Advances in Bijective Combinatorics

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Abstract of the Ph.D. Thesis

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Introduction

My Ph. D. thesis is based on the works [3], [4], and [5]. [3] is joint work with Peter Hajnal. The aim of the thesis is to evaluate the role of bijections as one of the special method of enumerative combinatorics. In addition, I emphasize with this work the importance and essence of bijective proofs. My goal is to show the beauty of bijective combinatorics beside the usefulness of the method.

The subject of combinatorics is the study of discrete mathematical structures. One aim of it is to reveal the connections between mathematical objects. Enumerative combinatorics provide quantitative characterizations. Sometimes the same number sequence enumerates different kind of objects, but the cause of it is not obvious. These cases require to find the characteristic common properties of the different objects in order to explain why do the sets have the same size.

Combinatorics contributes to the understanding of connections with its special method, the bijective proof. A bijection establishes a one–to–one correspondence between two sets and demonstrates this way that the two sets are equinumerous. If the size of one set is known then the bijection derives that the same formula gives the answer to the size of the other set, too. So a bijection is an appropriate method for enumeration of a set bringing it into connection with an other set. In addition it points out the joint structural characteristic property that both sets posses and this way it gives an explanation of properties of the number sequence also.

There are $n!$ bijections between two sets with n elements. The enumeration of a set often can be refined according to some special parameters. The statistical distributions of two sets according to parameters are sometimes equal. Some bijections are suitable to show this correspondence because some bijections reveal the finer structure and keep the values of the parameters. So some bijections are regarded in some sense as better and deeper. It is usual in bijective combinatorics that in the case of classical sets the researcher publish new bijections and are seeking for bijections that satisfy a particular purpose.

This work investigates problems from different active studied fields. It consists of new theorems and new combinatorial proofs of known results.

After an introductory chapter the second chapter is devoted to the investigations of combinatorial interpretations and properties of poly–Bernoulli numbers, the third chapter to the enumeration of monotone labellings of plane trees, the fourth to inversions of 312–avoiding permutations.

This work should be an instance of the fact that the purpose of mathematics is rather the understanding of the world around us captured by the special abstract way of mathematics than the collecting informations and formulating theorems. Because of this reason I think that it is important to prove a theorem using different methods. I'm convinced that the combinatorial approach I used in my thesis contributes substantially to the understanding of the underlying structures/problems.

Poly–Bernoulli numbers

As the name indicates poly–Bernoulli numbers are the generalizations of the classical and in many questions central Bernoulli numbers. Kaneko [14] introduced poly–Bernoulli numbers in 1997 during the investigations of multiple zeta values or Euler Zaiger sums. Multiple zeta values are nested generalizations of Riemann zeta functions.

Definition 1 ([14]) *Let* $\{B_n^{(k)}\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ *denote poly–Bernoulli numbers that are defined by the next generating function:*

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1-e^{-x})}{1-e^{-x}}$$

where

$$Li_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k}.$$

If $k \leq 0$ $B_n^{(k)}$ is a positive integer.

Table 1: The poly–Bernoulli numbers

	0	1	2	3	4	5
0	1	1	1	1	1	1
-1	1	2	4	8	16	32
-2	1	4	14	46	146	454
-3	1	8	46	230	1066	4718
-4	1	16	146	1066	6902	41506
-5	1	32	454	4718	41506	329462

For this case Kaneko derived a compact formula also:

Theorem 1 ([1]) *For* $k \in \mathbb{N}$

$$B_n^{(-k)} = \sum_{m=0}^{\min(n,k)} m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} m! \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\}, \tag{1}$$

where $\left\{ {n \atop k} \right\}$ is the number of the partitions of $[n]$ (or any set with n elements) with k nonempty blocks, i. e. the Stirling number of the second kind.

The following question is natural: is there beyond the obvious combinatorial interpretation a combinatorial problem in that these numbers arise?

First Brewbaker showed that the number of $n \times k$ lonesum matrices are exactly the poly–Bernoulli numbers $B_n^{(-k)}$. A *lonesum matrix* is a 01 matrix that is uniquely reconstructible from given row and column sums.

