Vector Sum Problems in Convex and Discrete Geometry

Thesis Booklet

Rainie Heck

Supervisor: Gergely Ambrus, PhD

Doctoral School of Mathematics

Bolyai Institute

University of Szeged,

Faculty of Science and Informatics

2026

Szeged

1 Introduction

This dissertation addresses two interrelated problems regarding vector sums from discrete and convex geometry: the vector balancing problem and the Steinitz problem. We study a generalization of the vector balancing problem and a reduction of the Steinitz problem using tools from probability, linear algebra, and geometry. The work in this dissertation is based on the following two publications:

- Gergely Ambrus and Rainie Bozzai. Colourful vector balancing. Mathematika, 70(4), August 2024.
- Gergely Ambrus and Rainie Heck. A note on the Steinitz constant. Accepted for publication; Mathematika, 2026.

2 Colorful Vector Balancing

In Chapters 2 and 3 we consider a generalization of the *vector balancing problem*, which can be stated as follows: given symmetric convex bodies $K, L \in \mathcal{K}_o^d$ with associated Minkowski norms $\|\cdot\|_K, \|\cdot\|_L$ and any collection of vectors $v_1, ..., v_n \in K$, select signs $\varepsilon_1, ..., \varepsilon_n \in \{\pm 1\}$ so that $\left\| \sum_{i \in [n]} \varepsilon_i v_i \right\|_L$ is minimal. The term *vector balancing* is readily motivated by the following interpretation: placing the vectors into the two plates of a scale according to their associated signs, the problem asks for achieving a nearly equal balance, that is, forcing the sum of the vectors in the plates to be as close as possible.

In order to facilitate the coming work, we introduce the notion of *vector balancing constants* of $K, L \in \mathcal{K}_o^d$. To this end, we define the n-vector balancing constant:

$$\mathrm{vb}(K, L, n) = \max_{v_1, ..., v_n \in K} \min_{\varepsilon_1, ..., \varepsilon_n \in \{\pm 1\}} \|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n\|_L. \tag{1}$$

One fascinating aspect of the problem is that even though $\mathrm{vb}(K, L, n)$ depends on the bodies $K, L \in \mathcal{K}_o^d$ and the two parameters $d, n \in \mathbb{N}$, the optimal bounds turn out to be independent of the number of vectors n in the case $n \geq d$ [5, 16, 21]. In light of this observation, we can define the *vector balancing constant* of K and L,

$$\mathrm{vb}(K, L) = \sup_{n \geq d} \mathrm{vb}(K, L, n).$$

The vector balancing problem is over six decades old. It was first introduced by Dvoretzky [11], who asked for bounds specifically in the ℓ_p setting for $p \geq 1$ and $d \in \mathbb{N}$. The problem has been studied extensively for many different convex bodies; here we highlight two results of particular importance to our work. The Euclidean case was settled independently by Sevast'yanov [19], Bárány (unpublished at the time, for the proof, see [8]), Spencer [21] and also, perhaps, by V.V. Grinberg [9], who all showed that $\mathrm{vb}(B_2^d, B_2^d) = \sqrt{d}$.

The case of the ℓ_∞-norm proved to be much more challenging, but it was later solved by Spencer in 1985 [22], who showed that $\mathrm{vb}(B_\infty^d, B_\infty^d) = O(\sqrt{d})$. This estimate is sharp in terms of order of magnitude, as one can see using random constructions involving Hadamard matrices.

Our work focuses on a natural "colorful" generalization of the vector balancing problem: given an origin-symmetric convex body $B \in \mathcal{K}_o^d$ and vector families $V_1, ..., V_n \subseteq B$ satisfying the condition that $0 \in \mathrm{conv}\ V_1 + \cdots + \mathrm{conv}\ V_n$, select one vector from each family, $v_i \in V_i$, so that $\|v_1 + \cdots + v_n\|_B$ is minimal.

