Solution sets and centralizers

Megoldáshalmazok és centralizátorok

PhD dissertation

Author:
Endre Tóth

Advisor:
Dr. Tamás Waldhauser
Associate professor
Department of Algebra and Number Theory

Doctoral School of Mathematics
Bolyai Institute, University of Szeged
2024

Acknowlegdements

First and foremost, I would like to thank my supervisor, for his constant help and continuous patience. He has been guiding me for several years into the depths of mathematics, his ideas, perseverance and way of thinking motivated me throughout my research. I would also like to thank my institute and all the friendly and quite capable colleagues and teachers, who always welcome the questions and interest of any colleague or student with open arms. Lastly, I would like to thank my family, who has been excitedly waiting for my doctorate. My mother always helped me with mathematics, she was my first teacher in this beautiful field. There is no doubt she was the one who helped me steer me in the direction of mathematics.

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Chapter 1

Introduction

We study solution sets of systems of equations over finite algebras. The study of (systems of) equations dates back as far as 1800 BC; two archaeological finds that hint at this are the Berlin Papyrus 6619, made by Egyptian mathematicians, and the Plimpton 322 clay tablet, created by Babylonians. Since then systems of equations became essential and really basic in several branches of mathematics. Some important examples could be systems of linear equations, polynomial equations, differential equations or in general nonlinear equations. Finding a solution to systems of equations is usually the main goal of investigating them, however, in this thesis we take a different approach; we investigate them from a universal algebraic perspective.

In classical algebraic geometry, one investigates solution sets of systems of equations over a field. These sets are called algebraic sets, as they are analogues of algebraic varieties[1]. In the late 90s, several mathematicians, for example B. Plotkin [Plo96], O., Kharlampovich and A. Myasnikov [KM98a; KM98b] begun studying algebraic sets over other, different algebraic structures. Thus, the field universal algebraic geometry was born. Kharlampovich and Myasnikov for example investigated the algebraic geometry of groups, and found a proof for Tarski's two conjectures from 1945 about the elementary theory of non-Abelian free groups. In their results they proved that two non-Abelian free groups with different ranks have the same elementary theory, and that the elementary theory of a free group is decidable. Universal algebraic geometry has been an active area of research ever since. Let us mention just one recent result: in 2023 E. Aichinger and B. Rossi [AR23] proved that there are continuum many algebraic geometries over finite sets with at least three elements. (On the two-element set there are only 25 algebraic geometries, see Remark 3.0.3). In this thesis we aim to contribute to the field of universal algebraic geometry; we investigate the solution sets (i.e., algebraic sets) in terms of closure conditions.

Let A be a finite nonempty set and F a set of operations on A. We allow

[1] Note that the word *variety* has a different meaning in universal algebra: a variety is an equationally definable class of algebras, or, equivalently, a class of algebras that is closed under homomorphic images, subalgebras and direct products.

operations from F in our equations, and since we can use these operations several times, we can build composite operations. This means that every equation in n variables can be written as $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$, where f and g are obtained as compositions of operations from F. The set of all such operations is denoted by $[F]$, and it is called the *clone* generated by F (see sections 1.1 and 1.2 for the precise definitions). Elements of the clone $[F]$ are also called *term functions* of the algebraic structure $\mathbb{A} = (A; F)$, and our equations are the same as equations over $\mathbb{A}$ in the sense of universal algebra.

If two sets of operations generate the same clone, then they produce the same equations, thus it is natural to investigate equations over a clone C. For every clone C, one can assign a clone C^* (called the *centralizer* of C) such that if $T \subseteq A^n$ is the set of all solutions of a system of equations over C, then T is closed under C^* (see Theorem 2.1.1). In certain special cases, such as in the case of (homogeneous) linear equations (see Example 2.1.2), being closed under C^* is sufficient for being the solution set of a system of C-equations. Unfortunately, this does not hold in general (see Example 2.1.3). If for a given algebra $\mathbb{A} = (A; F)$ (or the clone $C = [F]$) solution sets are characterized by being closed under the centralizer, then we will say that $\mathbb{A}$ (or C) has property (SDC). (See Definition 2.1.4)

In Chapter 2 we show the connection between solution sets and centralizer clones. We also prove that an algebra (or the associated clone) has property (SDC) if and only if there is quantifier elimination for certain primitive positive formulas. In Chapter 3 we prove that every clone of Boolean functions (thus, every two-element algebra) has property (SDC). In Chapter 4 we investigate centralizers of (the clone generated by the basic operations of) semilattices, distributive lattices and arbitrary lattices, and also give an insight on the number of (essentially) n-ary operations in these centralizer clones. We also obtain a simple proof for Kuznetsov's description [Kuz79] of primitive positive clones on the two-element set. In Chapter 5 we describe semilattices and lattices having property (SDC). In Chapter 6 we prove that for algebras property (SDC) is equivalent to the property called polymorphism-homogeneity (or polyhomogeneity). We also investigate connections between polymorphism-homogeneity of certain relational structures assigned to an algebra, injectivity of the algebra in certain classes and property (SDC).

1.1 Operations

Let A be an arbitrary finite set with at least two elements. By an *operation* on A we mean a map $f\colon A^n \to A$; the positive integer $n \in \mathbb{N}$ is called the *arity* of the operation f. (We also allow nullary operations, but by definition we consider them as n-ary operations with $n \geq 1$ that do not depend on their variables. This is the case throughout the whole thesis, with the exception of Chapter 4.) The set of all operations on A is denoted by $\mathcal{O}_A$. Operations on $A = \{0, 1\}$ are called *Boolean functions*, and we will also use the notation $\Omega = \mathcal{O}_{\{0,1\}}$ for the set of all Boolean functions (see Appendix 1.1 for some background on Boolean functions). For a set $F \subseteq \mathcal{O}_A$ of operations, by $F^{(n)}$ we mean the set of n-ary members of F. In particular, $\mathcal{O}_A^{(n)}$ stands for the set of all n-ary operations on A.

We will denote tuples by boldface letters, and we will use the corresponding plain letters with subscripts for the components of the tuples. For example, if $\mathbf{a} \in A^n$, then a_i denotes the i-th component of $\mathbf{a}$, i.e., $\mathbf{a} = (a_1, \ldots, a_n)$. In particular, if $f \in \mathcal{O}_A^{(n)}$, then $f(\mathbf{a})$ is a short form for $f(a_1, \ldots, a_n)$. If $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in A^n$ and $f \in \mathcal{O}_A^{(m)}$, then $f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})$ denotes the n-tuple obtained by applying f to the tuples $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}$ componentwise:

$$f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) = \Big(f(t_1^{(1)}, \ldots, t_1^{(m)}), \ldots, f(t_n^{(1)}, \ldots, t_n^{(m)})\Big).$$

We say that $T \subseteq A^n$ is *closed under* $C \subseteq \mathcal{O}_A$, if for all $m \in \mathbb{N}, \mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in T$ and for all $f \in C^{(m)}$ we have $f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$.

1.2 Clones and relational clones

Let $f \in \mathcal{O}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{O}_A^{(k)}$. By the *composition* of f by $g_1, \ldots, g_n$ we mean the operation $h \in \mathcal{O}_A^{(k)}$ defined by

$$h(\mathbf{x}) = f(g_1, \ldots, g_n)(\mathbf{x}) = f\Big(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})\Big) \text{ for all } \mathbf{x} \in A^k.$$

If a class $C \subseteq \mathcal{O}_A$ of operations is closed under composition and contains the *projections* $\pi_i : (x_1, \ldots, x_n) \mapsto x_i$ for all $1 \leq i \leq n \in \mathbb{N}$, then C is said to be a *clone* (notation: $C \leq \mathcal{O}_A$). Notable examples include all continuous operations on a topological space, all monotone operations on an ordered set, all polynomial operations of a ring (or any algebraic structure), etc. (see also Example 1.3.1). For an arbitrary set F of operations on A, there is a least clone $[F]$ containing F, called the clone *generated* by F. The elements of this clone are those operations that can be obtained from members of F and from projections by finitely many compositions. For example, for any algebra $\mathbb{A} = (A; F)$, the term operations of $\mathbb{A}$ form a clone, which is exactly $[F]$. Moreover, we can obtain any clone C in such a way, by setting any generating set of C to be F (obviously $C = F$ also suffices).

The set of all clones on A is a lattice under inclusion; the greatest element of this lattice is $\mathcal{O}_A$, and the least element is the *trivial clone* consisting of projections only. There are countably infinitely many clones on the two-element set; these have been described by Post [Pos41], hence the lattice of clones on $\{0, 1\}$ is called the *Post lattice*. In Appendix 1.1 we present the Post lattice and we define Boolean clones that we need in the proof of some of our results. If A is a finite set with at least three elements, then there is a continuum of clones on A [JM59], and it is a very difficult open problem to describe all clones on A even for $|A| = 3$.

A *k-ary partial operation* on A is a map $h\colon \operatorname{dom} h \to A$, where the *domain* of h can be any set $\operatorname{dom} h \subseteq A^k$. The set of all partial operations on A is denoted by $\mathcal{P}_A$, and the set of all k-ary partial operations on A is denoted by $\mathcal{P}_A^{(k)}$. A *strong partial clone* is a set of partial operations that is closed under composition, contains the projections, and contains all restrictions of its members to arbitrary subsets of their domains. Note that if $C \subseteq \mathcal{O}_A$ is a clone, then the least strong partial clone $\operatorname{Str}(C)$ containing C consists of all restrictions of elements of C, i.e., $h \in \mathcal{P}_A$ belongs to $\operatorname{Str}(C)$ if and only if h can be extended to a total operation $\widehat{h} \in C$.

The set of all strong partial clones on A is also a lattice under inclusion; the greatest element of this lattice is $\mathcal{P}_A$, and the least element is the *trivial strong partial clone* consisting of projections and their restrictions only. However, even on the two-element set there are continuum many strong partial clones [Fre66], and we can not say much more of this lattice.

An n-ary *relation* on A is a subset of A^n; the set of all relations (of arbitrary arities) on A is denoted by $\mathcal{R}_A$. Given a set of relations $R \subseteq \mathcal{R}_A$, a *primitive positive formula* $\Phi(x_1, \ldots, x_n)$ over R is an existentially quantified conjunction:

$$\Phi(x_1, \ldots, x_n) \equiv \exists y_1 \cdots \exists y_m \mathop{\&}_{i=1}^{t} \rho_i\Big(z_1^{(i)}, \ldots, z_{r_i}^{(i)}\Big), \tag{1.2.1}$$

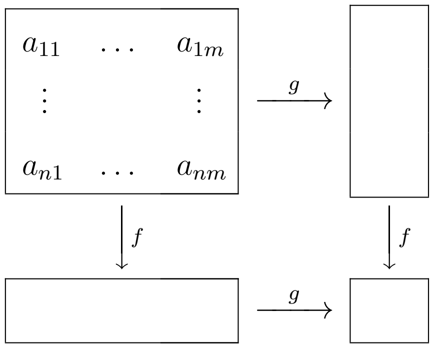
where $\rho_i \in R \cup \{=\}$ is a relation of arity r_i, and each $z_j^{(i)}$ is a variable from the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ for $i = 1, \ldots, t,\ j = 1, \ldots, r_i$. (Since the equality sign can appear in the formula, we use the symbol $\equiv$ to denote equality of formulas, to avoid confusion.) The relation $\rho = \{(a_1, \ldots, a_n) : \Phi(a_1, \ldots, a_n) \text{ is true}\} \subseteq A^n$ is then said to be *defined by the primitive positive formula* Φ. The set of all primitive positive definable relations over R is denoted by $\langle R \rangle_{\exists}$, and such sets of relations are called *relational clones*. If we allow only quantifier-free primitive positive formulas, then we obtain the *weak relational clone* $\langle R \rangle_{\nexists}$.

1.3 Centralizer clones

We say that the operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ *commute* (notation: $f \perp g$) if

$$\begin{aligned} f\Big(g(a_{11}, a_{12}, \ldots, a_{1m}), \ldots, g(a_{n1}, a_{n2}, \ldots, a_{nm})\Big) \\ = g\Big(f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{1m}, a_{2m}, \ldots, a_{nm})\Big) \end{aligned}$$

holds for all $a_{ij} \in A$ $(1 \leq i \leq n, 1 \leq j \leq m)$. This can be visualized as follows: for every $n \times m$ matrix $Q = (a_{ij})$, first applying g to the rows of Q and then applying f to the resulting column vector yields the same result as first applying f to the columns of Q and then applying g to the resulting row vector:



Denoting by $\mathbf{c}^{(j)} \in A^n$ $(j = 1, \ldots, m)$ the j-th column vector of Q, we can express the commutation property more compactly:

$$f\Big(g(\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)})\Big) = g\Big(f(\mathbf{c}^{(1)}), \ldots, f(\mathbf{c}^{(m)})\Big). \tag{1.3.1}$$

It is easy to verify that if $f, g_1, \ldots, g_n$ all commute with an operation h, then the composition $f(g_1, \ldots, g_n)$ also commutes with h. This implies that for any $F \subseteq \mathcal{O}_A$, the set $F^* := \{g \in \mathcal{O}_A \mid f \perp g \text{ for all } f \in F\}$ is a clone, called the *centralizer* of F (we will also say that F^* is the centralizer of the algebra $(A; F)$). Although there are countably infinitely many clones on the two-element set [Pos41] and uncountably many clones on sets with at least three elements [JM59], only finitely many clones are of the form F^* on a finite set [BW87]. These are the so-called *primitive positive clones*; the complete list of primitive positive clones is known only for $|A| \leq 3$ [Kuz79; Dan78].

It is useful to note that if $C = [F]$, then $C^* = F^*$. This implies that in order to compute the centralizer of a clone C, it is sufficient to determine the operations commuting with a (preferably small) generating set of C.

Example 1.3.1. Let K be a field, and let L be the clone of all operations over K that are represented by a linear polynomial:

$$L := \{a_1x_1 + \cdots + a_kx_k + c \mid k \geq 0, a_1, \ldots, a_k, c \in K\}.$$

Since L is generated by the operations $x + y$, ax $(a \in K)$ and the constants $c \in K$, the centralizer L^* consists of those operations f over K that commute with $x + y$ and ax (i.e., f is additive and homogeneous), and also commute with the constants (i.e., $f(c, \ldots, c) = c$ for all $c \in K$):

$$L^* := \{a_1x_1 + \cdots + a_kx_k \mid k \geq 1, a_1, \ldots, a_k \in K \text{ and } a_1 + \cdots + a_k = 1\}.$$

Similarly, one can verify that $L_0^* = L_0$ for the clone

$$L_0 := \{a_1x_1 + \cdots + a_kx_k \mid k \geq 0, a_1, \dots, a_k \in K\}.$$

1.4 Equations and solution sets

Let us fix a clone $C \leq \mathcal{O}_A$ and a positive integer n. By an n-ary *equation over* C (*C-equation* for short) we mean an equation of the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$, where $f, g \in C^{(n)}$. We will often simply write this equation as a pair (f, g). By an n-ary *equation over an algebra* $\mathbb{A} = (A; F)$ we mean an n-ary C-equation, where $[F] = C$. A *system of C-equations* is a finite set of C-equations of the same arity:

$$\mathcal{E} := \Big\{(f_1, g_1), \dots, (f_t, g_t)\Big\}, \text{ where } f_i, g_i \in C^{(n)} \ (i = 1, \dots, t).$$

We define the *solution set of* $\mathcal{E}$ as the set

$$\mathrm{Sol}(\mathcal{E}) := \Big\{\mathbf{a} \in A^n \mid f_i(\mathbf{a}) = g_i(\mathbf{a}) \text{ for } i = 1, \dots, t\Big\}.$$

For $\mathbf{a} \in A^n$ we denote by $\mathrm{Eq}_C(\mathbf{a})$ the set of C-equations satisfied by $\mathbf{a}$:

$$\mathrm{Eq}_C(\mathbf{a}) := \Big\{(f, g) \mid f, g \in C^{(n)} \text{ and } f(\mathbf{a}) = g(\mathbf{a})\Big\}.$$

Let $T \subseteq A^n$ be an arbitrary set of tuples. We denote by $\mathrm{Eq}_C(T)$ the set of C-equations satisfied by T:

$$\mathrm{Eq}_C(T) := \bigcap_{\mathbf{a} \in T} \mathrm{Eq}_C(\mathbf{a}).$$

Example 1.4.1. Considering the "linear" clones of Example 1.3.1, L-equations are linear equations and L_0-equations are homogeneous linear equations.

Remark 1.4.2. For any given $n \in \mathbb{N}$ and $C \leq \mathcal{O}_A$, the operators Sol and Eq_C give rise to a Galois connection between sets of n-tuples and systems of n-ary equations. In particular, if T is the solution set of a system of equations (i.e., T is Galois closed), then $T = \mathrm{Sol}(\mathrm{Eq}_C(T))$; moreover, $\mathcal{E} = \mathrm{Eq}_C(T)$ is the largest system of equations with $T = \mathrm{Sol}(\mathcal{E})$.

1.5 The (p)Pol-Inv Galois connection

If $M = (m_{ij}) \in A^{n \times k}$ is an $n \times k$ matrix over the set A, then we denote the i-th row and the j-th column of M by M_{i*} and M_{*j}, respectively:

$$M_{i*} = (m_{i1}, \dots, m_{ik}) \qquad (i = 1, \dots, n),$$
$$M_{*j} = (m_{1j}, \dots, m_{nj}) \qquad (j = 1, \dots, k).$$

If $h \in \mathcal{P}_A^{(k)}$ is a partial operation of arity k such that the rows of M are in the domain of h, then we can apply h to each row of M. The resulting n-tuple is the same as the one obtained by applying h to the k columns of M componentwise:

$$\Big(h(M_{1*}), \dots, h(M_{n*})\Big) = h(M_{*1}, \dots, M_{*k}).$$

We will often use the above equality without further mention.

We say that a k-ary (partial) operation h *preserves* the relation $\rho \subseteq A^n$, denoted as $h \rhd \rho$, if for every matrix $M \in A^{n \times k}$ such that each column of M belongs to ρ (and each row of M is in the domain of h), we have $h(M_{*1}, \dots, M_{*k}) \in \rho$. If R is a set of relations, then we write $h \rhd R$ to indicate that h preserves all elements of R. In other words, $h \rhd R$ holds if and only if h is a *(partial) polymorphism* of the relational structure $\mathcal{A} = (A, R)$, i.e., h is a homomorphism from (the substructure $\operatorname{dom} h$ of) $\mathcal{A}^k$ to $\mathcal{A}$. The set of all (partial) operations preserving each relation of R is denoted by $\operatorname{Pol} R$ ($\operatorname{pPol} R$), and the set of all relations preserved by each member of a set F of (partial) operations is denoted by $\operatorname{Inv} F$:

$$\begin{aligned}
\operatorname{Pol} R &= \Big\{h \in \mathcal{O}_A : h \rhd \rho \text{ for every } \rho \in R\Big\};\\
\operatorname{pPol} R &= \Big\{h \in \mathcal{P}_A : h \rhd \rho \text{ for every } \rho \in R\Big\};\\
\operatorname{Inv} F &= \Big\{\rho \in \mathcal{R}_A : h \rhd \rho \text{ for every } h \in F\Big\}.
\end{aligned}$$

Note that $\operatorname{Pol} R = \operatorname{pPol} R \cap \mathcal{O}_A$.

The closed sets under the Galois connection $\operatorname{Pol} - \operatorname{Inv}$ ($\operatorname{pPol} - \operatorname{Inv}$) between (partial) operations and relations are exactly the (strong partial) clones and the (weak) relational clones; this makes these Galois connections fundamental tools in clone theory.

Theorem 1.5.1. [Bod+69; Gei68] *For any set of operations $F \subseteq \mathcal{O}_A$ and any set of relations $R \subseteq \mathcal{R}_A$, we have $[F] = \operatorname{Pol} \operatorname{Inv} F$ and $\langle R \rangle_{\exists} = \operatorname{Inv} \operatorname{Pol} R$.*

Theorem 1.5.2. [Rom81] *For any set of partial operations $F \subseteq \mathcal{P}_A$ and any set of relations $R \subseteq \mathcal{R}_A$, we have $\operatorname{Str}(F) = \operatorname{pPol} \operatorname{Inv} F$ and $\langle R \rangle_{\nexists} = \operatorname{Inv} \operatorname{pPol} R$.*

In the next chapter we investigate the connection between solution sets and centralizers.

Chapter 2

Connections between solution sets and centralizers

Looking for a characterization of solution sets by means of closure conditions, we would like to determine operations under which solution sets of C-equations are closed. In Section 2.1 we show that solution sets are always closed under the centralizer, and related to this fact we introduce a property called property (SDC). In Section 2.2 we show that the newly introduced property is equivalent to quantifier-elimination of certain type of formulas.

2.1 Property (SDC)

The following theorem shows that the solution set is always closed under operations in the centralizer C^*.

Theorem 2.1.1. [TW17] *For any clone $C \leq \mathcal{O}_A$, the solution set of a system of C-equations is closed under C^*.*

Proof. Let $C \leq \mathcal{O}_A$ be a clone and let $\mathcal{E}$ be a system of n-ary C-equations with solution set $T = \mathrm{Sol}(\mathcal{E}) \subseteq A^n$. Let $\Phi \in C^*$ be an arbitrary m-ary operation, and let $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in T$; we need to prove that $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$. Consider an arbitrary equation $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ from $\mathcal{E}$. Since $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}$ are solutions of $\mathcal{E}$, we have $f(\mathbf{t}^{(j)}) = g(\mathbf{t}^{(j)})$ for $j = 1, \ldots, m$. This implies that

$$\Phi(f(\mathbf{t}^{(1)}), \ldots, f(\mathbf{t}^{(m)})) = \Phi(g(\mathbf{t}^{(1)}), \ldots, g(\mathbf{t}^{(m)})). \tag{2.1.1}$$

Let us consider the $n \times m$ matrix $Q = (t_i^{(j)})$ obtained by writing the tuples $\mathbf{t}^{(j)}$ next to each other as column vectors. Then the left hand side of (2.1.1) is obtained by applying f to the columns of Q and then applying Φ to the resulting row vector. Since Φ and f commute, we get the same by applying first Φ row-wise and then applying f column-wise, and the result in this case is $f(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}))$ (cf. also (1.3.1)). Rewriting similarly the right hand side of (2.1.1), we can conclude that

$$f(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})) = g(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})).$$

This means that the tuple $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})$ also satisfies the equation (f, g). This holds for every equation of $\mathcal{E}$, thus we have $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$. ■

Example 2.1.2. Let us consider once more the case of linear equations (we use the notation of Examples 1.3.1 and 1.4.1). A set of tuples (vectors) $T \subseteq K^n$ is closed under the clone L^* if and only if T is an affine subspace of K^n, and T is closed under $L_0^* = L_0$ if and only if T is a subspace of K^n. Thus in this case T is the solution set of a system of L-equations (L_0-equations) if and only if T is closed under L^* (L_0^*).

Theorem 2.1.1 gives a necessary condition for a set $T \subseteq A^n$ to be the set of all solutions of a system of C-equations. In the case of (homogeneous) linear equations this condition is sufficient as well (see the example above). Later (in Chapter 3) we will prove that if A is a two-element set then for every clone $C \leq \mathcal{O}_A$, every set of tuples that is closed under C^* is the solution set of some system of C-equations. However, for a three-element underlying set this is not always the case.

Example 2.1.3. [TW17] Let us consider the (nonassociative) binary operation $f(x, y) = x \otimes y$ on $A = \{0, 1, 2\}$ defined by the following operation table:

$\otimes$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	0

Observe that $x \otimes x = 0$ and $x \otimes 0 = 0 \otimes x = 0$ hold identically, hence the only unary operations in the clone $C = [f]$ are $g_0(x) = 0$ and $g_1(x) = x$. Therefore, the only nontrivial C-equation of arity $n = 1$ is (g_0, g_1), whose solution set is $\{0\}$. Thus there are only two subsets $T \subseteq A$ that are solution sets of (systems of) unary C-equations, namely $T = \{0\}$ and $T = \{0, 1, 2\}$. However, the set $\{0, 1\}$ is also closed under C^*. Indeed, if $\Phi \in C^*$ is an m-ary operation and $a_1, \ldots, a_m \in \{0, 1\}$, then, observing that $a_i = a_i \otimes 2$, we can compute $\Phi(\mathbf{a}) = \Phi(a_1, \ldots, a_m)$ as follows:

$$\Phi(\mathbf{a}) = \Phi(a_1 \otimes 2, \ldots, a_m \otimes 2) = \Phi(\mathbf{a}) \otimes \Phi(\mathbf{2}) = f(\Phi(\mathbf{a}), \Phi(\mathbf{2})). \qquad (2.1.2)$$

In the second step we use that Φ belongs to the centralizer of $\{\otimes\}$, meaning that it commutes with $\otimes$, and using the definition of commutation for the 2 by 2 matrix $\begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ 2 & 2 & \ldots & 2 \end{pmatrix}$ we obtain $\Phi(a_1 \otimes 2, \ldots, a_m \otimes 2) = \Phi(\mathbf{a}) \otimes \Phi(\mathbf{2})$.

Since the range of f contains only the elements 0 and 1, we see that the right hand side of (2.1.2) belongs to $\{0, 1\}$. We can conclude that the set $\{0, 1\}$ is closed under C^*, yet it is not the solution set of any system of C-equations.

We will say that an algebra (or the clone defined by the algebra) has property (SDC) if closure under the centralizer characterizes the solution sets (here SDC stands for "Solution sets are Definable by closure under the Centralizer").

Definition 2.1.4. [TW20] Let $\mathbb{A} = (A; F)$ be an algebra with $C = [F]$. We say that $\mathbb{A}$ (or C) has property (SDC) if the following are equivalent for all $n \in \mathbb{N}$ and $T \subseteq A^n$:

(a) there exists a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$;

(b) the set T is closed under C^*.

The main goal of this thesis is investigating which algebras have property (SDC). In Chapter 3 we prove that every two-element algebra (thus every clone on the two-element set) has property (SDC). In Chapter 5 we show that only those semilattices have property (SDC) that are semilattice reducts of distributive lattices, and that among lattices only Boolean lattices have property (SDC). In Chapter 6 we investigate Abelian groups, monounary algebras and three-element groupoids in terms of property (SDC). Since the centralizer has an important role in this property, it is only natural to also investigate the centralizer clones of known algebras themselves. Hence, in Chapter 4 we study centralizer clones of semilattices, distributive lattices and arbitrary lattices.

In the next section we investigate a property that turns out to be equivalent to poperty (SDC).

2.2 Quantifier elimination and property (SDC)

In this section we give a connection between property (SDC) and quantifier elimination of certain primitive positive formulas. We also show that for systems of equations over a clone C, if all solution sets can be described by closure under a clone D, then D must be the centralizer of C.

For $f \in \mathcal{O}_A^{(n)}$, we define the following relation on A, called the *graph of* f:

$$f^\bullet = \{(a_1, \ldots, a_n, b) \mid f(a_1, \ldots, a_n) = b\} \subseteq A^{n+1}.$$

For $F \subseteq \mathcal{O}_A$, let $F^\bullet = \{f^\bullet \mid f \in F\}$. It is not hard to see that for any $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$, the function f commutes with g if and only if f preserves the graph of g (or equivalently, if and only if g preserves the graph of f). Therefore, for any $F \subseteq \mathcal{O}_A$ we have $F^* = \mathrm{Pol}(F^\bullet)$.

For $F \subseteq \mathcal{O}_A$, let F° denote the set of all relations that are solution sets of some equation over F:

$$F^\circ = \left\{ \mathrm{Sol}(f,g) \,\middle|\, n \in \mathbb{N},\ f,g \in F^{(n)} \right\} \subseteq \mathcal{R}_A.$$

The following remark shows that the graph of an operation $f \in F$ also belongs to F°.

Remark 2.2.1. Let $f \in \mathcal{O}_A^{(n)}$, and define $\widetilde{f} \in \mathcal{O}_A^{(n+1)}$ by $\widetilde{f}(x_1, \ldots, x_n, x_{n+1}) := f(x_1, \ldots, x_n)$. Then we have

$$\begin{aligned}
&\mathrm{Sol}\left(\widetilde{f}, \pi_{n+1}\right) = \\
&\quad = \left\{ (a_1, \ldots, a_n, b) \in A^{n+1} \,\middle|\, \widetilde{f}(a_1, \ldots, a_n, b) = \pi_{n+1}(a_1, \ldots, a_n, b) \right\} \\
&\quad = \left\{ (a_1, \ldots, a_n, b) \in A^{n+1} \,\middle|\, f(a_1, \ldots, a_n) = b \right\} = f^\bullet.
\end{aligned}$$

The following three lemmas (together with Theorem 2.1.1) prepare the proof of Theorem 2.2.6, which gives us an equivalent condition to property (SDC) that we will use in sections 5.1 and 5.2.

Lemma 2.2.2. [TW20] *For every clone $C \leq \mathcal{O}_A$, we have $C^\bullet \subseteq C^\circ$ and $\langle C^\bullet \rangle_\exists = \langle C^\circ \rangle_\exists$.*

Proof. In accordance with Remark 2.2.1, for all $f \in C$ we have $\mathrm{Sol}(\widetilde{f}, \pi_{n+1}) = f^\bullet \in C^\circ$. Therefore $C^\bullet \subseteq C^\circ$, which implies that $\langle C^\bullet \rangle_\exists \subseteq \langle C^\circ \rangle_\exists$. To prove the reversed containment, let us consider an arbitrary relation $\rho = \mathrm{Sol}(f,g) \in C^\circ$ with $f, g \in C^{(n)}$. Then, for any $(x_1, \ldots, x_n) \in A^n$, we have

$$\begin{aligned}
(x_1, \ldots, x_n) \in \rho &\iff f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \\
&\iff \exists y \colon f(x_1, \ldots, x_n) = y \ \&\ g(x_1, \ldots, x_n) = y \\
&\iff \exists y \colon (x_1, \ldots, x_n, y) \in f^\bullet \ \&\ (x_1, \ldots, x_n, y) \in g^\bullet.
\end{aligned}$$

This means that ρ can be defined by a primitive positive formula over $\{f^\bullet, g^\bullet\}$, therefore $\rho \in \langle C^\bullet \rangle_\exists$. Thus, we obtain $C^\circ \subseteq \langle C^\bullet \rangle_\exists$, and this implies that $\langle C^\circ \rangle_\exists \subseteq \langle \langle C^\bullet \rangle_\exists \rangle_\exists = \langle C^\bullet \rangle_\exists$; therefore $\langle C^\bullet \rangle_\exists = \langle C^\circ \rangle_\exists$. ∎

Lemma 2.2.3. [TW20] *For every clone $C \leq \mathcal{O}_A$ and $T \subseteq A^n$, there is a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$ if and only if $T \in \langle C^\circ \rangle_\nexists$.*

Proof. Let Φ be an arbitrary quantifier-free primitive positive formula over C°. By definition, Φ is of the form

$$\Phi(x_1, \ldots, x_n) \equiv \mathop{\&}_{j=1}^{t} \left(f_j\left(z_1^{(j)}, \ldots, z_{r_j}^{(j)}\right) = g_j\left(z_1^{(j)}, \ldots, z_{r_j}^{(j)}\right) \right),$$

where $n, t \in \mathbb{N}$, $f_j, g_j \in C^{(r_j)}$ and $z_1^{(j)}, \ldots, z_{r_j}^{(j)} \in \{x_1, \ldots, x_n\}$ for all $j = 1, \ldots, t$. We define the operations $\widetilde{f_j}(x_1, \ldots, x_n) := f_j(z_1^{(j)}, \ldots, z_{r_j}^{(j)})$ and $\widetilde{g_j}(x_1, \ldots, x_n) :=$

$g_j(z_1^{(j)}, \dots, z_{r_j}^{(j)})$ (by identifying variables and by adding fictitious variables) for all $j = 1, \dots, t$.

Then Φ is equivalent to the formula

$$\Psi(x_1, \dots, x_n) \equiv \mathop{\&}_{j=1}^{t} \Big(\widetilde{f_j}\big(x_1, \dots, x_n\big) = \widetilde{g_j}\big(x_1, \dots, x_n\big) \Big),$$

and $\widetilde{f_j}, \widetilde{g_j} \in C^{(n)}$ for all $j = 1, \dots, t$. Since Φ and Ψ are equivalent, they define the same set $T \subseteq A^n$, and it is obvious that the set defined by Ψ is the solution set of the system $\{(\widetilde{f_1}, \widetilde{g_1}), \dots, (\widetilde{f_t}, \widetilde{g_t})\}$. Conversely, it is clear that every solution set can be defined by a quantifier-free primitive positive formula of the form of Ψ. ∎

Lemma 2.2.4. [TW20] *For every clone $C \leq \mathcal{O}_A$, we have* $\operatorname{Inv}(C^*) = \langle C^\bullet \rangle_\exists$. *Consequently, a set $T \subseteq A^n$ is closed under C^* if and only if $T \in \langle C^\circ \rangle_\exists$.*

Proof. Using that $F^* = \operatorname{Pol}(F^\bullet)$ and that $\operatorname{Inv}(\operatorname{Pol}(R)) = \langle R \rangle_\exists$ (see Theorem 1.5.1), we have

$$\operatorname{Inv}(C^*) = \operatorname{Inv}(\operatorname{Pol}(C^\bullet)) = \langle C^\bullet \rangle_\exists.$$

The second statement of the lemma follows immediately from Lemma 2.2.2 by observing that T is closed under C^* if and only if $T \in \operatorname{Inv}(C^*)$. ∎

Remark 2.2.5. By Lemma 2.2.4, in Example 2.1.3 we could also prove that the set $\{0, 1\}$ is closed under C^* by showing that it is definable by a primitive positive formula over C°. This is easy to do, since the range of f is $\{0, 1\}$, hence the primitive positive formula $\Phi(x) \equiv \exists u \, \exists v \; f(u, v) = \pi_1(x, u)$ defines exactly the set $\{0, 1\}$.

Theorem 2.1.1 shows that in property (SDC), condition (a) implies (b). Therefore, for all clones $C \leq \mathcal{O}_A$, it suffices to investigate the implication (b) $\Longrightarrow$ (a). As a consequence of lemmas 2.2.2, 2.2.3 and 2.2.4, we obtain the promised equivalent reformulation of property (SDC) in terms of quantifier elimination.

Theorem 2.2.6. [TW20] *For every clone $C \leq \mathcal{O}_A$, the following five conditions are equivalent:*

(i) *C has property* (SDC)*;*

(ii) $\langle C^\circ \rangle_\nexists = \operatorname{Inv}(C^*)$*;*

(iii) $\langle C^\circ \rangle_\nexists = \langle C^\circ \rangle_\exists$*;*

(iv) *every primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula over C°;*

(v) *$\langle C^\circ \rangle_\nexists$ is a relational clone.*

Proof. (i) $\iff$ (ii): By Lemma 2.2.3, T is the solution set of some system of equations over C if and only if $T \in \langle C^\circ \rangle_{\nexists}$.

(ii) $\iff$ (iii): This follows from (the proof of) Lemma 2.2.4.

(iii)) $\iff$ (iv): This is trivial.

(iii) $\iff$ (v): This follows from the fact that the relational clone generated by $\langle C^\circ \rangle_{\nexists}$ is $\langle C^\circ \rangle_{\exists}$. ■

In the following corollary, we will see that Theorem 2.2.6 implies that C^* is the only clone that can describe solution sets over C (if there is such a clone at all). Thus, the abbreviation SDC can also stand for "Solution sets are Definable by closure under any Clone".

Corollary 2.2.7. [TW20] *Let $C \leq \mathcal{O}_A$ be a clone, and assume that there is a clone $D \leq \mathcal{O}_A$ such that for all $n \in \mathbb{N}$ and $T \subseteq A^n$ the following equivalence holds:*

$$T \text{ is the solution set of a system of } C\text{-equations} \iff T \text{ is closed under } D.$$

Then we have $D = C^*$.