Though only this one interpretation becomes known my research show that there are numerous substantially different combinatorial interpretations. Poly–Bernoulli numbers arise in many places in mathematics. Its role is not as significant as that of Catalan numbers. So we find often comments on internet without publications concerning poly–Bernoulli numbers. Researcher who investigated a combinatorial problem often didn't have any knowledge about poly–Bernoulli numbers and so did not recognize that the answer they found is the poly–Bernoulli number sequence.

In the second chapter in sections 4.–9. I collected and listed systematically the interpretations that I found in the literature. However my survey is not simple a list. In every case I explain the connections with the explicit description of a bijection or a sketch of a bijective proof. Such analysis doesn't exist in the literature.

My starting point is the natural interpretation of the formula (1) using basic combinatorial principles:

Consider two sets with elements $n+1$ and $k+1$. In both sets there is a unique special element. Partition both sets into the same number of blocks (m). The blocks with the special elements is a special block. Order the ordinary blocks. The formula counts the number of the so defined ordered pairs of partitions of these two underlying sets.

This structure is reflected in the Callan permutations [7]. Let

$$\widehat{N} = \{0, 1 \ldots, n\} \quad \text{and} \quad \widehat{K} = \{n+1, \ldots, n+k, n+k+1\}.$$

Consider the permutations of the set $\widehat{N} \cup \widehat{K}$ that has 0 as the first element and $n+k+1$ as the last element. In addition, if elements that follow each other directly are of the same set (of $\widehat{N}$ resp. $\widehat{K}$) than they are in increasing order. Permutations with this property is called *Callan* permutations.

Permutations with ascending–to–max property can be regarded as the dual of Callan permutations. These permutations play important role in the theory of suffix arrays. There is a characterization theorem that states which restrictions a permutation has to fulfill to be a suffix array of a word over a binary alphabet. One of these restrictions is the Ascending–To–Max property. The original definition of this property can be modified in a way that the value–position duality to Callan permutations become obvious.

Consider again the permutations of the set $\widehat{N} \cup \widehat{K}$. In this case we require that if two elements with consecutive values are both in the first $n+1$ positions (resp. in the last

$k+1$ positions) then they should follow each other directly/ have consecutive positions in the permutations. This property is called *ascending–to–max property*. The reformulation and the duality proves obviously the next new theorem:

Theorem 2 ([3]) *Let $\mathcal{A}_n^{(k)}$ denote the permutations of the set $\{0,1,2,\ldots,n+k+1\}$ with the ascending–to–max property. Then*

$$|\mathcal{A}_n^{(k)}| = B_n^{(-k)}.$$

There is another from these permutations substantially different permutation class that is enumerated by the poly–Bernoulli numbers. The enumeration of permutations with restriction on their image set is a general problem. One of the most natural constraint is a bound of the distance between an element and its image. Vesztergombi [21] studied this problem and obtained a general formula. Launois recognized that special values of this formula coincides with poly–Bernoulli numbers.

I present a new combinatorial proof of this statement using Lovász' method. Let $\mathcal{V}_n^{(k)}$ denote the set of permutations $\pi \in S_{n+k}$ such that

$$-n \leq i - \pi(i) \leq k.$$

Then

Theorem 3 ([21],[17],[3])

$$|\mathcal{V}_n^{(k)}| = B_n^{(-k)}$$

The investigation of the acyclic orientations of a graph is an active research area with applications. The following extremal question is natural: Which graph with a given number of vertices (n) and edges (m) has the most/ the least number of acyclic orientations?

Linial answered the question concerning the minimalizations problem. Cameron [8] investigated the problem of maximal number of acyclic orientations. He conjectured that if m is an integer such that it is the number of the edges of the Turán graph with even parts then this Turán graph is the one with maximal number of acyclic orientations. He and his coauthors computed numerically the number of acyclic orientations in the case when the Turán graph has two parts. During these investigations Cameron recognized that the number of the acyclic orientations of the complete bipartite graph is given by the poly–Bernoulli numbers.

