkedését vizsgáljuk, ahol a olyan valós függvény, hogy $0 < a(t) < 1$, b egy tetszőleges valós függvény, p pedig egy olyan pozitiv valós függvény, hogy minden $T > t_0$ esetén van olyan $\delta > 0$, hogy $p(t) < t - \delta$, $t_0 \leq t \leq T$, és $\lim_{t\to\infty} p(t) = \infty$. Az eredmények alkalmazást nyernek a késleltetésfüggvény néhány speciális esetére.
A 5.2. alfejezet tételei alsó és felső becslést adnak a megoldásokra. A kapott eredményeket felhasználva külön foglalkozunk a $p_1 t \leq p(t) \leq p_2 t$ esettel, ahol $0 < p_1 \leq p_2 < 1$ és a becslőfüggvény egy hatványfüggvény és külön a $\sqrt[p_2]{t} \leq p(t) \leq \sqrt[p_1]{t}$ esettel, ahol $1 < p_1 \leq p_2$ természetes számok, és logaritmusfüggvénnyel lehet a megoldást felülről megbecsülni.

A 5.3. alfejezetben az adott kezdeti érték problémához tartozó karakterisztikus egyenlet szerepel, amelynek segítségével a megoldásokra újabb aszimptotikus becsléseket lehet adni arra az esetre, amikor a késleltetésfüggvény két konstans késleltetés között van, azaz $t-p_2 \leq p(t) \leq t-p_1$, $1 < p_1 \leq p_2$ valós számok esetén. Megállapítjuk, hogy a megoldás most egy exponenciálisan csökkenő függvénnyel becsülhető felülről.
A 5.4. alfejezetben a 5.2. és 5.3. alfejezetekben kapott eredményeket az

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t))$$

több késleltetésfüggvényt tartalmazó egyenletre általánosítjuk, ahol a olyan valós függvény, hogy $0 < a(t) < 1$, b_i, $i = 1, 2, ..., m$, tetszőleges valós függvények, p_i, $i = 1, 2, ..., m$, pedig olyan pozitiv valós függvények, hogy minden $T > t_0$ esetén van olyan $\delta > 0$, hogy $p_i(t) < t - \delta$, $t_0 \leq t \leq T$, és $\lim_{t\to\infty} p_i(t) = \infty$, $i = 1, 2, ..., m$. Alkalmazzuk az eredményeket a $p_i(t) = (1/p_i)t$, $i = 1, 2, ..., m$, és $p_i(t) = \sqrt[p_i]{t}$, $i = 1, 2, ..., m$, esetekre, ahol $1 < p_1 < p_2 < ... < p_m$ természetes számok. A megfelelő karakterisztikus egyenlet felhasználásával aszimptotikus becslést kapunk a több késleltetésfüggvényt tartalmazó egyenletre és alkalmazzuk ezt az eredményt a $p_i(t) = t - p_i$, $i = 1, 2, ..., m$, esetre, ahol $1 \leq p_1 < p_2 < ... < p_m$ valós számok.
A 5.5. alfejezetben egy sor alakú reprezentációt adunk az

$$x(t) = ax(t-1) + bx(pt)$$

egyenletre, ahol a, b, p valós állandók és $0 < p < 1$ és ez a megoldás szükséges és elégséges konvergenciakritériumot ad.
A 5.6. alfejezetben feltételeket adunk az

$$x(t) = (1-c(t))x(t-1) + c(t)x(p(t))$$

skaláris differenciaegyenlet korlátos megoldásainak létezésére, ahol c olyan valós függvény, hogy $0 < c(t) < 1$, p pedig olyan pozitiv valós függvény, hogy minden $T > t_0$ esetén van olyan $\delta > 0$, hogy $p(t) < t - \delta$, $t_0 \leq t \leq T$, és $\lim_{t\to\infty} p(t) = \infty$. Megfogalmazunk egy nyitott kérdést a korlátos megoldások konvergenciájával kapcsolatban.
Az 6. fejezetben olyan eredményeket fogalmazunk meg, amelyek a

$$x_{n+1} - x_n = -a_n x_n + b_n x_{p_n}$$

diszkrét differenciaegyenletre vonatkoznak, ahol a_n, b_n adott valós számok, $n = n_0, n_0 + 1, ..$, $\{p_n\}$, $n = n_0, n_0 + 1, ..$, pedig természetes számoknak egy olyan sorozata, hogy $p_n \leq n$, $n = n_0, n_0 + 1, ..$, és $\lim_{n\to\infty} p_n = \infty$.

