Spherically symmetric, static solutions in k-essence theory and minimally coupled scalar field fluid description in the nonorthogonal 2+1+1 formalism of spacetime decomposition

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1 Introduction

Various observations (galaxy rotation curves, cosmic microwave background, Ia supernovae) confirm that the universe contains dark matter and dark energy. It was once assumed that dark matter consists of compact, astrophysical objects (MACHOs) distributed within galactic halos, as well as asteroid-like primordial black holes. Observations developed to detect these objects, such as gamma-ray bursts, microlensing, X-ray pulsar lensing, and gravitational wave measurements, have not confirmed that they can constitute the entire amount of dark matter [1, 2]. Experiments relying on the participation of dark matter in weak interaction also ended unsuccessfully concerning the observation of dark matter particles [3]-[6].

Dark energy, associated with the accelerating expansion of the universe, could be a cosmological constant Λ, which has been proposed to represent the universe vacuum energy density. However, the measured value of the cosmological constant Λ differs by orders of magnitude from the prediction given by quantum field theory [7, 8]. Hence, the origin of such a constant remained unclear. Numerous models have been developed to reproduce the measurements related to the dark energy. For example, in Unified Dark Energy and Unified Dark Matter theories, dark matter and dark energy are identified with different states of an exotic fluid [9, 10].

Modified theories of gravity altering general relativity have been introduced to explain the observational results linked to dark matter and dark energy. The simplest representative of modified theories of gravity is the Ostrogradsky-instability-free Horndeski theory, which extends the metric degrees of freedom with a scalar field. These theories are constrained by observations. Due to the restrictions obtained from the GW170817 gravitational wave and GRB170817A gamma-ray burst events, the Kinetic Gravity Braiding subclass of Horndeski theory provides a viable model [11, 12], which also includes the k-essence theory appearing in the title of the dissertation [13].

Alongside the problem of identifying dark matter and dark energy, there has also been a growing demand for a theory that unifies the four fundamental interactions in physics. This requires a quantum field theory reformulation of the gravitational interaction currently described by general relativity, known as the theory of quantum gravity. Nevertheless, general relativity is not renormalizable [14, 15]. Considering quantum field theory, the 'correct' classical theory of gravity – regardless it is the general relativity or a modified gravity model – may originate from a low-energy effective field theory (EFT) approximation of a yet-to-be-formulated quantum gravity theory [16, 17]. From this perspective, modified theories of gravity, such as Kinetic Gravity Braiding and the k-essence theory, can be regarded as theories containing higher-order corrections to the general relativity [18, 19].

The derivation of spacetimes containing black holes or other objects and analyzing their stability, the examination of gravitational waves focusing on the ringdown phase, and the fitting of cosmological models to observational data without material dark matter and dark energy all provide opportunities for investigating new theories and testing their viability [20]-[22]. The derivation and study of exact solutions in the individual theories requires complex and intricate calculations. To make the calculations more manageable, new mathematical techniques have emerged, one of which concentrates on the decomposition of the 4-dimensional spacetime. From the perspective of this dissertation, the spacetime decomposition methods to be highlighted are the $1 + 1 + 2$ dimensional covariant formalism introducing kinematic quantities [23, 24], and the orthogonal $s+1+1$ dimensional double foliation employing metric variables [25]. The latter is based on the $3+1$ dimensional ADM (Arnowitt–Deser–Misner) formalism developed for the Hamiltonian formulation of gravity [26].