This interpretation is in strong connection with the inclusion–exclusion formula (2) of the poly–Bernoulli numbers also.

Theorem 4 ([14])

$$B_n^{(-k)} = (-1)^n \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} (m+1)^k \tag{2}$$

I presented a new proof of this theorem using the well known fact that the number of acyclic orientations of a graph is equal to the absolute value of the chromatic polynomial of the graph evaluated at -1.

The main result of this chapter, section 10. is the presentation of a new combinatorial interpretation. This new interpretation is the only one that gives a transparent explanation for the recursive formula of poly–Bernoulli numbers.

Definition 2 *Let $\mathcal{G}_n^{(k)}$ be the set of $n \times k$ 01 matrices in that none of the following matrices appear as submatrices.*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We call these matrices Γ–free *matrices.*

The restriction means that in the matrix there are no three 1s in positions that create a Γ. (NE, NW and SE). The study of such matrices appeared in extremal combinatorics. Füredi and Hajnal [10] established the number of 1s in Γ–free matrices. In the case of an $n \times k$ matrix it is $n+k-1$. This means that Γ–free matrices differ from lonesum matrices since the set of lonesum matrices includes for instance the all 1 matrix, too. In spite of this the size of the sets of lonesum and Γ–free matrices are equal.

Theorem 5 ([3])

$$|\mathcal{G}_n^{(k)}| = B_n^{(-k)}.$$

The proof that I present in [3] is of bijective nature, but the bijection is not between the sets of these two matrix classes. In the previous cases the correspondences with the standard set (the obvious interpretation) is more or less simple. In the case of the Γ–free matrices our bijection is more complicated and includes technical parts. The advantage of the theorem is that for Γ–free matrices it is natural to give a recursion.

Actually I could prove combinatorially every known combinatorial property of the poly–Bernoulli numbers.

Theorem 6 ([14])

$$B_n^{(-k)} = B_k^{(-n)}.$$

The symmetry is trivial according to any set corresponding to the poly–Bernoulli numbers. This proof of combinatorial nature is underlined already in Brewbakers work [6].

The next theorem was derived during the analytical investigations of poly–Bernoulli numbers.

Theorem 7 ([2], [11])

$$B_n^{(-k)} = B_n^{(-(k-1))} + \sum_{i=1}^{n} \binom{n}{i} B_{n-(i-1)}^{(-(k-1))}. \tag{3}$$

The proof of Kaneko does not explain the simple form of the formula. My combinatorial proof provides a satisfying explanation.

Through the study of the table of the poly–Bernoulli numbers one can easily recognize the following identity. It is possible to derive it using algebraic manipulations. I use the combinatorial approach: interpret it combinatorially and prove it by a bijection.

Theorem 8 ([3])

$$\sum_{\substack{n,k \in \mathbb{N} \\ n+k=N}} (-1)^n B_n^{(-k)} = 0.$$

The combinatorial content of this theorem is that among the Callan permutations of $N(= n + k)$ the number of those in that n is even and those in that n is odd are equal. In order to demonstrate this fact I formulate a bijection between these two sets.

There are many open problems that are related to poly–Bernoulli numbers. At the end of this chapter I mention some open questions, to that my work may contribute.

On one hand - because of the strong connections to multiple zeta values - researcher defined various generalizations algebraically, on the other hand in the combinatorial interpretations parameters appear that lead to natural ge- neralizations. An interesting problem is to find nontrivial connections between these generalizations.

Hamahata and Masubuchi defined algebraically the *multi–poly–Bernoulli numbers* and introduced the *special multi–poly–Bernoulli numbers* [12]. The combinatorial nature of the formulas require combinatorial interpretations.

The relation of the Stirling numbers of the second kind and first kind and the interesting properties of poly–Bernoulli numbers inspired Komatsu to define analogously the *poly–Cauchy numbers* [16]. At specific parameters the poly–Cauchy numbers are natural numbers also. Komatsu derived numerous identities between poly–Cauchy and poly–Bernoulli numbers and asked for combinatorial interpretation of poly–Cauchy numbers.