A kapott eredményeket alkalmazzuk a klasszikus pantográf egyenletnek megfelelő esetre, amikor $p_n = [n/p]$ és arra az esetre, amikor $p_n = [\sqrt[p]{n}]$, ahol p adott természetes szám. Most egy megfelelő ρ_n sorozat megadásával a megoldássorozat felülről megbecsülhető és ezáltal meghatározható a konvergencia sebessége is. Két példát adunk, amelyek megoldásainak grafikus képei jól összehasonlítják a diszkrét differenciaegyenlet megoldásainak viselkedését ebben a két klasszikus esetben. Ugyanakkor ezek az illusztrációk jól megmutatják hogy megfelelő feltételek megadásával a diszkrét és folytonos argumentumú differenciaegyenletek megoldásai hasonlóan viselkednek.

Végül egy mellékletet csatolunk az értekezéshez, amely a *MATHEMATICA* programcsomaggal készített fejlesztéseket és számításokat tartalmazza. A bemutatott grafikonok szemléltetik a disszertáció főbb eredményeit, kiemelve a diszkrét és folytonos differenciaegyenletek megoldásai közötti jellegzetes hasonlóságokat és különbözőségeket. A melléklet mindenképpen rámutat a számítógép alkalmazásának sokrétűségére és a matematikai kutatásokban való felhasználhatóságára.

References

[1] J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Birkhäuser Verlag Basel und Stuttgart, 1961.

[2] J. Aczél, *Functional Equations: history, applications and theory*, D. Reidel Publishing Company, Dordrecht - Boston - Lancaster, 1984.

[3] J. Aczél, *A Short Course on Functional Equations*, D. Reidel Publishing Company, Dordrecht - Boston - Lancaster - Tokyo, 1987.

[4] J. Aczél, J. G. Dhombres, *Functional Equations containing Several Variables*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 1989.

[5] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, Inc., New York - Basel - Hong Kong, 1992.

[6] O. Arino, I. Györi, *Necessary and Sufficient Condition for Oscillation of a Neutral Differential System with Several Delays*, J. Differential Equations **81** (1989), 98-105.

[7] F. V. Atkinson, J. R. Haddock, *Criteria for Asymptotic Constancy of Solutions of Functional Differential Equations*, J. Math. Anal. Appl. **91** (1983), 410-423.

[8] M. G. Blizorukov, *On the Construction of Solutions of Linear Difference Systems with Continuous Time*, Differ. Uravn. **32** (1996), No.1, 127-128, translation in Differential Equations **32** (1996), No.1, 133-134.

[9] M. D. Buhmann, A. Iserles, *On the Dynamics of a Discretized Neutral Equation*, Technical Report DAMTP 1990/NA8, University of Cambridge.

[10] M. D. Buhmann, A. Iserles, *Numerical Analysis of Functional Equations with a Variable Delay*, Technical Report DAMTP 1991/NA8, University of Cambridge.

[11] T. A. Burton, G. Makay, *Asymptotic Stability for Functional Differential Equations*, Acta Math. Hungar. **65(3)** (1994), 243-251.

[12] S. N. Busenberg, K. L. Cooke, *Stability Conditions for Linear Non autonomous Delay Differential Equations*, Quart. Appl. Math. **42** (1984), 295-306.

[13] S. N. Busenberg, K. L. Cooke, *Models of Vertically Transmitted Diseases with Sequential - Continuous Dynamics*, "Nonlinear Phenomena in Mathematical Sciences", Academic Press, New York, 1982.

[14] J. Carr, J. Dyson, *The Functional Differential Equation* $y'(x) = ay(\lambda x) + by(x)$, Proc. Roy. Soc. Edinburgh, **74A**, 13 (1974/75), 165-174.

[15] J. Carr, J. Dyson, *The Matrix Functional Differential Equation* $y'(x) = Ay(\lambda x) + By(x)$, Proceedings of the Royal Society of Edinburgh, **75A**, 2 (1975/76) 5-22.

[16] L. A. V. Carvalho, *On Quadratic Liapunov Functionals for Linear Difference Equations*, Linear Algebra Appl., **240**, (1996), 41-64.

[17] J. Čermák, *The Asymptotic Bounds of Solutions of Linear Delay Systems*, J. Math. Anal. Appl. **225** (1998), 373-388.

[18] J. Čermák, *On the Delay Differential Equation* $y'(x) = ay(\tau(x)) + by(x)$, Ann. Polon. Math. LXXI.**2** (1999), 161-169.