2 Motivations and goals

Linear order perturbations of spherically symmetric and static spacetimes in Horndeski theory and in its extension the Gleyzes–Langlois–Piazza–Vernizzi (GLPV) theory [28] were analized in Ref. [27]. The perturbation variables and equations were decomposed into even- and odd-parity sectors. The perturbation equations were derived from an EFT action using the orthogonal $s+1+1$ dimensional spacetime decomposition ($s=2$). An additional Helmholtz-like decomposition of the two-dimensional vectors and the metric tensor was also applied. A conformal and radial unitary gauge was chosen, reducing the number of odd-parity perturbation variables from three to two. By introducing a Lagrange multiplier and spherical harmonics, two dynamical equations were derived for the odd-parity variables. Thanks to a suitable variable transformation a master equation was obtained for the odd-parity gravitational sector. Similar calculations were planned to be carried out for the even-parity sector. There was a constraint during the gauge fixing namely, that the orthogonality of the $2+1+1$ spacetime decomposition had to be maintained. This led to an ambiguous gauge fixing, as one of the even-parity perturbation variables contained an arbitrary function.

For the reasons mentioned above, it is necessary to generalize the orthogonal $2+1+1$ dimensional spacetime decomposition, aiming to develop a mathematical formalism, where the 3-dimensional hypersurfaces are not orthogonal to each other. Returning to the problem described in Ref. [27], one of the goals of generalizing the orthogonal $2+1+1$ formalism is to achieve an unambiguous gauge fixing. The Hamiltonian formalism of general relativity was discussed in Ref. [29] using the orthogonal $s+1+1$ decomposition introduced in Ref. [25]. It is also worthwhile to derive the canonical equations of general relativity using the nonorthogonal $2+1+1$ formalism.

Ref. [27] includes the derivation of the field equations for Horndeski theory using the EFT approximation in the case of spherically symmetric and static background. The Horndeski Lagrangians L_2^{H}, L_3^{H} and L_4^{H} were expressed in terms of variables used in the orthogonal $2+1+1$ spacetime decomposition. The functional dependence of the EFT action were chosen accordingly, and the formal field equations were derived from this EFT action. However, Ref. [27] does not address the analysis of spacetime solutions. Therefore, by applying the nonorthogonal 2+1+1 dimensional spacetime decomposition and choosing conformal and radial unitary gauge, it is useful to determine the field equations for spherically symmetric and static backgrounds in the case of an EFT action and to compare them with the results of [27]. It is also worth considering the derivation of spacetime solutions in the viable Horndeski subtheories, like in k-essence theories based on the obtained field equations.

Ref. [30] revealed that the energy-momentum tensor of a minimally coupled Klein–Gordon scalar field with a timelike gradient can be described as an ideal fluid. The metric's $1+3$ decomposed form was used during these calculations. The gradient of the scalar field was identified with the timelike unit vector u^a used in the spacetime decomposition. It was proven that the resulting energy-momentum tensor with isotropic pressure p^{PF} and energy density ρ^{PF} can equivalently be derived from an $L_1 = p^{PF}$ or an $L_2 = -\rho^{PF}$ Lagrangian, provided that the scalar field is either massless and free (with $V(\phi) = 0$ and $X = \partial_a\phi\partial^a\phi/2 \neq 0$) or purely potential (with $X = 0$ and $V(\phi) \neq 0$). The characteristics of the energy-momentum tensor associated with a minimally coupled scalar field having a spacelike gradient were studied in Ref. [31]. The formulae (4) and (5) of Ref. [31], derived by using the $3+1$ spacetime decomposition and relied on a scalar field with a timelike gradient, contain errors. Ref. [32] pointed out that the timelike unit vector u^a cannot be associated with spacelike or null scalar field gradient. The equations (4) and (5) of Ref. [31] with regard to the Klein–Gordon scalar field were corrected in Ref. [32], and it was demonstrated that when the scalar field gradient is spacelike or null, the corresponding energy-momentum tensor describes an imperfect fluid.

Since an imperfect fluid features different tangential and radial pressure components, it may be necessary to use the $2+1+1$ spacetime decomposition. Hence, it is recommended to revisit the energy-momentum tensor of the minimally coupled scalar field with timelike, spacelike, and null scalar field gradients with the application of the $2+1+1$ decomposition.