Proof. The condition in the corollary gives us by Lemma 2.2.3 that for all $T \subseteq A^n$, we have $T \in \langle C^\circ \rangle_{\nexists}$ if and only if $T \in \mathrm{Inv}\,(D)$. This means that $\langle C^\circ \rangle_{\nexists} = \mathrm{Inv}\,(D)$, thus $\langle C^\circ \rangle_{\nexists}$ is a relational clone. Therefore, by Theorem 2.2.6, this is equivalent to the condition $\mathrm{Inv}\,(C^*) = \langle C^\circ \rangle_{\nexists} = \mathrm{Inv}\,(D)$. Applying the operator Pol to the last equality we get that

$$C^* = \mathrm{Pol}\,(\mathrm{Inv}\,(C^*)) = \mathrm{Pol}\,(\mathrm{Inv}\,(D)) = D. \qquad ■$$

Chapter 3

Solution sets over 2-element algebras

In this chapter we consider exclusively Boolean equations, that is, from now on our underlying set is $A = \{0, 1\}$. In accordance with Subsection 1.1, we denote the n-tuple $(1, 1, \ldots, 1)$ by $\mathbf{1}$, and similarly the n-tuple $(0, 0, \ldots, 0)$ by $\mathbf{0}$ (the length of the tuple shall be clear from the context). We will again use the notation of the appendix; in particular, $\Omega = \mathcal{O}_{\{0,1\}}$ stands for the set of all Boolean functions (see Figure 1.1 and Table 1.1). By proving a converse of Theorem 2.1.1, we will establish the following characterization of solution sets of Boolean equations.

Theorem 3.0.1. [TW17] *For any clone of Boolean functions $C \leq \Omega$ and $T \subseteq \{0, 1\}^n$, the following two conditions are equivalent:*

(i) *there is a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$;*

(ii) *T is closed under C^*.*

Thus clones of Boolean functions (i.e., two-element algebras) always have property (SDC). The implication (i) $\Longrightarrow$ (ii) of Theorem 3.0.1 follows from Theorem 2.1.1, so we only need to prove that (ii) implies (i). Since all Boolean clones are known (see the appendix), we could do this one by one for every single Boolean clone. However, many clones have the same centralizer, therefore, as the following remark shows, it suffices to prove Theorem 3.0.1 for a few clones (note that this remark is valid for any set A, not just for the two-element set).

Remark 3.0.2. Let $C_1 \leq C_2 \leq \mathcal{O}_A$ and $C_1^* = C_2^* = C$. Assume that C_1 has property (SDC), and let $T \subseteq A^n$ be closed under C. Then there is a system of C_1-equations such that $T = \mathrm{Sol}(\mathcal{E})$. From $C_1 \subseteq C_2$ it follows that $\mathcal{E}$ is also a system of C_2-equations. Thus property (SDC) holds for C_2 as well.

We can further reduce the number of cases by considering Boolean functions up to duality. The *dual* of $f \in \Omega^{(n)}$ is the Boolean function f^d defined by $f^d(x_1, \ldots, x_n) = \neg f(\neg x_1, \ldots, \neg x_n)$, and the dual of a Boolean clone C is

$C^d = \{f^d \mid f \in C\}$. Note that dualizing means just interchanging 0 and 1, hence if property (SDC) holds for C, then it is obviously valid for C^d, too.

Considering the observations above as well as the list of centralizers of Boolean clones given in Appendix 1.2, it suffices to prove the implication (ii) $\Longrightarrow$ (i) of Theorem 3.0.1 for the following 18 cases:

1. $L^* = L_{01},\ L_0^* = L_0,\ L_{01}^* = L$, $SL^* = SL$;
2. $M^* = [x]$, $(U^\infty M)^* = [0]$, $(U_{01}^\infty M)^* = [0,1]$, $S^* = [\neg]$, $SM^* = \Omega^{(1)}$;
3. $\Lambda^* = \Lambda_{01},\ {\Lambda_0}^* = \Lambda_0,\ {\Lambda_1}^* = \Lambda_1$, ${\Lambda_{01}}^* = \Lambda$;
4. $(\Omega^{(1)})^* = S_{01}$, $[\neg]^* = S$, $[0,1]^* = \Omega_{01}$, $[0]^* = \Omega_0$, $[x]^* = \Omega$.

We will present the proof through a sequence of 18 lemmas. These are grouped into four subsections by the methods used in their proofs, according to the numbering above.

Remark 3.0.3. According to Theorem 3.0.1, there are exactly as many algebraic geometries over the 2-element set as there are primitive positive clones, that is, 25 (see Appendix 1.2).

3.1 Linear clones

In this section we deal with the "first row" of the clones we need to investigate, that is, we prove Theorem 3.0.1 for the following clones: L_0, L_{01}, L and SL.

Lemma 3.1.1. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone ${L_0}^* = L_0$, then there exists a system $\mathcal{E}$ of L_0-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. This is a special case of Example 2.1.2 for the two-element field. ■

Lemma 3.1.2. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone ${L_{01}}^* = L$, then there exists a system $\mathcal{E}$ of L_{01}-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${L_{01}}^* = L$. Since T is closed under $L = [x+y, 1]$, it is a subspace in $\{0,1\}^n$, and we also have $\mathbf{1} \in T$. Therefore there exists a system of homogeneous linear equations $\mathcal{E}$ such that the set of solutions of $\mathcal{E}$ is exactly T. It only remains to verify that $\mathcal{E}$ is equivalent to a system of L_{01}-equations. Recall that $L_{01} = \{x_1 + \cdots + x_n \mid n \text{ is odd}\}$.

An equation in $\mathcal{E}$ is of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_m} = 0$. Since $\mathbf{1} \in T$, the tuple $\mathbf{1}$ satisfies this equation, hence it follows that $2 \mid m$. Adding x_{i_1} to both sides, we obtain the equivalent equation $x_{i_2} + \cdots + x_{i_m} = x_{i_1}$. Since there is an odd number of variables on both sides, this is an L_{01}-equation. ■

Lemma 3.1.3. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $L^* = L_{01}$, then there exists a system $\mathcal{E}$ of L-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. This is a special case of Example 2.1.2 for the two-element field. ■

Lemma 3.1.4. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $SL^* = SL$, then there exists a system $\mathcal{E}$ of SL-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $SL^* = SL$. Note that

$$SL = [x + y + z, x + 1] = \{x_1 + \cdots + x_n + c \mid n \text{ is odd, and } c \in \{0,1\}\}.$$

Since $SL \supseteq L_{01}$ we see that T is an affine subspace in $\{0,1\}^n$, hence there exists a system $\mathcal{E}$ of linear equations such that $T = \mathrm{Sol}(\mathcal{E})$. Moreover, since $x + 1 \in SL$, we have $\mathbf{x} \in T \Rightarrow \neg\mathbf{x} \in T$. It only remains to verify that $\mathcal{E}$ is equivalent to a system of SL-equations.

An equation in $\mathcal{E}$ is of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_m} = c$. Since $\mathbf{x} \in T$ implies that $\neg\mathbf{x} \in T$, it follows that $2 \mid m$. Our equation is equivalent to $x_{i_2} + \cdots + x_{i_m} = x_{i_1} + c$, and since at both sides of the equation there is an odd number of variables, it follows that this is an SL-equation. ■

3.2 Clones with unary centralizers

In this section we deal with the "second row" of the clones we need to investigate, that is, we prove Theorem 3.0.1 for the following clones: $M, U^\infty M, U^\infty_{01} M$, S and SM.

Lemma 3.2.1. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $M^* = [x]$, then there exists a system $\mathcal{E}$ of M-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Note that every subset of $\{0,1\}^n$ is closed under $[x]$. For every $T \subsetneq \{0,1\}^n$, we have

$$T = \bigcap_{\mathbf{v} \notin T} T_{\mathbf{v}}, \tag{3.2.1}$$

where $T_{\mathbf{v}} = \{0,1\}^n \setminus \{\mathbf{v}\}$. Therefore it suffices to show that for every $\mathbf{v} \in \{0,1\}^n$, there exists an M-equation (f, g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f, g)\})$.

Let $\mathbf{v} \in \{0,1\}^n$ be an arbitrary n-tuple. Let f and g be the following functions:

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} > \mathbf{v}; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad g(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \geq \mathbf{v}; \\ 0, & \text{otherwise.} \end{cases}$$

Figure 3.1 shows a schematic view of the Hasse diagram of $\{0,1\}^n$. Grey color indicates points where the value of the corresponding function is 1; on the remaining

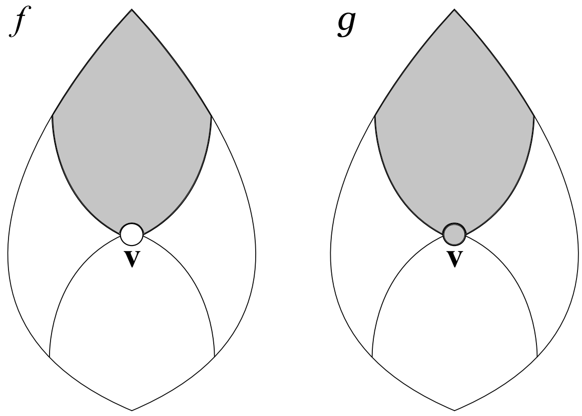


Figure 3.1: The functions f and g in the proof of Lemma 3.2.1.

tuples the values are 0. It is easy to see that $f, g \in M$ and that for all $\mathbf{v} \in \{0,1\}^n$, we have $f(\mathbf{x}) = g(\mathbf{x})$ if and only if $\mathbf{x} \neq \mathbf{v}$, therefore the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_{\mathbf{v}}$. ■

Lemma 3.2.2. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $(U^\infty M)^* = [0]$, then there exists a system $\mathcal{E}$ of $U^\infty M$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. A set $T \subseteq \{0,1\}^n$ is closed under $[0]$ if and only if $\mathbf{0} \in T$. Thus, similarly to the proof of Lemma 3.2.1, it suffices to show that for every $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}\}$ there exists a $U^\infty M$-equation (f, g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f,g)\})$. (We can exclude $\mathbf{v} = \mathbf{0}$ from the intersection (3.2.1) because $\mathbf{0} \in T$.)

Let $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}\}$ be an arbitrary n-tuple, and let f and g be the same functions, as defined in the proof of Lemma 3.2.1. We have seen that f and g are monotone and $\mathrm{Sol}(\{(f,g)\}) = T_{\mathbf{v}}$. Hence it only remains to verify that $f, g \in U^\infty$, that is, there exists $k \in \{1, 2, \ldots, n\}$ such that for all $\mathbf{x} \in \{0,1\}^n$, if $f(\mathbf{x}) = 1$ ($g(\mathbf{x}) = 1$), then $x_k = 1$. We may assume (after a permutation of coordinates) that $\mathbf{v}$ is of the form $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$. Since $\mathbf{v} \neq \mathbf{0}$, at least one 1 appears in $\mathbf{v}$, i.e., $v_n = 1$. If $f(\mathbf{x}) = 1$, then $\mathbf{x} > \mathbf{v}$, hence $x_n = 1$, thus $f \in U^\infty$. Similarly, $x_n = 1$ whenever $g(\mathbf{x}) = 1$, so $g \in U^\infty$. ■

Lemma 3.2.3. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $(U_{01}^\infty M)^* = [0,1]$, then there exists a system $\mathcal{E}$ of $U_{01}^\infty M$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. The proof is almost identical to those of the previous two lemmas. Here we have $\mathbf{0}, \mathbf{1} \in T$, hence we can assume that $\mathbf{v} \notin \{\mathbf{0}, \mathbf{1}\}$, and we only need to show that in this case the functions f and g defined in the proof of Lemma 3.2.1 are 0-preserving as well as 1-preserving. By the definition of the functions f and g, it is obvious that $f(\mathbf{0}) = 0$ and $g(\mathbf{1}) = 1$. Moreover, $\mathbf{v} \neq \mathbf{0}$ implies that $g(\mathbf{0}) = 0$ and $\mathbf{v} \neq \mathbf{1}$ implies that $f(\mathbf{1}) = 1$. Thus $f, g \in U_{01}^{\infty}M$, as claimed. ■

Lemma 3.2.4. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $S^* = [\neg]$, then there exists a system $\mathcal{E}$ of S-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. For every $T \subsetneq \{0,1\}^n$ that is closed under the clone $[\neg]$, we have

$$T = \bigcap_{\mathbf{v} \notin T} T_{\mathbf{v}},$$

where $T_{\mathbf{v}} = \{0,1\}^n \setminus \{\mathbf{v}, \neg\mathbf{v}\}$. (Note that we are changing the notation of the previous three lemmas.) Therefore it suffices to show that for every $\mathbf{v} \in \{0,1\}^n$, there exists an S-equation (f, g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f, g)\})$.

Let $\mathbf{v} \in \{0,1\}^n$ be an arbitrary n-tuple, and let $f \in S$ be an arbitrary n-ary self-dual function. Define the function g by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \notin \{\mathbf{v}, \neg\mathbf{v}\}; \\ \neg f(\mathbf{x}), & \text{if } \mathbf{x} \in \{\mathbf{v}, \neg\mathbf{v}\}. \end{cases}$$

Clearly, the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_{\mathbf{v}}$, and it is straightforward to verify that g is self-dual. ■

Lemma 3.2.5. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $SM^* = \Omega^{(1)}$, then there exists a system $\mathcal{E}$ of SM-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Using the notation of Lemma 3.2.4, we need to show that for every $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$ there exists an SM-equation (f, g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f, g)\})$. (We exclude $\mathbf{0}$ and $\mathbf{1}$ since T is closed under $\Omega^{(1)} = [0, 1, \neg x]$.)

Let $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, and let $h \in SM$ be an arbitrary n-ary self-dual monotone function. Define the function f by

$$f(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \leq \mathbf{v} \text{ or } \mathbf{x} < \neg\mathbf{v}; \\ 1, & \text{if } \mathbf{x} > \mathbf{v} \text{ or } \mathbf{x} \geq \neg\mathbf{v}; \\ h(\mathbf{x}), & \text{otherwise.} \end{cases}$$

Since $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$, the tuples $\mathbf{v}$ and $\neg\mathbf{v}$ are incomparable, hence the three cases in the definition of f are mutually exclusive and thus f is well defined. Define the function g by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \notin \{\mathbf{v}, \neg\mathbf{v}\}; \\ \neg f(\mathbf{x}), & \text{if } \mathbf{x} \in \{\mathbf{v}, \neg\mathbf{v}\}. \end{cases}$$

Let H be the set of tuples $\mathbf{x} \in \{0,1\}^n$ that are incomparable to both $\mathbf{v}$ and $\neg\mathbf{v}$. (Note that H is closed under negation.) The colors on Figure 3.2 indicate the value

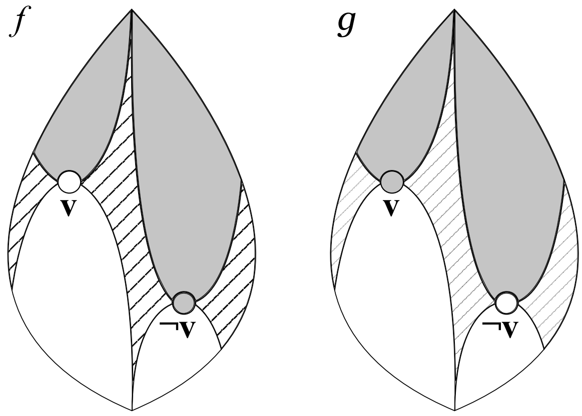


Figure 3.2: The functions f and g in the proof of Lemma 3.2.5.

of the corresponding function as in the proof of Lemma 3.2.1. The striped area represents the set H. From the definition of the function g it is clear that the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_{\mathbf{v}}$.

It only remains to verify that $f, g \in SM$, that is, f and g are both monotone and self-dual. We present the details for f only; the proof for g is similar.

Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-tuples with $\mathbf{x} \leq \mathbf{y}$. To verify that $f \in M$, we consider four cases:

1. If $\mathbf{x}, \mathbf{y} \in H$, then $f(\mathbf{x}) = h(\mathbf{x}) \leq h(\mathbf{y}) = f(\mathbf{y})$, as $h \in SM$.
2. If $\mathbf{x}, \mathbf{y} \notin H$, then from the definition of the function f we have $f(\mathbf{x}) \leq f(\mathbf{y})$.
3. If $\mathbf{x} \in H$ and $\mathbf{y} \notin H$, then $\mathbf{y}$ is comparable to $\mathbf{v}$ or $\neg\mathbf{v}$. If $f(\mathbf{y}) = 1$, then obviously $f(\mathbf{x}) \leq f(\mathbf{y})$. If $f(\mathbf{y}) = 0$, then $\mathbf{y} \leq \mathbf{v}$ or $\mathbf{y} < \neg\mathbf{v}$. However, in this case $\mathbf{x} \leq \mathbf{y}$ implies that $\mathbf{x}$ is comparable to $\mathbf{v}$ or to $\neg\mathbf{v}$, contradicting the assumption $\mathbf{x} \in H$.
4. The case $\mathbf{x} \notin H$, $\mathbf{y} \in H$ can be verified similarly to the previous case.

For self-duality, let $\mathbf{x} \in \{0,1\}^n$ be an arbitrary n-tuple; we need to show that $f(\mathbf{x}) = \neg f(\neg\mathbf{x})$. We distinguish two cases:

1. If $\mathbf{x} \notin H$, then $\neg\mathbf{x} \notin H$. If $f(\mathbf{x}) = 0$, then either $\mathbf{x} \leq \mathbf{v}$ or $\mathbf{x} < \neg\mathbf{v}$. In the first case, we have $\neg\mathbf{x} \geq \neg\mathbf{v}$, and in the second case, we have $\neg\mathbf{x} > \mathbf{v}$. In both cases, $f(\neg\mathbf{x}) = 1$. Similarly, $f(\mathbf{x}) = 1$ implies that $f(\neg\mathbf{x}) = 0$.
2. If $\mathbf{x} \in H$, then $\neg\mathbf{x} \in H$, therefore $f(\mathbf{x}) = h(\mathbf{x}) = \neg h(\neg\mathbf{x}) = \neg f(\neg\mathbf{x})$, as $h \in SM$. ■

3.3 Clones generated by conjunction and constants

In this section we deal with the "third row" of the clones we need to investigate, that is, we prove Theorem 3.0.1 for the following clones: Λ, Λ_0, Λ_1 and Λ_{01}.

Lemma 3.3.1. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $\Lambda^* = \Lambda_{01}$, then there exists a system $\mathcal{E}$ of Λ-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Note that $\Lambda = [x \wedge y, 0, 1]$, and that $\Lambda_{01} = [x \wedge y]$. Let $T \subseteq \{0,1\}^n$ be closed under the clone $\Lambda^* = \Lambda_{01}$, and let $\mathcal{E} = \mathrm{Eq}_\Lambda(T)$. We will show that $T = \mathrm{Sol}(\mathcal{E})$. Since $T \subseteq \mathrm{Sol}(\mathcal{E})$ is trivial, it suffices to prove that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ implies $\mathbf{v} \in T$ for all $\mathbf{v} \in \{0,1\}^n$.

Let $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$, and suppose first that $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$. We may assume without loss of generality that $\mathbf{v}$ is of the form $(1,1,\ldots,1,0,0,\ldots,0)$, where $v_1 = \cdots = v_k = 1$ and $v_{k+1} = \cdots = v_n = 0$ ($k \in \{1,\ldots,n-1\}$). Let us consider the following Λ-equation:

$$x_1 \wedge \cdots \wedge x_k = x_1 \wedge \cdots \wedge x_k \wedge x_{k+1}. \tag{3.3.1}$$

It is clear that $\mathbf{v}$ does not satisfy (3.3.1), thus the equation (3.3.1) does not appear in $\mathcal{E}$. Hence, there exists an n-tuple $\mathbf{t}^{(1)} \in T$ such that $\mathbf{t}^{(1)}$ does not satisfy (3.3.1), i.e., $t_1^{(1)} = \cdots = t_k^{(1)} = 1$ and $t_{k+1}^{(1)} = 0$. Similarly, for all $m \in \{1,\ldots,n-k\}$ we may consider the Λ-equation

$$x_1 \wedge \cdots \wedge x_k = x_1 \wedge \cdots \wedge x_k \wedge x_{k+m}. \tag{3.3.2}$$

Just like (3.3.1), the equation (3.3.2) does not appear in $\mathcal{E}$, thus there exists $\mathbf{t}^{(m)} \in T$ such that $t_1^{(m)} = \cdots = t_k^{(m)} = 1$ and $t_{k+m}^{(m)} = 0$. We know that T is closed under the clone Λ_{01}, in particular, T is closed under conjunctions. Therefore $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(n-k)} \in T$ implies that

$$\mathbf{t}^{(1)} \wedge \cdots \wedge \mathbf{t}^{(n-k)} = (1,1,\ldots,1,0,0,\ldots,0) = \mathbf{v} \in T.$$

It only remains to consider the cases $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = \mathbf{1}$. If $\mathbf{v} = \mathbf{0}$ satisfies $\mathcal{E}$, then let us consider the following Λ-equations for all $i \in \{1,\ldots,n\}$:

$$x_i = 1. \tag{3.3.3}$$

Since $\mathbf{v} = \mathbf{0}$ does not satisfy (3.3.3), this equation does not belong to $\mathcal{E}$. Thus T contains a counterexample $\mathbf{t}^{(i)}$ to (3.3.3) such that $t_i^{(i)} = 0$. Therefore we have

$$\mathbf{t}^{(1)} \wedge \cdots \wedge \mathbf{t}^{(n)} = (0,0,\ldots,0) = \mathbf{v} \in T.$$

If $\mathbf{v} = \mathbf{1}$ satisfies $\mathcal{E}$, then we consider the following Λ-equation:

$$x_1 \wedge \cdots \wedge x_n = 0. \tag{3.3.4}$$

Similarly as above, T contains a counterexample to (3.3.4), and the only such counterexample is $\mathbf{1}$. ■

Lemma 3.3.2. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_0}^* = \Lambda_0$, then there exists a system $\mathcal{E}$ of Λ_0-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_0}^* = \Lambda_0$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_0}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{0}$, then the same argument as in Lemma 3.3.1 proves that $\mathbf{v} \in T$. It only remains to consider the case $\mathbf{v} = \mathbf{0}$. Since T is closed under the clone Λ_0 and $\mathbf{0} \in \Lambda_0$, it follows that $\mathbf{0} \in T$. ■

Lemma 3.3.3. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_1}^* = \Lambda_1$, then there exists a system $\mathcal{E}$ of Λ_1-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_1}^* = \Lambda_1$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_1}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{1}$, then the same argument as in Lemma 3.3.1 proves that $\mathbf{v} \in T$. Since T is closed under the clone Λ_1 and $\mathbf{1} \in \Lambda_1$, it follows that $\mathbf{1} \in T$. ■

Lemma 3.3.4. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_{01}}^* = \Lambda$, then there exists a system $\mathcal{E}$ of Λ_{01}-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_{01}}^* = \Lambda$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_{01}}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$, then the same argument as in Lemma 3.3.1 proves that $\mathbf{v} \in T$. Since T is closed under the clone Λ and $\mathbf{0}, \mathbf{1} \in \Lambda$, it follows that $\mathbf{0}, \mathbf{1} \in T$. ■

3.4 Unary clones

In this section we deal with the "last row" of the clones we need to investigate, that is, we prove Theorem 3.0.1 for the following clones: $[x], [0], [0,1], [\neg]$ and $\Omega^{(1)}$.

Lemma 3.4.1. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $[x]^* = \Omega$, then there exists a system $\mathcal{E}$ of $[x]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[x]^* = \Omega$, and let $\mathcal{E} = \mathrm{Eq}_{[x]}(T)$. We will show that $T = \mathrm{Sol}(\mathcal{E})$. Since $T \subseteq \mathrm{Sol}(\mathcal{E})$ is trivial, it suffices to prove that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ implies $\mathbf{v} \in T$ for all $\mathbf{v} \in \{0,1\}^n$.

Let $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$, and let $T = \{\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}\}$, where $m = |T|$. Let us consider the matrix $Q = (t_i^{(j)}) \in \{0,1\}^{n \times m}$ whose j-th column vector is $\mathbf{t}^{(j)}$. Let $\mathbf{r}_i =$

$(t_i^{(1)}, \dots, t_i^{(m)})$ be the i-th row of Q, and let $R = \{\mathbf{r}_1, \dots, \mathbf{r}_n\}$ be the set of row vectors of Q. Define the m-ary function Φ by

$$\Phi(\mathbf{x}) = \begin{cases} v_i, & \text{if } \mathbf{x} = \mathbf{r}_i; \\ 0, & \text{if } \mathbf{x} \notin R. \end{cases}$$

Note that Φ is defined in such a way that $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)})$. However, we need to verify that Φ is a well-defined function. Assume that $\mathbf{r}_i = \mathbf{r}_j$ and $v_i \neq v_j$ for some $i, j \in \{1, \dots, n\}$. From $\mathbf{r}_i = \mathbf{r}_j$ it follows that T satisfies the $[x]$-equation $x_i = x_j$, hence this equation belongs to $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = v_j$, which is a contradiction. Therefore the function Φ is well defined, and obviously $\Phi \in \Omega$. The set T is closed under the clone Ω, hence $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)}) \in T$. ∎

Lemma 3.4.2. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $[0]^* = \Omega_0$, then there exists a system $\mathcal{E}$ of $[0]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[0]^* = \Omega_0$, let $\mathcal{E} = \mathrm{Eq}_{[0]}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$, R and Φ as in the proof of Lemma 3.4.1. The proof of Lemma 3.4.1 shows that Φ is well defined; we only need to verify that $\Phi \in \Omega_0$. If $\mathbf{0} \notin R$, then $\Phi(\mathbf{0}) = 0$ follows from the definition of Φ. If $\mathbf{r}_i = \mathbf{0}$ for some i, then the $[0]$-equation $x_i = 0$ holds in T, thus $(x_i, 0) \in \mathcal{E}$. Therefore $\mathbf{v}$ satisfies this equation as well, hence $\Phi(\mathbf{0}) = \Phi(\mathbf{r}_i) = v_i = 0$. This shows that $\Phi \in \Omega_0$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under Ω_0. ∎

Lemma 3.4.3. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $[0,1]^* = \Omega_{01}$, then there exists a system $\mathcal{E}$ of $[0,1]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. The proof is almost identical to that of Lemma 3.4.2; we just need to modify the definition of Φ such that $\Phi(\mathbf{1}) = 1$ if $\mathbf{1} \notin R$. Taking equations of the form $x_i = 0$ and $x_i = 1$ into account, we can prove that $\Phi \in \Omega_{01}$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under Ω_{01}. ∎

Lemma 3.4.4. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $[\neg]^* = S$, then there exists a system $\mathcal{E}$ of $[\neg]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[\neg]^* = S$, let $\mathcal{E} = \mathrm{Eq}_{[\neg]}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$ and R as in the proof of Lemma 3.4.1 and let $R' = \{\neg\mathbf{r}_1, \dots, \neg\mathbf{r}_n\}$. Let $h \in S$ be an arbitrary m-ary self-dual function and define the function $\Phi \in \Omega^{(m)}$ by

$$\Phi(\mathbf{x}) = \begin{cases} v_i, & \text{if } \mathbf{x} = \mathbf{r}_i; \\ \neg v_i, & \text{if } \mathbf{x} = \neg\mathbf{r}_i; \\ h(\mathbf{x}), & \text{if } \mathbf{x} \notin R \cup R'. \end{cases}$$

We show that the function Φ is well defined. We distinguish two cases:

1. If $\mathbf{r}_i = \mathbf{r}_j$ and $v_i \neq v_j$ for some $i, j \in \{1, \ldots, n\}$, then T satisfies the $[\neg]$-equation $x_i = x_j$, hence this equation belongs to $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = v_j$, which is a contradiction.

2. If $\mathbf{r}_i = \neg\mathbf{r}_j$ and $v_i \neq \neg v_j$ for some $i, j \in \{1, \ldots, n\}$, then T satisfies the $[\neg]$-equation $x_i = \neg x_j$, hence this equation appears in $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = \neg v_j$, which is a contradiction.

It only remains to verify that $\Phi \in S$. Let $\mathbf{a}$ be an arbitrary n-tuple. If $\mathbf{a} \notin R \cup R'$, then $\Phi(\mathbf{a}) = h(\mathbf{a}) = \neg h(\neg\mathbf{a}) = \neg\Phi(\neg\mathbf{a})$, since the function h is self-dual. If $\mathbf{a} = \mathbf{r}_i$ for some $i \in \{1, \ldots, n\}$, then $\neg\mathbf{a} = \neg\mathbf{r}_i$, thus $\Phi(\neg\mathbf{a}) = \neg v_i = \neg\Phi(\mathbf{a})$. This shows that $\Phi \in S$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under S. ■

Lemma 3.4.5. [TW17] *If $T \subseteq \{0,1\}^n$ is closed under the clone $(\Omega^{(1)})^* = S_{01}$, then there exists a system $\mathcal{E}$ of $\Omega^{(1)}$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $(\Omega^{(1)})^* = S_{01}$, let $\mathcal{E} = \mathrm{Eq}_{\Omega^{(1)}}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$, R and R' as in the proof of Lemma 3.4.4, and let us also define Φ in the same way as there, but this time choosing the function h from S_{01}. We can follow the same argument as before, but we also need to verify that $\Phi \in \Omega_{01}$. If $\mathbf{0} \notin R \cup R'$, then $\Phi(\mathbf{0}) = 0$, since $h \in S_{01}$. If $\mathbf{0} \in R$, and $\mathbf{0} = \mathbf{r}_i$, then the $\Omega^{(1)}$-equation $x_i = 0$ holds in $\mathcal{E}$, thus $v_i = 0$. Therefore, from the definition of the function Φ, we have $\Phi(\mathbf{0}) = 0$. If $\mathbf{0} \in R'$, and $\mathbf{0} = \neg\mathbf{r}_i$, then the $\Omega^{(1)}$-equation $\neg x_i = 0$ holds in $\mathcal{E}$, thus $\neg v_i = 0$, hence $\Phi(\mathbf{0}) = 0$. This proves that $\Phi \in \Omega_0$, and a similar argument shows that $\Phi \in \Omega_1$. Therefore $\Phi \in S_{01}$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under S_{01}. ■

Chapter 4

Centralizers of finite lattices and semilattices

In this chapter we study operations commuting with (semi)lattice operations, and we aim for concrete descriptions of these operations that allow us to classify (and in some cases also count) them according to the number of their essential variables. As a "byproduct", we also obtain a simple proof for Kuznetsov's description [Kuz79] of primitive positive clones on the two-element set.

We say that the i-th variable of $f \in \mathcal{O}_A^{(n)}$ is *essential* (or that f *depends* on its i-th variable) if there exist tuples $\mathbf{a}, \mathbf{a}' \in A^n$ differing only in their i-th component such that $f(\mathbf{a}) \neq f(\mathbf{a}')$. The number of essential variables of f is called the *essential arity* of f. To simplify notation, we often assume that operations do not have inessential variables, thus we say that f is an essentially n-ary operation on A if $f \in \mathcal{O}_A^{(n)}$ and f depends on all of its variables. In this definition we allow n to be 0, and thus in this chapter we also consider essentially nullary operations (contrary to other chapters).

In Section 4.1 we present two different characterizations of the essentially n-ary members of the centralizer of a finite semilattice (one of them is a slight variation of a result of Larose [Lar95]). We use these to give a general formula for the number of essentially n-ary operations commuting with the join operation of a finite lattice, and we illustrate this with the example of finite chains. This is a generalization of one of the results of [MR18], where this counting problem was solved for the three-element chain. We study operations commuting with both the join and the meet operation of a lattice in sections 4.2 and 4.3. Since the essential arity of operations in the centralizer is bounded for every finite lattice, here we focus on the existence of essentially n-ary operations instead of counting them. In Section 4.2 we give an explicit description of the elements of the centralizer of a finite distributive lattice, and then we provide two characterizations of finite distributive lattices having an essentially n-ary operation in their centralizers. In Section 4.3 we investigate thoroughly all results of the previous section to see which ones remain valid for nondistributive finite lattices. As a tool aiding this investigation, we give an upper bound for the essential arity of operations in the centralizer of a finite algebra generating a congruence distributive variety (following an idea of [CGL18; CM18]). Finally, in Section 4.4 we describe the centralizer clones of Boolean functions. Primitive positive clones on the two-element set were described already by Kuznetsov [Kuz79], and later also in [Her08], and probably many of the readers of this thesis have also computed these by themselves at some point. This is not a difficult task using the Post lattice (see Figure 1.1 in the appendix), but it involves some case-by-case analysis. We offer a “painless” proof that covers all cases by just three general theorems. Besides presenting the list of the 25 primitive positive clones, we also give the centralizer of each Boolean clone in Table 1.2.

4.1 Centralizers of finite semilattices

In this section we give two different characterizations of the centralizer clone of a join-semilattice $\mathbb{S} = (S; \vee)$. Since we are interested in counting the essentially n-ary operations in the centralizer, we assume that S is finite, but some of our results are also valid for infinite complete semilattices.

4.1.1 Characterizations

If $\mathbb{S}$ is a finite join-semilattice then it has a greatest element (denoted by 1), and if $\mathbb{S}$ also has a least element (denoted by 0), then there is a meet operation on S such that $(S; \vee, \wedge)$ is a lattice. In the latter case the centralizer clone $[\vee]^*$ is generated by its unary members (i.e., endomorphisms of $(S; \vee)$) together with the join operation. This was proved by B. Larose [Lar95]; in the following theorem we reprove this result, and we extend it by providing a unique expression for any $f \in [\vee]^*$ as a join of endomorphisms, and we also determine the necessary and sufficient condition for f to depend on all of its variables.

Theorem 4.1.1. [Lar95; TW21] *Let $\mathbb{S} = (S; \vee)$ be a finite semilattice with a least element 0 and greatest element 1. An n-ary operation $f \in \mathcal{O}_S$ belongs to the centralizer $[\vee]^*$ if and only if there exist unary operations $u_1, \ldots, u_n \in [\vee]^*$ such that*

$$f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n) \text{ and } u_1(0) = u_2(0) = \cdots = u_n(0).$$

The above expression for f is unique, and f depends on all of its variables if and only if none of the u_i are constant, i.e., $u_i(0) \neq u_i(1)$ for all $i \in \{1, \ldots, n\}$.

Proof. The “if” part is clear: the join operation commutes with itself, hence if f can be written as a composition of $\vee$ with $u_1, \ldots, u_n \in [\vee]^*$, then $f \in [\vee]^*$, as $[\vee]^*$ is closed under composition.