We make two remarks about this problem statement. First, to motivate the name "colorful" vector balancing, note that one can interpret the families as color classes, in which case the problem asks for a colorful sum of vectors of minimal norm. Second, to see that this problem is indeed a generalization of the original vector balancing problem, note that the original problem is retrieved by setting $V_i = \{\pm v_i\}$ for $i \in [n]$.

The colorful vector balancing problem was first introduced by Bárány and Grinberg, who proved the following result.

Theorem 2.1 (Bárány, Grinberg [9]). *Assume that $B \subset \mathcal{K}_o^d$ is an origin-symmetric convex body, and $V_1, \ldots, V_n \subseteq B$ are vector families so that $0 \in \sum_{i\in[n]} \mathrm{Conv} V_i$. Then there exists a selection of vectors $v_i \in V_i$ for $i \in [n]$ such that*

$$\Big\| \sum_{i\in[n]} v_i \Big\|_B \leq d. \tag{2}$$

Taking $B = B_1^d$, $n = d$, and $V_i = \{\pm e_i\}$ for $i \in [n]$ shows that Theorem 2.1 is sharp. Yet, for specific norms, asymptotically stronger estimates may hold. In light of the fact that $\mathrm{vb}(B_2^d, B_2^d) = \sqrt{d}$ and $\mathrm{vb}(B_\infty^d, B_\infty^d) = O(\sqrt{d})$, it is plausible to conjecture that for the Euclidean and the maximum norms, the sharp estimate is of order $O(\sqrt{d})$. For the case of the Euclidean norm, it is mentioned in [9] that V. V. Grinberg proved the sharp bound of $\sqrt{d}$, although this has never been published (or verified) – and 25 years later, the statement was again referred to as a conjecture [6]. Bárány and Grinberg [9] also note that “from the point of view of applications, it would be interesting to know more about” the case of the ℓ_∞-norm.

The colorful vector balancing problem also appears in the following result.

Theorem 2.2 (Bansal, Dadush, Garg, Lovett [4]). *Let $V_1, ..., V_n \subseteq B_2^d$ be vector families with $0 \in \mathrm{conv}\ V_i$ for each $i \in [n]$. Then for any convex body K with $\gamma_d(K) \geq 1/2$, there exist vectors $v_i \in V_i$ such that $\sum_{i=1}^n v_i \in cK$, where $c > 0$ is an absolute constant.*

Note that the condition $0 \in \sum_{i\in[n]} \mathrm{Conv} V_i$ is weaker than requiring $0 \in \mathrm{Conv} V_i$ for each i – by applying a shift of each family, the more general estimate can be derived from the statement under this more restrictive condition, albeit with the loss of a factor 2 compared to the above bound.

Applying Theorem 2.2 to the Euclidean norm, one retrieves a sum of norm at most $C\sqrt{d}$ for some constant $C > 1$, and for the maximum norm one obtains a bound of $O(\sqrt{d \ln d})$ (the latter can also be obtained by a straightforward application of the probabilistic method). We note that their proof method, which is based on the techniques of Lovász, Spencer, and Vesztergombi [15], can be modified to show that the bound in the colorful setting is at most twice the original vector balancing constant, which implies $O(\sqrt{d})$ bounds for both the Euclidean and maximum norm. This asymptotically matches the estimates that we prove, up to constants; we provide the details in Section 2.3.

In Chapter 3, we prove our two main results:

Theorem 2.3 ([1], Theorem 1.4). *Given vector families $V_1, ..., V_n \subseteq B_2^d$ with*

$$0 \in \sum_{i \in [n]} \mathrm{Conv}\ V_i,$$

one can select vectors $v_i \in V_i$ for $i \in [n]$ such that $\|v_1 + \cdots + v_n\|_2 \leq \sqrt{d}$.

Theorem 2.4 ([1], Theorem 1.5). *Given vector families $V_1, ..., V_n \subseteq B_\infty^d$ with*

$$0 \in \sum_{i \in [n]} \mathrm{Conv}\ V_i,$$

one can select vectors $v_i \in V_i$ for $i \in [n]$ such that $\|v_1 + \cdots + v_n\|_\infty \leq C\sqrt{d}$, where $C = 22$ suffices.