[19] C. V. Coffman, *Asymptotic Behaviour of Solutions of Ordinary Difference Equations*, Trans. Amer. Math. Soc. **110** (1964), 22-51.

[20] K. L. Cooke, L. A. C. Ladeira, *Applying Carvalho's Method to Find Periodic Solutions of Difference Equations*, J. Math. Anal. Appl. **2** (1996), 105-115.

[21] J. G. Dhombres, *Quelques aspects de l'histoire des équations fonctionnelles liés á l'évolution du concept de fonction*, Archive for History of Exact Sciences **36** (1986), 151-74.

[22] J. G. Dhombres, *On the historical role of functional equations*, Aequationes Math. **25** (1982), 293-9.

[23] J. Diblik, *Asymptotic Behaviour of Solutions of Linear Differential Equations with Delay*, Ann. Polon. Math. **58** (1993), 131-137.

[24] J. Diblik, *Asymptotic Representation of Solutions of Equation* $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$, J. Math. Anal. Appl., **217** (1998), 200-215.

[25] J. Diblik, *Behaviour of Solutions of Linear Differential Equations with Delay*, Archivum mathematicum (Brno), **34** (1998), 31-47.

[26] Y. Domshlak, *Oscillatory Properties of Linear Difference Equations with Continuous Time*, Differential Equations and Dynamical Systems, **1** No. 4 (1993), 311-324.

[27] L. Edelstein-Keshet, *Mathematical Models in Biology*, Birkhäuser Mathematics Series, 1987.

[28] S. Elaydi, *An Introduction to Difference Equations*, 2nd edn., Springer-Verlag, New York, 1999.

[29] S. Elaydi, I. Győri, *Asymptotic Theory for Delay Difference Equations*, Journal of Difference Equations and Applications, **1** (1995), 99-116.

[30] S. Elaydi, *An Extension of Levinson's Theorem to Asymptotically Jordan Difference Equations* , Journal of Difference Equations and Applications, **1** (1995), 369-390.

[31] S. Elaydi, *Difference Equations in Combinatorics, Number Theory, and Orthogonal Polynomials*, Journal of Difference Equations and Applications, **00** (1999), 1-14.

[32] S. Elaydi, *Asymptotics for Linear Difference Equations I: Basic Theory*, Journal of Difference Equations and Applications, **00** (1999), 1-27.

[33] S. Elaydi, S. Zhang, *Stability and Periodicity of Difference Equations with Finite Delay*, Funkcialaj Ekvacioj, **37** (1994), 401-413.

[34] M. Evgrafov, *The Asymptotic Behaviour of Solutions of Difference Equations*, Dokl. Acad. Nauk. SSSR **121** (1958), 26-29.

[35] J. M. Ferreira, A. M. Pedro, *Oscillations of Delay Difference Systems*, J. Math. Anal. Appl. **221** (1998), 364-383.

[36] L. Fox, D. F. Mayers, J. R. Ockendon, A. B. Taylor, *On a Functional Differential Equation*, J. Inst. Maths Applics, **8** (1971), 271-307.

[37] H. I. Freedman, J. Wu, *Periodic Solutions of Single-species Models with Periodic Delay*, SIAM J. Math. Anal., **23** 3 (1992), 689-701.

[38] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics* Kluwer Academic Publishers, 1992.

[39] J. R. Graef, C. Qian, *Asymptotic Behaviour of Forced Delay Equations with Periodic Coefficients* Comm. Appl. Anal. (to appear)

[40] J. R. Graef, C. Qian, *Global Stability in a Nonlinear Difference Equation* (to appear).

[41] I. Győri, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press - Oxford, 1991.

[42] I. Győri, M. Pituk, *Asymptotic Formulae for the Solutions of a Linear Delay Difference Equations*, J. Math. Anal. Appl. **195** (1995), 376-392.

[43] I. Győri, M. Pituk, *Comparison Theorems and Asymptotic Equilibrium for Delay Differential and Difference Equations*, Dynam. Systems Appl. **5** (1996) 277-302.

[44] J. R. Haddock, T. Krisztin, J. Terjéki, J. Wu, *An invariance principle of Ljapunov-Razumikhin type of neutral equations*, J. Diff. Eqns., **107** (1994), 395-417.

[45] J. Hale, *Theory of Functional Differential Equations*, Springer - Verlag, New York, 1977.

[46] J. K. Hale, M. A. Cruz *Stability of Functional Differential Equations of Neutral Type*, J. Differential Equations **7** (1970), 334-355.