The hook formula

The third chapter of my thesis is devoted to the enumeration of the monotone labellings of plane trees. The tree structure is basic, it plays for instance a central role in computer science. The knowledge of the combinatorial pro- perties of special tree classes are often necessary in analysis of algorithm.

Plane trees are rooted trees in which the order of the subtrees is essential. This structure defines naturally a partial order on the set of nodes. It is a basic question, how many ways can this partial order extended to linear order? In other words, how many monotone labellings of the nodes are there? (A monotone labelling is a labelling where if the node v is a descendant of u then the label of v is greater than the label of u.) Let f_T denote this number. The following theorem is a classical result of Knuth.

Theorem 9 ([15])

$$f_T = \frac{n!}{\prod_{v \in V(T)} h_v}, \tag{4}$$

where h_v is the number of descendants of v, including itself. The combinatorial nature of the formula becomes conspicuous by multiplying both sides with the denominator.

$$f_T \times \prod_{v \in V(T)} h_v = n!$$

This identity can be proven with a bijection between the permutations of $[n]$ and pairs (S, H), where S is a monotone labelling of the nodes of T and H is a hook function on the set of nodes. A hook function is a function that maps to each node a positive integer at most the number of the descendants of the node.

In my work I present two bijective proofs of the theorem 9.

Hook formulas has a long story. Frame, Robinson and Thrall discovered first the hook formula investigating standard Young tableaux [9]. Many authors presented different proofs of this result. Novelli, Pak and Stoyanovskii presented a bijective proof [19]. This method can be used to prove the formula for the case of shifted Young tableaux. My first bijective proof of theorem 9. is in the spirit of this method.

I define the bijection in the form of an algorithm. I fix a total order on the nodes of the tree. The algorithm visits the nodes in this order and - if its necessary - changes the label of the actual node. The procedure is similar to the concept of ,,jeu de taquin", the well known concept in the theory of Young tableaux. The moves of the labels are coded in the hook function.

Theorem 10 ([5]) *The algorithm terminates after visiting all nodes of the tree and its output is a unique pair of a monotone labelling of the nodes and of a hook function.*

My second bijection uses the special tree structure, instead of following the method of Novelli, Pak and Stoyanovskii.

I fix a total order in this case also, in that this algorithm visits the nodes. This algorithm moves the actual node in the poset of its descendants. The hook function determines the necessary moves.

Theorem 11 ([5]) *The algorithm terminates and assigns a unique permutation to a given pair of a monotone labelling and a hook function on the nodes of the tree.*

During the enumerations of the classes of trees new hook formulas are discovered. At the end of this chapter I mention some actual new results that are not well understood from combinatorial point of view. I think that my work contributes to the combinatorial proofs of these identity.

312–**avoiding permutations**

The subject of the fourth chapter fits into the research about Catalan numbers. The Catalan number sequence is basic in combinatorics. More than 200 objects are known that are enumerated by these sequence.

The main result of this chapter is a bijection between the 312–avoiding permutations and the triangulations of a polygon. This bijection does not appear in the literature and reveals deeper correspondences between the two sets.

The *triangulation* of a polygon is the dissection of the polygon into triangles by non-intersecting diagonals. We call a $\pi = \pi_1\pi_2\cdots\pi_n$ permutation 312–avoiding iff there are no π_i, π_j, π_k elements with $i < j < k$ but $\pi_j < \pi_k < \pi_i$.

Let $\{P_0, \ldots, P_{n+1}\}/\{0, 1 \ldots, n+1\}$ be the vertices of the polygon. Then every triangle in the dissection has a „middle" vertex.

Lemma 12 ([4]) *In each triangulation, for every $i \in \{1, 2, \ldots, n\}$ there is exactly one triangle where the middle vertex is i.*

I define an algorithm w that visits the vertices of the triangulated polygon in clockwise order and records the labels of the triangles which meet the actual vertex with its third vertex.