For the "only if" part let us assume that f is an n-ary operation in $[\vee]^*$, and define $u_1, \dots, u_n$ by $u_1(x) = f(x, 0, \dots, 0), \dots, u_n(x) = f(0, \dots, 0, x)$. Then obviously $u_1(0) = \cdots = u_n(0) = f(0, \dots, 0)$; furthermore, $u_1, \dots, u_n \in [\vee]^*$, as f and the constant 0 operation belong to $[\vee]^*$. Since f commutes with $\vee$, applying the definition of commutation to the $n \times n$ matrix

$$\begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_n \end{pmatrix},$$

we can conclude that

$$\begin{aligned} f(x_1, \dots, x_n) &= f\Big(x_1 \vee 0 \vee \cdots \vee 0, \ \dots, \ 0 \vee \cdots \vee 0 \vee x_n\Big) \\ &= f(x_1, 0, \dots, 0) \vee \cdots \vee f(0, \dots, 0, x_n) \\ &= u_1(x_1) \vee \cdots \vee u_n(x_n). \end{aligned}$$

To prove uniqueness, assume that $f(x_1, \dots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and $u_1(0) = u_2(0) = \cdots = u_n(0)$. Then we have

$$f(x_1, 0, \dots, 0) = u_1(x_1) \vee u_2(0) \vee \cdots \vee u_n(0) = u_1(x_1) \vee u_1(0) \vee \cdots \vee u_1(0) = u_1(x_1),$$

as u_1 is monotone. Thus u_1 is indeed uniquely determined by f, and the above equality also shows that if $u_1(0) \neq u_1(1)$, then f depends on its first variable: $f(0, 0, \dots, 0) = u_1(0) \neq u_1(1) = f(1, 0, \dots, 0)$. The statements about the uniqueness of u_i and about the essentiality of the i-th variable for $i = 2, \dots, n$ can be proved in an analogous way. ■

Example 4.1.2. Let $\mathbb{S} = (\{0, 1, 2, 3, 4\}; \vee)$ be a five-element chain (regarded as a join-semilattice), and let the unary operations u_1 and u_2 be defined by

$$\begin{aligned} &u_1(0) = 0, \ u_1(1) = 1, \ u_1(2) = 1, \ u_1(3) = 3, \ u_1(4) = 4; \\ &u_2(0) = 0, \ u_2(1) = 1, \ u_2(2) = 2, \ u_2(3) = 4, \ u_2(4) = 4. \end{aligned}$$

Figure 4.1 shows the values of the binary operation $f(x_1, x_2) = u_1(x_1) \vee u_2(x_2)$. We can see that each of the sets $\{(a_1, a_2) \in S^2 : f(a_1, a_2) \leq b\}$ $(b = 0, \dots, 4)$ is a "lower rectangle". We will see later that for every join-semilattice S and every $b \in S$, the set $\{\mathbf{a} \in S^n : f(\mathbf{a}) \leq b\}$ has a similar structure for all $f \in [\vee]^*$; moreover, we will characterize the operations in the centralizer in terms of these "down-sets".

The following theorem shows that the assumption about $\mathbb{S}$ having a least element cannot be dropped from Theorem 4.1.1: if a finite join-semilattice does not have a least element, then its centralizer cannot be generated by unary operations and the join operation.

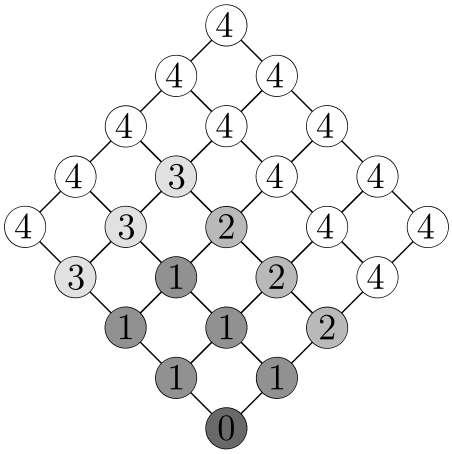


Figure 4.1: A binary operation in the centralizer of the chain $(\{0,1,2,3,4\};\vee)$

Theorem 4.1.3. [TW21] *The centralizer* $[\vee]^*$ *of a finite semilattice* $\mathbb{S} = (S;\vee)$ *is generated by its unary part and the join operation if and only if* $\mathbb{S}$ *has a least element (i.e., if* $\mathbb{S}$ *is the join reduct of a lattice).*

Proof. The "if" part follows from Theorem 4.1.1; for the "only if" part assume that $\mathbb{S} = (S;\vee)$ is a finite semilattice without a least element. Then there are distinct minimal elements $a, b \in S$. We define a binary operation f on S by

$$f(x_1,x_2) = \begin{cases} a, & \text{if } (x_1,x_2) = (a,b); \\ b, & \text{if } (x_1,x_2) = (b,b); \\ a \vee b, & \text{otherwise.} \end{cases}$$

In order to prove that f commutes with the join operation, we need to verify the following identity:

$$f(x_1,x_2) \vee f(y_1,y_2) = f(x_1 \vee y_1, x_2 \vee y_2). \tag{4.1.1}$$

Since a is a minimal element, the only way of writing a as the join of two elements is $a = a \vee a$. Therefore, the left hand side of (4.1.1) is a if and only if $f(x_1,x_2) = f(y_1,y_2) = a$, and, by the definition of f, this holds only for $x_1 = y_1 = a$ and $x_2 = y_2 = b$. The right hand side of (4.1.1) equals a if and only if $x_1 \vee y_1 = a$ and $x_2 \vee y_2 = b$, and this is also equivalent to $x_1 = y_1 = a$ and $x_2 = y_2 = b$. Similarly, both the left hand side and the right hand side of (4.1.1) take the value b if and only if $x_1 = y_1 = x_2 = y_2 = b$. For all other inputs, both sides of (4.1.1) give $a \vee b$. Thus f belongs to the centralizer $[\vee]^*$, indeed.

If a binary operation can be obtained as a composition of the join operation and endomorphisms of $\mathbb{S}$, then it can be written as $u_1(x_1) \vee u_2(x_2)$, where u_1 and

u_2 are endomorphisms. (The proof of this fact is a routine term induction; we leave it to the reader.) Assume, for contradiction, that our operation f can be expressed in this form: $f(x_1, x_2) = u_1(x_1) \vee u_2(x_2)$. Then we have $a = f(a, b) = u_1(a) \vee u_2(b)$, and this implies $u_1(a) = u_2(b) = a$. On the other hand, $b = f(b, b) = u_1(b) \vee u_2(b) = u_1(b) \vee a \geq a$, which is a contradiction. ■

Next we derive another kind of characterization of the centralizer of the clone $[\vee]$, which describes, in some sense, the "distribution" of the values of an n-ary operation $f \in [\vee]^*$ in S^n, as illustrated by Figure 4.1. For this characterization we will not need the assumption that $\mathbb{S}$ has a least element. Nevertheless, we will consider the lattice $\mathbb{S}_\perp$ obtained from $\mathbb{S}$ by adding a new element $\perp$ to the bottom of $\mathbb{S}$. Thus let $\mathbb{S}_\perp = \mathbb{S} \cup \{\perp\}$, where $\perp$ is an element not contained in S, and we define the partial order on $\mathbb{S}_\perp$ so that $\perp < a$ for all $a \in S$, and we keep the original ordering on the elements of S. Note that we add a new bottom element even if $\mathbb{S}$ happens to have a least element 0; in this case $\perp$ is the unique lower cover of 0.

If $\mathbb{S}$ is a join-semilattice, then $f \in [\vee]^*$ if and only if f is a join-homomorphism from $\mathbb{S}^n$ to $\mathbb{S}$. This motivates us to consider the set $\mathrm{Hom}_\vee(\mathbb{A}, \mathbb{B})$ of all join-homomorphisms from A to B, where $\mathbb{A}$ and $\mathbb{B}$ are finite join-semilattices. We use the notation $\mathrm{Hom}^1_\vee(\mathbb{A}, \mathbb{B})$ for the set of all join-homomorphisms from $\mathbb{A}$ to $\mathbb{B}$ that preserve the greatest element: $\mathrm{Hom}^1_\vee(\mathbb{A}, \mathbb{B}) := \{f \in \mathrm{Hom}_\vee(\mathbb{A}, \mathbb{B}) : f(1) = 1\}$. Similarly, if $\mathbb{A}$ and $\mathbb{B}$ have a least element, denoted by 0, then let $\mathrm{Hom}^0_\vee(\mathbb{A}, \mathbb{B})$ denote the set of join-homomorphisms preserving the least element, and let $\mathrm{Hom}^{01}_\vee(\mathbb{A}, \mathbb{B})$ be the set of join-homomorphisms preserving both boundary elements. Note that the least element of $\mathbb{A}_\perp$ and $\mathbb{B}_\perp$ is denoted by $\perp$ (not by 0), therefore in this case we will use the notation $\mathrm{Hom}^{\perp,1}_\vee(\mathbb{A}_\perp, \mathbb{B}_\perp)$ instead of $\mathrm{Hom}^{01}_\vee(\mathbb{A}_\perp, \mathbb{B}_\perp)$. For meet-semilattices $\mathbb{A}$ and $\mathbb{B}$, the sets $\mathrm{Hom}_\wedge(\mathbb{A}, \mathbb{B})$, $\mathrm{Hom}^0_\wedge(\mathbb{A}, \mathbb{B})$, etc. are defined analogously.

We need to introduce one more notation: for an element a in a partially ordered set $(A; \leq)$, the *principal ideal* and the *principal filter* generated by a are defined and denoted as follows:

$$\downarrow a = \{c \in A : c \leq a\}, \qquad \uparrow a = \{c \in A : c \geq a\}.$$

Observe that in Figure 4.1, the elements labeled by numbers less than or equal to b form a principal ideal for any $b \in \{0, \ldots, 4\}$. This is a special case of the following lemma, which states that for all $f \in \mathrm{Hom}_\vee(\mathbb{A}, \mathbb{B})$ and $b \in B$, the set $f^{-1}(\downarrow b) = \{a \in A : f(a) \leq b\}$ is a principal ideal whenever it is not empty.

Lemma 4.1.4. [TW21] *Let $\mathbb{A} = (A; \vee)$ and $\mathbb{B} = (B; \vee)$ be finite semilattices. If $f \colon \mathbb{A} \to \mathbb{B}$ is a homomorphism, then $f^{-1}(\downarrow b) \subseteq A$ is either empty or a principal ideal for all $b \in B$.*

Proof. Assume that $f^{-1}(\downarrow b)$ is not empty. Join-homomorphisms are monotone, hence it is clear that $f^{-1}(\downarrow b)$ is an ideal, i.e., $a_1 \leq a_2 \in f^{-1}(\downarrow b)$ implies that

$a_1 \in f^{-1}(\downarrow b)$. Moreover, $f^{-1}(\downarrow b)$ is closed under joins: if $a_1, a_2 \in f^{-1}(\downarrow b)$, then $f(a_1 \vee a_2) = f(a_1) \vee f(a_2) \leq b \vee b = b$, hence $a_1 \vee a_2 \in f^{-1}(\downarrow b)$. Since A is finite, we can take the join $a = \bigvee f^{-1}(\downarrow b)$ of all elements of $f^{-1}(\downarrow b)$, and from the above considerations it follows that $f^{-1}(\downarrow b)$ is the principal ideal generated by a. ■

In the next theorem we give a canonical bijection between the sets $\mathrm{Hom}_{\vee}(\mathbb{A}, \mathbb{B})$ and $\mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp})$, which will be the main tool for the promised characterization of the operations in the centralizer of a finite join-semilattice. Recall that $\mathbb{B}_{\perp}$ and $\mathbb{A}_{\perp}$ are lattices, and $\mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp})$ denotes the set of all meet-homomorphisms $g\colon \mathbb{B}_{\perp} \to \mathbb{A}_{\perp}$ satisfying $g(\perp) = \perp$ and $g(1) = 1$.

Theorem 4.1.5. [TW21] *Let $\mathbb{A}, \mathbb{B}$ be finite join-semilattices, and for every $f \in \mathrm{Hom}_{\vee}(\mathbb{A}, \mathbb{B})$ and $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp})$, let us define the maps $f^{\triangleleft}\colon B_{\perp} \to A_{\perp}$ and $g^{\triangleright}\colon A \to B$ by*

$$f^{\triangleleft}(b) = \begin{cases} \bigvee f^{-1}(\downarrow b), & \text{if } f^{-1}(\downarrow b) \neq \emptyset, \\ \perp, & \text{if } f^{-1}(\downarrow b) = \emptyset, \end{cases} \qquad g^{\triangleright}(a) = \bigwedge g^{-1}(\uparrow a).$$

Then the following two maps are mutually inverse bijections:

$$\begin{aligned} \mathrm{Hom}_{\vee}(\mathbb{A}, \mathbb{B}) &\to \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp}), \quad f \mapsto f^{\triangleleft}, \\ \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp}) &\to \mathrm{Hom}_{\vee}(\mathbb{A}, \mathbb{B}), \quad g \mapsto g^{\triangleright}. \end{aligned}$$

Proof. First let us show that if f is a join-homomorphism from $\mathbb{A}$ to $\mathbb{B}$, then $f^{\triangleleft}$ is a meet-homomorphism from $\mathbb{B}_{\perp}$ to $\mathbb{A}_{\perp}$. By Lemma 4.1.4, $f^{-1}(\downarrow b)$ is either empty or a principal ideal, and in the latter case $f^{\triangleleft}(b)$ is the greatest element of $f^{-1}(\downarrow b)$ by the definition of $f^{\triangleleft}$. Therefore, we can reformulate the definition of $f^{\triangleleft}$ as follows:

$$\forall a \in A \; \forall b \in B_{\perp}\colon \; f(a) \leq b \iff a \leq f^{\triangleleft}(b). \tag{4.1.2}$$

(Note that if $f^{-1}(\downarrow b) = \emptyset$, then $f(a) \leq b$ does not hold for any $a \in A$. In this case we have $f^{\triangleleft}(b) = \perp$, and the only element $a \in A_{\perp}$ satisfying $a \leq f^{\triangleleft}(b) = \perp$ is $a = \perp$, thus the right hand side of (4.1.2) does not hold for any $a \in A$ either.) From (4.1.2) we can deduce the following chain of equivalences for all $a \in A$ and $b_1, b_2 \in B_{\perp}$:

$$\begin{aligned} a \leq f^{\triangleleft}(b_1 \wedge b_2) &\iff f(a) \leq b_1 \wedge b_2 \\ &\iff f(a) \leq b_1 \text{ and } f(a) \leq b_2 \\ &\iff a \leq f^{\triangleleft}(b_1) \text{ and } a \leq f^{\triangleleft}(b_2) \\ &\iff a \leq f^{\triangleleft}(b_1) \wedge f^{\triangleleft}(b_2). \end{aligned}$$

Thus we have $a \leq f^{\triangleleft}(b_1 \wedge b_2) \iff a \leq f^{\triangleleft}(b_1) \wedge f^{\triangleleft}(b_2)$ for every $a \in A$, and this implies that $f^{\triangleleft}(b_1 \wedge b_2) = f^{\triangleleft}(b_1) \wedge f^{\triangleleft}(b_2)$, i.e., $f^{\triangleleft}$ is a meet-homomorphism. To verify $f^{\lhd}(1_{\mathbb{B}}) = 1_{\mathbb{A}}$, we just need to observe that $f^{-1}(\downarrow 1_{\mathbb{B}}) = f^{-1}(B) = A$, and the greatest element of $\mathbb{A}$ is indeed $1_{\mathbb{A}}$. Since $f^{-1}(\downarrow \bot) = \emptyset$, we have $f^{\lhd}(\bot) = \bot$. This completes the proof of the claim $f^{\lhd} \in \mathrm{Hom}_{\wedge}^{\bot,1}(\mathbb{B}_\bot, \mathbb{A}_\bot)$.

Now assume that $g\colon \mathbb{B}_\bot \to \mathbb{A}_\bot$ is a meet-homomorphism such that $g(1_{\mathbb{B}}) = 1_{\mathbb{A}}$ and $g(\bot) = \bot$. Then $1_{\mathbb{B}} \in g^{-1}(\uparrow a)$ for all $a \in A$, hence $g^{-1}(\uparrow a)$ is never empty. Therefore, the dual of Lemma 4.1.4 shows that $g^{-1}(\uparrow a)$ is a principal filter in $\mathbb{B}_\bot$; moreover, $g(\bot) = \bot$ implies that $\bot \notin g^{-1}(\uparrow a)$ for every $a \in A$. This shows that the map $g^{\rhd}\colon A \to B, a \mapsto \bigwedge g^{-1}(\uparrow a)$ is well defined. One can prove, by an argument similar to that of the previous paragraph, that $g^{\rhd}$ is a join-homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

It remains to prove that the maps $f \mapsto f^{\lhd}$ and $g \mapsto g^{\rhd}$ are inverses of each other. This follows immediately from the fact that $g = f^{\lhd}$ and $f = g^{\rhd}$ are both equivalent to

$$\forall a \in A\ \forall b \in B_\bot\colon\ f(a) \le b \iff a \le g(b). \tag{4.1.3}$$

for all $f \in \mathrm{Hom}_{\vee}(\mathbb{A}, \mathbb{B})$ and $g \in \mathrm{Hom}_{\wedge}^{\bot,1}(\mathbb{B}_\bot, \mathbb{A}_\bot)$. (Let us mention that a pair (f, g) of maps satisfying (4.1.3) is called a *monotone Galois connection.*) ■

Remark 4.1.6. Let us give a categorical interpretation of Theorem 4.1.5. Let $\mathcal{J}$ denote the category of finite join-semilattices (with join-homomorphisms), and let $\mathcal{L}$ denote the category of finite lattices (with meet-homomorphisms that preserve the boundary elements). Then the following two maps are mutually inverse functors, thus $\mathcal{J}$ and $\mathcal{L}$ are isomorphic categories:

$$F\colon \mathcal{J} \to \mathcal{L},\ \mathbb{A} \mapsto \mathbb{A}_\bot,\ f \mapsto f^{\lhd};$$
$$G\colon \mathcal{L} \to \mathcal{J},\ \mathbb{B} \mapsto \mathbb{B} \setminus \{0_{\mathbb{B}}\},\ g \mapsto g^{\rhd}.$$

(Here $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ is the join-semilattice obtained by removing the bottom element of the lattice $\mathbb{B}$. Of course, if $\mathbb{B}$ is given as $\mathbb{B} = F(\mathbb{A}) = \mathbb{A}_\bot$, then $0_{\mathbb{B}} = \bot$.)

Theorem 4.1.5 can be useful if $\mathbb{A}$ is (much) larger than $\mathbb{B}$, as in this case it might be an easier task to determine the meet-homomorphisms from $\mathbb{B}_\bot$ to $\mathbb{A}_\bot$ than describing the join-homomorphisms from $\mathbb{A}$ to $\mathbb{B}$. This is the case when $\mathbb{A} = \mathbb{S}^n$ and $\mathbb{B} = \mathbb{S}$, where $\mathbb{S}$ is a finite join-semilattice: as mentioned before, the n-ary operations in $[\vee]^*$ are the join-homomorphisms from $\mathbb{S}^n$ to $\mathbb{S}$, and these can be described in terms of the 1- and $\bot$-preserving meet-homomorphisms from $\mathbb{S}_\bot$ to $(\mathbb{S}^n)_\bot$, with the help of Theorem 4.1.5. We formulate this characterization in the next corollary, and we complement it with the necessary and sufficient condition for the operation to depend on all of its variables.

Corollary 4.1.7. [TW21] *Let $\mathbb{S} = (S; \vee)$ be a finite semilattice, and let n be a nonnegative integer. The n-ary members of $[\vee]^*$ are exactly the operations f of the form*

$$f\colon S^n \to S,\ \mathbf{x} \mapsto \bigwedge g^{-1}(\uparrow \mathbf{x}),$$

where $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{S}_{\perp}, (\mathbb{S}^n)_{\perp})$*; here* g *is uniquely determined by* f*. The operation* f *depends on all of its variables if and only if for each* $i \in \{1, \dots, n\}$*, the range of* g *contains an element of* S^n *whose* i*-th component is different from* 1*. If* $\mathbb{S}$ *has a least element (i.e., if* $\mathbb{S}$ *is a lattice), then the latter condition is satisfied if and only if the range of* g *contains a tuple from* $(S \setminus \{1\})^n$*.*

Proof. An n-ary operation $f \in \mathcal{O}_S$ belongs to $[\vee]^*$ if and only if f is a join-homomorphism from $\mathbb{S}^n$ to $\mathbb{S}$. Applying Theorem 4.1.5 with $\mathbb{A} = \mathbb{S}^n$ and $\mathbb{B} = \mathbb{S}$, we see that these operations can be uniquely written as $f(\mathbf{x}) = g^{\triangleright}(\mathbf{x}) = \bigwedge g^{-1}(\uparrow \mathbf{x})$ with $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{B}_{\perp}, \mathbb{A}_{\perp})$.

We prove that f depends on its i-th variable if and only if the range of g contains a tuple $\mathbf{s}^{(i)} \in S^n$ such that the i-th component of $\mathbf{s}^{(i)}$ is not equal to 1. First assume that f depends on the i-th variable; this means that there exist elements $\mathbf{a}, \mathbf{a}' \in S^n$ differing only in their i-th component such that $b := f(\mathbf{a})$ and $b' := f(\mathbf{a}')$ are different. We can assume without loss of generality that either $b < b'$ or b and b' are incomparable. In both cases we can conclude that $\mathbf{a} \in f^{-1}(\downarrow b)$ and $\mathbf{a}' \notin f^{-1}(\downarrow b)$. By Theorem 4.1.5, we have $g = f^{\triangleleft}$, thus $g(b) = \bigvee f^{-1}(\downarrow b)$. Therefore, $\mathbf{a} \leq g(b)$ and $\mathbf{a}' \nleq g(b)$. This implies that the i-th component of the tuple $\mathbf{s}^{(i)} := g(b) \in S^n$ (which certainly belongs to the range of g) is strictly less than 1.

Conversely, let us suppose that there is an element $b \in S$ such that the i-th component of $\mathbf{s}^{(i)} := g(b)$ is less than 1. Letting $\mathbf{t}^{(i)}$ be the tuple obtained from $\mathbf{s}^{(i)}$ by changing its i-th component to 1, we have $\mathbf{s}^{(i)} < \mathbf{t}^{(i)}$. Now $b \in g^{-1}(\uparrow \mathbf{s}^{(i)})$ but $b \notin g^{-1}(\uparrow \mathbf{t}^{(i)})$, therefore $g^{-1}(\uparrow \mathbf{s}^{(i)}) \neq g^{-1}(\uparrow \mathbf{t}^{(i)})$. Since $f = g^{\triangleright}$, this implies that

$$f(\mathbf{s}^{(i)}) = g^{\triangleright}(\mathbf{s}^{(i)}) = \bigwedge g^{-1}(\uparrow \mathbf{s}^{(i)}) \neq \bigwedge g^{-1}(\uparrow \mathbf{t}^{(i)}) = g^{\triangleright}(\mathbf{t}^{(i)}) = f(\mathbf{t}^{(i)}).$$

Taking into account that $\mathbf{s}^{(i)}$ and $\mathbf{t}^{(i)}$ differ only at the i-th component, we can conclude that f does depend on its i-th variable.

If $\mathbb{S}$ is a lattice, then $\mathbf{s}^{(1)} \wedge \dots \wedge \mathbf{s}^{(n)}$ is a tuple in the range of g, and all of its components are less than 1. ∎

4.1.2 Counting

Using the characterizations presented in the previous subsection, we can determine the exact number of n-ary operations in the centralizers of certain semilattices. First we count the essentially n-ary operations commuting with the join operation of the smallest non-lattice semilattice.

Proposition 4.1.8 ([TW21])**.** *Let* $\mathbb{S} = (\{a, b, 1\}, \vee)$ *be the join-semilattice with* $a \vee b = 1$*. The number of essentially* n*-ary operations in the centralizer of* $\mathbb{S}$ *is* $8^n - 6^n + 2 \cdot 2^n + 0^n$*.*

Proof. By Corollary 4.1.7, we need to count the elements $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{S}_{\perp}, (\mathbb{S}^n)_{\perp})$ such that the range of g contains a tuple from $S^{i-1} \times \{a, b\} \times S^{n-i}$ for every $i \in \{1, \ldots, n\}$. For an arbitrary map $g\colon S_{\perp} \to (S^n)_{\perp}$, we have $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(\mathbb{S}_{\perp}, (\mathbb{S}^n)_{\perp})$ if and only if $g(1) = 1$, $g(\perp) = \perp$ and $g(a) \wedge g(b) = \perp$; moreover, such a map is uniquely determined by $g(a)$ and $g(b)$. We distinguish four cases upon these values (we denote by $f = g^{\triangleright}$ the element of $[\vee]^*$ corresponding to g in Corollary 4.1.7).

1. If $g(a) = \perp = g(b)$, then f is constant 1, hence f is essentially n-ary if and only if $n = 0$. Thus the number of essentially n-ary operations of this type is 0^n, i.e., it is 1 if $n = 0$ and 0 if $n > 0$ (here it is convenient to use the convention $0^0 = 1$; for more justification, see [Knu92]).

2. If $g(a) \neq \perp = g(b)$, then f depends on all of its variables if and only if $g(b) \in \{a, b\}^n$, thus the number of essentially n-ary operations $f \in [\vee]^*$ of this type is 2^n (for $n = 0$ we get the constant a function).

3. If $g(a) = \perp \neq g(b)$, then, similarly to the previous case, we have 2^n functions (here $n = 0$ corresponds to the constant b function).

4. If $g(a) \neq \perp \neq g(b)$, then let $\mathbf{s} := g(a) \in S^n$ and $\mathbf{t} := g(b) \in S^n$. Writing these two tuples below each other, we get the $2 \times n$ matrix $\left(\begin{smallmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{smallmatrix}\right)$. Now f depends on all of its variables if and only if no column of this matrix is $(1, 1)^{\mathsf{T}}$, and there are 8^n such matrices. However, some of the corresponding maps g will violate the condition $g(a) \wedge g(b) = \perp$: we must exclude those matrices that contain neither $(a, b)^{\mathsf{T}}$ nor $(b, a)^{\mathsf{T}}$ as a column. The number of such matrices is 6^n, so we obtain $8^n - 6^n$ essentially n-ary operations $f \in [\vee]^*$ in this case.

Summing up the four cases, we see that the number of essentially n-ary operations in $[\vee]^*$ is $8^n - 6^n + 2 \cdot 2^n + 0^n$. ■

Remark 4.1.9. It is easy to see that if the number of essentially n-ary operations in a clone C is p_n, then the number of all operations of arity n in C is $\sum_{k=0}^{n} \binom{n}{k} p_k$. Thus, by Proposition 4.1.8 (and by the binomial theorem), the number of n-ary operations in the centralizer of the join operation of the semilattice $(\{a, b, 1\}, \vee)$ is

$$\sum_{k=0}^{n} \binom{n}{k} (8^k - 6^k + 2 \cdot 2^k + 0^k) = 9^n - 7^n + 2 \cdot 3^n + 1^n.$$

Next we provide a general formula for the number of essentially n-ary operations commuting with the join operation of a finite lattice, and then we apply it to the case of finite chains.

Theorem 4.1.10. [TW21] *Let $\mathbb{S} = (S; \vee, \wedge)$ be a finite lattice, and let n be a nonnegative integer. The number of essentially n-ary operations in $[\vee]^*$ is*

$$\sum_{b \in S} (|\mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)| - 1)^n = \sum_{b \in S} (|\mathrm{Hom}^1_\wedge(\uparrow b, \mathbb{S})| - 1)^n.$$

Proof. According to Theorem 4.1.1, the essentially n-ary members of the centralizer are in a one-to-one correspondence with the tuples $(u_1, \ldots, u_n) \in \mathrm{Hom}_\vee(\mathbb{S}, \mathbb{S})^n$ such that $u_1(0) = \cdots = u_n(0)$ and none of the u_i are constant. Let $b := u_1(0)$, then each u_i maps S to $\uparrow b$ in such a way that the least element of $\mathbb{S}$ is mapped to the least element of $\uparrow b$, i.e., $u_i \in \mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)$ for $i = 1, \ldots, n$. However, we need to exclude the constant b function, hence the number of choices for each u_i is $|\mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)| - 1$, and this gives the first formula. (Note that for $b = 1$, the principal filter $\uparrow b$ has just one element, thus $|\mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)| - 1 = 0$. Therefore, the contribution of $b = 1$ to the sum is 0^n, and this could be omitted if $n > 0$. However, for $n = 0$, we need to keep the term $0^0 = 1$ in order to get the correct number of nullary operations, which is clearly $|S|$.)

The second formula follows from the first one by applying Theorem 4.1.5 to $\mathbb{A} = \mathbb{S}$ and $\mathbb{B} = \uparrow b$. If $f \in \mathrm{Hom}_\vee(\mathbb{S}, \uparrow b)$ and $g \in \mathrm{Hom}^{\perp,1}_\wedge((\uparrow b)_\perp, \mathbb{S}_\perp)$ correspond to each other under the bijections of Theorem 4.1.5, then $f(0) = b$ if and only if $g(b) \neq \perp$ (i.e., g does not take the value $\perp$ except for $g(\perp) = \perp$). Therefore, we obtain a bijection from $\mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)$ to $\mathrm{Hom}^1_\wedge(\uparrow b, \mathbb{S})$ by restricting $f^\triangleleft$ to the set $\uparrow b$ for each $f \in \mathrm{Hom}^0_\vee(\mathbb{S}, \uparrow b)$. ∎

Remark 4.1.11. The second formula of the above theorem can be also derived directly from Corollary 4.1.7 as follows. We need to count the meet-homomorphisms $g \in \mathrm{Hom}^{\perp,1}_\wedge(\mathbb{S}_\perp, (\mathbb{S}^n)_\perp)$ whose range satisfies the conditions of Corollary 4.1.7. If $\mathbb{S}$ is a lattice and g is such a meet-homomorphism, then there is a least element $b \in S$ such that $g(b) \neq \perp$. Restricting g to $\uparrow b$, we get a 1-preserving meet-homomorphism from $\uparrow b$ to $\mathbb{S}^n$, which can be viewed as an n-tuple $(g_1, \ldots, g_n)$ of 1-preserving meet-homomorphisms from $\uparrow b$ to $\mathbb{S}$. The range of g contains a tuple from $(S \setminus \{1\})^n$ if and only if $g(b) \in (S \setminus \{1\})^n$, which holds if and only if none of the g_i are constant 1. Therefore, there are exactly $(|\mathrm{Hom}^1_\wedge(\uparrow b, \mathbb{S})| - 1)^n$ such tuples $(g_1, \ldots, g_n)$, and this proves the second formula of Theorem 4.1.10. This argument does not work if $\mathbb{S}$ is not a lattice, even though Corollary 4.1.7 holds in that case, too. The problem is that there may be no least element b with $g(b) \neq \perp$; in fact, $g^{-1}(S^n)$ is not necessarily closed under meets. Nevertheless, as we have seen in Proposition 4.1.8, Corollary 4.1.7 can be used to count the essentially n-ary operations in a semilattice even if it is not a lattice.

Corollary 4.1.12 ([TW21]). *The number of essentially n-ary operations commuting with the join operation of a chain of cardinality ℓ is*

$$\sum_{i=1}^{\ell}\left[\binom{\ell+i-2}{\ell-1}-1\right]^n.$$

Proof. First, as an auxiliary result, let us count the join-homomorphisms from an r-element chain $\mathbb{A}=\{a_1<\cdots<a_r\}$ to an s-element chain $\mathbb{B}=\{b_1<\cdots<b_s\}$. Clearly, the join-homomorphisms in this case are just the monotone maps, thus an element of $\operatorname{Hom}_\vee(\mathbb{A},\mathbb{B})$ can be given by a nondecreasing sequence $f(a_1)\leq\cdots\leq f(a_r)$ in $\mathbb{B}$. These sequences can be viewed as r-combinations with repetitions from the elements $b_1,\ldots,b_s$, and the number of such combinations is $\binom{s+r-1}{r}=\binom{s+r-1}{s-1}$.

Now let $\mathbb{S}$ be a chain of cardinality ℓ, and let $b\in S$. The 0-preserving join-homomorphisms from $\mathbb{S}$ to $\uparrow b$ are in a one-to-one correspondence with the join-homomorphisms from $\mathbb{S}\setminus\{0\}$ to $\uparrow b$ (by restricting to $S\setminus\{0\}$). Note that $\mathbb{S}\setminus\{0\}$ is a chain of size $\ell-1$, and if b is the i-th element from the top in $\mathbb{S}$, then $\uparrow b$ is an i-element chain. Thus, by the considerations made in the first paragraph, we have

$$|\operatorname{Hom}_\vee^0(\mathbb{S},\uparrow b)|=|\operatorname{Hom}_\vee(\mathbb{S}\setminus\{0\},\uparrow b)|=\binom{\ell+i-2}{\ell-1}.$$

Applying Theorem 4.1.10 completes the proof: we just need to substitute the above formula into the first sum of Theorem 4.1.10 (replacing the summation variable b by i). ∎

Remark 4.1.13. For an arbitrary poset $\mathbb{A}$, the number of monotone maps from $\mathbb{A}$ to an s-element chain is a polynomial in s, called the *order polynomial* of $\mathbb{A}$. As we have seen in the proof of the above proposition, the order polynomial of the r-element chain is $\binom{s+r-1}{r}$. This fact, and much more about order polynomials can be found in [Sta12].

Remark 4.1.14. Similarly to Remark 4.1.9, we can derive from Corollary 4.1.12 that the number of n-ary operations in the centralizer of the join (or meet) operation of an ℓ element chain is

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=1}^{\ell}\left[\binom{\ell+i-2}{\ell-1}-1\right]^k=\sum_{i=1}^{\ell}\sum_{k=0}^{n}\binom{n}{k}\left[\binom{\ell+i-2}{\ell-1}-1\right]^k=\sum_{i=1}^{\ell}\binom{\ell+i-2}{\ell-1}^n.$$

Example 4.1.15. For the two-element chain (regarded as a semilattice), Corollary 4.1.12 gives the formula 1^n+0^n for the number of essentially n-ary operations (recall that 0^0 is defined as 1), and Remark 4.1.14 gives 2^n+1^n for the number of all n-ary operations in the centralizer (which are of course also easily verified without our results). Similarly, for the three-element chain, we have $5^n+2^n+0^n$ essentially

n-ary operations and $6^n + 3^n + 1^n$ operations of arity n in the centralizer of the join operation; for the four-element chain we get the numbers $19^n + 9^n + 3^n + 0^n$ and $20^n + 10^n + 4^n + 1^n$, etc. A formula for the number of n-ary operations in the centralizer of the meet operation the of three-element chain appeared already in [MR18]:

$$3^{n+1} + \sum_{\substack{0 \leq p < n \\ 0 \leq q \leq n-p}} \binom{n}{p}\binom{n-p}{q}(3^p 2^q - 1).$$

This can be simplified to $6^n + 3^n + 1^n$ with the help of the binomial theorem.

4.2 Centralizers of finite distributive lattices

In this section $\mathbb{L} = (L; \vee, \wedge)$ denotes a finite distributive lattice with greatest element 1 and least element 0. The centralizer clone $[\vee, \wedge]^*$ can be described by a slight variation of Theorem 4.1.1.