[47] J. K. Hale, M. A. Cruz *Exponential Estimates and the Saddle Point Property for Neutral Functional Differential Equations*, J. Math. Anal. Appl. **34** (1971), 267-288.

[48] J. K. Hale, *Parametric Stability in Difference Equations*, Bol. Un. Mat. It. **(4)10** (1974).

[49] L. Hatvani, *Annulus Arguments in the Stability Theory for Functional Differential Equations*, Differential and Integral Equations **10** (1997), 975-1002.

[50] M. L. Heard, *A Change of Variables for Functional Differential Equations*, J. Differential Equations, **18** (1975), 1-10.

[51] D, Henry, *Linear Autonomous Neutral Functional Differential Equations*, J. Differential Equations, **15** (1974), 106-128.

[52] L. Huang, J. Yu, B. Dai, *Asymptotic Behaviour of Solutions for a Class of Difference Equations*, J. Math. Anal. Appl. **204** (1996), 830-839.

[53] L. Huang, J. Yu, B. Dai, *Asymptotic Behaviour of Solutions for a Class of Retarded Difference Equations*, Acta Mathematica Sinica, New Series **14**, No. 3 (1998), 419-422.

[54] N. Karydas, *Asymptotic Behaviour of the Solutions of a Neutral Delay Differential Equation with Piecewise Constant Argument*, Differential Equations and Dynamical Systems, **3** 2 (1995), 217-224.

[55] T. Kato, J. B. McLeod, *The Functional-differential Equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Am. Math. Soc. **77** (1971), 891-937.

[56] T. Kato, *Asymptotic Behavior of Solutions of the Functional Differential Equation* $y'(x) = ay(\lambda x)+by(x)$,in "Delay and Functional Differential Equations and Their Applications" (Klaus Schmitt, Ed.),Academic Press, New York (1972), 197-217.

[57] J. Kato, *Stability Criteria for Difference Equations Related with Liapunov Functions for Delay-Differential Equations*, Dynamic Systems and Applications, **3** (1994), 75-84.

[58] W. G. Kelley, A. C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press, Inc., Harcourt Brace Jovanovich Publishers, San Diego, 1991.

[59] T. Krisztin, *Convergence and Periodicity in Difference Equations Arising in Compartmental Systems*, Proceedings of the Second International Conference of Difference Equations and Applications, Veszprém, 1995, 371-380.

[60] T. Krisztin, J. Wu, *Asymptotic Behaviors of Solutions of Scalar Neutral Functional Differential Equations*, Differential Equations and Dynamical Systems, **4** No.3/4 (1996), 351-366.

[61] M. Kuczma, B. Chochewski, R. Ger, *Iterative Functional Equations*, in 'Encyclopedia of Mathematics and Its Applications', Cambridge University Press, Cambridge, 1990.

[62] M. Kuczma, *A Survey on the theory of functional equations*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **130** (1964), 64 pp.

[63] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat. **46** Polish Scientific Publishers, Warsaw, 1968.

[64] G. Ladas, *Recent Development in the Oscillation of Delay Difference Equations*, International Conference of Differential Equations, Colorado Springs, USA, June 1989.

[65] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, Inc., New York, 1987.

[66] E. - B. Lim, *Asymptotic Behavior of Solutions of the Functional Differential Equation* $x'(t) = Ax(\lambda t)+Bx(t)$, $\lambda > 0$, J. Math. Anal. Appl. **55** (1976), 794-806.

[67] E. - B. Lim, *Asymptotic Bounds of Solutions of the Functional Differential Equation* $x'(t) = ax(\lambda t) + bx(t) + f(t)$, $\lambda > 0$, SIAM J. Math. Anal. **9** No. 5 (1978), 915-920.

[68] K. Mahler, *On a Special Functional Equation*, J. London Math. Soc. **15** (1940), 115-123.

[69] G. Makay, J. Terjéki, *On the Asymptotic Behaviour of the Pantograph Equation*, E. J. Qualitative Theory of Diff. Equ. No. **2** (1998), 1-12.

[70] A. Máté, P. Nevai, *A Generalization of Poincaré's Theorem for Reccurence Equations*, J. Approximation Theory **63** (1990),92-97.

[71] J. B. McLeod, *The Functional- differential Equation* $y'(x) = ay(\lambda x) + by(x)$ *and generalizations*, Lecture Notes Math. **280** (1972), 308-313.