The proof of both results relies on a reduction of the colorful vector balancing problem to a vertex approximation problem for direct products of simplices. The proof of this reduction, which utilizes linear algebra and the theory of basic feasible solutions from linear programming [23], is the subject of Section 2.1. The precise result is stated below:

Corollary 2.5 ([1], Corollary 2.5). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ with unit ball B. Suppose there exists a constant $C(d)$ such that given any collection of $k \leq d$ families $U = \{U_1, \ldots, U_k\}$ in B satisfying $|U_1| + \cdots + |U_k| \leq k + d$, and any $\lambda \in \Delta_U$, there exists a selection vector $\mu \in \Delta_U$ such that*

$$\|V\lambda - V\mu\| \leq C(d).$$

Then given any collection of families $V_1, \ldots, V_n \subseteq B$ with $0 \in \sum_{i\in[n]} \mathrm{Conv} V_i$, there exists a selection of vectors $v_i \in V_i$ for $i \in [n]$ such that

$$\Big\| \sum_{i\in[n]} v_i \Big\| \leq C(d).$$

Corollary 2.5 reduces the proofs of Theorems 2.3 and 2.4 to vertex approximation problems in the Euclidean and maximum norms, respectively. For the proof of Theorem 3 (see Section 3.1), we generalize the probabilistic approach used by Spencer [21] to prove the analogous result for the vector balancing problem. In particular, given vector families $V_1, ..., V_k \in B_2^d$ and a point $x \in \mathrm{conv}\ V_1 + \cdots + \mathrm{conv}\ V_k$, we use the convex coefficients defining x to define a probability distribution over the vectors in each family and select a vector $v_i \in V_i$ randomly for each $i \in [k]$. By analyzing the expected Euclidean distance between x and $v_1 + \cdots + v_k$, we conclude that in expectation $\|v_1 + \cdots + v_k - x\|_2^2$ is bounded by d, hence there must exist a corresponding choice of vectors satisfying this bound.

As in the vector balancing case, the proof in the maximum norm setting is much more challenging. Our proof generalizes Lovett and Meka's [14] algorithmic proof that $\mathrm{vb}(B_\infty^d, B_\infty^d) = O(\sqrt{d})$. The algorithm defines a Gaussian random walk inside a direct product of simplices, representing the space of convex coefficients for the families. In this way, vertices of the direct product of simplices exactly correspond to selections of one vector from each family. The walk is further restricted by additional linear constraints that prevent the approximation error from growing too large and that depend on the vector families $V_1, ..., V_k$. A careful analysis of the random walk shows that with high probability, if one initializes at any point lying in the direct product of simplices (in particular, corresponding to a point in the sum of the convex hulls of the families $V_1, ..., V_k$), then the walk terminates on a lower dimensional face of the simplex product, specifically reducing the number of coordinates lying in $(0, 1)$ by at least a constant factor. Geometrically, this means that we narrow down to fewer vectors with non-zero coordinates in each of the families. This algorithm can then be iterated until only one vector remains in each family, and the error incurred can be bounded by the triangle inequality over iterations of the algorithm. By choosing parameters appropriately, this algorithm proves the bound in Theorem 2.4.

The algorithm is described and used to prove Theorem 2.4 via iteration in Section 3.2; the in-depth analysis of the algorithm is contained in Section 3.3.

3 The Steinitz Problem for 'Almost-Unit' Vectors

In Chapter 4 we turn our attention to the Steinitz problem, which arises in connection with a famous theorem that will be familiar to all mathematicians: the Riemann rearrangement theorem [17]. This theorem, a classic result in analysis, tells us that a conditionally convergent series can be rearranged to converge to any real number. Formulated through a different lens, for any real series, consider the set of all sums of its possible rearrangements. The Riemann rearrangement theorem tells us that this set is either empty, i.e. the series is divergent; a single point, i.e.

the series is absolutely convergent; or the entire real line, i.e. the series is conditionally convergent. A natural question is what happens if one studies sequences of complex numbers, or even more generally, sequences of vectors in $\mathbb{R}^d$. This problem was first addressed by Lévy in 1905 [13].