Theorem 4.2.1. [TW21] *Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite distributive lattice and $f \in \mathcal{O}_L^{(n)}$. Then the following are equivalent:*

(i) *$f \in [\vee, \wedge]^*$;*

(ii) *there exist unary operations $u_1, \ldots, u_n \in [\vee, \wedge]^*$ such that $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and for all $i, j \in \{1, \ldots, n\}, i \neq j$ we have $u_i(1) \wedge u_j(1) = u_1(0) = \cdots = u_n(0)$.*

Furthermore, the operation f given by (ii) *depends on all of its variables if and only if none of the unary operations u_i are constant.*

Proof. (i) $\Rightarrow$ (ii): As in the proof of Theorem 4.1.1, we define the unary operations $u_1, \ldots, u_n$ as $u_1(x) = f(x, 0, \ldots, 0), \ldots, u_n(x) = f(0, \ldots, 0, x)$. By that theorem, for these unary operations we have $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and $u_1(0) = \cdots = u_n(0) = f(0, \ldots, 0)$. Since f and the constant 0 operation belong to $[\vee, \wedge]^*$, we have $u_1, \ldots, u_n \in [\vee, \wedge]^*$. It only remains to show that for all $i, j \in \{1, \ldots, n\}, i \neq j$ we have $u_i(1) \wedge u_j(1) = f(0, \ldots, 0)$. For notational simplicity, let us assume that $i = 1$ and $j = 2$; the proof of the general case is similar. Using that f commutes with the operation $\wedge$, with the help of the $2 \times n$ matrix $\left(\begin{smallmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{smallmatrix}\right)$ we can conclude that the following equality holds:

$$\begin{aligned} f(0, \ldots, 0) &= f\Big(1 \wedge 0,\ 0 \wedge 1,\ 0 \wedge 0,\ \ldots,\ 0 \wedge 0) \\ &= f(1, 0, 0, \ldots, 0) \wedge f(0, 1, 0, \ldots, 0) = u_1(1) \wedge u_2(1). \end{aligned}$$

(ii) $\Rightarrow$ (i): Since $\vee \in [\vee]^*$ and $u_1, \ldots, u_n \in [\vee, \wedge]^*$, we have $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n) \in [\vee]^*$. Therefore, to complete the proof we have to show that f commutes with $\wedge$:

$$f(x_1, \ldots, x_n) \wedge f(y_1, \ldots, y_n) = f(x_1 \wedge y_1, \ldots, x_n \wedge y_n).$$

Thus we need to prove that for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ we have

$$\Big(u_1(x_1) \vee \cdots \vee u_n(x_n)\Big) \wedge \Big(u_1(y_1) \vee \cdots \vee u_n(y_n)\Big) = u_1(x_1 \wedge y_1) \vee \cdots \vee u_n(x_n \wedge y_n).$$

Using the notation $c_i := u_i(x_i), d_i := u_i(y_i) (i = 1, \ldots, n)$, the above equality can be written as

$$(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) = u_1(x_1 \wedge y_1) \vee \cdots \vee u_n(x_n \wedge y_n).$$

Since $\mathbb{L}$ is distributive, we have

$$(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) = \bigvee_{i,j=1}^{n} (c_i \wedge d_j). \tag{4.2.1}$$

From $u_1, \ldots, u_n \in [\vee, \wedge]^*$ it follows that these operations are monotone, hence $u_1(0) = u_i(0) \leq c_i, d_i \leq u_i(1)$ for every i; moreover, (ii) also implies that for all $i \neq j$, we have

$$u_1(0) = u_i(0) \wedge u_j(0) \leq c_i \wedge d_j \leq u_i(1) \wedge u_j(1) = u_1(0).$$

Thus $c_i \wedge d_j = u_1(0) \leq c_i \wedge d_i$ whenever $i \neq j$, so we can omit $c_i \wedge d_j$ from the join on the right hand side of (4.2.1), and using the fact that each u_i commutes with $\wedge$ we obtain the desired equality:

$$\begin{aligned}(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) &= \bigvee_{i=1}^{n} (c_i \wedge d_i) \\ &= \bigvee_{i=1}^{n} (u_i(x_i) \wedge u_i(y_i)) \\ &= \bigvee_{i=1}^{n} (u_i(x_i \wedge y_i)). \qquad \blacksquare\end{aligned}$$

In the next lemma we investigate when the centralizer contains essentially n-ary operations.

Lemma 4.2.2. [TW21] *Let* $\mathbb{L} = (L; \vee, \wedge)$ *be a finite distributive lattice. Then the following are equivalent:*

(Ess) *there exists an essentially* n*-ary operation in* $[\vee, \wedge]^*$*;*

(Sub) *there exists a sublattice of* $\mathbb{L}$ *that is isomorphic to* $\mathbb{2}^n$*.*

Proof. (Ess) $\Rightarrow$ (Sub): Let $f \in \mathcal{O}_L$ be an essentially n-ary operation. Then by Theorem 4.2.1 we have $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and $u_i(1) \wedge u_j(1) = u_1(0) = \cdots = u_n(0)$ for all $i \neq j$. Let us introduce the notation $a_i = u_i(1)$ and $b = u_1(0)$. Then for all $i \neq j$ we have $a_i \wedge a_j = b$ and since f is essentially n-ary, we also have $a_i > b$. From this it is not hard to deduce using distributivity that $\mathcal{P}(\{1, \ldots, n\}) \hookrightarrow \mathbb{L},\ I \mapsto \bigvee\{a_i : i \in I\}$ is an embedding. (Alternatively, one can verify with the help of Theorem 360 of [Grä11] that $\{a_1, \ldots, a_n\}$ is an independent set in the sublattice $\uparrow b$, hence it generates a sublattice isomorphic to $\mathcal{P}(\{1, \ldots, n\})$.)

(Sub) $\Rightarrow$ (Ess): Let us suppose that there is a sublattice of $\mathbb{L}$ isomorphic to $\mathbb{2}^n$, let b be the least element and let a_i $(i \in \{1, \ldots, n\})$ be the atoms of this cube. Then we have $b = a_i \wedge a_j$ for all $i \neq j$. Let us define the operations $u_1, \ldots, u_n$ as $u_i(x_i) := (x_i \vee b) \wedge a_i$ for all $i \in \{1, \ldots, n\}$. Since $\mathbb{L}$ is distributive, u_i is an endomorphism of $\mathbb{L}$, i.e., we have $u_i \in [\vee, \wedge]^*$. Note that $u_i(0) = b$ and $u_i(1) = a_i$, therefore we have $u_1(0) = \cdots = u_n(0)$ and also $u_i(1) \wedge u_j(1) = u_1(0)$ for all $i \neq j$. By Theorem 4.2.1 this means that the operation $f \in \mathcal{O}_L$ defined as $f(x_1, \ldots, x_n) := u_1(x_1) \vee \cdots \vee u_n(x_n)$ belongs to $[\vee, \wedge]^*$. Since none of the u_i are constant, Theorem 4.2.1 also implies that f is essentially n-ary. ∎

Corollary 4.2.3. [TW21] *For a finite distributive lattice* $\mathbb{L} = (L; \vee, \wedge)$ *the following are equivalent:*

(i) *every operation in* $[\vee, \wedge]^*$ *is essentially at most unary;*

(ii) $\mathbb{L}$ *is a chain.*

Although the next lemma follows from the description of projective and injective distributive lattices [Bal67], we provide a short proof.

Lemma 4.2.4. [TW21] *Let* $\mathbb{L} = (L; \vee, \wedge)$ *be a finite distributive lattice. Then the following are equivalent:*

(Sub) *there exists a sublattice of* $\mathbb{L}$ *that is isomorphic to* $\mathbb{2}^n$*;*

(Quo) *there exists a congruence* ϑ *of* $\mathbb{L}$ *such that* $\mathbb{L}/\vartheta$ *is isomorphic to* $\mathbb{2}^n$*.*

Proof. Instead of $\mathbb{2}^n$, it will be more convenient to use the lattice $\mathbb{K}_n := \mathcal{P}(\{1, ..., n\})$, which is clearly isomorphic to $\mathbb{2}^n$. To prove (Sub) $\Rightarrow$ (Quo), assume that $\mathbb{L}$ has a sublattice that is isomorphic to $\mathbb{2}^n$. Identifying this sublattice with $\mathbb{K}_n$, we may assume without loss of generality that $\mathbb{K}_n$ itself is a sublattice of $\mathbb{L}$. For any $i \in \{1, \ldots, n\}$, the principal ideal generated by $\{1, \ldots, n\} \setminus \{i\}$ does not contain $\{i\}$, hence, by the prime ideal theorem for distributive lattices, there is a prime ideal P_i of $\mathbb{L}$ that does not contain $\{i\}$ (see Corollary 116 in [Grä11]). Consequently, there is a homomorphism $\varphi_i \colon \mathbb{L} \to \mathbb{2}$ mapping P_i to 0 and $\mathbb{L} \setminus P_i$ to 1. In particular, we have $\varphi_i(\{i\}) = 1$ and $\varphi_i(\{j\}) = 0$ for all $j \neq i$. Combining these maps we obtain a homomorphism $\varphi \colon \mathbb{L} \to \mathbb{2}^n, \ x \mapsto (\varphi_1(x_1), \ldots, \varphi_n(x_n))$. We have $\varphi(\{i\}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the i-th coordinate. These elements generate $\mathbb{2}^n$, hence φ is surjective, and this proves (Sub).

For (Quo) $\Rightarrow$ (Sub), let us suppose that ϑ is a congruence of $\mathbb{L}$ such that $\mathbb{L}/\vartheta$ is isomorphic to $\mathbb{K}_n$. For every $I \in \mathbb{K}_n$, let C_I denote the congruence class of ϑ corresponding to I at this isomorphism. Let a be the greatest element of $C_\emptyset$, and let b_i be the least element of $C_{\{i\}}$ for all $i \in \{1, \ldots, n\}$. Then $c_i := a \vee b_i$ belongs to $C_{\{i\}}$, and $c_i \wedge c_j$ belongs to $C_\emptyset$ whenever $i \neq j$. Moreover, $c_i \wedge c_j = (a \vee b_i) \wedge (a \vee b_j) \geq a$, hence $c_i \wedge c_j = a$, as a is the greatest element of $C_\emptyset$. From this it follows using the same argument as in the proof of Lemma 4.2.2 that $\mathbb{K}_n \to \mathbb{L}, \ I \mapsto \bigvee\{c_i : i \in I\}$ is an embedding. ■

The previous two lemmas together give us the following characterization of the existence of essentially n-ary operations in the centralizer of a finite distributive lattice.

Theorem 4.2.5. [TW21] *Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite distributive lattice. Then the following are equivalent:*

(Ess) *there exists an essentially n-ary operation in $[\vee, \wedge]^*$;*

(Sub) *there exists a sublattice of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$;*

(Quo) *there exists a congruence ϑ of $\mathbb{L}$ such that $\mathbb{L}/\vartheta$ is isomorphic to $\mathbb{2}^n$.*

4.3 Centralizers of finite lattices

In this section our goal is to investigate whether the results proved in Section 4.2 hold for arbitrary lattices. Let us look at Theorem 4.2.1 first. Note that in the proof of (i) $\Rightarrow$ (ii) we did not use that the lattice $\mathbb{L}$ was distributive neither that $\mathbb{L}$ was finite, and thus the proof provided there shows that this implication holds for arbitrary bounded lattices. However, in the proof of (ii) $\Rightarrow$ (i) we used distributivity, and the following example shows that this implication does not hold for arbitrary lattices.

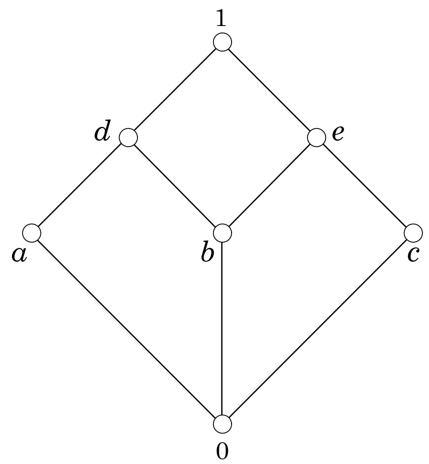


Figure 4.2: A nondistributive counterexample to the implication (2) ⇒ (1) of Theorem 4.2.1.

Example 4.3.1. Let $\mathbb{L}$ be the lattice shown on Figure 4.2. Let us define the operations u_1 and u_2 as

$$u_1(x) = \begin{cases} 0, & \text{if } x \in \{0, b\}, \\ a, & \text{if } x \in \{a, d\}, \\ b, & \text{if } x \in \{c, e\}, \\ d, & \text{if } x \in \{1\}; \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \in \{0, a, b, d\}, \\ c & \text{if } x \in \{c, e, 1\}. \end{cases}$$

It is easy to check that $\ker(u_1)$ is a congruence of $\mathbb{L}$, and u_1 establishes an isomorphism from the quotient lattice $\mathbb{L}/\ker(u_1) = \{\{0, b\}, \{a, d\}, \{c, e\}, \{1\}\}$ to the sublattice $\{0, a, b, d\}$. Therefore u_1 is an endomorphism of $\mathbb{L}$. Similarly, one can verify that $u_2 \in [\vee, \wedge]^*$. Let $f(x_1, x_2) = u_1(x_1) \vee u_2(x_2)$; we will show that f does not belong to $[\wedge]^*$. Let us suppose that f commutes with $\wedge$. Then applying the definition of commutation to the 2×2 matrix $\left(\begin{smallmatrix} a & c \\ c & c \end{smallmatrix}\right)$, we have the following equality:

$$f(a \wedge c, c \wedge c) = f(a, c) \wedge f(c, c);$$

that is,

$$u_1(a \wedge c) \vee u_2(c \wedge c) = (u_1(a) \vee u_2(c)) \wedge (u_1(c) \vee u_2(c)).$$

However, the left hand side evaluates to c, and the value of the right hand side is e;

$$c = 0 \vee c = u_1(0) \vee u_2(c) = (a \vee c) \wedge (b \vee c) = 1 \wedge e = e$$

which is a contradiction.

We have seen that the implication (ii) $\Rightarrow$ (i) of Theorem 4.2.1 does not hold for arbitrary lattices, but, interestingly, for finite simple lattices we have (i) $\Leftrightarrow$ (ii). Moreover, as we shall see in the following remark, for a finite simple lattice $\mathbb{L} = (L; \vee, \wedge)$, every operation in $[\vee, \wedge]^*$ depends on at most one variable.

Remark 4.3.2. If $\mathbb{L}$ is a finite simple lattice, then (ii) $\Rightarrow$ (i) holds even if $\mathbb{L}$ is not distributive. For one-element lattices (ii) $\Rightarrow$ (i) holds trivially. Assume that $\mathbb{L}$ is a finite simple lattice with at least two elements, $u_1, \ldots, u_n$ are endomorphisms of L such that $u_1(0) = \cdots = u_n(0)$ and $u_i(1) \wedge u_j(1) = u_1(0)$ whenever $i \neq j$, and let $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$. Simplicity of $\mathbb{L}$ implies that the kernel of every u_i is one of the two relations L^2 or $\{(a, a) \mid a \in L\}$; therefore, every u_i is either a constant operation or an automorphism of $\mathbb{L}$. We distinguish three cases on the number of automorphisms occurring in $u_1(x_1) \vee \cdots \vee u_n(x_n)$.

1. If each u_i is constant, then f is also constant, hence $f \in [\vee, \wedge]^*$.
2. If u_i and u_j are automorphisms with $i \neq j$, then $1 = u_i(1) \wedge u_j(1) = u_i(0) = 0$, which is a contradiction.
3. In the remaining cases we can assume without loss of generality that u_1 is an automorphism and $u_2, \ldots, u_n$ are all constants. Then $u_1(0) = 0$, thus $u_2(0) = \cdots = u_n(0) = 0$ and $f(x_1, \ldots, x_n) = u_1(x_1)$.

Therefore, if (ii) holds for a finite simple lattice, then f depends on at most one variable, and f is equivalent to an automorphism or a constant, hence $f \in [\vee, \wedge]^*$. Thus (i) $\Leftrightarrow$ (ii) holds for finite simple lattices; moreover, every operation in $[\vee, \wedge]^*$ depends on at most one variable.

Before investigating further which results of Section 4.2 hold for arbitrary finite lattices, we need to recall some facts from universal algebra. First we define the so-called product congruence; this definition can also be found in [BS81].

Definition 4.3.3. [BS81] Let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be algebras of the same type and for all $i \in \{1, \ldots, n\}$ let $\vartheta_i \in \operatorname{Con}(\mathbb{A}_i)$. The *product congruence* $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$ on the set $A_1 \times \cdots \times A_n$ is defined by

$$\Big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\Big) \in \vartheta \Leftrightarrow \forall i \in \{1, \ldots, n\} \colon (a_i, b_i) \in \vartheta_i.$$

Next we give two lemmas about congruences of direct products in congruence distributive varieties. The first one is a special case of Lemma 11.10 of [BS81]; the second one is implicit in [CM18; CGL18], but we include the proof for the sake of self-containedness. The variety of lattices is congruence distributive, hence we can use them in our study of centralizers of lattices. These two lemmas will also be helpful later in describing the centralizers of the clones over the two-element set (see Section 4.4).

Lemma 4.3.4. [BS81] *Let $\mathcal{V}$ be a congruence distributive variety, $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$ and $\vartheta \in \mathrm{Con}(\mathbb{A}_1 \times \cdots \times \mathbb{A}_n)$. Then there exist $\vartheta_i \in \mathrm{Con}(\mathbb{A}_i)$ for all $i \in \{1, \dots, n\}$ such that $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$.*

Lemma 4.3.5. [CM18; CGL18] *Let $\mathcal{V}$ be a congruence distributive variety and $\mathbb{A} \in \mathcal{V}$. Then the following are equivalent:*

(i) *There exists an essentially n-ary operation $f \in \mathcal{O}_A$ that is a homomorphism from $\mathbb{A}^n$ to $\mathbb{A}$.*

(ii) *There exist $\vartheta_i \in \mathrm{Con}(\mathbb{A})$ such that $\vartheta_i \neq A^2$ for all $i \in \{1, \dots, n\}$, and $\mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n$ embeds into $\mathbb{A}$.*

Proof. (i) $\Rightarrow$ (ii): Let $f\colon \mathbb{A}^n \to \mathbb{A}$ be a homomorphism. Then by Lemma 4.3.4 there exist $\vartheta_1, \dots, \vartheta_n \in \mathrm{Con}(\mathbb{A})$ such that $\ker(f) = \vartheta_1 \times \cdots \times \vartheta_n$. Since f is essentially n-ary, we have that $\vartheta_i \neq A^2$ for all $i \in \{1, \dots, n\}$. By the homomorphism theorem we have $\mathbb{A}^n/\ker(f) = \mathbb{A}^n/(\vartheta_1 \times \cdots \times \vartheta_n) = \mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n \cong f(\mathbb{A}^n)$, and since $f(\mathbb{A}^n)$ is a subalgebra of $\mathbb{A}$, it is clear that $\mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n$ is embeddable into $\mathbb{A}$.

(ii) $\Rightarrow$ (i): Let $\phi\colon \mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n \hookrightarrow \mathbb{A}$ be an embedding, let $\vartheta \in \mathrm{Con}(\mathbb{A}^n)$ be the product congruence $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$ and let ν denote the natural homomorphism from $\mathbb{A}^n$ to $\mathbb{A}^n/\vartheta$. Then we define the operation $f \in \mathcal{O}_A^{(n)}$ as $f(x_1, \dots, x_n) = \phi(x_1/\vartheta_1, \dots, x_n/\vartheta_n) = \phi(\nu(x_1, \dots, x_n))$. Thus $f = \phi \circ \nu$ is a homomorphism, and since $\vartheta_i \neq A^2$ for all i, we have that f is essentially n-ary. ■

The following corollary of Lemma 4.3.5 gives an upper bound for the essential arity of operations in the centralizer of a finite algebra in a congruence distributive variety.

Corollary 4.3.6. [TW21] *Let $\mathcal{V}$ be a congruence distributive variety, let $\mathbb{A} \in \mathcal{V}$ be a finite algebra and let C denote the clone of term operations of $\mathbb{A}$. If there is an essentially n-ary operation in C^*, then we have $n \leq \log_2 |A|$. In other words, the essential operations in C^* are at most $\log_2 |A|$-ary.*

Proof. By Lemma 4.3.5, if we have an essentially n-ary operation in C^*, then there exist $\vartheta_1, \dots, \vartheta_n \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n$ is embeddable into $\mathbb{A}$. Therefore we have $|\mathbb{A}/\vartheta_1 \times \cdots \times \mathbb{A}/\vartheta_n| = |\mathbb{A}/\vartheta_1| \cdot \ldots \cdot |\mathbb{A}/\vartheta_n| \leq |A|$, and since for every i we have $\vartheta_i \neq A^2$, it follows that $2^n = 2 \cdot \ldots \cdot 2 \leq |\mathbb{A}/\vartheta_1| \cdot \ldots \cdot |\mathbb{A}/\vartheta_n| \leq |A|$. Thus $n = \log_2(2^n) \leq \log_2(|A|)$. ■

Now we will focus on Theorem 4.2.5, or more precisely, we investigate which implications between (Ess), (Sub) and (Quo) hold for arbitrary finite lattices. Using that the variety of lattices is congruence distributive, and also that every lattice (except the one-element lattice) has a two-element sublattice, as a corollary of

Lemma 4.3.5 we can conclude that the implication (Ess) $\Rightarrow$ (Sub) of Theorem 4.2.5 also holds for arbitrary finite lattices.

However, (Sub) $\Rightarrow$ (Ess) does not hold in general; partition lattices provide counterexamples to this implication. Indeed, every finite lattice (in particular, $\mathbb{2}^n$) embeds into a large enough finite partition lattice (see Theorem 413 in [Grä11]), and partition lattices are simple (see Theorem 404 in [Grä11]), hence by Remark 4.3.2, (Ess) holds only for $n \leq 1$.

Now we show that for arbitrary lattices neither (Sub) $\Rightarrow$ (Quo) nor (Quo) $\Rightarrow$ (Sub) holds in general. Using partition lattices again, we can give counterexamples to (Sub) $\Rightarrow$ (Quo). The lattice $\mathbb{2}^n$ embeds into a large enough finite partition lattice, and since partition lattices are simple, we have that $\mathbb{2}^n$ is not a homomorphic image of a partition lattice.

To disprove (Quo) $\Rightarrow$ (Sub), let $\mathbb{K}_n = \mathcal{P}(\{1, \ldots, n\}) \cong \mathbb{2}^n$ for some $n \geq 4$, and define a partial order on the set $L := K_n \times \{0, 1\}$ as follows. For $(a, i), (b, j) \in K_n$, let $(a, i) \leq (b, j)$ iff either $a \leq b$, or $a = b$ and $i \leq j$. Note that this is the lexicographic order on L, and this makes $\mathbb{L}$ a lattice with the following lattice operations (here $\parallel$ stands for incomparability in $\mathbb{K}_n$):

$$(a,i)\vee(b,j) = \begin{cases} (a, i \vee j), & \text{if } a = b, \\ (a,i), & \text{if } a > b, \\ (b,j), & \text{if } a < b, \\ (a \vee b, 0), & \text{if } a \parallel b; \end{cases} \qquad (a,i)\wedge(b,j) = \begin{cases} (a, i \wedge j), & \text{if } a = b, \\ (b,j), & \text{if } a > b, \\ (a,i), & \text{if } a < b, \\ (a \wedge b, 1), & \text{if } a \parallel b. \end{cases}$$

Now $\mathbb{K}_n$ is a homomorphic image of $\mathbb{L}$ under the homomorphism $\mathbb{L} \to \mathbb{K}_n,\ (a, i) \mapsto a$. To see that $\mathbb{K}_n$ does not occur as a sublattice of $\mathbb{L}$, note that $\{1, 2\}$ is a doubly reducible element in $\mathbb{K}_n$, i.e., it can be written as a join as well as a meet of two incomparable elements: $\{1, 2\} = \{1\} \vee \{2\} = \{1, 2, 3\} \wedge \{1, 2, 4\}$. However, there is no doubly reducible element in $\mathbb{L}$, since a nontrivial join in $\mathbb{L}$ is always of the form $(a, 0)$, and a nontrivial meet is always of the form $(a, 1)$. (It is not necessary to double each element of K_n: with a more careful argument, one can construct a counterexample of only $2^n + 1$ elements.)

Note that the assumption $n \geq 4$ was essential in the construction of this counterexample, since for $n \leq 3$, there are no doubly reducible elements in $\mathbb{K}_n$. In fact, one can prove that (Quo) $\Rightarrow$ (Sub) holds for all lattices (distributive or not) for $n \leq 3$ (see Lemma 73 in [Grä11].

Summarizing the results up to this point we know that (Ess) $\Rightarrow$ (Sub) holds, but none of the implications (Sub) $\Rightarrow$ (Ess), (Sub) $\Rightarrow$ (Quo) or (Quo) $\Rightarrow$ (Sub) hold for arbitrary finite lattices. This immediately implies that (Quo) $\Rightarrow$ (Ess) can not hold in general. The lattice $\mathbb{L} = M_3^n$ shows that (Ess) $\Rightarrow$ (Quo) does not hold, either. It is straightforward to verify that the operation $f \in \mathcal{O}_L^{(n)}$ defined by

$$f\Big((x_{11}, \ldots, x_{1n}), \ldots, (x_{n1}, \ldots, x_{nn})\Big) = (x_{11}, \ldots, x_{n1})$$

commutes with $\vee$ and $\wedge$ and depends on all of its variables, hence (Ess) holds for $\mathbb{L}$. By Lemma 4.3.4, every quotient of M_3^n is isomorphic to a product of quotients of M_3. Since M_3 is simple, $\mathbb{L}$ only has the quotients $M_3, M_3^2, \ldots, M_3^n$, and therefore $\mathbb{2}^n$ does not appear as a quotient algebra of $\mathbb{L}$.

It is also an interesting question to investigate whether any two of the three statements (Ess), (Sub) and (Quo) (of Theorem 4.2.5) imply the third statement in general. First, it is easy to see that (Ess) and (Sub) together do not imply (Quo), but (Ess) and (Quo) imply (Sub), since we have (Ess) $\Rightarrow$ (Sub) and (Ess) $\not\Rightarrow$ (Quo). We will show that (Sub) and (Quo) together imply (Ess) for arbitrary finite lattices. Let us suppose that (Sub) and (Quo) hold for a finite lattice $\mathbb{L}$. Then by (Quo), there is a quotient of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$, and since $\mathbb{2}$ is a homomorphic image of $\mathbb{2}^n$, we have that $\mathbb{2}$ is a homomorphic image of $\mathbb{L}$. Let Φ denote a surjective homomorphism $\Phi\colon \mathbb{L} \to \mathbb{2}$. Then obviously $\ker(\Phi) \neq L^2$ and by (Sub) we have that $\mathbb{L}/\ker(\Phi) \times \cdots \times \mathbb{L}/\ker(\Phi) = \big(\mathbb{L}/\ker(\Phi)\big)^n \cong \mathbb{2}^n$ embeds into $\mathbb{L}$. Therefore, by Lemma 4.3.5, there exists an essentially n-ary operation in $\mathcal{O}_L$.

This section gave us some insight to the appearance of n-ary operations in the centralizer of an arbitrary finite lattice. We summarize these results in the following proposition.

Proposition 4.3.7. [TW21] *Let $\mathbb{L} = (L; \vee, \wedge)$ be an arbitrary finite lattice. Then the following are true:*

- *For any n-ary operation $f \in [\vee, \wedge]^*$, there exist unary operations $u_1, \ldots, u_n \in [\vee, \wedge]^*$ such that $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and for all $i, j \in \{1, \ldots, n\}, i \neq j$ we have $u_i(1) \wedge u_j(1) = u_1(0) = \cdots = u_n(0)$.*
- *If there is an essentially n-ary operation in $[\vee, \wedge]^*$, then there is a sublattice of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$.*
- *If there is a sublattice $\mathbb{L}_s$ and a quotient $\mathbb{L}_q$ of $\mathbb{L}$ that are both isomorphic to $\mathbb{2}^n$, then there is an essentially n-ary operation in $[\vee, \wedge]^*$.*

4.4 Centralizer clones over the two-element set

As promised earlier, in this section we are going to determine the centralizer of each clone on $\{0, 1\}$ in a fairly simple way. We use the Post lattice and the notation for Boolean clones from the appendix (see Figure 1.1 and Table 1.1). First let us record two entirely obvious facts, just for reference:

Fact 4.4.1. *For any clones $C_1, C_2 \leq \mathcal{O}_A$ we have $(C_1 \vee C_2)^* = C_1^* \wedge C_2^*$. (Here $\vee$ and $\wedge$ denote the join and meet operations of the clone lattice over A, i.e., $C_1 \vee C_2 = [C_1 \cup C_2]$ and $C_1 \wedge C_2 = C_1 \cap C_2$.) This implies that if $C_1 \leq C_2$ then $C_2^* \leq C_1^*$.*

Fact 4.4.2. *By the definition of the clones Ω_0, Ω_1 and S, for any clone $C \leq \mathcal{O}_{\{0,1\}}$ we have*

- $0 \in C^* \iff C \leq \Omega_0$,
- $1 \in C^* \iff C \leq \Omega_1$,
- $\neg \in C^* \iff C \leq S$.

We will see that all centralizers over $\{0,1\}$ can be computed using three tools: Theorem 4.1.1, Corollary 4.3.6, and Proposition 4.4.3 below. This proposition can be found in [Sze86, Proposition 2.1], but, for the reader's convenience, we include a proof, which is very similar to the proof of Theorem 4.1.1.

Proposition 4.4.3. *Let $\mathbb{A} = (A; +)$ be an Abelian group, and let $m(x,y,z) = x - y + z$. An n-ary operation $f \in \mathcal{O}_A$ belongs to the centralizer $[m]^*$ if and only if there exist unary operations $u_1, \ldots, u_n \in [m]^*$ such that*

$$f(x_1, \ldots, x_n) = u_1(x_1) + \cdots + u_n(x_n).$$

Proof. It is easy to see that the addition commutes with m, hence if f is a sum of endomorphisms of $(A; m)$, then $f \in [m]^*$.

Conversely, assume that f is an n-ary operation in $[m]^*$, and define $u_1, \ldots, u_n$ the same way as in the proof of Theorem 4.1.1: $u_1(x) = f(x, 0, \ldots, 0), \ldots, u_n(x) = f(0, \ldots, 0, x)$. Then we have $u_1, \ldots, u_n \in [m]^*$, as f and the constant 0 operation commute with m. Let us consider the $(n+1)$-ary operation $g(x_1, \ldots, x_n, y) = x_1 + \cdots + x_n - (n-1)y$. The following expression shows that $g \in [m]$:

$$\begin{aligned} g(x_1, \ldots, x_n, y) &= x_1 - y + x_2 - y + x_3 - \cdots - y + x_n \\ &= m(\cdots m(m(x_1, y, x_2), y, x_3), \ldots, y, x_n). \end{aligned}$$

(Actually, it is well known and also easy to verify that the elements of $[m]$ are the operations of the form $a_1 x_1 + \cdots + a_n x_n$ ($n \in \mathbb{N}, a_i \in \mathbb{Z}, \sum a_i = 1$), but for the purposes of this proof we only need the operation g above.) Since $g \in [m]$ and $f \in [m]^*$, the operations f and g commute. Applying the definition of commutation to the $(n+1) \times n$ matrix

$$\begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

we can conclude that

$$\begin{aligned} f(x_1,\dots,x_n) &= f\Big(g(x_1,0,\dots,0,0),\ \dots,\ g(0,\dots,0,x_n,0)\Big) \\ &= g\Big(f(x_1,0,\dots,0),\dots,f(0,\dots,0,x_n),f(0,\dots,0)\Big) \\ &= u_1(x_1)+\dots+u_n(x_n)-(n-1)\cdot f(0,\dots,0). \end{aligned}$$

This is almost the required form of f; we only need to deal with the constant term $(n-1)\cdot f(0,\dots,0)$. However, since m is idempotent, every constant commutes with m, thus $u'_n(x_n) := u_n(x_n)-(n-1)\cdot f(0,\dots,0)$ belongs to $[m]^*$. Then we can write f as $f(x_1,\dots,x_n) = u_1(x_1)+\dots+u_{n-1}(x_{n-1})+u'_n(x_n)$, and this completes the proof. ■

Theorem 4.4.4. [TW21] *The centralizers of the clones of Boolean functions are as indicated in Table 1.2 in the appendix. The clones are grouped by their centralizer clones; the first column shows the 25 primitive positive clones over the two-element set, and the second column lists all clones having the given primitive positive clone as their centralizer.*

Proof. Let us recall that a variety $\mathcal{V}$ is congruence distributive if and only if it has, for some n, a sequence of terms $J_0(x,y,z),\dots,J_n(x,y,z)$ satisfying the following identities:

$$\begin{aligned} &J_0(x,y,z) = x, \\ &J_n(x,y,z) = z, \\ &J_i(x,y,x) = x \text{ for each } 0 \le i \le n, \\ &J_i(x,x,y) = J_{i+1}(x,x,y) \text{ if } i \text{ is even}, \\ &J_i(x,y,y) = J_{i+1}(x,y,y) \text{ if } i \text{ is odd}. \end{aligned}$$

These terms J_i are called *Jónsson terms.* If $C \le \mathcal{O}_{\{0,1\}}$ is a clone, then the existence of a sequence of Jónsson terms in the clone C guarantees that the variety generated by the algebra $(\{0,1\};C)$ is congruence distributive. By Corollary 4.3.6, this implies that $C^* \le \Omega^{(1)}$.

The clone SM of self-dual monotone Boolean functions is generated by the majority operation $\mu(x,y,z) = xy+xz+yz$, which immediately gives us a sequence of Jónsson terms with $J_0(x,y,z) = x$, $J_1(x,y,z) = \mu(x,y,z)$ and $J_2(x,y,z) = z$. We provide a sequence of Jónsson terms in $U_{01}^{\infty}M$ in Table 4.1; the duals of these operations are Jónsson terms in $W_{01}^{\infty}M$. Thus if C contains at least one of the three clones $U_{01}^{\infty}M, W_{01}^{\infty}M$ and SM as a subclone, then C^* contains only essentially at most unary functions by Corollary 4.3.6, and then C^* is easy to find using Fact 4.4.2. This covers the first six rows of Table 1.2.

x	y	z	$x = J_0$	J_1	J_2	J_3	$J_4 = z$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	1	1
1	0	0	1	0	0	0	0
1	0	1	1	1	1	1	1
1	1	0	1	1	0	0	0
1	1	1	1	1	1	1	1

Table 4.1: A sequence of Jónsson terms in the clone $U_{01}^{\infty}M$.

After having determined clones with essentially unary centralizers, there are finitely many clones left to investigate. It is easy to see that these clones appear as joins of some of the clones $[0], [1], [\neg], V_{01}, \Lambda_{01}$ and L_{01}. According to Fact 4.4.1, it suffices to determine the centralizers of these six clones. It follows immediately from the definition of Ω_0, Ω_1 and S that $[0]^* = \Omega_0$, $[1]^* = \Omega_1$ and $[\neg]^* = S$.

Theorem 4.1.1 gives us the centralizer of V_{01}: every operation in V_{01}^* is of the form $u_1(x_1) \vee \cdots \vee u_n(x_n)$, where $u_i(x_i) = x_i$ or u_i is constant for all $i = 1, \ldots, n$. Thus, we have $V_{01}^* = [\vee, 0, 1] = V$, and dually, $\Lambda_{01}^* = [\wedge, 0, 1] = \Lambda$. Finally, Proposition 4.4.3 shows that the centralizer of $L_{01} = [x - y + z] = [x + y + z]$ consists of sums of unary functions, hence $L_{01}^* = \{x_1 + x_2 + \cdots + x_n + c \mid c \in \{0,1\}, n \in \mathbb{N}_0\} = L$. ■

The following remark makes it easier to remember the centralizers of all Boolean clones.

Remark 4.4.5. We can group the clones on $\{0, 1\}$ by the "type" of their centralizers. These groups give a partition of the Post lattice into five blocks:

- $\mathcal{C}_{\text{cd}} := \{C \leq \mathcal{O}_{\{0,1\}} \mid U_{01}^{\infty}M \leq C \text{ or } W_{01}^{\infty}M \leq C \text{ or } SM \leq C\}$;
- $\mathcal{C}_{\vee} := \{V, V_0, V_1, V_{01}\}$;
- $\mathcal{C}_{\wedge} := \{\Lambda, \Lambda_0, \Lambda_1, \Lambda_{01}\}$;
- $\mathcal{C}_{\text{lin}} := \{L, L_0, L_1, L_{01}, SL\}$;
- $\mathcal{C}_{\text{un}} := \{C \leq \mathcal{O}_{\{0,1\}} \mid C \leq \Omega^{(1)}\}$.