Lemma 13 ([4]) *The word $w(T)$ is a 312–avoiding permutation of $[n]$ (we consider permutations as words).*

The bijection is based on the properties of inversions of 312–avoiding permutations. We call a (π_i, π_j) pair inversion if $i < j$ and $\pi_i > \pi_j$. The s–vector of a permutation is $\underline{s} = (s_1, \ldots, s_n)$, where s_k is the number of elements in the permutation, that are greater than k and precede k in the permutation.

$$s_k = |\{\pi_i | \pi_i > k = \pi_j \text{ and } i < j\}|.$$

Observation 14 ([4]) *The inversion table of a* 312*–avoiding permutation* $\pi = \pi_1\pi_2\cdots\pi_n$ *satisfies the following condition:*

$$s_{k+i} \leq s_k - i \quad \textit{for} \quad 1 \leq k \leq n-2 \quad \textit{and} \quad 1 \leq i \leq s_k.$$

Furthermore for an inversion table with this additional property the corresponding permutation is a 312*–avoiding permutation.*

The next observation underlines the relation between the s–vector and the corresponding triangulation.

Observation 15 ([4]) *Let* T *be a triangulation. Take the triangle labelled by* i*. Its* $[B_i, C_i]$ *side determines the* i*–th condition of the* s*–vector of the permutation* $w(T)$*:*

$$s_i = l(C_i) - l(B_i) - 1.$$

This observation allows to build a corresponding triangulation to a 312–avoiding permutation according the s–vector of the permutation.

These observations and lemmas lead to the main theorem of the third chapter of my thesis.

Theorem 16 ([4]) *The map* w *is a bijection between the set of triangulations of a convex* $(n+2)$*-gon and the set of 312-avoiding permutations of* $[n]$*.*

My bijection has more advantages. I mention first the possibility of a generalization to the case of k–triangulations.

A k–triangulation is a maximal set of diagonals in a polygon, such that there are no $k+1$ diagonals that mutually intersect. A k–triangulation can be viewed also as a union of $(2k+1)$–stars. I supplement this point of view with the observation that the $2k+1$–stars can be labelled, for instance by the position of their middle vertex. Similarly to my bijection an algorithm can be defined that by visiting the vertices of the polygon records the labels of the stars. This way we can order to a k–triangulation a permutation of the multiset $\{1^k 2^k \cdots n^k\}$. Through the understanding of this permutation we can investigate k–triangulations in a bijective way. Unfortunately this program covers many open questions.

My bijection underlines the central role of inversions in a permutation. It brings new ideas to combinatorial solutions of further enumeration results. I work out one instance of it in details in my thesis.

The inversions of a permutation can be recorded basically in two different ways. The one is, that I call s–vector, the other that I call c–vector. A c–vector is a vector of non-negative integers $(c_1, \ldots, c_n)$, where c_k is the number of the elements in the permutation that is less than k, but appears after k in the permutation.

$$c_k = |\{\pi_i : \pi_i < k = \pi_j \quad \text{and} \quad i > j\}|.$$

I define an inversion diagram in order to visualize the special conditions and the relations of these two vectors corresponding to a 312–avoiding permutation. The posets that are defined by the natural order on these two vectors are the well known Tamari resp. Dyck lattice.

I mention in my work some combinatorial problems, that are enumerated by the intervals of these lattices. It is an open question whether there are simple bijections between these combinatorial objects and pairs of 312–avoiding permutations. I close the third chapter of my work with a description of one example.

The problem of pattern avoidance was generalized in many ways. Researcher investigated perfect matchings on ordered graphs without appearing of special patterns. (*A perfect matching on an ordered graph* is a graph where there's is a total order on the nodes and the degree of each node is exactly one.) The 312–avoiding permutations are strongly connected with perfect matchings without the pattern *abccab* (see Figure 1.).