3 Thesis points

T1 I developed the formalism of the nonorthogonal $2+1+1$ spacetime decomposition. With the help of this formalism, I achieved an unambiguous gauge fixing for the description of spherically symmetric, static spacetime perturbations in Horndeski theory. Additionally, I derived the canonical equations of motion for general relativity within the framework of this formalism. (publications: A1, A2, A3, A4)

T1/a For the nonorthogonal double foliation of the 4-dimensional spacetime, I determined the (n^a, m^a) and (k^a, l^a) bases adapted to the families of 3-dimensional hypersurfaces characterized by t =const. (denoted as S_t) and χ =const. (denoted as $\mathfrak{M}_\chi$) respectively, along with the relations between the two bases using duality relations. The transformation between the two bases corresponds to a hyperbolic rotation, where the rapidity is proportional to the metric variable $\mathcal{N}$. When the foliation is orthogonal, $\mathcal{N}=0$. I gave the forms of the 4-dimensional metric, together with the evolution vectors $(\partial/\partial t)^a$ and $(\partial/\partial\chi)^a$ in both bases. I derived the geometric quantities characterizing the embedding of the 2-dimensional surface $\Sigma_{t\chi}$ into the 4-dimensional spacetime in both the (n^a, m^a) and (k^a, l^a) bases. In the formalism, the embedding of $\Sigma_{t\chi}$ is determined by four extrinsic curvatures (K_{ab}, L^*_{ab}, L_{ab}, K^*_{ab}), two normal fundamental forms ($\mathcal{K}_a$, $\mathcal{L}_a$), two additional forms ($\mathcal{K}^*_a$, $\mathcal{L}^*_a$), four normal fundamental scalars ($\mathcal{K}$, $\mathcal{L}^*$, $\mathcal{L}$, $\mathcal{K}^*$), and four accelerations ($\mathfrak{a}_a$, $\mathfrak{b}^*_a$, $\mathfrak{b}_a$, $\mathfrak{a}^*_a$).

T1/b I determined the 2- and 3-dimensional vorticities of the hypersurface normals n^a and l^a, and of the vectors m^a and k^a, through calculating Lie brackets of the bases and considering the Frobenius theorem. The projections of the 3-dimensional vorticities of m^a and k^a along the normals are related to the forms $\mathcal{L}^*_a$ and $\mathcal{K}^*_a$, therefore, these forms do not exhibit the symmetry properties of the normal fundamental forms.

T1/c I derived the dependence of the embedding variables expressed in the (n^a, m^a) and (k^a, l^a) bases from the metric variables and their derivatives.

T1/d I derived gauge transformation rules for the even-parity ($\delta\phi$, δN, δM, N, P, V, A, B) and odd-parity (Q, W, C) perturbation variables of spherically symmetric, static background within the framework of nonorthogonal $2+1+1$ spacetime decomposition (similarly to Ref. [27]). I chose radial unitary and conformal gauge by fixing $\widehat{\delta\phi} = 0$ and $\widehat{B} = 0, \widehat{C} = 0$. Then I achieved an unambiguous gauge fixing by imposing an additional condition: $\widehat{P} = 0$.

T1/e I expressed the Ricci scalar in terms of the quantities used in the (n^a, m^a) and (k^a, l^a) bases for the Hamiltonian formulation of gravity. I found that, for studying the time evolution of the canonical variables, it is advantageous to continue the calculations in the (n^a, m^a) basis. To express the Lagrangian in Liouville form, I formulated the Hamiltonian and the two momentum constraints. I replaced the embedding variables K_{ab}, $\mathcal{K}^a$, $\mathcal{K}$ with the canonical momenta π^{ab}, p_a, p, which were then substituted back into the Lagrangian. I determined the equations of motion for the canonical coordinates g_{ab}, M^a, M. I derived the equations of motion for the canonical momenta π^{ab}, p_a, p by calculating the Poisson brackets defined for the canonical pairs.