The "centralizing" operation $C \mapsto C^*$ preserves this partition: for every $C \leq \mathcal{O}_{\{0,1\}}$, we have

- if $C \in \mathcal{C}_{\text{cd}}$ then $C^* \in \mathcal{C}_{\text{un}}$ (i.e., $C^* \leq \Omega^{(1)}$);
- if $C \in \mathcal{C}_{\vee}$ then $C^* \in \mathcal{C}_{\vee}$;
- if $C \in \mathcal{C}_{\wedge}$ then $C^* \in \mathcal{C}_{\wedge}$;
- if $C \in \mathcal{C}_{\text{lin}}$ then $C^* \in \mathcal{C}_{\text{lin}}$;
- if $C \in \mathcal{C}_{\text{un}}$, then $C^* \geq S_{01}$ (and thus $C \in \mathcal{C}_{\text{cd}}$).

Figure 1.1 in the appendix shows the above partition of the Post lattice (the five blocks are indicated by different symbols) with primitive positive clones marked by a symbol having an outline. Observe that primitive positive clones belonging to the same block have different unary parts most of the time, the only exception being $\Omega_{01} \cap \Omega^{(1)} = S_{01} \cap \Omega^{(1)} = [x]$. Thus the observations above together with Fact 4.4.2 allow us to find the centralizer of any clone with ease.

Chapter 5

Solution sets over finite lattices and semilattices

In this chapter we describe all finite lattices and semilattices with property (SDC). In Chapter 3 we used the centralizer clones to prove that every two-element algebra has property (SDC). In theory, we might be able to use our results for centralizers of finite lattices and semilattices – that we obtained in Chapter 4 – in a similar fashion. However, we will take a different approach.

In Section 2.2 we gave a connection between property (SDC) and quantifier elimination of certain primitive positive formulas. In this chapter we rely on this connection in our investigation. Section 5.1 contains the full description of finite lattices with property (SDC): a finite lattice has property (SDC) if and only if it is a Boolean lattice. In Section 5.2 finite semilattices having property (SDC) are described as semilattice reducts of distributive lattices.

5.1 Systems of equations over finite lattices

In this, and in the following section $\mathbb{L} = (L; \wedge, \vee)$ denotes a finite lattice, with meet operation $\wedge$ and join operation $\vee$. Furthermore, $0_{\mathbb{L}}$ denotes the least and $1_{\mathbb{L}}$ denotes the greatest element of $\mathbb{L}$ (that is, $0_{\mathbb{L}} = \bigwedge L$ and $1_{\mathbb{L}} = \bigvee L$).

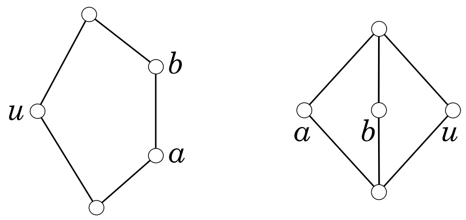
The following lemma shows that property (SDC) does not hold for nondistributive lattices, i.e., solution sets of systems of equations over a nondistributive lattice can not be characterized via closure conditions.

Theorem 5.1.1. [TW20] *Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite lattice. If property* (SDC) *holds for $C = [\wedge, \vee]$, then $\mathbb{L}$ is a distributive lattice.*

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a nondistributive finite lattice and $C = [\wedge, \vee] \leq \mathcal{O}_L$. By Lemma 2.2.4, the set

$$T = \{(x, y) \mid \exists u \in L \colon u \wedge x = u \wedge y \text{ and } u \vee x = u \vee y\} \subseteq L^2$$

is closed under C^*. We prove that T is not the solution set of a system of equations over C, hence property (SDC) does not hold for C. Suppose that there exists a



$=$	x	y	$x \wedge y$	$x \vee y$
x	–	(a,b)	(b,a)	(a,b)
y	–	–	(a,b)	(b,a)
$x \wedge y$	–	–	–	(a,b)
$x \vee y$	–	–	–	–

Figure 5.1: Counterexamples showing that the equations in the proof of Theorem 5.1.1 do not belong to $\mathcal{E}$.

system of C-equations $\mathcal{E}$ such that $T = \mathrm{Sol}(\mathcal{E})$. Since $\mathbb{L}$ is not distributive, by Birkhoff's theorem we know that there is a sublattice of $\mathbb{L}$, which is isomorphic either to N_5 or M_3. Now neither of the equations

$$x = y \ (\iff x \wedge y = x \vee y), \quad x = x \wedge y, \quad x = x \vee y, \quad y = x \wedge y, \quad y = x \vee y$$

belong to $\mathcal{E}$; we prove this by presenting a counterexample for each equation. These counterexamples are shown in Figure 5.1, where we choose the elements a and b as presented in the figure. (Note that an element u, chosen like on the figure, shows that $(a,b),(b,a) \in T$. In the table, the entry (x_1,y_1) in the line labeled by the term $s(x,y)$ and column labeled by the term $t(x,y)$ witnesses that (x_1,y_1) is not a solution of $s(x,y) = t(x,y)$.)

There are no other nontrivial 2-variable equations over C, therefore we get that T satisfies only trivial equations, hence $T = L^2$. This is a contradiction, since $(0_{\mathbb{L}}, 1_{\mathbb{L}}) \notin T$. ∎

The following lemma will help us to prove that property (SDC) can only hold for Boolean lattices. Before the lemma, for a distributive lattice $\mathbb{L}$, we define the *median* of the elements $x, y, z \in L$ as

$$m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

Lemma 5.1.2. [TW20] *Let $\mathbb{L} = (L; \wedge, \vee)$ be a distributive lattice, and for all $x, y, z, u \in L$, let*

$$p(x,y,z,u) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \vee (u \wedge x) \vee (u \wedge y) \vee (u \wedge z).$$

Then for all $x, y, z, u \in L$, we have

$$p(x,y,z,u) = x \vee y \vee z \vee u \iff m(x,y,z) \vee u = x \vee y \vee z.$$

Proof. Let $x, y, z, u \in L$ be arbitrary elements. Let us denote $m(x,y,z)$ simply by m and $p(x,y,z,u)$ by p for better readability.

First let us suppose that $p = x \vee y \vee z \vee u$. It is easy to see that $p \leq x \vee y \vee z$ always holds (since every meet in p is less than or equal to $x \vee y \vee z$). Since $p = x \vee y \vee z \vee u$, we get that $p \leq x \vee y \vee z \leq x \vee y \vee z \vee u = p$, hence $p = x \vee y \vee z$. Observe that by the distributivity of $\mathbb{L}$, p can be rewritten as $p = m \vee (u \wedge (x \vee y \vee z))$, and from $p = x \vee y \vee z \vee u = x \vee y \vee z$ we can see that $u \leq x \vee y \vee z$, therefore we have $p = m \vee u$. Thus $m \vee u = p = x \vee y \vee z$.

For the other direction, suppose that $m \vee u = x \vee y \vee z$. Using that $\mathbb{L}$ is distributive, we get that $p = m \vee (u \wedge (x \vee y \vee z)) = (m \vee u) \wedge (m \vee (x \vee y \vee z))$, and by the assumption this implies that $p = x \vee y \vee z$. Our assumption also implies that $u \leq x \vee y \vee z$, therefore we have $p = x \vee y \vee z \vee u$. ■

Theorem 5.1.3. [TW20] *Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite distributive lattice. If property* (SDC) *holds for $C = [\wedge, \vee]$, then $\mathbb{L}$ is a Boolean lattice.*

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite distributive lattice and let $C = [\wedge, \vee] \leq \mathcal{O}_L$. Since $\mathbb{L}$ is distributive, by Birkhoff's representation theorem, $\mathbb{L}$ can be embedded into a Boolean lattice $\mathbb{B}$, hence we may suppose without loss of generality that $\mathbb{L}$ is already a sublattice of $\mathbb{B}$. We can also assume that $0_{\mathbb{L}} = 0_{\mathbb{B}}$ and $1_{\mathbb{L}} = 1_{\mathbb{B}}$, and in the sequel we omit the subscripts and write only 0 and 1 for the boundary elements. Let us denote the complement of an element $x \in \mathbb{B}$ by x'. We define the dual of $p = p(x, y, z, u)$ (from Lemma 5.1.2) as $p^d = q = q(x, y, z, u)$, i.e.,

$$q(x, y, z, u) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \wedge (u \vee x) \wedge (u \vee y) \wedge (u \vee z).$$

Let T be the following set:

$$T = \Big\{(x, y, z) \in L^3 \;\Big|\; \exists u \in L\colon p(x, y, z, u) = x \vee y \vee z \vee u \text{ and } q(x, y, z, u) = x \wedge y \wedge z \wedge u\Big\}.$$

By Lemma 2.2.4, the set T is closed under C^*. Let $(x, y, z) \in T$ be arbitrary with an element $u \in L$ witnessing that $(x, y, z) \in T$. From Lemma 5.1.2 it follows that $p(x, y, z, u) = x \vee y \vee z \vee u$ if and only if $m \vee u = x \vee y \vee z$. Meeting both sides of the latter equality by m', we get

$$u \wedge m' = (m \wedge m') \vee (u \wedge m') = (m \vee u) \wedge m' = (x \vee y \vee z) \wedge m'. \tag{5.1.1}$$

By the dual of Lemma 5.1.2, we know that $q(x, y, z, u) = x \wedge y \wedge z \wedge u$ if and only if $m \wedge u = x \wedge y \wedge z$. Then joining the last equality and (5.1.1), we get that

$$\begin{aligned} u &= u \wedge 1_{\mathbb{L}} = u \wedge (m' \vee m) = (u \wedge m') \vee (u \wedge m) \\ &= ((x \vee y \vee z) \wedge m') \vee (x \wedge y \wedge z). \end{aligned}$$

It is not hard to derive from the defining identities of Boolean algebras that the latter formula is in fact the symmetric difference $x \vartriangle y \vartriangle z$ in $\mathbb{B}$. Alternatively, using Stone's representation theorem for Boolean algebras, we may assume that x, y and z are sets, and that the operations $\wedge, \vee, '$ are the set-theoretic intersection, union and complementation. Then m corresponds to the set of elements that belong to at least two of the sets x, y and z. Thus $(x \vee y \vee z) \wedge m'$ consists of those elements that belong to exactly one of x, y and z, and $((x \vee y \vee z) \wedge m') \vee (x \wedge y \wedge z)$ contains those elements that belong to one or three of the sets x, y and z, and this is indeed $x \vartriangle y \vartriangle z$ in $\mathbb{B}$.

We have proved that the element u witnessing that $(x, y, z) \in T$ can only be $x \vartriangle y \vartriangle z$:

$$\forall x, y, z \in L\colon (x, y, z) \in T \iff \exists u \in L\colon u = x \vartriangle y \vartriangle z \iff x \vartriangle y \vartriangle z \in L. \tag{5.1.2}$$

It is easy to see that $\{0, 1\}^3 \subseteq T$, and using the main theorem of [Grä64], we get that if $(f, g) \in \mathrm{Eq}(T)$, then $f = g$ must hold. (In our case this theorem says that every term function of $\mathbb{L}$ is uniquely determined by its restriction to $\{0, 1\}^3$.) Therefore, only trivial equations can appear in $\mathrm{Eq}(T)$, hence $T = L^3$. Then (5.1.2) implies that $\mathbb{L}$ is closed under the ternary operation $x \vartriangle y \vartriangle z$. In particular, for any $x \in L$ we have $x \vartriangle 0 \vartriangle 1 = x' \in L$, which means that $\mathbb{L}$ is a Boolean lattice. ■

We will need the following lemmas for the proof of Theorem 5.1.7, which states that Boolean lattices have property (SDC). This will complete the characterization of lattices with property (SDC).

Lemma 5.1.4. [TW20] *Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite distributive lattice and let $C = [\wedge, \vee] \leq \mathcal{O}_L$. Then every system of C-equations is equivalent to a system of inequalities $\{p_1 \leq q_1, \dots, p_l \leq q_l\}$, where $p_i \in [\wedge]$ and $q_i \in [\vee]$ $(i = 1, \dots, l)$.*

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite distributive lattice, let $C = [\wedge, \vee] \leq \mathcal{O}_L$ and let

$$\mathcal{E} = \{f_1 = g_1, \dots, f_t = g_t\}$$

be a system of C-equations. For arbitrary $a, b \in L$ we have $a = b$ if and only if $a \leq b$ and $b \leq a$, therefore $\mathcal{E}$ is equivalent to the system of inequalities

$$\mathcal{E}' = \{f_1 \leq g_1, g_1 \leq f_1, \dots, f_t \leq g_t, g_t \leq f_t\}.$$

Denote the disjunctive normal forms of the left hand sides of the inequalities in $\mathcal{E}'$ as DNF_j, and denote the conjunctive normal forms of the right hand sides of the inequalities in $\mathcal{E}'$ as CNF_j $(j = 1, \dots, 2t)$. Then $\mathcal{E}'$ is equivalent to the system of inequalities

$$\{DNF_1 \leq CNF_1, \dots, DNF_{2t} \leq CNF_{2t}\}.$$

Each DNF_j is a join of some meets, and each CNF_j is a meet of some joins. Therefore, for every j, the inequality $DNF_j \leq CNF_j$ holds if and only if every meet in DNF_j is less than or equal to every join in CNF_j. This means that there exists a system of inequalities $\{p_1 \leq q_1, \ldots, p_l \leq q_l\}$ equivalent to $\mathcal{E}$, such that $p_i \in [\wedge]$ and $q_i \in [\vee]$ $(i = 1, \ldots, l)$. ■

Lemma 5.1.5. [TW20] *Let* $\mathbb{B} = (B, \wedge, \vee, ')$ *be a Boolean algebra. Then for every* $a, b, c, d, u \in B$, *we have*

(i) $a \wedge u \leq b \iff u \leq a' \vee b$;

(ii) $b \leq a \vee u \iff u \geq a' \wedge b$;

(iii) $a \wedge b' \leq c' \vee d \iff a \wedge c \leq b \vee d$.

Proof. Let $a, b, c, d, u \in B$ be arbitrary elements. For the proof of (i), let us first suppose that $a \wedge u \leq b$. Joining both sides of the inequality by a', we get

$$a' \vee (a \wedge u) = (a' \vee a) \wedge (a' \vee u) = 1_{\mathbb{B}} \wedge (a' \vee u) = a' \vee u \leq a' \vee b,$$

and from this, $u \leq a' \vee b$ follows.

For the other direction, if $u \leq a' \vee b$ holds, then meeting both sides by a, we get that

$$a \wedge u \leq a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b) = 0_{\mathbb{B}} \vee (a \wedge b) = a \wedge b,$$

and from this, $a \wedge u \leq b$ follows.

The second statement is the dual of (i).

For the proof of (iii) let us use (i) with $u = a \wedge b'$, and then we get that

$$a \wedge b' \leq c' \vee d \iff c \wedge (a \wedge b') = (c \wedge a) \wedge b' \leq d.$$

Then using (ii) with $u = d$, we get

$$(c \wedge a) \wedge b' \leq d \iff c \wedge a \leq b \vee d,$$

which proves (iii). ■

Helly's theorem from convex geometry states that if we have k $(> d)$ convex sets in $\mathbb{R}^d$, such that any $d + 1$ of them have a nonempty intersection, then the intersection of all k sets is nonempty as well. The following lemma says something similar for intervals in lattices (with $d = 1$).

Lemma 5.1.6. [TW20] *Let* $\mathbb{L} = (L; \wedge, \vee)$ *be a lattice,* $c_i, d_i \in L$ $(i = 1, \ldots, k)$. *Then we have*

$$\bigcap_{i=1}^{k} [c_i, d_i] \neq \emptyset \iff \forall i, j \in \{1, \ldots, k\} \colon c_i \leq d_j.$$

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a lattice, and $c_i, d_i \in L$ $(i = 1, \dots, k)$. Then obviously,

$$\bigcap_{i=1}^{k} [c_i, d_i] = [c_1 \vee \cdots \vee c_k, d_1 \wedge \cdots \wedge d_k],$$

which is nonempty if and only if $c_1 \vee \cdots \vee c_k \leq d_1 \wedge \cdots \wedge d_k$, which holds if and only if $c_i \leq d_j$ for all $i, j \in \{1, ..., k\}$. ■

The last step in the characterization of finite lattices having property (SDC) is to show that Boolean lattices do indeed have property (SDC). For proving this, we will use the equivalence of this property with the quantifier eliminability for primitive positive formulas over $C^\circ = [\wedge, \vee]^\circ$ (see Theorem 2.2.6).

Theorem 5.1.7. [TW20] *If* $\mathbb{L} = (L; \wedge, \vee)$ *is a finite Boolean lattice, then property* (SDC) *holds for* $C = [\wedge, \vee]$.

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite Boolean lattice, and let $C = [\wedge, \vee]$. Let us denote the complement of an element $x \in \mathbb{L}$ by x'. By Theorem 2.2.6, property (SDC) holds for C if and only if any primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula. Let us consider a primitive positive formula with a single quantifier:

$$\Phi(x_1, \dots, x_n) \equiv \exists u \mathop{\&}_{j=1}^{t} \rho_j\left(z_1^{(j)}, \dots, z_{r_j}^{(j)}\right), \tag{5.1.3}$$

where $\rho_j \in (C^\circ)^{(r_j)}$, and $z_i^{(j)}$ $(j = 1, \dots, t,$ and $i = 1, \dots, r_j)$ are variables from the set $\{x_1, \dots, x_n, u\}$. We will show that Φ is equivalent to a quantifier-free primitive positive formula, and thus (by iterating this argument) every primitive positive formula is equivalent to a quantifier-free primitive positive formula. By Lemma 5.1.4, we can rewrite Φ to an equivalent formula

$$\exists u \mathop{\&}_{i=1}^{l} (p_i \leq q_i),$$

where $p_i \in [\wedge]$ and $q_i \in [\vee]$ $(i = 1, \dots, l)$.

Let a_i denote the meet of all variables from $\{x_1, \dots, x_n\}$ appearing in p_i, and let b_i denote the join of all variables from $\{x_1, \dots, x_n\}$ appearing in q_i. Then we can distinguish four cases for the i-th inequality:

(0) If u does not appear in the inequality, then the inequality is of the form $a_i \leq b_i$.

(1) If u appears only on the left hand side of the inequality, then the inequality is of the form $a_i \wedge u \leq b_i$.

(2) If u appears only on the right hand side of the inequality, then the inequality is of the form $a_i \leq b_i \vee u$.

(3) If u appears on both sides of the inequality, then the inequality is of the form $a_i \wedge u \leq b_i \vee u$, which always holds, since $a_i \wedge u \leq u \leq b_i \vee u$.

Let I_j denote the following set of indices:

$$I_j = \{i \mid \text{the inequality } p_i \leq q_i \text{ belongs to case (j)}\}$$

for $j = 0, 1, 2, 3$. The only cases we have to investigate are case (1) and case (2) (since u does not appear in case (0) and in case (3) there are only trivial inequalities). By Lemma 5.1.5,

$$\begin{aligned}&\text{for } i \in I_1 \text{ we have } a_i \wedge u \leq b_i \iff u \leq a_i' \vee b_i \iff u \in [0_{\mathbb{L}}, a_i' \vee b_i] =: [c_i, d_i];\\ &\text{for } i \in I_2 \text{ we have } a_i \leq b_i \vee u \iff u \geq b_i' \wedge a_i \iff u \in [a_i \wedge b_i', 1_{\mathbb{L}}] =: [c_i, d_i].\end{aligned}$$

Then we have

$$\exists u\, \forall i \in I_1 \cup I_2\colon p_i \leq q_i \iff \bigcap_{i \in I_1 \cup I_2} [c_i, d_i] \neq \emptyset \iff \forall i, j \in I_1 \cup I_2\colon c_i \leq d_j$$

by Lemma 5.1.6. Since u does not appear in the condition above, in principle, the quantifier has been eliminated. However, our formula still involves complements. Therefore, we use Lemma 5.1.5 to rewrite the formula. The only nontrivial case is if $c_i \neq 0_{\mathbb{L}}$ and $d_j \neq 1_{\mathbb{L}}$, that is, $c_i = a_i \wedge b_i'$ and $d_j = a_j' \vee b_j$ $(i \in I_2, j \in I_1)$. In this case $c_i \leq d_j$ if and only if $a_i \wedge a_j \leq b_i \vee b_j$ by Lemma 5.1.5.

Summarizing the observations above, we have

$$\begin{aligned}\Phi(x_1, \ldots, x_n) &\iff \exists u \mathop{\&}_{i=1}^{l} (p_i \leq q_i) \iff \mathop{\&}_{i \in I_0} (a_i \leq b_i) \;\&\; \mathop{\&}_{i,j \in I_1 \cup I_2} (c_i \leq d_j)\\ &\iff \mathop{\&}_{i \in I_0} (a_i \leq b_i) \;\&\; \mathop{\&}_{i \in I_2, j \in I_1} (a_i \wedge a_j \leq b_i \vee b_j),\end{aligned}$$

which is equivalent to a quantifier-free primitive positive formula over $[\wedge, \vee]^\circ$ (since for all $x, y \in L$, we have $x \leq y$ if and only if $x = x \wedge y$). ■

We can summarize the results of this section in the following corollary of theorems 5.1.1, 5.1.3 and 5.1.7.

Corollary 5.1.8. [TW20] *A finite lattice has property* (SDC) *if and only if it is a Boolean lattice.*

This means that for any finite lattice $\mathbb{L} = (L; \wedge, \vee)$, solution sets of systems of equations over $\mathbb{L}$ can be characterized via closure conditions if and only if $\mathbb{L}$ is a Boolean lattice.

$=$	x	y	$x \wedge y$
x	–	$(a, 0_{\mathbb{M}})$	$(a, 0_{\mathbb{M}})$
y	–	–	$(0_{\mathbb{M}}, a)$
$x \wedge y$	–	–	–

Table 5.1: Counterexamples showing that the equations in the proof of Theorem 5.2.1 do not belong to $\mathcal{E}$.

5.2 Systems of equations over finite semilattices

Similarly to Section 5.1, in this section $\mathbb{M} = (M; \wedge)$ denotes a finite semilattice with meet operation $\wedge$ and least element $0_{\mathbb{M}}$.

Theorem 5.2.1. [TW20] *Let* $\mathbb{M} = (M; \wedge)$ *be a finite semilattice. If* $\mathbb{M}$ *has no greatest element, then property* (SDC) *does not hold for* $C = [\wedge]$*.*

Proof. Let $\mathbb{M} = (M; \wedge)$ be a finite semilattice with no greatest element, and let $C = [\wedge] \leq \mathcal{O}_M$. The set

$$\begin{aligned} T &= \{(x, y) \mid \exists u \in L \colon x \wedge u = x \text{ and } y \wedge u = y\} = \\ &= \{(x, y) \mid \exists u \in L \colon x \leq u \text{ and } y \leq u\} \subseteq M^2 \end{aligned}$$

is closed under C^* by Lemma 2.2.4. Similarly to Theorem 5.1.1, we will prove that T is not the solution set of any system of equations over C. Suppose that there exists a system of C-equations $\mathcal{E}$ such that $T = \mathrm{Sol}(\mathcal{E})$. There are only three nontrivial 2-variable equations over C:

$$x = y, \quad x \wedge y = x, \quad x \wedge y = y.$$

As in Theorem 5.1.1, we prove that none of these equations can appear in $\mathcal{E}$ by presenting counterexamples to them (see Table 5.1). Note that since $\mathbb{M}$ is finite and it has no greatest element, there exist maximal elements $a \neq b$ in $\mathbb{M}$. We have that only trivial equations can appear in $\mathcal{E}$, thus $T = M^2$. But this is a contradiction, since $(a, b) \notin T$. ∎

If a finite semilattice $\mathbb{M} = (M; \wedge)$ has a greatest element, then for all $(a, b) \in M^2$, the set $H = \{x \in M \mid a \leq x \text{ and } b \leq x\}$ is not empty. Since $\mathbb{M}$ is a finite semilattice, it follows that $\bigwedge H$ exists for all $(a, b) \in M^2$. This means that we can define a join operation $\vee$ on M, such that $\mathbb{L} = (L; \wedge, \vee)$ is a lattice (with $L = M$). Therefore, from now on it suffices to investigate lattices (but the clone we use for the equations is still $C = [\wedge]$).

The following theorem shows that property (SDC) does not hold for nondistributive lattices (regarded as semilattices), i.e., solution sets of systems of equations over a nondistributive lattice (as a semilattice) can not be characterized via closure conditions.

Remark 5.2.2. A meet-semilattice $\mathbb{M}$ is *distributive* if for any $a, b_0, b_1 \in \mathbb{M}$, the inequality $a \geq b_0 \wedge b_1$ implies that there exist $a_0, a_1 \in \mathbb{M}$ such that $a_0 \geq b_0$, $a_1 \geq b_1$ and $a = a_0 \wedge a_1$ (see Section 5.1 in Chapter II of [Grä11]). From Lemma 184 of [Grä11] it follows that a finite semilattice is distributive if and only if it is a semilattice reduct of a distributive lattice.

Theorem 5.2.3. [TW20] *Let* $\mathbb{L} = (L; \wedge, \vee)$ *be a finite lattice. If* $\mathbb{L}$ *is not distributive, then property* (SDC) *does not hold for* $C = [\wedge]$.

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite lattice and let $C = [\wedge] \leq \mathcal{O}_L$. Since $\mathbb{L}$ is not distributive, we know that there exists a sublattice of $\mathbb{L}$ isomorphic to either N_5 or M_3. Let us denote these two cases as (N_5) and (M_3), respectively. The figures and tables we use in this proof can be found in Appendix 1.3. Let T be the set

$$\begin{aligned} T &= \{(x, y, z) \in L^3 \mid \exists u \in L\colon x \wedge y = u \wedge y \text{ and } u \wedge x = x \text{ and } u \wedge z = z\} \\ &= \{(x, y, z) \in L^3 \mid \exists u \in L\colon x \wedge y = u \wedge y \text{ and } u \geq x \text{ and } u \geq z\}, \end{aligned}$$

which is closed under C^* by Lemma 2.2.4. As in Theorem 5.1.1, we will prove that T is not the solution set of any system of equations over C.

Similarly to Theorem 5.1.1, we present counterexamples to nontrivial equations, the only difference is that here we prove that there can be only one nontrivial equation satisfied by T (see tables 1.3 and 1.4 for case (N_5) and (M_3), respectively).

We choose the elements a, b and c as presented in Figure 1.2 for case (N_5), and in Figure 1.3 for case (M_3). (Note that an element u, chosen like on the figures, shows that in case (N_5) we have $(a, c, b), (b, a, c) \in T$, and in case (M_3) we have $(a, b, c), (a, c, b) \in T$.)

So now we have that in both cases the only nontrivial equation that T can satisfy is the equation $y \wedge z = x \wedge y \wedge z$. One can verify that this equation holds on T: if $(x, y, z) \in T$, then we have

$$x \wedge y = u \wedge y \geq z \wedge y \implies x \wedge y \wedge z \geq y \wedge z,$$

which implies that $y \wedge z = x \wedge y \wedge z$. Therefore, we can conclude that the only nontrivial equation in $\operatorname{Eq}(T)$ is $y \wedge z = x \wedge y \wedge z$. We will prove that T is not the solution set of any system of equations by presenting a tuple $(x_1, y_1, z_1) \in \operatorname{Sol}(\operatorname{Eq}(T)) \setminus T$ (cf. Remark 1.4.2). Since there exists a sublattice of $\mathbb{L}$ isomorphic to N_5 or M_3, there exists a tuple (x_1, y_1, z_1) as shown in Figure 1.4, which satisfies $y_1 \wedge z_1 = x_1 \wedge y_1 \wedge z_1$, thus $(x_1, y_1, z_1) \in \operatorname{Sol}(\operatorname{Eq}(T))$. However, one can easily verify that (x_1, y_1, z_1) does not belong to T. Indeed, suppose that $(x_1, y_1, z_1) \in T$, then there exists $u \in L$ such that $u \geq x_1$, $u \geq z_1$ and $x_1 \wedge y_1 = u \wedge y_1$. But then we have $u \geq x_1 \vee z_1 > y_1$ (since N_5 or M_3 is a sublattice), therefore $x_1 \wedge y_1 < u \wedge y_1 = y_1$ gives us a contradiction. Thus, $T \neq \mathrm{Sol}(\mathrm{Eq}(T))$, hence, by Remark 1.4.2, T is not the solution set of any system of equations over C. ■

Theorem 5.2.1 and Theorem 5.2.3 prove that if $\mathbb{M} = (M; \wedge)$ has property (SDC), then it is the semilattice reduct of a distributive lattice $\mathbb{L} = (L; \wedge, \vee)$. To complete the characterization of finite semilattices with property (SDC), we prove that the clone $[\wedge]$ has property (SDC) whenever $\wedge$ is the meet operation of a finite distributive lattice.

Theorem 5.2.4. [TW20] *If* $\mathbb{L} = (L; \wedge, \vee)$ *is a finite distributive lattice, then property* (SDC) *holds for* $C = [\wedge]$.

Proof. Let $\mathbb{L} = (L; \wedge, \vee)$ be a finite distributive lattice and $C = [\wedge] \leq \mathcal{O}_L$. Since $\mathbb{L}$ is distributive, by Birkhoff's representation theorem $\mathbb{L}$ can be embedded into a Boolean lattice $\mathbb{B}$, hence we may suppose without loss of generality that $\mathbb{L}$ is already a sublattice of $\mathbb{B}$. We can also assume that $0_{\mathbb{L}} = 0_{\mathbb{B}}$ and $1_{\mathbb{L}} = 1_{\mathbb{B}}$. Let us denote the complement of an element $x \in \mathbb{B}$ by x'.

By Theorem 2.2.6, property (SDC) holds for C if and only if any primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula. Similarly to the proof of Theorem 5.1.7, it suffices to consider primitive positive formulas with a single existential quantifier. Let

$$\Phi(x_1, \ldots, x_n) \equiv \exists u \mathop{\&}_{j=1}^{t} \rho_j\left(z_1^{(j)}, \ldots, z_{r_j}^{(j)}\right), \tag{5.2.1}$$

where $\rho_j \in (C^\circ)^{(r_j)}$, and $z_i^{(j)}$ $(j = 1, \ldots, t,$ and $i = 1, \ldots, r_j)$ are variables from the set $\{x_1, \ldots, x_n, u\}$. We will show that Φ is equivalent to a quantifier-free primitive positive formula.

Since for all $a, b \in L$ we have $a = b$ if and only if $a \leq b$ and $b \leq a$, we can rewrite Φ to an equivalent formula

$$\exists u \mathop{\&}_{i=1}^{l} (p_i \leq q_i),$$

where $p_i, q_i \in [\wedge]$ $(i = 1, \ldots, l)$.

Let a_i denote the meet of all variables from $\{x_1, \ldots, x_n\}$ appearing in p_i, and let b_i denote the meet of all variables from $\{x_1, \ldots, x_n\}$ appearing in q_i. Then we can distinguish four cases for the i-th inequality:

(0) If u does not appear in the inequality, then the inequality is of the form $a_i \leq b_i$.

(1) If u appears only on the left hand side of the inequality, then the inequality is of the form $a_i \wedge u \leq b_i$.

(2) If u appears only on the right hand side of the inequality, then the inequality is of the form $a_i \leq b_i \wedge u$, which holds if and only if $a_i \leq b_i$ and $a_i \leq u$.

(3) If u appears on both sides of the inequality, then the inequality is of the form $a_i \wedge u \leq b_i \wedge u$, which holds if and only if $a_i \wedge u \leq b_i$ and $a_i \wedge u \leq u$, that is, $a_i \wedge u \leq b_i$.

Let I_j denote the following set of indices:

$$I_j = \{i \mid \text{the inequality } p_i \leq q_i \text{ belongs to case (j)}\}$$

for $j = 0, 1, 2, 3$. We investigate only cases (1), (2) and (3), since u does not appear in case (0). Moreover; in case (2), we only have to deal with the inequality $a_i \leq u$, since u does not appear in the inequality $a_i \leq b_i$.

By Lemma 5.1.5,

$$\begin{aligned}
&\text{for } i \in I_1 \text{ we have } a_i \wedge u \leq b_i \iff u \leq a_i' \vee b_i \iff u \in [0_{\mathbb{L}}, a_i' \vee b_i] =: [c_i, d_i];\\
&\text{for } i \in I_2 \text{ we have } a_i \leq u \iff u \in [a_i, 1_{\mathbb{L}}] =: [c_i, d_i];\\
&\text{for } i \in I_3 \text{ we have } a_i \wedge u \leq b_i \iff u \leq a_i' \vee b_i \iff u \in [0_{\mathbb{L}}, a_i' \vee b_i] =: [c_i, d_i].
\end{aligned}$$

Then we have

$$\bigcap_{i \in I_1 \cup I_2 \cup I_3} [c_i, d_i] \neq \emptyset \iff \forall i, j \in I_1 \cup I_2 \cup I_3 \colon c_i \leq d_j$$

by Lemma 5.1.6. Just as in the proof of Theorem 5.1.7, we apply Lemma 5.1.5 to eliminate complements and joins from the formula above. The only interesting case is if $c_i \neq 0_{\mathbb{L}}$ and $d_j \neq 1_{\mathbb{L}}$, that is, $c_i = a_i$ and $d_j = a_j' \vee b_j$ ($i \in I_2, j \in I_1 \cup I_3$). In this case $c_i \leq d_j$ if and only if $a_i \leq a_j' \vee b_j$, which holds if and only if $a_i \wedge a_j \leq b_j$ by Lemma 5.1.5 (with $u = a_i$).

Summarizing the observations above, we have

$$\begin{aligned}
\Phi(x_1, \ldots, x_n) &\iff \exists u \mathop{\&}_{i=1}^{l} (p_i \leq q_i) \iff \mathop{\&}_{i \in I_0 \cup I_2} (a_i \leq b_i) \,\&\, \mathop{\&}_{i,j \in I_1 \cup I_2 \cup I_3} (c_i \leq d_j)\\
&\iff \mathop{\&}_{i \in I_0 \cup I_2} (a_i \leq b_i) \,\&\, \mathop{\&}_{i \in I_2, j \in I_1 \cup I_3} (a_i \wedge a_j \leq b_j),
\end{aligned}$$

which is equivalent to a quantifier-free primitive positive formula over $[\wedge]^\circ$ (since for all $x, y \in L$, we have $x \leq y$ if and only if $x = x \wedge y$). ■

We can summarize the results of this section in the following corollary of theorems 5.2.1, 5.2.3 and 5.2.4.

Corollary 5.2.5. [TW20] *A finite semilattice has property* (SDC) *if and only if it is distributive.*

This means that for any finite semilattice $\mathbb{M}$, solution sets of systems of equations over $\mathbb{M}$ can be characterized (via closure conditions) if and only if $\mathbb{M}$ is a semilattice reduct of a distributive lattice (see Remark 5.2.2).

Chapter 6

Solution sets and polymorphism-homogeneity

Various notions of homogeneity appear in several areas of mathematics, such as model theory, group theory, combinatorics, etc. Roughly speaking, a structure $\mathcal{A}$ is said to be homogeneous if certain kinds of local morphisms (i.e., morphisms defined on "small" substructures of $\mathcal{A}$) extend to endomorphisms of $\mathcal{A}$. Specifying the kind of morphisms that are expected to be extendible, one can define many different versions of homogeneity. We consider a variant called polymorphism-homogeneity, introduced by C. Pech and M. Pech [PP15], that involves "multivariable" homomorphisms: we require extendibility of homomorphisms defined on finitely generated substructures of direct powers of $\mathcal{A}$ (see Section 6.1 for the precise definition).