Theorem 3.1 (Lévy-Steinitz Theorem)**.** *Given a series of vectors in $\mathbb{R}^d$, the set of all sums of its rearrangements is empty, or it forms an affine subspace of $\mathbb{R}^d$.*

Recall that an affine subspace of $\mathbb{R}^d$ is of the form $L + x$, where $L \subset \mathbb{R}^d$ is a linear subspace and $x \in \mathbb{R}^d$. The reader may notice that the theorem is also attributed to Steinitz: the reason for this is that Lévy's proof contained serious gaps in dimensions $d \geq 3$, which was pointed out and fixed by Steinitz in a series of works published in three parts [24, 25, 26], which is quite technical and covers much ground. The key step in his proof is the following, which is the birth of what we will call the Steinitz problem.

Theorem 3.2 ([24], p.171)**.** *Given any finite family of vectors $V \subset \mathbb{R}^d$ of Euclidean norm at most 1 summing to 0, one can order the elements of V as $v_1, ..., v_n$ so that for every $k = 1, ..., n$,*

$$\Big\| \sum_{i \in [k]} v_i \Big\|_2 \leq C, \tag{3}$$

where C is a constant that depends only on the dimension d.

Steinitz's proof shows that in fact $C \leq 2d$. It is natural to ask for the smallest value of C for which (3) holds, in general norms as well. This quantity will be called the *Steinitz constant*, and it is defined as follows.

Definition 3.3 (Steinitz constant)**.** *Let $B \in \mathcal{K}_o^d$. The* Steinitz constant of *B, denoted $S(B)$, is the smallest number C for which any finite family of* *vectors $V \subset B$ with $\Sigma(V) = 0$ has an ordering $V = \{v_1, \ldots, v_n\}$ along which each partial sum has norm at most C. That is, for every $k \in [n]$,*

$$\Big\| \sum_{i \in [k]} v_i \Big\|_B \leq C.$$

Note that the term 'constant' above refers to the fact that $S(B)$ depends only on the choice of B, but not on the vector family $V \subset B$. We make no reference to the dimension d, as the value of the Steinitz constant is independent of d as long as B can be embedded in $\mathbb{R}^d$.

One can also consider a generalized version of the Steinitz constant, where the zero-sum condition $\Sigma(V) = 0$ on the vector family is dropped:

Definition 3.4 (Relaxed Steinitz constant). *For $B \in \mathcal{K}_o^d$, let $S^*(B)$ denote the smallest constant C for which any finite family of vectors $V \subset B$ has an ordering $V = \{v_1, \ldots, v_n\}$ so that*

$$\Big\| \sum_{i \in [k]} v_i - \frac{k}{n}\Sigma(V) \Big\|_B \leq C \tag{4}$$

holds for every $k \in [n]$.

The relationship with the original Steinitz constant is given by the simple chain of inequalities

$$S(B) \leq S^*(B) \leq (1 + \rho(B))S(B), \tag{5}$$

where

$$\rho(B) := \max_{v \in B} \| - v\|_B$$

measures the asymmetry of B. Note that $\rho(B) = 1$ if B is symmetric. The lower bound in (5) is trivial; to see the upper estimate, one has to observe that starting from any family V of n vectors in B, the triangle inequality implies that $\|\Sigma(V)\|_B \leq n$, hence $\| - \frac{\Sigma(V)}{n}\|_B \leq \rho(B)$. Accordingly, the zero-sum vector family $\left\{v - \frac{\Sigma(V)}{n} : v \in V\right\}$ lies in $(1+\rho(B))B$, and the estimate readily follows.