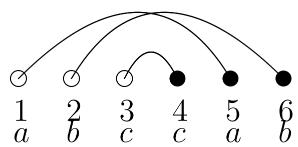

Figure 1: The pattern *abccab*

Let $M_n(abccab)$ denote these perfect matchings. Then

Theorem 17 ([13])

$$|M_n(abccab)| = \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix} \qquad (5)$$

I present a new combinatorial proof of this theorem. It is well known that (5) is the number of intervals in the Dyck lattice. I identify an interval with a pair of 312–avoiding permutations (π, σ), such that the c–vectors of the permutations fulfill the condition: $c(\pi) \leq c(\sigma)$. I define a map that order to a perfect matching two permutations.

Lemma 18 *The two permutations π and σ that the map order to a perfect matching is unique, avoid the pattern* 312 *and it is true that $c(\pi) \leq c(\sigma)$.*

This theorem proves Theorem 17. I think my bijection is in some sense very natural.

References

[1] T. Arakawa, M. Kaneko. On Poly–Bernoulli Numbers, *Comment Math. Univ. St. Paul* **48.2** (1999), 159–167.

[2] T. Arakawa, M. Kaneko. Multiple Zeta Values, Poly–Bernoulli Numbers and Related Zeta Functions, *Nagoya Math. J.* **153** (1999), 189–209.

[3] B. Bényi, P. Hajnal. Combinatorics of poly–Bernoulli numbers, accepted for publication in *Studia Scientarium Mathematicarum Hungarica*

[4] B. Bényi. A simple bijection between 312–avoiding permutations and triangulations, accepted for publication in *Journal of Combinatorial Mathematics and Combinatorial Computing*

[5] B. Bényi. Bijective proofs of the hook formula for rooted trees. *Ars Combinatoria* **106** (2012), 483–494.

[6] Ch. Brewbaker. A Combinatorial Interpretation of the Poly-Bernoulli numbers and two Fermat Analogue, *Integers* **8** (2008), A3

[7] D. Callan. Comment to A099594 in OEIS, July 22nd, (2008).

[8] P. Cameron, C. Glass, R. Schumacher. Poly–Bernoulli numbers, Peter Cameron's blog, URL http://cameroncounts.wordpress.com/2014/01/19/poly-bernoulli-numbers/ (2014).

[9] J. S. Frame, G. B. Robinson, R.M. Thrall. The hook graphs of the symmetric group, *Canad. J. Math* **6** (1954), 316–325.

[10] Z. Füredi, P. Hajnal. Davenport-Schinzel theory of matrices, *Discrete Mathematics* **103** (1992), 233–251.

[11] Y. Hamahata, H. Masubuchi. Recurrence Formulae for Multi–poly–Bernoulli Numbers, *Integers: Electronic Journal of Combinatorial Number Theory* **7** (2007), A46

[12] Y. Hamahata, H. Masubuchi. Special Multi–Poly–Bernoulli Numbers, *Journal of Integer Sequences* **10** (2007), Article 07.4.1, 6pp.

[13] V. Jelinek. Dyck paths and pattern avoiding matchings, *European Journal of Combinatorics* **28** (2007), 202–213.

[14] M. Kaneko. Poly-Bernoulli numbers. *Journal de Theorie des Nombres de Bordeaux* **9** (1997) 221–228.

[15] D. Knuth. *The Art of Computer Programming, Vol. III.*, Sorting and Searching, Addison Wesley Longman, (1998).

[16] T. Komatsu. Poly–Cauchy numbers, *Kyushu J. Math.* **67** (2013).

[17] S. Launois. Combinatorics of H–primes in quantum matrices, *Journal of Algebra* **309** (2007), 139–167.

[18] L. Lovász, *Combinatorial Problems and Excercises*, 2nd Edition, North-Holland Publishing Co, Amsterdam, (1993).

[19] J.-C. Novelli, I. Pak and A.V. Stoyanovskii. A direct bijective proof of the hook-length formula, *Discrete Math. Theoret. Computer Sience* **1** (1997), 53–67.

[20] V. Pilaud, F. Santos. Multitriangulations as complexes of star polygons, *Discrete and Computational Geometry* **41** (2009), 284–317.

[21] K. Vesztergombi. Permutations with restriction of middle strength, *Studia Scientiarium Mathematicarum Hungarica* **9** (1974), 181–185.