In this chapter we study polymorphism-homogeneity of finite algebraic structures and of certain relational structures constructed from algebras. Since homomorphisms depend on the term operations, not on the particular choice of basic operations, we work mainly with the clone $C = \mathrm{Clo}(\mathbb{A})$ of term operations of the algebraic structure $\mathbb{A} = (A; F)$ (i.e., C is the clone generated by F). Probably the most natural way to convert $\mathbb{A}$ into a relational structure is to consider the graphs of the operations of $\mathbb{A}$, i.e., $C^\bullet = \{f^\bullet \mid f \in C\}$. We will prove that if the relational structure $(A, C^\bullet)$ is polymorphism-homogeneous, then the algebra $\mathbb{A}$ is also polymorphism-homogeneous, but the converse is not true in general.

To construct a relational structure that is equivalent to $\mathbb{A}$ in terms of polymorphism-homogeneity, observe that the relation $f^\bullet$ is nothing else than the solution set of the equation $f(x_1, \dots, x_n) = x_{n+1}$. If we consider more general equations where the right hand side is not necessarily a single variable, but another (n-ary) operation from C, then we get the relational structure $C^\circ = \left\{\mathrm{Sol}(f, g) \;\middle|\; n \in \mathbb{N},\ f, g \in C^{(n)}\right\}$ on A (see Section 2.2). It turns out that (A, C°) is the "right" choice for a relational counterpart of $\mathbb{A}$: the algebra $\mathbb{A}$ is polymorphism-homogeneous if and only if the relational structure (A, C°) is polymorphism-homogeneous. We also show that these properties are equivalent to property (SDC) as well.

The categorical notion of injectivity also asks for extensions of certain homomorphisms, so it is not surprising that a finite algebra $\mathbb{A}$ is polymorphism-homogeneous

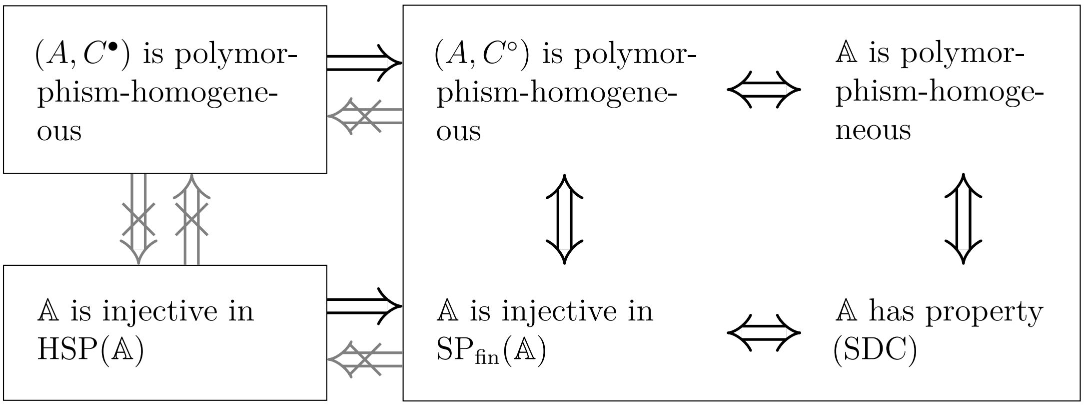


Figure 6.1: Relationships between property (SDC) and several variants of polymorphism-homogeneity and injectivity.

if and only if it is injective in a certain class of algebras, namely in the class of finite subpowers of $\mathbb{A}$ (see Section 6.1 for the definitions). Perhaps it is more natural to consider injectivity in the variety HSP $\mathbb{A}$ generated by $\mathbb{A}$, hence we will also investigate the relationship between this notion and polymorphism-homogeneity.

Figure 6.1 shows the six properties that we are concerned with in this chapter. In Section 6.2 we prove all the implications and equivalences indicated in the figure. It turns out that for finite algebras, four of the six conditions are equivalent, thus we have actually three different properties marked by the three boxes. In Section 6.3 we determine finite semilattices, lattices, Abelian groups and monounary algebras possessing these three properties, and these examples will justify all of the "non-implications" in Figure 6.1.

6.1 Polymorphism-homogeneity and injectivity

A first-order structure $\mathcal{A}$ (i.e., a set A equipped with relations and/or operations) is said to be *k-polymorphism-homogeneous*, if every homomorphism $h\colon \mathcal{B} \to \mathcal{A}$ defined on a finitely generated substructure $\mathcal{B} \leq \mathcal{A}^k$ extends to a homomorphism $\widehat{h}\colon \mathcal{A}^k \to \mathcal{A}$. (Considering only finite structures, the assumption that $\mathcal{B}$ is finitely generated can be omitted from the definition.) The case $k = 1$ gives the notion of *homomorphism-homogeneity* introduced by P. J. Cameron and J. Nešetřil [CN06]. If $\mathcal{A}$ is k-polymorphism-homogeneous for every positive integer k, then we say that $\mathcal{A}$ is *polymorphism-homogeneous* [PP15]. These two notions are linked by the following result, which was proved for relational structures by C. Pech and M. Pech [PP15] and for algebraic structures by Z. Farkasová and D. Jakubíková-Studenovská [FJS15], but the same proof works for arbitrary first-order structures.

Proposition 6.1.1. [PP15; FJS15] *A first-order structure $\mathcal{A}$ is polymorphism-homogeneous if and only if $\mathcal{A}^k$ is homomorphism-homogeneous for all positive integers k.*

In the next proposition we recall a useful result from [PP15] that relates polymorphism-homogeneity and quantifier elimination for finite relational structures; we give a short proof utilizing the Galois connections between (partial) operations and relations.

Proposition 6.1.2. [PP15] *A finite relational structure has quantifier elimination for primitive positive formulas if and only if it is polymorphism-homogeneous.*

Proof. A finite relational structure $\mathcal{A} = (A, R)$ has quantifier elimination for primitive positive formulas if and only if $\langle R \rangle_{\nexists} = \langle R \rangle_{\exists}$. Using the Galois connections $\operatorname{Pol} - \operatorname{Inv}$ (clones and relational clones, see Theorem 1.5.1) and $\operatorname{pPol} - \operatorname{Inv}$ (strong partial clones and weak relational clones, see Theorem 1.5.2), we can reformulate this condition in several steps to reach polymorphism-homogeneity:

$$\begin{aligned}
&\langle R \rangle_{\nexists} = \langle R \rangle_{\exists} \\
&\qquad \iff \operatorname{Inv} \operatorname{pPol} R = \operatorname{Inv} \operatorname{Pol} R \\
&\qquad \iff \operatorname{pPol} \operatorname{Inv} \operatorname{pPol} R = \operatorname{pPol} \operatorname{Inv} \operatorname{Pol} R \\
&\qquad \iff \operatorname{pPol} R = \operatorname{Str}(\operatorname{Pol} R) \\
&\qquad \iff \{h \in \mathcal{P}_A : h \rhd R\} = \{h \in \mathcal{P}_A : h \text{ extends to } \widehat{h} \in \mathcal{O}_A \text{ such that } \widehat{h} \rhd R\} \\
&\qquad \iff \mathcal{A} \text{ is polymorphism-homogeneous.}
\end{aligned}$$

■

Let $\mathcal{K}$ be a class of algebras and $\mathbb{A} \in \mathcal{K}$. We say that $\mathbb{A}$ is *injective* in $\mathcal{K}$ if every homomorphism $h \colon \mathbb{B} \to \mathbb{A}$ extends to a homomorphism $\widehat{h} \colon \mathbb{C} \to \mathbb{A}$ whenever $\mathbb{B}, \mathbb{C} \in \mathcal{K}$ and $\mathbb{B} \leq \mathbb{C}$. Clearly, if $\mathbb{A}$ is injective in $\mathcal{K}$, then $\mathbb{A}$ is also injective in every subclass of $\mathcal{K}$ that contains $\mathbb{A}$. Injectivity is most often considered in the largest relevant class $\mathcal{K}$; for example, if $\mathbb{A}$ is a group or a lattice, then $\mathcal{K}$ is usually chosen to be the class of all groups or lattices. In this thesis we shall consider smaller classes, namely the variety $\operatorname{HSP} \mathbb{A}$ generated by $\mathbb{A}$ and the set of finite subpowers $\operatorname{SP}_{\text{fin}} \mathbb{A}$ of $\mathbb{A}$ (the latter consists of all subalgebras of finite direct powers of $\mathbb{A}$). Let us mention that in [KN82] a group $\mathbb{A}$ is called *relatively injective* if it is injective in the variety $\operatorname{HSP} \mathbb{A}$.

6.2 Property (SDC) and polymorphism-homogeneity

First let us prove the equivalences shown on the right hand side of Figure 6.1. The equivalence of property (SDC) and polymorphism-homogeneity of (A, C°) follows immediately from Proposition 6.1.2.

Proposition 6.2.1. [TW22] *If* $\mathbb{A}$ *is a finite algebra and* $C = \mathrm{Clo}(\mathbb{A})$*, then* $\mathbb{A}$ *has property* (SDC) *if and only if* (A, C°) *is polymorphism-homogeneous.*

Proof. By Theorem 2.2.6, property (SDC) of $\mathbb{A}$ is equivalent to quantifier elimination for primitive positive formulas for the relational structure (A, C°), and the latter is equivalent to polymorphism-homogeneity of (A, C°) by Proposition 6.1.2. ■

In the next theorem we establish the connection between "algebraic" and "relational" polymorphism-homogeneity. We need two technical lemmas for the proof of this result.

Lemma 6.2.2. [TW22] *Suppose that* $\mathbb{A}$ *is a finite algebra,* $C = \mathrm{Clo}(\mathbb{A})$ *and* $h \in \mathcal{P}_A^{(k)}$ *is a* k*-ary partial operation on* A*. If* $\operatorname{dom} h$ *is a subalgebra of* $\mathbb{A}^k$*, then the following three conditions are equivalent:*

(a) $h \rhd C^\bullet$;

(b) $h \rhd C^\circ$;

(c) h *is a homomorphism from* $\operatorname{dom} h$ *to* $\mathbb{A}$*.*

Proof. To show that (a) implies (b), assume that $h \rhd C^\bullet$, and let $\rho = \mathrm{Sol}(f, g)$, where $f, g \in C^{(n)}$; we shall prove that $h \rhd \rho$. Let $M \in A^{n\times k}$ be a matrix such that each row of M belongs to $\operatorname{dom} h$ and each column of M belongs to ρ. Then we have $f(M_{1*}, \dots, M_{n*}) \in \operatorname{dom} h$, as $\operatorname{dom} h$ is a subalgebra of $\mathbb{A}^k$ and $f \in C$. Now let $M' \in A^{(n+1)\times k}$ be the matrix obtained by adding the row $f(M_{1*}, \dots, M_{n*})$ to the bottom of M. Since $f(M_{1*}, \dots, M_{n*}) = (f(M_{*1}), \dots, f(M_{*k}))$, every column of M' belongs to $f^\bullet$, hence applying h to each row of M', we obtain a tuple in $f^\bullet$, because h preserves $f^\bullet$ by our assumption. This means that

$$h(f(M_{*1}), \dots, f(M_{*k})) = f(h(M_{*1}, \dots, M_{*k})). \tag{6.2.1}$$

Using a similar argument, replacing f by g, we obtain

$$h(g(M_{*1}), \dots, g(M_{*k})) = g(h(M_{*1}, \dots, M_{*k})). \tag{6.2.2}$$

All columns of M were assumed to be in the relation $\rho = \mathrm{Sol}(f, g)$; therefore,

$$f(M_{*1}) = g(M_{*1}), \dots, f(M_{*k}) = g(M_{*k}). \tag{6.2.3}$$

Combining (6.2.1), (6.2.2) and (6.2.3), we can conclude that $f(h(M_{*1}, \dots, M_{*k})) = g(h(M_{*1}, \dots, M_{*k}))$, hence $h(M_{*1}, \dots, M_{*k}) \in \rho$, and this proves that $h \rhd \rho$.

Next suppose that (b) holds; to prove that h is an algebra homomorphism, consider an operation $f \in C^{(n)}$ and tuples $\mathbf{d}_1, \dots, \mathbf{d}_n \in \operatorname{dom} h$. Since $\operatorname{dom} h$ is a subalgebra, we have $f(\mathbf{d}_1, \dots, \mathbf{d}_n) \in \operatorname{dom} h$. Let $M \in A^{(n+1)\times k}$ be the matrix whose rows are $\mathbf{d}_1, \dots, \mathbf{d}_n, f(\mathbf{d}_1, \dots, \mathbf{d}_n)$. Then all columns of M belong to $f^\bullet$, hence $(h(\mathbf{d}_1), \dots, h(\mathbf{d}_n), h(f(\mathbf{d}_1, \dots, \mathbf{d}_n))) \in f^\bullet$, since h was assumed to preserve C° (recall that $C^\circ \supseteq C^\bullet$). Thus we have $f(h(\mathbf{d}_1), \dots, h(\mathbf{d}_n)) = h(f(\mathbf{d}_1, \dots, \mathbf{d}_n))$, proving that h is indeed a homomorphism.

Finally, assume (c) and let us verify (a). Let $f \in C^{(n)}$, and let $M \in A^{(n+1)\times k}$ be an arbitrary matrix whose rows and columns belong to $\operatorname{dom} h$ and to $f^\bullet$, respectively; in particular, the last row of M is $f(M_{1*}, \dots, M_{n*})$. We need to show that $h(M_{*1}, \dots, M_{*k}) \in f^\bullet$, which is equivalent to $f(h(M_{1*}), \dots, h(M_{n*})) = h(f(M_{1*}, \dots, M_{n*}))$. The latter equality is justified by the fact that h is a homomorphism. ■

Lemma 6.2.3. [TW22] *Let C be a clone on a finite set A, and let $h \in \mathcal{P}_A^{(k)}$ be a k-ary partial operation on A. If h preserves C°, then h can be extended to a partial operation $\widetilde{h} \in \mathcal{P}_A^{(k)}$ such that $\widetilde{h} \rhd C^\circ$ and* $\operatorname{dom} \widetilde{h} = [\operatorname{dom} h]$ *(the subalgebra of $\mathbb{A}^k$ generated by* $\operatorname{dom} h$*).*

Proof. If $\mathbf{a} \in [\operatorname{dom} h]$, then $\mathbf{a}$ can be obtained from the elements of $\operatorname{dom} h$ by an operation $t \in C$. Adding inessential variables to t if necessary, we can assume that actually all elements of $\operatorname{dom} h$ are used, and thus the arity of t is $m := |\operatorname{dom} h|$. Therefore, we can write $\mathbf{a} = t(\mathbf{d}_1, \dots, \mathbf{d}_m)$, where $\operatorname{dom} h = \{\mathbf{d}_1, \dots, \mathbf{d}_m\}$ and $t \in C^{(m)}$. We then define the desired extension of h at $\mathbf{a}$ by

$$\widetilde{h}(\mathbf{a}) = t(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)). \tag{6.2.4}$$

First we need to verify that $\widetilde{h}$ is well defined. Suppose that an element $\mathbf{a} \in [\operatorname{dom} h]$ can be written in more than one way in the above form: $\mathbf{a} = t_1(\mathbf{d}_1, \dots, \mathbf{d}_m) = t_2(\mathbf{d}_1, \dots, \mathbf{d}_m)$ with $t_1, t_2 \in C^{(m)}$. Setting $\rho = \operatorname{Sol}(t_1, t_2) \in C^\circ$, and letting $D \in A^{m\times k}$ be the matrix with rows $\mathbf{d}_1, \dots, \mathbf{d}_m$, every column of D belongs to the relation ρ. Since h preserves ρ, we have $h(D_{*1}, \dots, D_{*k}) \in \rho$, and therefore $t_1(h(D_{*1}, \dots, D_{*k})) = t_2(h(D_{*1}, \dots, D_{*k}))$ holds. This implies that $\widetilde{h}$ is well defined, as the value of $\widetilde{h}(\mathbf{a})$ in (6.2.4) does not depend on the particular choice of the operation t:

$$\begin{aligned} t_1(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) &= t_1(h(D_{1*}), \dots, h(D_{m*})) \\ &= t_1(h(D_{*1}, \dots, D_{*k})) \\ &= t_2(h(D_{*1}, \dots, D_{*k})) \\ &= t_2(h(D_{1*}), \dots, h(D_{m*})) \\ &= t_2(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)). \end{aligned}$$

Choosing the i-th projection $t(x_1, \dots, x_m) = x_i$ in (6.2.4), we see that $\widetilde{h}(\mathbf{d}_i) = h(\mathbf{d}_i)$ for all $i \in \{1, \dots, m\}$, thus $\widetilde{h}$ is an extension of h. It remains to prove that $\widetilde{h}$ preserves C°.

Let $f, g \in C^{(n)}$, let $\rho = \mathrm{Sol}(f, g) \in C^\circ$, and let $M \in A^{n\times k}$ be a matrix such that all rows of M are in $\mathrm{dom}\,\widetilde{h}$ and each column of M belongs to the relation ρ. Since $M_{i*} \in \mathrm{dom}\,\widetilde{h} = [\mathrm{dom}\,h]$, we can write $M_{i*} = t_i(\mathbf{d}_1, \dots, \mathbf{d}_m)$ for suitable operations $t_i \in C^{(m)}$ for $i = 1, \dots, n$. Since $M_{*j} \in \rho$, we have $f(M_{*j}) = g(M_{*j})$ for $j = 1, \dots, k$, or, equivalently, $f(M_{1*}, \dots, M_{n*}) = g(M_{1*}, \dots, M_{n*})$. Combining the latter two observations, we get that

$$\begin{aligned}
f(t_1, \dots, t_n)(\mathbf{d}_1, \dots, \mathbf{d}_m) &= f(t_1(\mathbf{d}_1, \dots, \mathbf{d}_m), \dots, t_n(\mathbf{d}_1, \dots, \mathbf{d}_m)) \\
&= f(M_{1*}, \dots, M_{n*}) \\
&= g(M_{1*}, \dots, M_{n*}) \\
&= g(t_1(\mathbf{d}_1, \dots, \mathbf{d}_m), \dots, t_n(\mathbf{d}_1, \dots, \mathbf{d}_m)) \\
&= g(t_1, \dots, t_n)(\mathbf{d}_1, \dots, \mathbf{d}_m).
\end{aligned}$$

Setting $f' = f(t_1, \dots, t_n)$ and $g' = g(t_1, \dots, t_n)$, we can summarize the above calculation as $f'(\mathbf{d}_1, \dots, \mathbf{d}_m) = g'(\mathbf{d}_1, \dots, \mathbf{d}_m)$, which means that the columns of the matrix D belong to the relation $\rho' := \mathrm{Sol}(f', g')$. The clone C is closed under composition, thus $f', g' \in C$, hence $\rho' \in C^\circ$. We assumed that $h \rhd C^\circ$; therefore, we have $(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) \in \rho'$, and this is equivalent to $f'(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) = g'(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m))$. Expanding this last equality using the definition of f' and g' together with (6.2.4), we obtain $\widetilde{h}(M_{*1}, \dots, M_{*k})) \in \rho$, which completes the proof of $\widetilde{h} \rhd C^\circ$:

$$\begin{aligned}
f(\widetilde{h}(M_{*1}, \dots, M_{*k})) &= f(\widetilde{h}(M_{1*}), \dots, \widetilde{h}(M_{n*})) \\
&= f\Big(\widetilde{h}(t_1(\mathbf{d}_1, \dots, \mathbf{d}_m)), \dots, \widetilde{h}(t_n(\mathbf{d}_1, \dots, \mathbf{d}_m))\Big) \\
&= f\Big(t_1(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)), \dots, t_n(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m))\Big) \\
&= f(t_1, \dots, t_n)(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) \\
&= f'(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) \\
&= g'(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) \\
&= g(t_1, \dots, t_n)(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)) \\
&= g\Big(t_1(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m)), \dots, t_n(h(\mathbf{d}_1), \dots, h(\mathbf{d}_m))\Big) \\
&= g\Big(\widetilde{h}(t_1(\mathbf{d}_1, \dots, \mathbf{d}_m)), \dots, \widetilde{h}(t_n(\mathbf{d}_1, \dots, \mathbf{d}_m))\Big) \\
&= g(\widetilde{h}(M_{1*}), \dots, \widetilde{h}(M_{n*})) \\
&= g(\widetilde{h}(M_{*1}, \dots, M_{*k})). \qquad \blacksquare
\end{aligned}$$

Theorem 6.2.4. [TW22] *If $\mathbb{A}$ is a finite algebra and $C = \mathrm{Clo}(\mathbb{A})$, then $\mathbb{A}$ is polymorphism-homogeneous if and only if (A, C°) is polymorphism-homogeneous.*

Proof. Assume first that $\mathbb{A}$ is polymorphism-homogeneous, and consider an arbitrary partial polymorphism h of (A, C°), i.e., let $h \in \mathcal{P}_A^{(k)}$ preserve C°. By

Lemma 6.2.3, h can be extended to a C°-preserving partial operation $\widetilde{h}$ defined on the subalgebra $[\operatorname{dom} h] \leq \mathbb{A}^k$. Applying Lemma 6.2.2 to $\widetilde{h}$, we see that $\widetilde{h}$ is an algebra homomorphism from $[\operatorname{dom} h]$ to $\mathbb{A}$. Since $\mathbb{A}$ is polymorphism-homogeneous, $\widetilde{h}$ extends to a homomorphism $\widehat{\widetilde{h}}\colon \mathbb{A}^k \to \mathbb{A}$. Using Lemma 6.2.2 again, we see that $\widehat{\widetilde{h}}$ preserves C°, hence it is a polymorphism of the relational structure (A, C°), and this proves that the latter is polymorphism-homogeneous.

Now suppose that (A, C°) is polymorphism-homogeneous, and let $h \in \mathcal{P}_A^{(k)}$ be a homomorphism from a subalgebra $\operatorname{dom} h \leq \mathbb{A}^k$ to $\mathbb{A}$. Lemma 6.2.2 shows that $h \rhd C^\circ$, i.e., h is a partial polymorphism of (A, C°). Since (A, C°) is polymorphism-homogeneous, h can be extended to a polymorphism $\widehat{h}$ of (A, C°). By Lemma 6.2.2, $\widehat{h}\colon \mathbb{A}^k \to \mathbb{A}$ is a homomorphism, and this proves that $\mathbb{A}$ is polymorphism-homogeneous. ■

To complete the proof of the equivalences in the box on the right hand side of Figure 6.1, we relate injectivity and polymorphism-homogeneity.

Proposition 6.2.5. [TW22] *If $\mathbb{A}$ is a finite algebra, then $\mathbb{A}$ is polymorphism-homogeneous if and only if $\mathbb{A}$ is injective in $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$.*

Proof. Assume that $\mathbb{A}$ is polymorphism-homogeneous, and let $\mathbb{B}, \mathbb{C} \in \mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$ such that $\mathbb{B} \leq \mathbb{C}$. Then we have $\mathbb{B} \leq \mathbb{C} \leq \mathbb{A}^k$ for some $k \in \mathbb{N}$; in particular, $\mathbb{B}$ is a subalgebra of $\mathbb{A}^k$. Therefore, if $h\colon \mathbb{B} \to \mathbb{A}$ is a homomorphism, then h extends to a homomorphism $\widehat{h}\colon \mathbb{A}^k \to \mathbb{A}$ by the polymorphism-homogeneity of $\mathbb{A}$. A restriction of $\widehat{h}$ then gives a homomorphism form $\mathbb{C}$ to $\mathbb{A}$ that extends h, thereby proving the injectivity of $\mathbb{A}$.

Conversely, if $\mathbb{A}$ is injective in $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$ and $h \in \mathcal{P}_A^{(k)}$ is a homomorphism from a subalgebra $\operatorname{dom} h \leq \mathbb{A}^k$ to $\mathbb{A}$, then the injectivity of $\mathbb{A}$ immediately yields an extension $\widehat{h}\colon \mathbb{A}^k \to \mathbb{A}$ of h, thus $\mathbb{A}$ is indeed polymorphism-homogeneous. ■

Corollary 6.2.6. [TW22] *If $\mathbb{A}$ is a finite algebra and $C = \mathrm{Clo}(\mathbb{A})$, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*

(iii) (A, C°) *is polymorphism-homogeneous;*

(iv) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*.*

Proof. Combine propositions 6.2.1 and 6.2.5 and Theorem 6.2.4. ■

It remains to verify the "one-way" implications in Figure 6.1. Since $\mathrm{HSP}(\mathbb{A}) \supseteq \mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$, it is trivial that if $\mathbb{A}$ is injective in $\mathrm{HSP}(\mathbb{A})$, then it is also injective in $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$. We end this section by proving the remaining implication; in fact, we formulate it in a bit more explicit form, which will be useful in the next section.

Proposition 6.2.7. [TW22] *If $\mathbb{A}$ is a finite algebra and $C = \mathrm{Clo}(\mathbb{A})$, then $(A, C^{\bullet})$ is polymorphism-homogeneous if and only if (A, C°) is polymorphism-homogeneous and $\langle C^{\bullet}\rangle_{\nexists} = \langle C^{\circ}\rangle_{\nexists}$.*

Proof. According to Proposition 6.1.2, we need to prove the following equivalence:

$$\langle C^{\bullet}\rangle_{\nexists} = \langle C^{\bullet}\rangle_{\exists} \iff \langle C^{\circ}\rangle_{\nexists} = \langle C^{\circ}\rangle_{\exists} \text{ and } \langle C^{\bullet}\rangle_{\nexists} = \langle C^{\circ}\rangle_{\nexists}.$$

This follows immediately from the following chain of trivial containments, and from the last equality, which is Lemma 2.2.2:

$$\langle C^{\bullet}\rangle_{\nexists} \subseteq \langle C^{\circ}\rangle_{\nexists} \subseteq \langle C^{\circ}\rangle_{\exists} = \langle C^{\bullet}\rangle_{\exists}. \qquad \blacksquare$$

6.3 Characterizations in special classes of algebras

We describe explicitly the finite algebras satisfying the properties considered in the previous section in certain well known varieties: semilattices, lattices, Abelian groups and monounary algebras, and we also take a look at three-element groupoids. These characterizations will provide counterexamples showing that the only valid implications among these properties are the ones shown in Figure 6.1.

6.3.1 Semilattices

If we consider finite semilattices, then it turns out that five of the six conditions of Figure 6.1 are equivalent, and these semilattices have already been determined in the literature.

Theorem 6.3.1. *If $\mathbb{A}$ is a finite semilattice and $C = \mathrm{Clo}(\mathbb{A})$, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*

(iii) (A, C°) *is polymorphism-homogeneous;*

(iv) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*

(v) $\mathbb{A}$ *is injective in* $\mathrm{HSP}(\mathbb{A})$*;*

(vi) $\mathbb{A}$ *is the semilattice reduct of a finite distributive lattice.*

Proof. We know that conditions (i)–(iv) are equivalent (see Corollary 6.2.6), and it was proved in Corollary 5.2.5 that (i) is equivalent to (vi). G. Bruns and H. Lakser [BL70] and, independently, A. Horn and N. Kimura [HK71] showed that the injective objects in the category of semilattices are the semilattice reducts of completely distributive lattices. Therefore, if $\mathbb{A}$ is the semilattice reduct of a finite distributive lattice, then $\mathbb{A}$ is injective in the variety of all semilattices, thus $\mathbb{A}$ is also injective in $\mathrm{HSP}(\mathbb{A})$. This proves that (vi) implies (v), and taking into account that (v) obviously implies (iv), the proof is complete. ■

The top left condition of Figure 6.1 is not equivalent to the others; in fact, there is no nontrivial finite semilattice for which $(A, C^\bullet)$ is polymorphism-homogeneous.

Lemma 6.3.2. [TW22] *Let* $\mathbb{A}$ *be a two-element semilattice and let* $C = \mathrm{Clo}(\mathbb{A})$*. Then the relational structure* $(A, C^\bullet)$ *is not polymorphism-homogeneous.*

Proof. We can assume without loss of generality that $\mathbb{A} = (\{0,1\}, \wedge)$ with the usual ordering $0 < 1$. Let us consider the equation $x \wedge y \wedge z = x \wedge y$. Obviously, the solution set $S = \{0,1\}^3 \setminus \{(1,1,0)\}$ of this equation is defined by a quantifier-free primitive positive formula over C°. The nontrivial 3-variable equalities that can appear in a quantifier-free primitive positive formula over $C^\bullet$ are the following:

$$\begin{array}{lllll} x = y, & x = x \wedge y, & x = x \wedge z, & x = y \wedge z, & x = x \wedge y \wedge z, \\ y = z, & y = x \wedge y, & y = x \wedge z, & y = y \wedge z, & y = x \wedge y \wedge z, \\ z = x, & z = x \wedge y, & z = x \wedge z, & z = y \wedge z, & z = x \wedge y \wedge z. \end{array}$$

It is easy to check that S does not satisfy any of the equalities above; therefore, S cannot be defined by a quantifier-free primitive positive formula over $C^\bullet$. Thus S belongs to $\langle C^\circ \rangle_{\nexists}$ but not to $\langle C^\bullet \rangle_{\nexists}$, hence $(A, C^\bullet)$ is not polymorphism-homogeneous by Proposition 6.2.7. ■

Theorem 6.3.3. [TW22] *If* $\mathbb{A}$ *is a nontrivial finite semilattice and* $C = \mathrm{Clo}(\mathbb{A})$*, then the relational structure* $(A, C^\bullet)$ *is not polymorphism-homogeneous.*

Proof. Let $a, b \in A$ such that $a < b$, and let us consider the same equation as in the proof of Lemma 6.3.2. Now for the solution set S of this equation we have that $S \cap \{a,b\}^3 = \{a,b\}^3 \setminus \{(b,b,a)\}$. The same argument as in the proof of Lemma 6.3.2 shows that S cannot be defined by a quantifier-free primitive positive formula over $C^\bullet$. ■

6.3.2 Lattices

For finite lattices the situation is very similar to the case of semilattices: five of the six conditions of Figure 6.1 are equivalent, and the sixth one is satisfied only by trivial lattices.

Theorem 6.3.4. *If* $\mathbb{A}$ *is a finite lattice and* $C = \mathrm{Clo}(\mathbb{A})$*, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*

(iii) (A, C°) *is polymorphism-homogeneous;*

(iv) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*

(v) $\mathbb{A}$ *is injective in* $\mathrm{HSP}(\mathbb{A})$*;*

(vi) $\mathbb{A}$ *is a finite Boolean lattice (i.e., a direct power of the two-element chain).*

Proof. Just as in the proof of Theorem 6.3.1, the equivalence of (i)–(iv) follows from Corollary 6.2.6, and the equivalence of (i) and (vi) is Corollary 5.1.8. (Let us mention that I. Dolinka and D. Mašulović [DM11] proved that a finite lattice is homomorphism-homogeneous if and only if it is a chain or a Boolean lattice. This together with Proposition 6.1.1 can also be used to prove that (ii) and (vi) are equivalent.) To complete the proof, it suffices to prove that (vi) implies (v). This follows immediately from a result of R. Balbes [Bal67]: the injective objects in the category of distributive lattices are the complete Boolean lattices (observe that if $\mathbb{A}$ is a nontrivial Boolean lattice, then $\mathrm{HSP}(\mathbb{A})$ is the variety of distributive lattices). ■

Lemma 6.3.5. [TW22] *Let* $\mathbb{A}$ *be a two-element lattice and let* $C = \mathrm{Clo}(\mathbb{A})$*. Then the relational structure* $(A, C^{\bullet})$ *is not polymorphism-homogeneous.*

Proof. We can assume without loss of generality that $\mathbb{A} = (\{0,1\}, \vee, \wedge)$ with the usual ordering $0 < 1$. Let us consider the equation $(x_1 \vee x_2) \wedge (x_3 \wedge x_4) = x_3 \wedge x_4$; the solution set $S = \{0,1\}^4 \setminus \{(0,0,1,1)\}$ of this equation is defined by a quantifier-free primitive positive formula over C°. If S can be defined by a quantifier-free primitive positive formula Φ over $C^{\bullet}$, then we can assume without loss of generality that Φ consists of a single equality, as S misses only one element of $\{0,1\}^4$ (in other words, S is meet-irreducible in the lattice of subsets of $\{0,1\}^4$). Thus S is the solution set of an equation of the form $f(x_1, x_2, x_3, x_4) = u$, where $u \in \{x_1, x_2, x_3, x_4\}$. Note that since f is generated by the lattice operations $\vee$ and $\wedge$, it is a monotone function. We consider four cases corresponding to the variable u.

1. If $u = x_1$, then $f(x_1, x_2, x_3, x_4) = x_1$ holds for all $(x_1, x_2, x_3, x_4) \in S$ and $f(0, 0, 1, 1) = 1$. In particular, we have $f(0, 1, 1, 1) = 0 < 1 = f(0, 0, 1, 1)$, contradicting the monotonicity of f.

2. If $u = x_2$, then $f(x_1, x_2, x_3, x_4) = x_2$ holds for all $(x_1, x_2, x_3, x_4) \in S$ and $f(0, 0, 1, 1) = 1$. In particular, we have $f(1, 0, 1, 1) = 0 < 1 = f(0, 0, 1, 1)$, contradicting the monotonicity of f.

3. If $u = x_3$, then $f(x_1, x_2, x_3, x_4) = x_3$ holds for all $(x_1, x_2, x_3, x_4) \in S$ and $f(0, 0, 1, 1) = 0$. In particular, we have $f(0, 0, 1, 0) = 1 > 0 = f(0, 0, 1, 1)$, contradicting the monotonicity of f.

4. If $u = x_4$, then $f(x_1, x_2, x_3, x_4) = x_4$ holds for all $(x_1, x_2, x_3, x_4) \in S$ and $f(0, 0, 1, 1) = 0$. In particular, we have $f(0, 0, 0, 1) = 1 > 0 = f(0, 0, 1, 1)$, contradicting the monotonicity of f.

We see that S cannot be defined by a quantifier-free primitive positive formula Φ over $C^\bullet$, hence $\langle C^\circ \rangle_{\nexists} \neq \langle C^\bullet \rangle_{\nexists}$, and thus $(A, C^\bullet)$ is not polymorphism-homogeneous by Proposition 6.2.7. ■

Theorem 6.3.6. [TW22] *If $\mathbb{A}$ is a nontrivial finite lattice and $C = \mathrm{Clo}(\mathbb{A})$, then the relational structure $(A, C^\bullet)$ is not polymorphism-homogeneous.*

Proof. Let $a, b \in A$ such that $a < b$, and let us consider the same equation as in the proof of Lemma 6.3.5. Now for the solution set S of this equation we have that $S \cap \{a, b\}^4 = \{a, b\}^4 \setminus \{(a, a, b, b)\}$. If S can be defined by a quantifier-free primitive positive formula Φ over $C^\bullet$, then at least one of the equalities in Φ defines the set $\{a, b\}^4 \setminus \{(a, a, b, b)\}$ when restricted to the sublattice $\{a, b\}$, and this leads to a contradiction using the same argument as in the proof of Lemma 6.3.5. ■

6.3.3 Abelian groups

For Abelian groups all six conditions of Figure 6.1 are equivalent, and these groups have already been determined, so we only need to combine some results from the literature to prove the following theorem.