Theorem 3.2, proved by Steinitz, justifies that $S(B_2^d)$ and, via (5), that $S^*(B_2^d)$ are well-defined. The proof can be extended to any symmetric norm. For asymmetric norms, the justification of Definitions 3.3 and 3.4 is implied by the following general bound, proved in 1978 by Sevastyanov [18] and by Grinberg and Sevastyanov [12] for not necessarily symmetric bodies by a simpler proof.

Theorem 3.5 (The Steinitz Lemma for general norms [12, 18]). *For any convex body $B \in \mathcal{K}_o^d$,*

$$S(B) \leq d. \tag{6}$$

The bound is tight for non-symmetric convex bodies, as is shown by taking B to be the regular simplex centered at the origin and choosing V to be the set of its vertices, whereas it is sharp by the order of magnitude for symmetric norms, which is confirmed by the inequality $S(B_1^d) \geq (d+1)/2$, see [12]. For symmetric $B \in \mathcal{K}_o^d$, the estimate in (6) can be strengthened to $d - 1 + \frac{1}{d}$, see [20].

Via (5), Theorem 3.5 readily implies the bound

$$S^*(B) \leq (1+\rho(B))d,$$

which also follows from the results in [12]. In particular, $S^*(B) \leq 2d$ holds for symmetric $B \in \mathcal{K}_o^d$. The story of the Steinitz constant is rich and interesting, and it is treated in more detail in Section 4.2.

The following long-standing conjecture of Bergström [7], be it confirmed, would yield a much stronger estimate on the Steinitz constant in the Euclidean case:

Conjecture 3.6. *For all $d \geq 1$, $S(B_2^d) = O(\sqrt{d})$.*

The same bound is expected to hold for the maximum norm. So far, Conjecture 3.6, which is sometimes also called the Euclidean Steinitz problem, has refuted all attempts. An explicit construction [10, 12] shows that $S(B_2^d) \geq \sqrt{d+3}/2$ must hold, meaning that no stronger estimate is possible. The exact value of the planar Euclidean Steinitz constant was determined by Banaszczyk [3], who proved that $S(B_2^2) = \sqrt{5}/2$, matching this lower bound.

Our work in this dissertation focuses on reducing the Steinitz constant to the restricted setting of 'nearly unit' vectors: the subscript 'ε' will mean that only families of vectors are considered whose members have norm in the interval $[1-\varepsilon, 1]$. To this end we introduce the following definition.

Definition 3.7 (ε-Steinitz constants). *For $B \in \mathcal{K}_o^d$ and $0 \leq \varepsilon \leq 1$, let $S_\varepsilon^*(B)$ denote the smallest constant C for which any finite family $V \subset \mathbb{R}^d$ consisting of vectors of $\|.\|_B$-norm in $[1-\varepsilon, 1]$ may be ordered as $V = \{v_1, \ldots, v_n\}$ so that*

$$\Big\| \sum_{i \in [k]} v_i - \frac{k}{n}\Sigma(V) \Big\|_B \leq C$$

holds for every $k = 1, \ldots, n$. Furthermore, let $S_\varepsilon(B)$ denote analogous quantity for vector families that satisfy the extra condition $\Sigma(V) = 0$.

Note that for any $0 \leq \varepsilon \leq 1$, $S_0(B) \leq S_\varepsilon(B) \leq S_1(B) = S(B)$, $S_0^*(B) \leq S_\varepsilon^*(B) \leq S_1^*(B) = S^*(B)$, and $S_\varepsilon(B) \leq S_\varepsilon^*(B)$. Thus, (5) ensures that

$$S_\varepsilon^*(B) \leq 2S(B) \tag{7}$$

for symmetric norms, while

$$S_\varepsilon^*(B) \leq (1 + \rho(B))S(B)$$

holds for arbitrary $B \in \mathcal{K}_o^d$.