T2 I derived the EFT field equations using the nonorthogonal $2+1+1$ dimensional spacetime decomposition. I showed that in the nonminimally coupled k-essence theory, the Schwarzschild solution emerges exclusively from the general relativity limit of the theory. The derived spacetime solutions can contain a singularity without a horizon, or a black hole with one/two horizons, or a black hole with a logarithmic-type singularity in addition to the central singularity, which located either outside the event horizon or between the two horizons. (publication: A6)

T2/a I expressed the Horndeski theory Lagrangian contributions $L_2^{\rm H}$, $L_3^{\rm H}$, $L_4^{\rm H}$ in terms of the quantities introduced when using either (n^a, m^a) or (k^a, l^a) bases in the formalism of the nonorthogonal $2+1+1$ spacetime decomposition. I found that due to the radial unitary gauge, the equations take a simpler form when using the (k^a, l^a) basis.

T2/b I assumed that the EFT action depends functionally on the variables $\mathcal{N}$, N, M, $\mathcal{K}^*$, K^*, $\mathcal{L}$, L, λ, R, ϕ at first order, which appear in the decomposed form of the Lagrangian $L_2^{\rm H}$, $L_3^{\rm H}$ and $L_4^{\rm H}$, when using the (k^a, l^a) basis. Contrarily, the action depends on the variables $\mathcal{N}$, N, M, $\mathcal{K}$, K, $\mathcal{L}^*$, L^*, λ^*, R, ϕ when using the (n^a, m^a) basis. I derived the field equations for the spherically symmetric and static background by varying the EFT action, employing both the (n^a, m^a) and (k^a, l^a) bases.

T2/c I specialized the EFT field equations to nonminimally coupled k-essence theories by choosing $\bar{G}_2(\phi, X)$, $\bar{G}_3(\phi, X) = 0$, $\bar{G}_4(\phi, X) = \bar{G}_4(\phi)$ and $\bar{G}_5(\phi, X) = 0$. I showed that the Schwarzschild solution can only be derived from the Einstein–Hilbert action within this subclass. An additional simplification is achieved by imposing the condition $\bar{N} = \bar{M}^{-1}$ on the metric functions. I derived the equations governing the functions $\bar{N}^2$, $\bar{G}_{2X}$, $\bar{G}_{2\phi}$ and $\bar{G}_{4\phi}$. These are necessary for finding the spacetime solutions. When a specific form of $\bar{G}_4(\phi)$ is given, the remaining functions can be computed. The given $\bar{G}_4(\phi)$ fixes the theory.

T2/d First, I considered the case, where $\bar{G}_4(\phi) = \phi^\alpha = r^\alpha$, with $\alpha \geq 0$. When $\alpha = 0$, I chose $\bar{G}_4(\phi) = (16\pi G)^{-1}$. Depending on the value of the integration constant Λ appearing during the integration of the $\bar{N}^2$ metric component, I obtained the Schwarzschild ($\Lambda = 0$) and the Schwarzschild-(anti) de Sitter solutions ($\Lambda \neq 0$). I have found different spacetime solutions depending on the signs of the integration constants Λ and Q in the case of $\alpha = 1$. The spacetime contains a black hole with two horizons for $\Lambda > 0$ and $Q < 0$. The spacetime contains a black hole with a single horizon, if $\Lambda > 0$, $Q > 0$, or $\Lambda < 0$, $Q < 0$, while the spacetime contains a naked singularity when $\Lambda < 0$, $Q > 0$. In case of $\alpha \geq 1$, I identified different solutions affected by the signs of the integration constants Λ and Q. I employed Descartes' rule of signs to establish these results. If $\Lambda = 0$ while $C \neq 0$, the spacetime contains a black hole with a single horizon. Regarding the number of horizons, I found similar results as for $\alpha = 1$. Given that $\Lambda \neq 0$ and $C \neq 0$, the spacetime contains the following objects: $\Lambda > 0$, $C < 0$ - a black hole with two horizons; $\Lambda > 0$, $C > 0$ or $\Lambda < 0$, $C < 0$ - a black hole with one horizon; $\Lambda < 0$, $C > 0$ - a naked singularity. None of the derived spacetime solutions are asymptotically flat, even if $\Lambda = 0$.