[72] W. R. Melvin, *Stability Properties of Functional Difference Equations*, J. Math, Anal. Appl. **48** (1974), 749-763.

[73] K. S. Miller, *An Introduction to the Calculus of Finite Differences and Difference Equations*, Henry Holt and Company, New York, 1960.

[74] J. D. Murray, *Mathematical Biology*, Springer - Verlag, New York - Berlin - Heidelberg, 1993.

[75] J. R. Ockendon, A. B. Taylor, *The Dynamics of a Current Collection System for an Electric Locomotive*, Proc. Roy. Soc. London, Ser A **322** (1971), 447-468.

[76] O. C. Oliveira Filho, L. A. V. Carvalho, *On periodic Solutions of* $x(t) = ax(t-1)+bx(t-2)$, Proceedings of the first international conference on difference equations, Trinity University, San Antonio, TX, USA, May 25-28, 1994, London, Gordon and Breach, 185-195 (1995).

[77] L. Pandolfi, *Some Observations on the Asymptotic Behaviour of the Solutions of the Equation* $x'(t) = A(t)x(\lambda t)+B(t)x(t), \lambda > 0$, J. Math. Anal. Appl. **67** (1979), 483-489.

[78] H. Péics, *Generalized Characteristic Equation of Linear Delay Difference Equations*, Proceedings of the Second International Conference of Difference Equations and Applications, Veszprém, 1995, 499-505.

[79] H. Péics, *Applications of Generalized Characteristic Equation of Linear Delay Difference Equations*, Matematički Vesnik, Beograd, **50** (1998), 31-36.

[80] H. Péics, *Representation of Solutions of Difference Equations with Continuous Time*, Proceedings of the Sixth Colloquium on the Qualitative Theory of Differential Equation, Szeged, 1999. August, 10-14. (to appear).

[81] H. Péics, *On the Asymptotic Behaviour of Difference Equations with Continuous Arguments*, Dynamics of Continuous, Discrete and Impulsive Systems (to appear).

[82] H. Péics, *On the Asymptotic Behaviour of Solutions of a System of Functional Equations*, Mathematica Periodica Hungarica (to appear).

[83] G. P. Pelyukh, *General Solution of Systems of Linear Difference Equations with Continuous Arguments*, Dopov. Akad. Nauk Ukr. **1** (1994), 16-21 (Russian).

[84] G. P. Pelyukh, *A Certain Representation of Solutions to Finite Difference Equations with Continuous Argument*, Differ. Uravn. **32** (1996), No.2, 256-264, translation in Differential Equations **32** (1996), No.2, 260-268.

[85] G. P. Pelyukh, *On Periodic Solutions of Difference Equations with Continuous Argument*, Ukrain. Mat. Zh. **48** No. 1 (1996), 140-145.

[86] O. Perron, *Über Summengleichungen und Poincaré Differenzengleichungen*, Math. Annalen **84** (1921), 1.

[87] H. Poincaré, *Sur les equation linéares aux differentelles ordinaires et au différence finies*, Amer. J. Math. **7** (1885), 203-258.

[88] A. N. Sharkovsky, Yu. L. Maistrenko, E. Yu. Romanenko, *Difference Equations and Their Applications*, Kluver Academic Publishers, 1993.

[89] J. H. Shen, *Comparison Theorems for the Oscillation of Difference Equations with Continuous Argument and Applications* Chin. Sci. Bull. **41** No. 18. (1996), 1506-1510 (in Chinese).

[90] J. Terjéki, *Representation of the Solutions to Linear Pantograph Equations*, Acta Sci. Math. Szeged, **60** (1995), 705-713.

[91] S. Zhang, *Stability Analysis of Delay Difference Systems*, Computers Math. Applic. **33** No. 10. (1997), 41-52.

[92] Y. Z. Zhang, J. R. Yan, *Oscillation Criteria for Difference Equations with Continuous Arguments*, Acta Math. SINICA, **38** No.3 (1995), 406-411 (in Chinese).

[93] Y. S. Zhang, *Forced Oscillation of Delay Difference Equations with Continuous Variables and Variant Coefficients*, J. Hunan Univ., Nat. Sci., **24** No. 5. (1997), 6-10 (in Chinese).

[94] Y. Y. Zhang, *An Algebraic Principle for the Stability of Difference Operators*, J. Differ. Equations, **136** No. 2. (1997), 236-247.

[95] Z. Zhou, J. S. Yu, *Decaying Solutions of Difference Equations with Continuous Arguments*, Annals of Differential Equations, **14** No.3. (1998), 576-582.