Furthermore, observe that setting $\varepsilon = 0$ restricts the problem to families of unit vectors. In the Euclidean case, a construction given by Damsteeg and Halperin [10] implies that

$$\Omega(\sqrt{d}) \leq S_0(B_2^d) \leq S_0^*(B_2^d) \leq S_\varepsilon^*(B_2^d). \tag{8}$$

In this dissertation we prove two results establishing reverse estimates of (7). The first result is specific to the Euclidean norm.

Theorem 3.8. *For any* $0 < \varepsilon < 1$ *and all* $d \geq 2$*,*

$$S(B_2^d) < \frac{1}{\varepsilon}\Big(S_\varepsilon^*(B_2^d) + 200\sqrt{\frac{d}{\log d}}\Big). \tag{9}$$

In particular, an $o(d)$ bound on $S_\varepsilon^*(B_2^d)$ for some fixed $0 < \varepsilon \leq 1$ would yield an $o(d)$ estimate on $S(B_2^d)$, hence improving the current strongest bound. Moreover, (8) and (9) imply that Conjecture 3.6 is equivalent to the statement that $S_\varepsilon^*(B_2^d) = O(\sqrt{d})$ for some constant $\varepsilon \in (0, 1]$.

The second result generalizes and simplifies the techniques of the proof of Theorem 3.8 and yields an even stronger estimate for general norms.

Theorem 3.9 ([2] Theorem 7)**.** *For all* $d \geq 2$*, for every convex body* $B \in \mathcal{K}_o^d$*, and* $0 < \varepsilon \leq 1$*,*

$$S(B) < \frac{1}{\varepsilon}\Big(S_\varepsilon^*(B) + 2\rho(B) + 1\Big). \tag{10}$$

In the case that B is symmetric, the bound simplifies to $\frac{1}{\varepsilon}(S_\varepsilon^*(B) + 3)$.

We prove Theorem 3.8 in Sections 4.2 and 4.3. The strategy of the proof is to pre-process the vectors to remove any short (i.e., of norm less than $1 - \varepsilon$) vectors. We do this by summing together short vectors within spherical caps until we get a vector that is sufficiently long; in doing so, we must be careful that all partial sums of the new long vectors remain sufficiently short, and we must deal with a handful of extra vectors that are left after pre-processing; these are the details of the technical lemmas in Section 4.3. The key challenge of the proof is balancing the contributions of the new partial sums introduced during pre-processing, whose norms shrink with the height of the spherical cap, and the partial sums of the remaining vectors after pre-processing, whose norms grow with the height of the spherical cap. Choosing the optimal spherical cap height to balance these two contributions is the crux of the proof.

In Section 4.4, we show that through a simple modification of the proof strategy for Theorem 3.8, we can obtain a simpler proof that works for general norms and gives a stronger bound, yielding Theorem 3.9. The key modification comes from selecting the pre-processed sums slightly more carefully, which makes half-spaces the optimal choice rather than spherical caps, and frees us of the dependence on sphere concentration results. Although Theorem 3.9 is strictly stronger than Theorem 3.8, we still include the proof of Theorem 3.8, as the Euclidean-specific techniques are of independent interest.

References

[1] Gergely Ambrus and Rainie Bozzai. Colourful vector balancing. *Mathematika*, 70(4), August 2024.

[2] Gergely Ambrus and Rainie Heck. A note on the Steinitz constant. *Accepted for publication; Mathematika*, 2026.

[3] Wojciech Banaszczyk. A note on the Steinitz constant of the Euclidean plane. *CR Math. Rep. Acad. Sci. Canada*, 12(4):97–102, 1990.

[4] Nikhil Bansal, Daniel Dadush, Shashwat Garg, and Shachar Lovett. The Gram–Schmidt walk: A cure for the Banaszczyk blues. *Theory of Computing*, 15(21):1–27, 2019.

[5] Imre Bárány. A Vector-sum Theorem and its Application to Improv-

ing Flow Shop Guarantees. *Mathematics of Operations Research*, 6(3):445–452, 1981.