T2/e Upon choosing $\bar{G}_4(\phi) = B(\phi + A)$, I derived additional spacetime solutions. After fixing one of the integration constants, I plotted the function $\bar{N}^2$ and used its disappearance to determine the locations of the event horizons. In the case of $Bm = 1$, if $\Lambda m^2 < 0$, the spacetime contains a black hole with one horizon, whereas for $\Lambda m^2 > 0$, a singularity without a horizon appears at $r = 0$. When choosing $Bm = -1$, if $\Lambda m^2 < 0$, in addition to the single horizon concealing the central singularity, a horizonless singularity also emerges. If $\Lambda m^2 > 0$, the central singularity is covered by two horizons, and a logarithmic type singularity is also present between the horizons. When choosing $\Lambda m^2 = 1$, the central singularity is always hidden behind a horizon, and for $Bm < 0$, an additional logarithmic-type singularity appears outside the horizon.

T3 I derived the energy-momentum tensor of the minimally coupled Klein–Gordon scalar field and the general scalar field defined by $L(\phi, X)$, using the $2+1+1$ decomposed form of the 4-dimensional metric. In either case, the following apply to the energy-momentum tensors: perfect fluid if the scalar field gradient is timelike; type I imperfect fluid if the scalar field gradient is spacelike; type II imperfect fluid if the scalar field gradient is null. In addition, I determined the energy conditions for the energy-momentum tensors. (publication: A5)

T3/a If the gradient of the Klein–Gordon scalar field is timelike, the weak energy condition implies $-\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq -2V$, the dominant energy condition $V \geq 0$, and the strong energy condition $-\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq V$. All energy conditions are satisfied if $0 \leq V \leq -\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi$.

T3/b If the Klein–Gordon scalar field has a spacelike gradient, I found the following energy conditions for the type I imperfect fluid energy-momentum tensor: i) weak: $\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq -2V$; ii) dominant: $V \geq 0$; iii) strong: $V \leq 0$; iv) all: $V = 0$, the latter implies $\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi \geq 0$. The derived energy-momentum tensor can be interpreted as a superposition of the energy-momentum tensors corresponding to an ideal fluid and two radiation (null dust). In the case of spherical symmetry, the null dusts represent an ingoing and an outgoing radiation.

T3/c Considering a Klein–Gordon scalar field with null gradient, the weak, dominant, and strong energy conditions imply $V(\phi) = 0$ for the type II imperfect fluid energy-momentum tensor, and then the energy-momentum tensor of the scalar field represents a null dust.

4 Publications

Publications related to the thesis

A1 C. Gergely, Z. Keresztes, L. Á. Gergely, *Hamiltonian Dynamics of Doubly-Foliable Space-Times*, Universe **2018**, 4(1), 9 [arXiv:2007.00983[gr-qc]] (2018), IF: 2,6

A2 C. Gergely, Z. Keresztes, and L. Á. Gergely, *Gravitational dynamics in a* $2+1+1$ *decomposed spacetime along nonorthogonal double foliations: Hamiltonian evolution and gauge fixing*, Phys. Rev. D **99**, 104071 [arXiv:1905.00039[gr-qc]] (2019), IF: 5,3

A3 C. Gergely, *Feketelyuk-perturbációk skalár-tenzor gravitáció-elméletekben*, Fizikai Szemle 70: 3 pp. 97-102. , 6 p. (2020)

A4 C. Gergely, Z. Keresztes, L. Á. Gergely, $2+1+1$ *General relativistic Hamiltonian dynamics and gauge fixing in Horndeski gravity*, Romanian Astron. J., Vol. **30**, No. 1, 45-54 (2020), IF: 1,9

