Dynamical *r*-matrices and their appearance in Calogero-Moser models

PhD thesis

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1. Preliminaries

The theory of integrable systems is a very rapidly developing branch of modern mathematical physics. From a physical point of view the importance of this subject is quite clear. A great amount of knowledge about nature is based on the use of special, exactly solvable models. The harmonic oscillator and the Kepler problem play a central role in classical mechanics as well as in quantum theory. The KdV and KP equations help us to understand the wave motion of shallow water and indicate the behaviour of `solitons' in general. The nonlinear Schrödinger equation found applications in the theory of optical fibres. The sine-Gordon equation serves as a theoretical laboratory for particle physicists. These selected examples are enough to convince us about the importance of integrable systems, even in their own right. Furthermore, it is a well-known fact that beside numerical simulations it is perturbation theory that allows us to get an insight into the details of physical phenomena. It is worth keeping in mind that perturbative calculations rely heavily on exactly solved problems. From a mathematical point of view it is much simpler and much more obvious to give grounds for the investigations of these models. First, they are challenging mathematical problems. Second, almost all branches of mathematics can be used during calculations. Not only the classical parts of mathematics are applicable, but the newest methods play a role, too.

2. Aims and methods

The Calogero-Moser models [1, 2, 3] are very important integrable hamiltonian systems. The Calogero-Moser type many particle systems have been much studied recently due to their fascinating mathematics and applications [4] ranging from solid state physics to Seiberg-Witten theory. The definition of these models involves a root system and a potential function depending on the inter-particle `distance'. The potential is given either by the Weierstrass *P*-function or one of its (hyperbolic, trigonometric or rational) degenerations. The classical equations of motion of the models admit Lax representations, which underlie their integrability. Babelon and Viallet [5] pointed out that Liouville integrability can be understood in general as a consequence of the Poisson brackets of the Lax matrix having an *r*-matrix form. The *r*-matrix may depend on the dynamical variables. When the *r*-matrix really does depend on the phase space variables, one says that it is `dynamical'. The *r*-matrix corresponding to the standard Lax representations of the degenerate Calogero-Moser models was studied by Avan and Talon [6], who found that it is necessarily dynamical, and may be chosen so as to be momentum independent. It is a natural question whether the equations of motion of these models admit such Lax representations, which can be characterized by non-dynamical *r*-matrices. The obvious method is to perform such gauge transformations on the Lax matrices that result in non-dynamical *r*-matrices. Proceeding in this manner, we managed to give a complete description of the non-dynamical, constant *r*-matrices of the standard

Calogero-Moser models associated with degenerate potential functions, which can be obtained by gauge transformations of their usual Lax representation.

It is a well-known fact that *r*-matrices play a central role in the modern theory of integrable systems. As we have mentioned, Avan and Talon pointed out that dynamical *r*-matrices appear naturally in the Calogero-Moser models. Recently, Li and Xu [7] found a close connection between dynamical *r*-matrices and certain generalized spin Calogero-Moser models. This observation served as one of the main motivations in our study of the theory of dynamical *r*-matrices. The dynamical *r*-matrices that arise in this context must satisfy the so-called modified classical dynamical Yang-Baxter equation (mCDYBE) [8]. Dynamical generalizations of the Yang-Baxter equations and the associated algebraic structures are in the focus of current interest due to their applications in the theory of integrable systems and other areas of mathematical physics and pure mathematics [9]. One of our aims was to give a direct proof of the mCDYBE for the so-called canonical *r*-matrix found earlier in [10, 11]. As opposed to the local power series expansion, we used the well-known holomorphic functional calculus of linear operators [12] to define the canonical *r*-matrix, and thus our proof is valid globally on the maximal domain of the `dynamical variable'. An advantage of our proof is that it also yields a uniqueness result for the holomorphic function that enters the definition of the *r*-matrix. Namely, the mCDYBE translates into a functional equation that has a unique solution under certain natural conditions.

Our purpose was also to further develop the construction of dynamical *r*-matrices building mainly on the seminal paper of Etingof and Varchenko [8]. Motived by the evaluation homomorphism technique found in their work, we succeeded in associating elliptic dynamical *r*-matrices with any finite-dimensional self-dual Lie algebra endowed with an automorphism of finite order. These *r*-matrices can be regarded as generalizations of Felder's [13] celebrated elliptic dynamical *r*-matrices.

3. Results

Concerning the degenerate Calogero-Moser models, our main results [17, 18] can be summarized as follows:

- We have determined the most general momentum independent *r*-matrices for the standard Lax representation of these systems.
- We have selected those *r*-matrices whose coordinate dependence can be gauged away.
- We have determined the constant *r*-matrices resulting from gauge transformation. These *r*-matrices are related to well-known *r*-matrices. In the hyperbolic/trigonometric case a non-dynamical *r*-matrix equivalent to a real/imaginary multiple of the Cremmer-Gervais [14] classical *r*-matrix is found. In the rational case the constant *r*-matrix corresponds to the antisymmetric solution of the classical Yang-Baxter equation associated with the Frobenius subalgebra of *gl(n)* consisting of the matrices with vanishing last row.

These claims are consistent with previous results of

Hasegawa [15] and others, which imply that Belavin's [16] elliptic *r*-matrix and its degenerations appear in the Calogero-Moser models. The advantages of our analysis are that it is elementary and also clarifies the extent to which the constant *r*-matrix is unique in the degenerate cases.

Concerning the theory of dynamical *r*-matrices, our main results can be summarized as follows:

- We have given a new, direct proof [19] of the statement that the `canonical' *r*-matrix satisfies the modified classical dynamical Yang-Baxter equation on arbitrary finite-dimensional self-dual Lie algebras. Our proof also shows the uniqueness of the complex holomorphic function that enters the definition of this *r*-matrix.
- We have associated a dynamical *r*-matrix with any graded self-dual Lie algebra subject to rather mild conditions [20, 21].
- We have applied this construction to the general class of affine Lie algebras corresponding to the automorphisms of the finite-dimensional self-dual Lie algebras that preserve the scalar product and are of finite order [20, 21].
- Motived by the work of Etingof and Varchenko [8], we have associated elliptic, spectral parameter dependent, dynamical *r*-matrices with every finite-dimensional self-dual Lie algebra endowed with an automorphism of finite order [20, 21]. We have also recovered Felder's [13] famous dynamical *r*-matrices.

The possible application of these *r*-matrices to spin Calogero-Moser type integrable systems is an interesting problem for future study.

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