Theorem 6.3.7. *If $\mathbb{A}$ is a finite Abelian group and $C = \mathrm{Clo}(\mathbb{A})$, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is homomorphism-homogeneous;*

(iii) $\mathbb{A}$ *is polymorphism-homogeneous;*

(iv) (A, C°) *is polymorphism-homogeneous;*

(v) $(A, C^\bullet)$ *is polymorphism-homogeneous;*

(vi) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*

(vii) $\mathbb{A}$ *is injective in* $\mathrm{HSP}(\mathbb{A})$*;*

(viii) *each Sylow-subgroup of* $\mathbb{A}$ *is homocyclic, i.e.,* $\mathbb{A} \cong \mathbb{Z}_{q_1}^{m_1} \times \cdots \times \mathbb{Z}_{q_k}^{m_k}$*, where* $q_1, \ldots, q_k$ *are powers of different primes and* $m_1, \ldots, m_k \in \mathbb{N}$*.*

Proof. Conditions (i), (iii), (iv) and (vi) are equivalent by Corollary 6.2.6. It is clear that (iv) is equivalent to (v), since we have $\langle C^\circ \rangle_{\nexists} = \langle C^\bullet \rangle_{\nexists}$ for groups: every equality can be written in an equivalent form where there is only a single variable on the right hand side. The equivalence of (ii) and (viii) follows from the description of quasi-injective Abelian groups presented as an exercise in [Fuc70] (for finite groups quasi-injectivity is equivalent to homomorphism-homogeneity). The class of groups given in (viii) is closed under taking finite direct powers, so we can conclude with the help of Proposition 6.1.1 that (iii) and (viii) are equivalent. It seems to be a folklore fact that the injective members of the variety of Abelian groups defined by the identity $nx = 0$ with $n = q_1 \cdot \ldots \cdot q_k$ are exactly the groups given by (viii) (see, e.g., [GL82]). Therefore, (viii) implies (vii), and this completes the proof, as (vii) trivially implies (vi). ■

6.3.4 Monounary algebras

A *monounary algebra* is an algebra $\mathbb{A} = (A, f)$ with a single unary operation $f \in \mathcal{O}_A^{(1)}$. An element $a \in A$ is *cyclic* if there is a positive integer k such that $f^k(a) = a$. (Here $f^k(a)$ stands for $f(\cdots f(a) \cdots)$ with a k-fold repetition of f, and we also use the convention $f^0(a) = a$.) If A is finite, then for every element $a \in A$, there is a least nonnegative integer $\mathrm{ht}(a)$, called the *height* of a, such that $f^{\mathrm{ht}(a)}(a)$ is cyclic. If $a \in A \setminus f(A)$, i.e., a has no preimage, then we say that a is a *source*. (Note that $\mathrm{ht}(a) = 0$ if and only if a is cyclic; in particular, $\mathrm{ht}(a) \geq 1$ for any source a.)

Polymorphism-homogeneous monounary algebras were characterized by Z. Farkasová and D. Jakubíková-Studenovská in [FJS15] using Proposition 6.1.1 and the description of homomorphism-homogeneous monounary algebras obtained by É. Jungábel and D. Mašulović [JM13]. As an illustration of the results of Section 6.2, we present a simple self-contained proof, which relies on the following technical lemma about quantifier elimination in monounary algebras.

Lemma 6.3.8. [TW22] *Let $\mathbb{A} = (A, f)$ be a finite monounary algebra, and let $C = \mathrm{Clo}(\mathbb{A})$. The algebra $\mathbb{A}$ is polymorphism-homogeneous if and only if for each positive integer k, there is a quantifier-free primitive positive formula $\Psi_k(x)$ over C° such that*

$$\forall a \in A\colon \Psi_k(a) \iff \exists a_0 \in A\colon a = f^k(a_0). \tag{6.3.1}$$

Proof. We use Theorem 6.2.4: we prove that the existence of Ψ_k is necessary and sufficient for polymorphism-homogeneity of (A, C°). By Proposition 6.1.2, the necessity is obvious; to prove sufficiency, let us consider an arbitrary primitive positive formula $\Phi(x_1, \dots, x_n)$ over C°. We show how to eliminate one quantifier; repeatedly applying this procedure we can eliminate all quantifiers from Φ. So we may assume without loss of generality that Φ involves only one quantifier, hence it has the following form:

$$\Phi(x_1, \dots, x_n) \equiv \exists y \mathop{\&}_{i=1}^{t} \Big(f^{r_i}(u_i) = f^{s_i}(v_i)\Big),$$

where t, r_i, s_i are nonnegative integers, and the variables u_i, v_i belong to the set $\{x_1, \dots, x_n, y\}$ for $i = 1, \dots, t$. We define the weight of Φ as

$$w(\Phi) = \sum_{\substack{i=1,\dots,t\\ u_i=y}} (r_i + 1) + \sum_{\substack{i=1,\dots,t\\ v_i=y}} (s_i + 1).$$

Informally speaking, $w(\Phi)$ shows how "deeply" y is involved in Φ.

If y occurs in at least two equalities in Φ, then we can use (at least) one of the following four types of substitutions to decrease the weight of the formula (we omit trivial equalities):

$$f^k(y) = f^\ell(x_i) \,\&\, f^m(y) = f^n(x_j) \quad\rightsquigarrow\quad f^{k-m+n}(x_j) = f^\ell(x_i) \,\&\, f^m(y) = f^n(x_j),$$
$$\text{if } k \geq m;$$
$$f^k(y) = f^\ell(x_i) \,\&\, f^m(y) = f^n(y) \quad\rightsquigarrow\quad f^{k-m+n}(y) = f^\ell(x_i) \,\&\, f^m(y) = f^n(y),$$
$$\text{if } k \geq m,\ m > n;$$
$$f^k(y) = f^\ell(x_i) \,\&\, f^m(y) = f^n(y) \quad\rightsquigarrow\quad f^{m-k+\ell}(x_i) = f^n(y) \,\&\, f^k(y) = f^\ell(x_i),$$
$$\text{if } k < m,\ m > n;$$
$$f^k(y) = f^\ell(y) \,\&\, f^m(y) = f^n(y) \quad\rightsquigarrow\quad f^{k-m+n}(y) = f^\ell(y) \,\&\, f^m(y) = f^n(y),$$
$$\text{if } k \geq m,\ k > \ell,\ m > n.$$

After finitely many steps we arrive at a formula Φ' such that Φ' is equivalent to Φ, and it is not possible to decrease the weight of Φ' any more using the substitutions above. This implies that the variable y appears in at most one equality in Φ'. We have one of the following three cases for Φ'.

1. If y does not appear at all, then we can simply drop the quantifier $\exists y$ from Φ'.

2. If y appears in an equality of the form $f^k(y) = f^\ell(y)$, then there is no "interaction" between y and the other variables. If there is an element $a \in A$ such that $f^k(a) = f^\ell(a)$, then we can again omit the quantifier $\exists y$ and the equality $f^k(y) = f^\ell(y)$ from Φ', and the resulting quantifier-free formula is equivalent to Φ' (hence also equivalent to Φ). If there is no element $a \in A$ such that $f^k(a) = f^\ell(a)$, then $\Phi'(x_1, \ldots, x_n)$ is never satisfied: it defines the empty n-ary relation. In this case the empty relation can be defined by the quantifier-free formula $f^k(x_1) = f^\ell(x_1)$, thus this formula is equivalent to Φ.

3. If y appears in an equality of the form $f^k(y) = f^\ell(x_i)$, then let $\Phi''(x_1, \ldots, x_n)$ be the formula that is obtained from Φ' by deleting the quantifier $\exists y$ and the equality $f^k(y) = f^\ell(x_i)$. Then Φ is equivalent to the quantifier-free formula $\Phi''(x_1, \ldots, x_n) \mathbin{\&} \Psi_k(f^\ell(x_i))$, according to (6.3.1). ■

Theorem 6.3.9. [FJS15],[TW22] *If* $\mathbb{A} = (A, f)$ *is a finite monounary algebra and* $C = \mathrm{Clo}(\mathbb{A})$*, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*

(iii) (A, C°) *is polymorphism-homogeneous;*

(iv) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*

(v) *Either* $\mathbb{A}$ *has no sources, or all sources of* $\mathbb{A}$ *have the same height:* $\forall a, b \in A \setminus f(A)\colon\ \mathrm{ht}(a) = \mathrm{ht}(b)$.

Proof. Conditions (i)–(iv) are equivalent by Corollary 6.2.6, so it suffices to prove the equivalence of (ii) and (v). As a preliminary observation, let us note that an element $a \in A$ is cyclic if and only if $a = f^\ell(a)$, where ℓ is the least common multiple of the lengths of the cycles of $\mathbb{A}$.

Suppose that (v) holds, and assume first that there are no sources in $\mathbb{A}$. Then every element is cyclic, thus the formula $x = x$ can be chosen for $\Psi_k(x)$ in (6.3.1) for all $k \in \mathbb{N}$. Suppose now that all sources in $\mathbb{A}$ have the same height $n \geq 1$. If $k \geq n$, then an arbitrary element $a \in A$ can be written as $a = f^k(a_0)$ for a suitable $a_0 \in A$ if and only if a is cyclic. Thus, the formula $x = f^\ell(x)$ can be chosen for $\Psi_k(x)$, whenever $k \geq n$. Similarly, if $k < n$, then $\exists a_0 \in A\colon a = f^k(a_0)$ holds if and only if $\mathrm{ht}(a) \leq n - k$, i.e., if $f^{n-k}(a)$ is cyclic. Therefore, we can put $f^{n-k}(x) = f^{n-k+\ell}(x)$ for $\Psi_k(x)$ in this case. This proves that (v) implies (ii), according to Lemma 6.3.8.

Conversely, assume that there exist formulas $\Psi_k(x)$ satisfying (6.3.1). We can write $\Psi_1(x)$ in the following form, and we can assume without loss of generality that $r_i < s_i$ for $i = 1, \ldots, t$:

$$\Psi_1(x) \equiv \mathop{\&}_{i=1}^{t} \Big(f^{r_i}(x) = f^{s_i}(x) \Big).$$

If a is a cyclic element, then $\Psi_1(a)$ must hold according to (6.3.1). This implies that $r_i \equiv s_i \pmod{\ell}$ for all $i \in \{1, \ldots, t\}$.

Now suppose for contradiction that there exist sources $a, b \in A \setminus f(A)$ with $\mathrm{ht}(a) < \mathrm{ht}(b)$. Clearly, $\Psi_1(f(b))$ must be true by (6.3.1), thus we have $f^{r_i}(f(b)) = f^{s_i}(f(b))$ for each i. This is equivalent to $f^{r_i+1}(b) = f^{s_i+1}(b)$, which implies that $f^{r_i+1}(b)$ is a cyclic element (recall that $r_i < s_i$), hence we have $\mathrm{ht}(b) \leq r_i + 1$. Since $\mathrm{ht}(a) \leq \mathrm{ht}(b) - 1$, we can conclude that $\mathrm{ht}(a) \leq r_i$, i.e., $f^{r_i}(a)$ is a cyclic element. The length of every cycle is a divisor of ℓ, and we know that $r_i \equiv s_i \pmod{\ell}$, thus $f^{r_i}(a) = f^{s_i}(a)$. This means that $\Psi_1(a)$ is true, contradicting the fact that a is a source. ■

Next we determine finite monounary algebras corresponding to the top left box of Figure 6.1.

Theorem 6.3.10. [TW22] *Let $\mathbb{A} = (A, f)$ be a finite monounary algebra, and let $C = \mathrm{Clo}(\mathbb{A})$. Then the relational structure $(A, C^{\bullet})$ is polymorphism-homogeneous if and only if f is either bijective or constant.*

Proof. If f is constant, then it is clear that $(A, C^{\bullet})$ is polymorphism-homogeneous. Assume now that f is bijective. Then the condition of Theorem 6.3.9 is satisfied (there are no sources at all), so $\mathbb{A}$ is polymorphism-homogeneous, and thus by Theorem 6.2.4 (A, C°) is polymorphism-homogeneous as well. Therefore, by Proposition 6.2.7, it suffices to show that $\langle C^{\circ} \rangle_{\nexists} = \langle C^{\bullet} \rangle_{\nexists}$. This is clear, as any equality of the form $f^k(x) = f^{\ell}(y)$ with $k < \ell$ is equivalent to $x = f^{\ell-k}(y)$, since f is bijective (here x and y might be the same variable).

For the other direction, let us suppose that $(A, C^{\bullet})$ is polymorphism-homogeneous. By Proposition 6.2.7, (A, C°) is also polymorphism-homogeneous, and then Theorem 6.3.9 (together with Theorem 6.2.4) implies that either there are no sources, or there is an integer $n \geq 1$ such that every source in $\mathbb{A}$ has height n. If there are no sources in $\mathbb{A}$, then every element is cyclic, and therefore f is bijective. From now on let us suppose that $\mathbb{A}$ has sources with a common height n. Proposition 6.2.7 shows that there exists a quantifier-free primitive positive formula $\Phi(x, y)$ over $C^{\bullet}$ such that $\Phi(x, y)$ is equivalent to $f(x) = f(y)$. We can write $\Phi(x, y)$ in the following form:

$$\Phi(x, y) \equiv \mathop{\&}_{i=1}^{t} \Big(f^{r_i}(u_i) = v_i \Big),$$

where t, r_i are nonnegative integers, and $u_i, v_i \in \{x, y\}$ for $i = 1, \ldots, t$. Obviously, $\Phi(a, a)$ holds for every element $a \in A$. Let us choose a to be of height n, i.e., let a be a source. Then $f^{r_i}(u_i) = v_i$ holds for $u_i = v_i = a$ if and only if $r_i = 0$, thus $\Phi(x, y)$ is equivalent either to $x = y$ or to $x = x$. Taking into account that $\Phi(x, y)$ is also equivalent to $f(x) = f(y)$, we can conclude that $f(x) = f(y) \iff x = y$ or $f(x) = f(y) \iff x = x$. In the first case f is a bijection, and in the second case f is constant. ∎

Injective objects in the category of all monounary algebras were determined by D. Jakubíková-Studenovská [JS98]; in the finite case these are exactly the monounary algebras $\mathbb{A} = (A, f)$ where f is bijective and has a fixed point. However, in order to complete the picture of Figure 6.1 for monounary algebras, we need to describe those monounary algebras $\mathbb{A}$ that are injective in the variety $\mathrm{HSP}\,\mathbb{A}$. This has been done by D. Jakubíková-Studenovská and G. Czédli, but this result appeared only in Hungarian in the masters thesis [Jeg00] of T. Jeges, a student of G. Czédli.

Theorem 6.3.11. [Jeg00] *A finite monounary algebra $\mathbb{A} = (A, f)$ is injective in the variety $\mathrm{HSP}(\mathbb{A})$ if and only if all of its sources have the same height and it has a one-element subalgebra (i.e., f has a fixed point).*

Let us note that comparing theorems 6.3.9, 6.3.10 and 6.3.11, one can construct examples illustrating each one of the "non-implications" of Figure 6.1.

6.3.5 Three-element groupoids

There are 19683 groupoids on the three-element set, but their number up to isomorphism is only 3330. In the paper [BB96] Joel Berman and Stanley Burris investigated 12 properties of three-element groupoids with the help of computers. They further reduced the number of groupoids that needed to be investigated: they studied the properties up to *clone equivalence* (also called term equivalence), which is an equivalence relation on the groupoids induced by the preorder given by

$$\mathbb{A} \leq \mathbb{B} \iff \mathrm{Clo}(\mathbb{A}) \subseteq \mathrm{Clo}(\mathbb{B}'), \text{ where } \mathbb{B}' \text{ is an isomorphic copy of } \mathbb{B}.$$

This equivalence relation has only 411 equivalence classes. Since polymorphism-homogeneity of an algebra $\mathbb{A} = (A, F)$ only depends on the clone $\mathrm{Clo}(\mathbb{A})$ generated by F, we only need to investigate representatives of these 411 cases. We use the same notation for the groupoids as in [BB96].

First we investigate polymorphism-homogeneity of special three-element groupoids. We consider Abelian group(s), affine algebras, semilattices and monounary algebras regarded as groupoids. We also formulate a conjecture about groupoids generating a congruence distributive variety.

Abelian group(s)

The only Abelian group on the three-element set is $\mathbb{Z}_3$ (up to isomorphism). The corresponding groupoid is groupoid (2124) in [BB96], which is polymorphism-homogeneous by Theorem 6.3.7.

Affine algebras

Affine algebras are special kinds of Mal'tsev algebras that play an important role in universal algebra. One way of defining them is the following. An n-ary operation $f \in \mathcal{O}_A$ is called *affine with respect to* an Abelian group $\mathbb{G} = (A; +, -, 0)$, if f commutes with the ternary operation $x - y + z$. An algebra is *affine* if and only if there exists an Abelian group $\mathbb{G} = (A; +, -, 0)$ such that $x - y + z$ is a term operation of the algebra and every basic operation of the algebra is affine with respect to $\mathbb{G}$. In this subsection we investigate affine algebras $\mathbb{A}$ such that the elements of $\mathrm{Clo}(\mathbb{A})$ are polynomials of a finite vector space. For these affine algebras we can write $\mathrm{Clo}(\mathbb{A})$ in the form of one of the following clones (see [Sze86]):

$$X(\mathbb{V}, S) = \Big\{\sum_{i=1}^{n} a_i\mathbf{x}_i + \mathbf{c} \;\Big|\; a_1, \dots, a_n \in \mathbb{K},\ \mathbf{c} = \mathbf{s} - (\sum_{i=1}^{n} a_i)\mathbf{s}' \text{ for some } \mathbf{s}, \mathbf{s}' \in S\Big\};$$

$$Y(\mathbb{V}, W) = \Big\{\sum_{i=1}^{n} a_i\mathbf{x}_i + \mathbf{c} \;\Big|\; a_1, \dots, a_n \in \mathbb{K}, \sum_{i=1}^{n} a_i = 1, \mathbf{c} \in W\Big\};$$

where $\mathbb{K}$ is a finite field, $\mathbb{V}$ is a finite dimensional vector space over $\mathbb{K}$, and W is a subspace and S is an affine subspace (i.e., a coset with respect to a subspace) of $\mathbb{V}$.

In the next theorem we show that for any finite vector space $\mathbb{V}$ and any subspace W of $\mathbb{V}$, the clone $Y(\mathbb{V}, W)$ has property (SDC), that is, all of the corresponding finite affine algebras are polymorphism-homogeneous.

Theorem 6.3.12. *Let $\mathbb{V}$ be a finite vector space and let W be a subspace of $\mathbb{V}$. The clone $Y(\mathbb{V}, W)$ has property* (SDC).

Proof. Let $C = Y(\mathbb{V}, W)$, where $\mathbb{V}$ is a finite dimensional vector space over the finite field $\mathbb{K} = (K; +, \cdot)$ and let W be a subspace of $\mathbb{V}$. By Theorem 2.2.6, C has property (SDC) if and only if every primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula. Let us consider a primitive positive formula over C° with only one quantifier. We prove that it can be eliminated, and iterating this argument completes the proof. Since C° consists of relations that are defined by single equations, a primitive positive formula over C° with one quantifier is of the form

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{u}) \equiv \exists \mathbf{u}$$
$$f_1(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) = g_1(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) \& \cdots \& f_t(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) = g_t(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}),$$

where $t \in \mathbb{N}$ and $f_i, g_i \in C$ for all $i \in \{1, \ldots, t\}$. We reformulate every equation $f_i(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{u}) = g_i(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{u})$ containing $\mathbf{u}$ to an equation in the form $h_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{u}$.

Let us consider an equation over C. This equation can be written in the form

$$\sum_{i=1}^{n+1} a_i \mathbf{x}_i + \mathbf{c} = \sum_{i=1}^{n+1} b_i \mathbf{x}_i + \mathbf{d},$$

where $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 1$ with $\mathbf{u} = \mathbf{x}_{n+1}$ and $\mathbf{c}, \mathbf{d} \in W$. We of course permit $a_i = 0$ or $b_i = 0$ (for any $i \in \{0, 1, \ldots, n+1\}$). We can write our equation in the form

$$\sum_{i=1}^{n} (a_i - b_i)\mathbf{x}_i + (\mathbf{c} - \mathbf{d}) = (b_{n+1} - a_{n+1})\mathbf{x}_{n+1}. \tag{6.3.2}$$

Since $\mathbb{K}$ is a field, the element $(b_{n+1} - a_{n+1})$ is either 0, or has a multiplicative inverse. In the first case the following equation (that omits $\mathbf{u}$)

$$\sum_{i=1}^{n} a_i \mathbf{x}_i + a_{n+1}\mathbf{x}_1 + \mathbf{c} = \sum_{i=1}^{n} b_i \mathbf{x}_i + b_{n+1}\mathbf{x}_1 + \mathbf{d},$$

is an equation over C, and it is equivalent to our original equation. In the second case the following equation is equivalent to (6.3.2)

$$(b_{n+1} - a_{n+1})^{-1}\Big(\sum_{i=1}^{n} (a_i - b_i)\mathbf{x}_i + (\mathbf{c} - \mathbf{d})\Big) = \mathbf{x}_{n+1} = \mathbf{u}.$$

If we reformulate the equation above, then we get

$$\sum_{i=1}^{n} \Big((b_{n+1} - a_{n+1})^{-1}(a_i - b_i)\mathbf{x}_i\Big) + (b_{n+1} - a_{n+1})^{-1}(\mathbf{c} - \mathbf{d}) = \sum_{i=1}^{n} 0 \cdot \mathbf{x}_i + 1 \cdot \mathbf{x}_{n+1} + \mathbf{0}. \tag{6.3.3}$$

Therefore, to prove that (6.3.3) is also an equation over C we only need to prove that $\sum_{i=1}^{n} (b_{n+1} - a_{n+1})^{-1}(a_i - b_i) = 1$ and that $(b_{n+1} - a_{n+1})^{-1}(\mathbf{c} - \mathbf{d})$ belongs to W. Since

$$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 1, \text{ we have}$$

$$a_{n+1} = 1 - \sum_{i=1}^{n} a_i \quad \text{and} \quad b_{n+1} = 1 - \sum_{i=1}^{n} b_i.$$

But then $b_{n+1} - a_{n+1} = \sum_{i=1}^{n} (a_i - b_i)$, hence the sum of the coefficients in the left-hand side of (6.3.3) is

$$(b_{n+1} - a_{n+1})^{-1} \sum_{i=1}^{n} (a_i - b_i) = (b_{n+1} - a_{n+1})^{-1}(b_{n+1} - a_{n+1}) = 1.$$

For proving that $(b_{n+1} - a_{n+1})^{-1}(\mathbf{c} - \mathbf{d})$ belongs to W, observe that since W is a subspace of $\mathbb{K}$ and $\mathbf{c}, \mathbf{d} \in W$, we have $(\mathbf{c} - \mathbf{d}) \in W$, and also $(b_{n+1} - a_{n+1})^{-1}(\mathbf{c} - \mathbf{d}) \in W$.

Let us suppose then that every equation containing $\mathbf{u}$ is of the form (6.3.3). There are two cases. Suppose first that there is only one equation that contains $\mathbf{u}$. Then this equation can be replaced with $\mathbf{x}_1 = \mathbf{x}_1$: for any n-tuple $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ it holds that there exists $\mathbf{u} \in \mathbb{V}$ such that (6.3.3) holds (defined by the left-hand side of (6.3.3)).

If there are more that one equations containing $\mathbf{u}$, then we replace $\mathbf{u}$ everywhere with the other side of the first equation containing $\mathbf{u}$ (that is, if the first equation containing $\mathbf{u}$ is of the form $h(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{u}$, then we replace $\mathbf{u}$ in every equation with $h(\mathbf{x}_1, \dots, \mathbf{x}_n)$). We can also delete the equation $h(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{u}$. And then we get a system of equations equivalent to the one defining Φ that omits $\mathbf{u}$. ■

Remark 6.3.13. Note that on the two-element set $L_{01} = Y(\mathbb{Z}_2, \{0\})$ and $SL = Y(\mathbb{Z}_2, \mathbb{Z}_2)$, thus Theorem 6.3.12 is essentially a generalization of lemmas 3.1.2 and 3.1.4.

The affine three-element groupoids (up to clone equivalence) are groupoids (2124), (2346) and (2934) (see [BB96]). We can use Theorem 6.3.12 to investigate these groupoids. Groupoid (2124) has already been considered, since it is (essentially) the Abelian group $\mathbb{Z}_3$. The other two groupoids are also polymorphism-homogeneous by the following corollary.

Corollary 6.3.14. *The three-element affine groupoids given by*

(2346)	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

(2934)	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

are polymorphism-homogeneous.

Proof. Notice that the basic operation of groupoid (2346) is $x * y = -x - y$, and that the basic operation of groupoid (2346) is $x \odot y = -x - y + 1$. It is easy to see that the clone generated by $*$ is

$$Y(\mathbb{Z}_3, \{0\}) = \Big\{ \sum_{i=1}^{n} a_i x_i \mid a_1, \dots, a_n \in \mathbb{Z}_3, \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N} \Big\},$$

and the clone generated by $\odot$ is

$$Y(\mathbb{Z}_3, \mathbb{Z}_3) = \Big\{ \sum_{i=1}^{n} a_i x_i + c \mid a_1, \dots, a_n, c \in \mathbb{Z}_3, \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N} \Big\}.$$

For both cases Theorem 6.3.12 can be applied. ■

Now we investigate the clones appearing in the form $X(\mathbb{V}, S)$. In the next theorem we show that for any finite field $\mathbb{V}$ and the coset S of any subspace of $\mathbb{V}$, the clone $X(\mathbb{V}, S)$ has property (SDC), that is, all of the corresponding finite affine algebras are polymorphism-homogeneous.

Theorem 6.3.15. *Let $\mathbb{V}$ be a finite vector space and let S be an affine subspace of $\mathbb{V}$. The clone $X(\mathbb{V}, S)$ has property* (SDC).

Proof. We will use a similar approach as in the proof of Theorem 6.3.12. Let $C = X(\mathbb{V}, S)$, where $\mathbb{V}$ is a vector space over the field $\mathbb{K} = (K; +, \cdot)$ and $S = \mathbf{z} + W$ for some subspace W in $\mathbb{V}$ and element $\mathbf{z} \in \mathbb{V}$. Let us consider a primitive positive formula over C° (with only one quantifier) of the form

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \mathbf{u}) \equiv \exists \mathbf{u}$$
$$f_1(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) = g_1(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) \& \cdots \& f_t(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) = g_t(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}),$$

where $t \in \mathbb{N}$ and $f_i, g_i \in C$ for all $i \in \{1, \dots, t\}$. We reformulate every equation $f_i(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u}) = g_i(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{u})$ containing $\mathbf{u}$ to an equation in the form $h_i(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{u}$.

Let us consider an equation over C. This equation can be written in the form

$$\sum_{i=1}^{n+1} a_i \mathbf{x}_i + \mathbf{c} = \sum_{i=1}^{n+1} b_i \mathbf{x}_i + \mathbf{d},$$

where $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{K}$, $\mathbf{u} = \mathbf{x}_{n+1}$, $\mathbf{c} = \mathbf{s}_1 - (\sum_{i=1}^{n+1} a_i)\mathbf{s}'_1$ and $\mathbf{d} = \mathbf{s}_2 - (\sum_{i=1}^{n+1} b_i)\mathbf{s}'_2$ for some $\mathbf{s}_1, \mathbf{s}'_1, \mathbf{s}_2, \mathbf{s}'_2 \in S$. We of course permit $a_i = 0$ or $b_i = 0$ (for any $i \in \{0, 1, \dots, n+1\}$). Then similarly to the proof of Theorem 6.3.12 if $b_{n+1} - a_{n+1} = 0$ we can easily reformulate our equation such that it omits $\mathbf{u}$, and if $b_{n+1} - a_{n+1} \neq 0$ we can reformulate the equation above as

$$\sum_{i=1}^{n} \Big((b_{n+1} - a_{n+1})^{-1} (a_i - b_i) \mathbf{x}_i \Big) + (b_{n+1} - a_{n+1})^{-1} (\mathbf{c} - \mathbf{d}) = \sum_{i=1}^{n} 0 \cdot \mathbf{x}_i + 1 \cdot \mathbf{x}_{n+1} + \mathbf{0}. \tag{6.3.4}$$

We show that equation (6.3.4) is an equation over C. The right-hand side is an operation belonging to C, since

$$\mathbf{u} = \mathbf{x}_{n+1} = \sum_{i=1}^{n} 0 \cdot \mathbf{x}_i + 1 \cdot \mathbf{x}_{n+1} + 0 = \sum_{i=1}^{n+1} q_i \cdot \mathbf{x}_i + \Big(\mathbf{s}_3 - (\sum_{i=1}^{n+1} q_i \cdot) \mathbf{s}_3 \Big),$$

where $q_1 = \cdots = q_n = 0$, $q_{n+1} = 1$ and $\mathbf{s}_3 \in S$ is arbitrary.

To prove that the left-hand side also belongs to C we need to show that $(b_{n+1} - a_{n+1})^{-1}(\mathbf{c} - \mathbf{d}) = \mathbf{s} - \Big(\sum_{i=1}^{n} (b_{n+1} - a_{n+1})^{-1} (a_i - b_i) \Big) \cdot \mathbf{s}'$ for some elements $\mathbf{s}, \mathbf{s}' \in S$. Now we have

$$\mathbf{c} - \mathbf{d} = \Big(\mathbf{s}_1 - (\sum_{i=1}^{n+1} a_i) \mathbf{s}'_1 \Big) - \Big(\mathbf{s}_2 - (\sum_{i=1}^{n+1} b_i) \mathbf{s}'_2 \Big) = (\mathbf{s}_1 - \mathbf{s}_2) - \sum_{i=1}^{n+1} (a_i - b_i) \cdot \mathbf{s}'_1 + \sum_{i=1}^{n+1} b_i (\mathbf{s}'_2 - \mathbf{s}'_1),$$

and thus

$$(b_{n+1}-a_{n+1})^{-1}(\mathbf{c}-\mathbf{d}) =$$

$$(b_{n+1}-a_{n+1})^{-1}(\mathbf{s}_1-\mathbf{s}_2)-\sum_{i=1}^{n+1}(b_{n+1}-a_{n+1})^{-1}(a_i-b_i)\cdot\mathbf{s}'_1+\sum_{i=1}^{n+1}(b_{n+1}-a_{n+1})^{-1}b_i(\mathbf{s}'_2-\mathbf{s}'_1). \tag{6.3.5}$$

Let us take a look at the subterm in the middle of (6.3.5):

$$-\sum_{i=1}^{n+1}(b_{n+1}-a_{n+1})^{-1}(a_i-b_i)\cdot\mathbf{s}'_1 =$$

$$(b_{n+1}-a_{n+1})^{-1}(b_{n+1}-a_{n+1})\cdot\mathbf{s}'_1-\sum_{i=1}^{n}(b_{n+1}-a_{n+1})^{-1}(a_i-b_i)\cdot\mathbf{s}'_1$$

$$= \mathbf{s}'_1-\sum_{i=1}^{n}(b_{n+1}-a_{n+1})^{-1}(a_i-b_i)\cdot\mathbf{s}'_1.$$

To complete proving that $\mathbf{c}-\mathbf{d}$ is in the desired form note that since $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}'_1, \mathbf{s}'_2 \in S = \mathbf{z}+W$, we have $\mathbf{s}_1-\mathbf{s}_2, \mathbf{s}'_2-\mathbf{s}'_1 \in W$, and thus $(b_{n+1}-a_{n+1})^{-1}(\mathbf{s}_1-\mathbf{s}_2) \in W$ and $\sum_{i=1}^{n+1}(b_{n+1}-a_{n+1})^{-1}b_i(\mathbf{s}'_2-\mathbf{s}'_1) \in W$. (So the first and third subterms of (6.3.5) belong to W.) Therefore, we can choose $\mathbf{s}$ as the sum of these two subterms and $\mathbf{s}'_1$, that is,

$$\mathbf{s} = \mathbf{s}'_1 + (b_{n+1}-a_{n+1})^{-1}(\mathbf{s}_1-\mathbf{s}_2)+\sum_{i=1}^{n+1}(b_{n+1}-a_{n+1})^{-1}b_i(\mathbf{s}'_2-\mathbf{s}'_1),$$

and $\mathbf{s}'$ as $\mathbf{s}'_1$. Then we indeed get that $\mathbf{s}, \mathbf{s}' \in S$ and $(b_{n+1}-a_{n+1})^{-1}(\mathbf{c}-\mathbf{d}) = \mathbf{s} - \left(\sum_{i=1}^{n}(b_{n+1}-a_{n+1})^{-1}(a_i-b_i)\right)\cdot\mathbf{s}'$.

So we proved that (6.3.4) is an equation over C. Then the same argument as in the end of the proof of Theorem 6.3.12 can be used. ∎

Remark 6.3.16. Note that on the two-element set $L_0 = X(\mathbb{Z}_2, \{0\})$ and $L = X(\mathbb{Z}_2, \mathbb{Z}_2)$, thus Theorem 6.3.15 is essentially a generalization of lemmas 3.1.1 and 3.1.3.

Semilattices

We investigate three-element semilattices with the help of Theorem 6.3.1. There are only two (meet) semilattices on the three-element set, namely the three-element chain and the "V-shaped" semilattice (see Figure 6.2). The groupoid that is essentially the second semilattice (groupoid (80)) is not polymorphism-homogeneous by Theorem 6.3.1, since it is not a semilattice reduct of a distributive lattice. The three-element chain on the other hand is polymorphism-homogeneous by Theorem 6.3.1. The corresponding groupoid is groupoid (105).

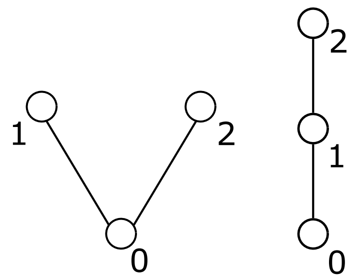


Figure 6.2: A schematic view of groupoids (80) and (105).

Monounary algebras

We consider all the three-element groupoids (up to clone equivalence) that have an essentially unary basic operation. These groupoids are groupoids (1), (14), (27), (275), (366), (2466) and (3242).

Lemma 6.3.17. *The following groupoids are polymorphism-homogeneous.*

(1)	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

(14)	0	1	2
0	0	0	0
1	0	0	0
2	1	1	1

(27)	0	1	2
0	0	0	0
1	0	0	0
2	2	2	2

(275)	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

(366)	0	1	2
0	0	0	0
1	2	2	2
2	1	1	1

(2466)	0	1	2
0	1	0	0
1	1	0	0
2	1	0	0

(3242)	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

Proof. By Theorem 6.3.9 we only need to check the height of the sources of each groupoid (regarded as a monounary algebra). Note that on the three-element set it is impossible to have two elements with different heights, but it is also easy to see that this holds for the groupoids above. Groupoid (1) has two sources, 1 and 2, and their heights are $\mathrm{ht}(1) = \mathrm{ht}(2) = 1$. The other groupoids either have only one source, or have no source at all. ∎

Groupoids generating a congruence distributive variety

If a groupoid generates a congruence distributive variety, then it will be called a CD-groupoid. We have 57 CD-groupoids out of the 411 groupoids (up to clone equivalence). By Corollary 4.3.6, we can say the following.

Corollary 6.3.18. *Let $\mathbb{G}$ be a three-element groupoid that generates a congruence distributive variety. If $\mathbb{G}$ has an essentially k-ary partial polymorphism for some $k \geq 2$, then $\mathbb{G}$ is not polymorphism-homogeneous.*

Proof. By Corollary 4.3.6, any total polymorphism of $\mathbb{G}$ is at most unary. Therefore, if $\mathbb{G}$ has an essentially k-ary partial polymorphism for some $k \geq 2$, then this partial polymorphism is obviously non-extendable. ■

Groupoid (2407) is a special groupoid among the CD-groupoids: it is primal, meaning that the clone generated by its basic operation contains every operation on the three-element set. It is not hard to see that this groupoid is polymorphism-homogeneous; in fact, every finite primal algebra is polymorphism-homogeneous.

Lemma 6.3.19. *Every finite primal algebra is polymorphism-homogeneous.*

Proof. The lemma is trivial for one-element algebras. Let $\mathbb{A} = (A; F)$ be a finite primal algebra with $A = \{0, \ldots, n\}$ $(n \geq 1)$, and let $C = \mathrm{Clo}(\mathbb{A})$. By Theorem 6.2.6 the algebra is polymorphism-homogeneous if and only if it has property (SDC); we will prove the latter one. Since $\mathbb{A}$ is primal, we have that the centralizer of $\mathbb{A}$ only contains the projections, and every set of n-tuples is closed under the projections (for any $n \in \mathbb{N}$). Therefore, we need to prove that any set of n-tuples is a solution set of some system of equations over C. We will use an argument similar to the one in the proof of Lemma 3.2.1.