A5 C. Gergely, Z. Keresztes, L. Á. Gergely, *Minimally coupled scalar fields as imperfect fluids*, Phys. Rev. D **102**, 024044 [arXiv:2007.01326[gr-qc]] (2020), IF: 5,3

A6 C. Nagy, Z. Keresztes, L. Á. Gergely, *Spherically symmetric, static black holes with scalar hair, and naked singularities in nonminimally coupled k-essence*, Phys. Rev. D **103**, 124056 [arXiv:2210.00287[gr-qc]] (2021), IF: 5,3

Conferences

B1 C. Nagy, Z. Keresztes, L. Á. Gergely, *Hamiltonian dynamics in doubly-foliable space-times*, BGL2017: 10th Bolyai-Gauss-Lobachevsky Conference on Non-Euclidean Geometry and Its Applications, Gyöngyös, Hungary, presentation (2017).

B2 C. Gergely, Z. Keresztes, L. Á. Gergely, *Gravitational waves and scalar perturbations in spherically symmetric scalar-tensor theories*, Solvay Workshop "Sugar2018: Searching for the sources of galactic and extragalactic cosmic rays", Université Libre Brussels, Belgium, poster (2018).

B3 C. Gergely, Z. Keresztes, L. Á. Gergely, *Hamiltonian dynamics and gauge-choices in doubly-foliable spacetimes*, Cosmology group of Szczecin, Szczecin, Poland, presentation (2018).

B4 C. Gergely, Z. Keresztes, L. Á. Gergely, *Doubly-foliable spacetimes and gauge-fixing of scalar-tensor perturbations*, 15. Marcel Grossmann (MG15) conference, University of La Sapienza, Rome, Italy, presentation (2018), Conference proceeding: C. Gergely, Z. Keresztes, L. Á. Gergely, *Doubly-foliable space-times and gauge-fixing of perturbations in scalar-tensor gravity theories,* The Fifteenth Marcel Grossmann Meeting, pp. 386-391 (2022).

B5 C. Gergely, Z. Keresztes, L. Á. Gergely, *Gauge-fixing of black hole perturbations in the beyond Horndeski theories*, FUture of Gravitational Alternatives (FUGA) conference, University of Valencia, Valencia, Spain, presentation (2018).

B6 C. Gergely, Z. Keresztes, L. Á. Gergely, *Black hole perturbations and gauge fixing in generalised kinetic braiding theories*, Texas2019: 30th Texas Symposium on Relativistic Astrophysics: Gravity – Modified gravity (B6), Portsmouth, United Kingdom, presentation (2019).

B7 C. Gergely, Z. Keresztes, L. Á. Gergely, *EFT action for spherically symmetric, static black holes*, QG-MM: COST CA18108: First annual conference, Granada, Spain, poster (2020).

B8 C. Nagy, Z. Keresztes, L. Á. Gergely, *Spherically symmetric, static black holes in nonminimally coupled k-essence theory*, GRAVITEX2021: International Conference on Gravitation: Theory an Experiment, online presentation (2021).

B9 C. Nagy, Z. Keresztes, *Can high frequency gravitational waves modify the Vainshtein mechanism?*, GRAVITEX2021: International Conference on Gravitation: Theory an Experiment, online poster presentation (2021)

B10 C. Nagy, Z. Keresztes, L. Á. Gergely, *Spherically symmetric, static black holes with scalar hair, and naked singularities in nonminimally coupled k-essence*, CERS12: 12th Central European Relativity Seminar, Budapest, Hungary, poster (2022).

B11 C. Nagy, Z. Keresztes, L. Á. Gergely, *Spherically symmetric, static spacetimes and their perturbations in the effective field theory of gravity*, QG-MM: "Third Annual Conference – Napoli", Naples, Italy, presentation (2022).

B12 C. Nagy, Z. Keresztes, L. Á. Gergely, *Black hole perturbations in the effective field theory of gravity*, DarkCosmoGrav: "New Frontiers in Particle Physics, Gravity and Cosmology", University of Pisa, Pisa, Italy, poster (2023).

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