[6] Imre Bárány and Benjamin Doerr. Balanced partitions of vector sequences. *Linear algebra and its applications*, 414(2-3):464–469, 2006.

[7] Viktor Bergström. Zwei Sätze über ebene Vectorpolygone. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 8, pages 206–214. Springer, 1931.

[8] Imre Bárány. On the power of linear dependencies. In *Building bridges: Between Mathematics and Computer Science*, volume 19 of *Bolyai Society Mathematical Studies*, pages 31–45. Bolyai Society and Springer, 2010.

[9] Imre Bárány and Victor Grinberg. On some combinatorial questions in finite dimensional spaces. *Linear Algebra and its Applications*, 41:1–9, 1981.

[10] Ira Damsteeg and Israel Halperin. The Steinitz-Gross theorem on sums of vectors. *Trans. Roy. Soc. Canada Sect. III*, 44:31–35, 1950.

[11] Aryeh Dvoretzky. Problem. In *Convexity*, volume 7 of *Proceedings of Symposia in Pure Mathematics*, page 496. Amer. Math. Soc., 1963.

[12] Victor Grinberg and Sergey Sevast'yanov. Value of the Steinitz Constant. *Functional Analysis and its Applications*, 14:125–126, 1980.

[13] Paul Lévy. Sur les séries semi-convergentes. *Nouvelles annales de Mathématiques: journal des candidats aux Écoles Polytechnique et Normale*, 5:506–511, 1905.

[14] Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. *SIAM Journal on Computing*, 44(5):1573–1582, 2015.

[15] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancy of

set-systems and matrices. *European Journal of Combinatorics*, 7(2):151–160, 1986.

[16] Victor Reis and Thomas Rothvoss. Vector balancing in Lebesgue spaces. *Random Structures & Algorithms*, 08 2022.

[17] Bernhard Riemann. *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, volume 13. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1868.

[18] Sergey V. Sevast'yanov. On Approximate Solutions of Scheduling Problems. *Metody Discretnogo Analiza*, 32:66–75, 1978.

[19] Sergey V. Sevast'yanov. On the approximate solution of the problem of calendar planning. *Upravlyaemye Systemy*, 20:49–63, 1980. (in Russian).

[20] Sergey V. Sevastyanov. On a Compact Vector Summation. *Diskretnaya Matematika*, 3:66–72, 1991. (in Russian).

[21] Joel Spencer. Balancing unit vectors. *Journal of Combinatorial Theory*, 30:349–350, 1981.

[22] Joel Spencer. Six standard deviations suffice. *Transactions of the American Mathematical Society*, 289(2):679–706, 1985.

[23] Ross M. Starr. Quasi-equilibria in markets with non-convex preferences. *Econometrica*, 37(1):25–38, 1969.

[24] Ernst Steinitz. Bedingt Konvergente Reihen und Konvexe Systeme. *Journal für die reine und angewandte Mathematik*, 143:128–176, 1913.

[25] Ernst Steinitz. Bedingt Konvergente Reihen und Konvexe Systeme. *Journal für die reine und angewandte Mathematik*, 144:1–40, 1914.

[26] Ernst Steinitz. Bedingt Konvergente Reihen und Konvexe Systeme. *Journal für die reine und angewandte Mathematik*, 146:1–52, 1916.

Co-Author Declaration

Hereby, I, Gergely Ambrus, declare that I have not used and will not use the following 2 joint papers with Rainie Heck in order to obtain a scientific degree.

- Gergely Ambrus and Rainie Bozzai. Colourful vector balancing. Mathematika, 70(4), August 2024.
- Gergely Ambrus and Rainie Heck. A note on the Steinitz constant. Accepted for publication; Mathematika, 2026.

The contribution of Rainie Heck was essential to the papers listed above, in particular, approximately 50 % in the first paper and 50% in the second paper.

Dr. Gergely Ambrus
Bolyai Institute
University of Szeged
Associate Professor