Let $T \subseteq A^n$. For any $\mathbf{v} = (a_1, \ldots, a_n) \in A^n$, we give an equation over C such that the solution set of this equation is $T_{\mathbf{v}} = A^n \setminus \{\mathbf{v}\}$. Then the solution set of the system of these equations is exactly $T = \bigcap_{\mathbf{v} \notin T} T_{\mathbf{v}}$. Let $\mathbf{v} = (a_1, \ldots, a_n) \in A^n \setminus T$. Let $f \in \mathcal{O}_A^{(n)}$ be the constant 0 operation and let us define the n-ary operation g at $\mathbf{v}$ as 1, and as 0 everywhere else. Since $\mathbb{A}$ is primal, these operations belong to C. Then the solution set of the equation $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ is exactly $T_{\mathbf{v}}$. ■

Corollary 6.3.20. *The three-element CD-groupoid given by*

(2407)	0	1	2
0	1	0	0
1	0	2	0
2	0	0	0

is polymorphism-homogeneous.

For the other groupoids that were not mentioned so far, a computer program was used to find non-extendable partial polymorphisms. Using this program, we see that 3 of the CD-groupoids, namely groupoids (219), (222) and (239) are not polymorphism-homogeneous. But the other 54 CD-groupoids have neither a unary, nor a binary non-extendable partial polymorphism. After some discussion with Mike Behrisch, he offered to help me in my investigations; he used a SAT-solver to see if the CD-groupoids (all 57) have any essentially ternary partial polymorphisms. His finding was that none of the 57 groupoids has an essentially ternary local polymorphism. Therefore, the following conjecture is given.

Conjecture 6.3.21. The three-element CD-groupoids except for (219), (222) and (239) are polymorphism-homogeneous.

The general case

For every three-element groupoid a computer program was used to find non-extendable partial polymorphisms. The program gave us counterexamples for 291 of the 411 equivalence classes investigated (including the “V-shaped” semilattice and the three CD-groupoids mentioned earlier). These results were proven by hand as well, but the proofs are long and tedious, hence we omit them. We do present however the partial polymorphisms proving that these groupoids are not polymorphism-homogeneous, in Appendix 1.4. The operation tables for these groupoids are shown in Appendix 1.5.

Overall, our investigations show that groupoids belonging to 291 of the equivalence classes are not polymorphism-homogeneous, and in this chapter we showed that for other 12 equivalence classes we have polymorphism-homogeneity. For the remaining 108 equivalence classes we can not say more for now.

Summary

1 Introduction

We study solution sets of systems of equations over arbitrary finite algebras. The essence of our investigation is characterizing the solution sets with a certain type of closure condition. Following the example of systems of linear equations, for any algebra, we can find a set of operations such that the solution sets are always closed under these operations. If this closure is sufficient as well (that is, every closed set is also a solution set of some system of equations), then we will say that the investigated algebra has property (SDC). This thesis studies algebras with property (SDC) and properties that are equivalent to property (SDC).

Let A be a finite set. An n-ary *operation* defined on a set A is a function from A^n to A. The set of all operations on A is denoted by $\mathcal{O}_A$. If a set of operations is closed under composition and it contains the projections, then it is called a *clone*. For an arbitrary set F of operations on A, there is a least clone $[F]$ containing F, called the clone *generated* by F. For any algebra $\mathbb{A} = (A; F)$, the term operations of $\mathbb{A}$ form a clone, which is exactly $[F] = \mathrm{Clo}(\mathbb{A})$. If C is a clone, then a *C-equation* is an equation of the form $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$, where $f, g \in C$. The *solution set* of a system of C-equations $\mathcal{E}$ is denoted by $\mathrm{Sol}(\mathcal{E})$. We say that an n-ary operation f *commutes* with an m-ary operation g, if f is a homomorphism from $(A; g)^n$ to $(A; g)$. The *centralizer* of a set of operations F is the set of those operations that commute with every operation in F (notation: F^*). The centralizer of any set of operations is always a clone.

Let R be a set of relations. The set of all primitive positive definable relations over R is denoted by $\langle R\rangle_{\exists}$, and such sets of relations are called relational clones. If we allow only quantifier-free primitive positive formulas, then we obtain the weak relational clone $\langle R\rangle_{\nexists}$.

2 Connections between solution sets and centralizers

There is a connection between solution sets of systems of equations over a clone and the centralizer of the clone.

Theorem 1. [TW17] *For any clone $C \leq \mathcal{O}_A$, the solution set of a system of C-equations is closed under C^*.*

The other direction does not always hold, but when it does, we will say that the clone (or the associated algebra $(A; C)$) has property (SDC). SDC here stands for "Solution sets are Definable by closure under the Centralizer".

Definition 2. [TW20] Let $\mathbb{A} = (A; F)$ be an algebra with $C = [F]$. We say that $\mathbb{A}$ (or C) has property (SDC) if the following are equivalent for all $n \in \mathbb{N}$ and $T \subseteq A^n$:

(a) there exists a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$;

(b) the set T is closed under C^*.

Let C° denote the set of those sets of tuples that are solution sets of a system of equations containing only one equation (over C). The main result of this chapter is that a clone (or the associated algebra) has property (SDC) if and only if it has quantifier elimination for primitive positive formulas over C°.

Theorem 3. [TW20] *For every clone $C \leq \mathcal{O}_A$, the following five conditions are equivalent:*

(i) *C has property* (SDC)*;*

(ii) *every primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula over C°;*

(iii) *$\langle C^\circ \rangle_{\nexists}$ is a relational clone.*

As a corollary we can see that the centralizer clone is the only clone that can describe solution sets by a closure condition.

Corollary 4. [TW20] *Let $C \leq \mathcal{O}_A$ be a clone, and assume that there is a clone D such that for all $n \in \mathbb{N}$ and $T \subseteq A^n$, the following equivalence holds:*

$$T \text{ is the solution set of a system of } C\text{-equations} \iff T \text{ is closed under } D.$$

Then we have $D = C^*$.

3 Solution sets over 2-element algebras

After making some preliminary observations, we prove that every clone of Boolean functions (or equivalently, every two-element algebra) has property (SDC).

Theorem 5. [TW17] *For any clone of Boolean functions $C \leq \mathcal{O}_{\{0,1\}}$ and $T \subseteq \{0,1\}^n$, the following two conditions are equivalent:*

(i) *there is a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$;*

(ii) *T is closed under C^*.*

4 Centralizers of finite lattices and semilattices

In this chapter we focus on describing the centralizers of finite semilattices and lattices. We also investigate the number of essentially n-ary operations, and their occurrence in the centralizer.

Our first result describes the operations commuting with the join operation of a semilattice $\mathbb{S} = (S; \vee)$ in terms of meet-homomorphisms from $\mathbb{S}_\perp$ to $\mathbb{S}_\perp^n$, where $\mathbb{S}_\perp$ and $\mathbb{S}_\perp^n$ denote the lattices obtained from $\mathbb{S}$ and $\mathbb{S}^n$ by adding an external bottom element.

Theorem 6. [TW21] *Let $\mathbb{S} = (S; \vee)$ be a finite semilattice, and let n be a non-negative integer. The n-ary members of $[\vee]^*$ are exactly the operations f of the form*

$$f : S^n \to S, \ \mathbf{x} \mapsto \bigwedge g^{-1}(\uparrow \mathbf{x}),$$

where $g : \mathbb{S}_\perp \to \mathbb{S}_\perp^n$ is a meet-homomorphism preserving the boundary elements $\perp$ and 1. Here g is uniquely determined by f, and the operation f depends on all of its variables if and only if for each $i \in \{1, \ldots, n\}$, the range of g contains an element of S^n whose i-th component is different from 1.

The theorem above allows us to count the essentially n-ary operations in the centralizer $[\vee]^*$. For finite chains we obtain the following explicit formula.

Corollary 7. [TW21] *The number of essentially n-ary operations commuting with the join operation of a chain of cardinality ℓ is*

$$\sum_{i=1}^{\ell} \left[\binom{\ell + i - 2}{\ell - 1} - 1 \right]^n .$$

For finite distributive lattices we can characterize the existence of an essentially n-ary operation in the centralizer $[\vee, \wedge]^*$ in terms of sublattices as well as in terms of quotient lattices.

Proposition 8. [TW21] *Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite distributive lattice. Then the following are equivalent:*

(Ess) *there exists an essentially n-ary operation in $[\vee, \wedge]^*$;*

(Sub) *there exists a sublattice of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$;*

(Quo) *there exists a quotient lattice of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$.*

If our lattice is not distributive, then (Ess), (Sub) and (Quo) are not necessarily equivalent; only the implications presented in the following theorem are true in general (for all other implications we provide counterexamples).

Theorem 9. [TW21] *Let $\mathbb{L} = (L; \vee, \wedge)$ be an arbitrary finite lattice. Then the following are true:*

- *If there is an essentially n-ary operation in $[\vee, \wedge]^*$, then there is a sublattice of $\mathbb{L}$ that is isomorphic to $\mathbb{2}^n$.*
- *If there is a sublattice of $\mathbb{L}$ isomorphic to $\mathbb{2}^n$ and a quotient of $\mathbb{L}$ isomorphic to $\mathbb{2}^n$, then there is an essentially n-ary operation in $[\vee, \wedge]^*$.*

With the help of the results we got in this chapter we also show that instead of a long case-by-case analysis, we can obtain all centralizer clones on the two-element set as the corollary of five general theorems.

5 Solution sets over finite lattices and semilattices

Using Theorem 3 we characterize lattices and semilattices having property (SDC).

Theorem 10. [TW20] *A finite lattice has property* (SDC) *if and only if it is a Boolean lattice.*

Theorem 11. [TW20] *A finite semilattice has property* (SDC) *if and only if it is distributive.*

6 Solution sets and polymorphism-homogeneity

In this chapter we prove that an algebra has property (SDC) if and only if it is polymorphism-homogeneous. We can assign several relational structures to a given algebra. The most natural would be the relational structure $(A; C^\bullet)$, where $C^\bullet$ is the set of graphs of the operations in C. It turns out that the structure $(A; C^\circ)$ is more relevant, since its polymorphism-homogeneity is equivalent to polymorphism-homogeneity of the algebra (this is not the case for $(A; C^\bullet)$).

We obtain the following equivalence between property (SDC), polymorphism-homogeneity and injectivity.

Theorem 12. [TW22] *If $\mathbb{A}$ is a finite algebra and $C = \mathrm{Clo}(\mathbb{A})$, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*

(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*

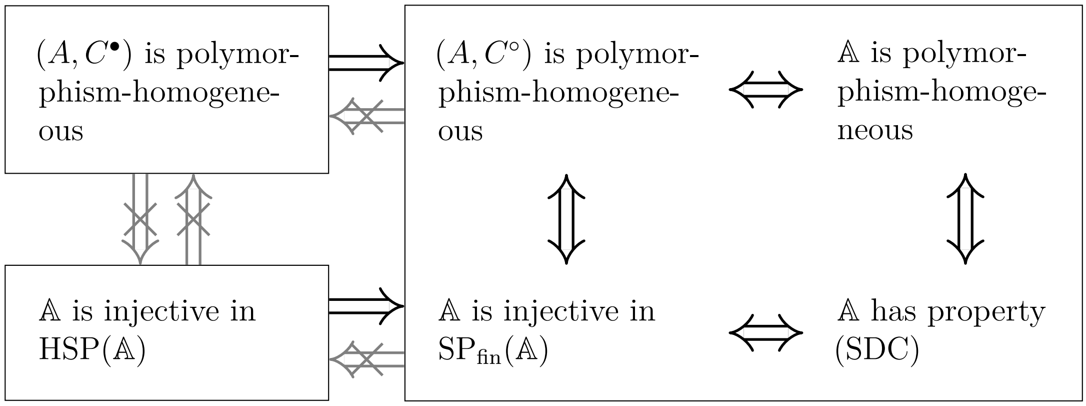


Figure 6.1: Relationships between property (SDC) and several variants of polymorphism-homogeneity and injectivity.

(iii) (A, C°) *is polymorphism-homogeneous;*

(iv) $\mathbb{A}$ *is injective in the class of finite subpowers of* $\mathbb{A}$.

On top of these four equivalent properties we also investigate polymorphism-homogeneity of the relational structure $(A; C^\bullet)$ and injectivity of $\mathbb{A}$ in the variety $\mathrm{HSP}(\mathbb{A})$ for an arbitrary finite algebra $\mathbb{A} = (A; F)$ and $C = [F]$. Our results for the connections between these six properties are shown on Figure 6.1.

We already described polymorphism-homogeneous lattices and semilattices (theorems 10 and 11). In the final part of the dissertation we characterize polymorphism-homogeneous algebras among Abelian groups and monounary algebras as well, moreover, we also investigate the two remaining properties shown on the left side of Figure 6.1. Finally, we study polymorphism-homogeneity of three-element groupoids.

Összefoglaló

1 Bevezetés

Tetszőleges véges algebra feletti egyenletrendszerek megoldáshalmazait vizsgáljuk. A kutatásunk lényege, hogy leírjuk a megoldáshalmazokat egyfajta zártsági feltétellel. A lineáris egyenletrendszerek példájára tetszőleges algebra esetén található egy olyan függvényhalmaz, melyre a megoldáshalmazok mindig zártak kell legyenek, és amennyiben ez a zártság elégséges feltétel is (tehát minden zárt halmaz megoldáshalmaz is), akkor azt mondjuk, hogy a vizsgált algebra (SDC) tulajdonságú. A disszertáció az (SDC) tulajdonságú algebrákat, és az (SDC) tulajdonsággal ekvivalens tulajdonságokat kutatja.

Legyen A véges halmaz. Az A^n-ből A-ba menő függvényeket n-változós *műveleteknek* nevezzük. Az A-n értelmezett műveletek halmazát $\mathcal{O}_A$-val jelöljük. Ha műveletek egy C halmaza zárt a függvényösszetételre és tartalmazza a projekciókat, akkor a C művelethalmaz *klón*. Tetszőleges A halmazon értelmezett F művelethalmaz esetén van egy, az F-et tartalmazó legszűkebb $[F]$ klón, amelyet az F által *generált* klónnak nevezünk. Tetszőleges $\mathbb{A} = (A; F)$ algebra termfüggvényei klónt alkotnak, mely nem más, mint az $[F] = \mathrm{Clo}(\mathbb{A})$ klón. Ha C klón, akkor egy C feletti egyenlet egy $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ alakú egyenlet, ahol $f, g \in C$. Egy C feletti $\mathcal{E}$ egyenletrendszer *megoldáshalmazát* $\mathrm{Sol}(\mathcal{E})$ jelöli. Ha f egy n-változós művelet, g pedig m-változós művelet, valamint f az $(A; g)^n$ algebrából az $(A; g)$ algebrába menő homomorfizmus, akkor azt mondjuk, hogy f és g *felcserélhető*. Egy F művelethalmaz *centralizátorán* azon műveletek halmazát értjük, melyek felcserélhetőek az összes F-beli művelettel (jelölés: F^*). Tetszőleges művelethalmaz centralizátora klón.

Legyen R relációk halmaza. Jelölje $\langle R \rangle_\exists$ az R feletti primitív pozitív formulával definiálható relációk halmazát, és jelölje $\langle R \rangle_\nexists$ az R feletti kvantormentes primitív pozitív formulával definiálható relációk halmazát. (Előbbi nem más, mint az R által generált relációs klón.)

2 Megoldáshalmazok és centralizátorok közötti kapcsolatok

Tetszőleges klón esetén a következő kapcsolat van a klón centralizátora és a klón feletti egyenletrendszerek megoldáshalmazai között.

1. Tétel. [TW17] *Tetszőleges $C \leq \mathcal{O}_A$ klón esetén a C feletti egyenletrendszerek megoldáshalmazai zártak C^*-ra.*

A másik irányú implikáció nem feltétlenül teljesül. Ha azonban egy C klón esetén teljesül, akkor azt mondjuk, hogy a C klón (vagy a hozzá tartozó $(A; C)$ algebra) (SDC) tulajdonságú. Itt SDC a „Solution sets are Definable by closure under the Centralizer” rövidítése.

2. Definíció. [TW20] Legyen $\mathbb{A} = (A; F)$ algebra és $C = [F]$. Azt mondjuk, hogy $\mathbb{A}$ (vagy C) (SDC) tulajdonságú, ha a következők ekvivalensek tetszőleges $n \in \mathbb{N}$ és $T \subseteq A^n$ esetén:

(a) létezik olyan $\mathcal{E}$ egyenletrendszer C felett, amelyre $T = \mathrm{Sol}(\mathcal{E})$;

(b) T zárt C^*-ra.

Azon relációk halmazát, melyek előállnak egyetlen C feletti egyenlet megoldáshalmazaként, C°-val fogjuk jelölni. Ennek a fejezetnek a fő eredménye, hogy egy klón (illetve a kapcsolódó algebra) pontosan akkor (SDC) tulajdonságú, ha a C° feletti primitív pozitív formulák kvantoreliminálhatóak.

3. Tétel. [TW20] *Minden $C \leq \mathcal{O}_A$ klónra ekvivalensek a következő állítások:*

(i) C (SDC) *tulajdonságú;*

(ii) *minden C° feletti primitív pozitív formula ekvivalens valamely C° feletti kvantormentes primitív pozitív formulával;*

(iii) $\langle C^\circ \rangle_{\nexists}$ *relációs klón.*

A tétel következményeként megmutatjuk, hogy az egyetlen klón, mely zártsági feltételekkel leírhatja a megoldáshalmazokat, a centralizátor klón.

4. Következmény. [TW20] *Legyen $C \leq \mathcal{O}_A$ klón, és tegyük fel, hogy létezik olyan D klón, melyre minden $n \in \mathbb{N}$ és $T \subseteq A^n$ esetén a következő ekvivalencia teljesül:*

$$T \text{ megoldáshalmaza valamely } C \text{ feletti egyenletrendszernek} \iff T \text{ zárt } D\text{-re.}$$

Ekkor $D = C^*$.

3 Kételemű algebrák feletti megoldáshalmazok

Néhány előzetes észrevétel segítségével igazoljuk, hogy a kételemű halmazon minden klón (illetve minden kételemű algebra) (SDC) tulajdonságú.

5. Tétel. [TW17] *Boole-függvények tetszőleges C klónja és tetszőleges $T \subseteq \{0,1\}^n$ halmaz esetén ekvivalensek a következők:*

(i) *van olyan $\mathcal{E}$ egyenletrendszer C felett, melynek T a megoldáshalmaza;*

(ii) *T zárt C^*-ra.*

4 Véges hálók és félhálók centralizátorai

Ebben a fejezetben félhálók és hálók centralizátorait vizsgáljuk. Megnézzük azt is, hogy hány lényegében n-változós művelet van a centralizátorban, illetve ezek előfordulását is kutatjuk.

Első eredményünk leírja egy $\mathbb{S} = (S, \vee)$ félháló egyesítés műveletével felcserélhető műveleteket, mint az $\mathbb{S}_\perp$-ből $\mathbb{S}_\perp^n$-ba menő metszet-homomorfizmusokat. Itt $\mathbb{S}_\perp$ és $\mathbb{S}_\perp^n$ azon hálókat jelöli, melyeket $\mathbb{S}$-ből és $\mathbb{S}^n$-ből úgy kapjuk, hogy hozzájuk illesztünk egy (új) legkisebb elemet.

6. Tétel. [TW21] *Legyen $\mathbb{S} = (S; \vee)$ véges félháló, és legyen n pozitív egész szám. A $[\vee]^*$ centralizátor n-változós részében pontosan azon f műveletek vannak, melyek a következő alakúak:*

$$f\colon S^n \to S,\ \mathbf{x} \mapsto \bigwedge g^{-1}(\uparrow \mathbf{x}),$$

ahol $g : \mathbb{S}_\perp \to \mathbb{S}_\perp^n$ egy, a $\perp$ és 1 korlátelemeket megőrző metszet-homomorfizmus. Itt g-t egyértelműen meghatározza f, továbbá az f művelet pontosan akkor függ minden változójától, ha mindegyik $i \in \{1, \ldots, n\}$ esetén g értékkészlete tartalmaz olyan S^n-beli elemet, melynek i-edik komponense különbözik 1-től.

A fenti tétel segítségével meg tudjuk adni a $[\vee]^*$ centralizátorban lévő lényegében n-változós műveletek számát. Véges láncokra a következő explicit formulát kapjuk:

7. Következmény. [TW21] *Egy l-elemű lánc (mint félháló) centralizátorában lévő lényegében n-változós műveletek száma*

$$\sum_{i=1}^{\ell} \left[\binom{\ell + i - 2}{\ell - 1} - 1 \right]^n .$$

Véges disztributív hálók esetén karakterizálhatjuk a centralizátorban lévő lényegében n-változós művelet létezését részhálók, illetve faktorhálók segítségével.

8. Állítás. [TW21] *Legyen* $\mathbb{L} = (L; \vee, \wedge)$ *véges disztributív háló. A következők ekvivalensek:*

(Ess) *Van* n*-változós művelet az* $[\vee, \wedge]^*$ *centralizátorban;*

(Sub) $\mathbb{L}$*-nek van olyan részhálója, mely izomorf a* $\mathbb{2}^n$ *hálóval;*

(Quo) $\mathbb{L}$*-nek van olyan faktorhálója, mely izomorf a* $\mathbb{2}^n$ *hálóval.*

Ha a hálónk nem disztributív, akkor az (Ess), (Sub) és (Quo) tulajdonságok nem feltétlenül ekvivalensek; csak a következő tételben szereplő implikációk teljesülnek általában (minden más implikációra adtunk ellenpéldát is).

9. Tétel. [TW21] *Legyen* $\mathbb{L} = (L; \vee, \wedge)$ *tetszőleges véges háló. A következők teljesülnek:*

- *Ha van lényegében* n*-változós művelet az* $[\vee, \wedge]^*$ *centralizátorban, akkor* $\mathbb{L}$*-nek van olyan részhálója, mely izomorf a* $\mathbb{2}^n$ *hálóval.*
- *Ha* $\mathbb{L}$*-nek van a* $\mathbb{2}^n$ *hálóval izomorf részhálója és a* $\mathbb{2}^n$ *hálóval izomorf faktorhálója, akkor létezik lényegében* n*-változós művelet az* $[\vee, \wedge]^*$ *centralizátorban.*

A fejezetben eddig kapott eredmények segítségével azt is megmutatjuk, hogy a kételemű halmaz feletti összes centralizátor hosszadalmas esetvizsgálat helyett öt általános tétel következményeként is megadható.

5 Véges hálók és félhálók feletti megoldáshalmazok

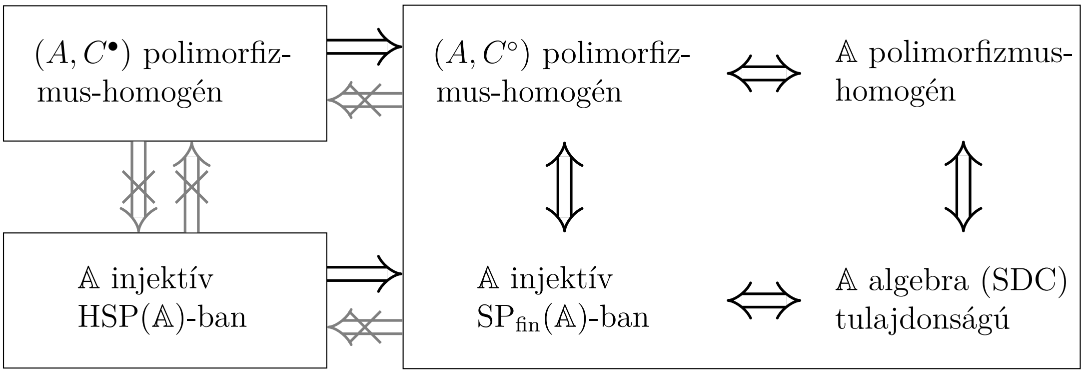
A 3. Tétel segítségével leírjuk az (SDC) tulajdonságú félhálókat és hálókat.

10. Tétel. [TW20] *Egy véges háló pontosan akkor* (SDC) *tulajdonságú, ha Boole-háló.*

11. Tétel. [TW20] *Egy véges félháló pontosan akkor* (SDC) *tulajdonságú, ha disztributív.*

6 Megoldáshalmazok és polimorfizmus-homogenitás

Ebben a fejezetben megmutatjuk, hogy egy algebra pontosan akkor (SDC) tulajdonságú, ha polimorfizmus-homogén. Egy algebrához több relációstruktúrát is



6.1. ábra: Az (SDC) tulajdonság és különböző polimorfizmus-homogenitások és injektivitások közötti kapcsolatok.

lehet társítani. A legtermészetesebb az $(A; C^\bullet)$ relációstruktúra lenne, ahol $C^\bullet$ a C-beli műveletek gráfjainak halmaza. Kiderül azonban, hogy relevánsabb az $(A; C^\circ)$ struktúra, mert ennek polimorfizmus-homogenitása ekvivalens az algebra polimorfizmus-homogenitásával (míg $(A; C^\bullet)$ esetén nem ez a helyzet).

Az alábbi ekvivalenciát kapjuk az (SDC) tulajdonság, a polimorfizmus-homogenitás és az injektivitás között.

12. Tétel. [TW22] *Ha* $\mathbb{A}$ *véges algebra és* $C = \mathrm{Clo}(\mathbb{A})$*, akkor ekvivalensek a következők:*

(i) $\mathbb{A}$ (SDC) *tulajdonságú;*

(ii) $\mathbb{A}$ *polimorfizmus-homogén;*

(iii) (A, C°) *polimorfizmus-homogén;*

(iv) $\mathbb{A}$ *injektív az* $\mathbb{A}$ *algebra véges részhatványainak osztályában.*

A fenti négy ekvivalens tulajdonságon felül tetszőleges $\mathbb{A} = (A; F)$ algebra esetén vizsgáljuk az $(A; C^\bullet)$ relációstruktúra polimorfizmus-homogenitását, valamint $\mathbb{A}$ injektivitását a $\mathrm{HSP}(\mathbb{A})$ varietásban. Ezen hat tulajdonság közötti kapcsolatokra vonatkozó eredményeinket a 6.1. ábra mutatja.

A félhálók és hálók körében már leírtuk a polimorfizmus-homogéneket (10. és 11. Tétel). A disszertáció befejező részében Abel-csoportok és monounér algebrák körében is karakterizáljuk a polimorfizmus-homogén algebrákat, valamint vizsgáljuk a 6.1. ábra bal oldalán szereplő tulajdonságokat is. Végül a háromelemű grupoidok polimorfizmus-homogenitását kutatjuk.

1.1 The Post lattice

The lattice of all clones on the set $\{0, 1\}$ is shown in Figure 1.1. Different symbols are used according to the partition defined in Remark 4.4.5; primitive positive clones are indicated by a symbol having an outline, while the gray circles without an outline indicate clones that are not primitive positive. In Table 1.1 we give the definitions of the clones that are labeled on the diagram; the remaining clones can be obtained as intersections of some of these clones.

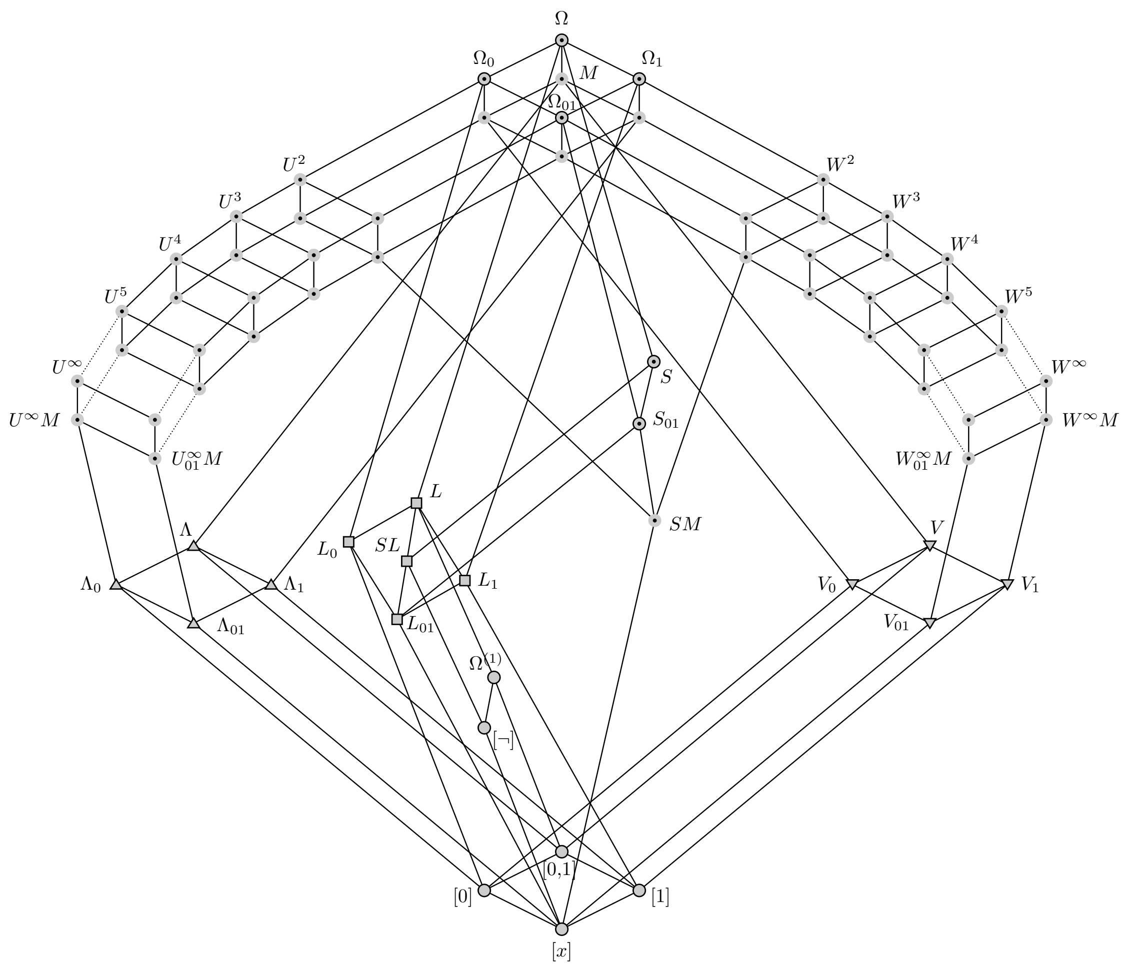


Figure 1.1: The Post lattice.

Ω is the clone of all Boolean functions: $\Omega = \mathcal{O}_{01}$.
$\Omega_0 = \{f \in \Omega \mid f(0,\dots,0) = 0\}$ is the clone of 0-preserving functions. $\Omega_1 = \{f \in \Omega \mid f(1,\dots,1) = 1\}$ is the clone of 1-preserving functions. $\Omega_{01} = \Omega_0 \cap \Omega_1$ is the clone of idempotent functions. In general, if C is a clone, then $C_0 = C \cap \Omega_0$, $C_1 = C \cap \Omega_1$, and $C_{01} = C_0 \cap C_1$.
$\Omega^{(1)}$ is the clone of all essentially at most unary functions: $\Omega^{(1)} = [x, \neg x, 0, 1]$. $[x]$ is the trivial clone containing only projections. $[0]$, $[1]$ and $[0, 1]$ are the clones generated by constant operations. $[\neg]$ is the clone generated by negation.
$M = \{f \in \Omega \mid \mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\}$ is the clone of monotone functions.
$U^\infty = \{f \in \Omega^{(n)} \mid n \in \mathbb{N}_0, \exists k \in \{1,\dots,n\}\colon f(\mathbf{x}) = 1 \implies x_k = 1\}$, and $U^\infty M = U^\infty \cap M$, and $U^\infty_{01} M = U^\infty \cap \Omega_{01} \cap M$.
$W^\infty = \{f \in \Omega^{(n)} \mid n \in \mathbb{N}_0, \exists k \in \{1,\dots,n\}\colon f(\mathbf{x}) = 0 \implies x_k = 0\}$, and $W^\infty M = W^\infty \cap M$ and $W^\infty_{01} M = W^\infty \cap \Omega_{01} \cap M$.
$S = \{f \in \Omega \mid \neg f(\neg \mathbf{x}) = f(\mathbf{x})\}$ is the clone of self-dual functions. $SM = S \cap M = [\mu]$ where $\mu(x, y, z)$ is the majority function on $\{0, 1\}$.
$\Lambda = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [0, 1] = [\wedge, 0, 1]$. $\Lambda_0 = \Lambda \cap \Omega_0 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [0] = [\wedge, 0]$. $\Lambda_1 = \Lambda \cap \Omega_1 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [1] = [\wedge, 1]$. $\Lambda_{01} = \Lambda \cap \Omega_{01} = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} = [\wedge]$.
$V = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}\} \cup [0, 1] = [\vee, 0, 1]$. $V_0 = V \cap \Omega_0 = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}\} \cup [0] = [\vee, 0]$. $V_1 = V \cap \Omega_1 = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}\} \cup [1] = [\vee, 1]$. $V_{01} = V \cap \Omega_{01} = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}\} = [\vee]$.
$L = \{x_1 + \cdots + x_n + c \mid c \in \{0, 1\}, n \in \mathbb{N}_0\} = [x + y, 1]$. $L_0 = L \cap \Omega_0 = \{x_1 + \cdots + x_n \mid n \in \mathbb{N}_0\} = [x + y]$. $L_1 = L \cap \Omega_1 = \{x_1 + \cdots + x_n + (n + 1 \mod 2) \mid n \in \mathbb{N}_0\} = [x + y + z, 1]$. $L_{01} = L \cap \Omega_{01} = \{x_1 + \cdots + x_n \mid n \text{ is odd}\} = [x + y + z]$. $SL = S \cap L = \Big\{x_1 + \cdots + x_n + c \mid n \text{ is odd and } c \in \{0, 1\}\Big\} = [x + y + z, x + 1]$.

Table 1.1: Definitions of some clones of Boolean functions.

1.2 Centralizer clones of Boolean clones

P	all clones $C \leq \mathcal{O}_{\{0,1\}}$ such that $C^* = P$
$[x]$	Ω, M
$[0]$	$\Omega_0, M_0, U^k, U^\infty, U^k M, U^\infty M$ (for all $k \in \mathbb{N}$)
$[1]$	$\Omega_1, M_1, W^k, W^\infty, W^k M, W^\infty M$ (for all $k \in \mathbb{N}$)
$[0,1]$	$\Omega_{01}, M_{01}, U_{01}^k, U_{01}^\infty, U_{01}^k M, U_{01}^\infty M, W_{01}^k, W_{01}^\infty, W_{01}^k M, W_{01}^\infty M$ (for all $k \in \mathbb{N}$)
$[\neg]$	S
$\Omega^{(1)}$	S_{01}, SM
L_{01}	L
L_0	L_0
L_1	L_1
L	L_{01}
SL	SL
Λ_{01}	Λ
Λ_0	Λ_0
Λ_1	Λ_1
Λ	Λ_{01}
V_{01}	V
V_0	V_0
V_1	V_1
V	V_{01}
S_{01}	$\Omega^{(1)}$
S	$[\neg]$
Ω_{01}	$[0,1]$
Ω_0	$[0]$
Ω_1	$[1]$
Ω	$[x]$

Table 1.2: The centralizers of all clones of Boolean functions.

1.3 Figures and tables for the proof of Theorem 5.2.3

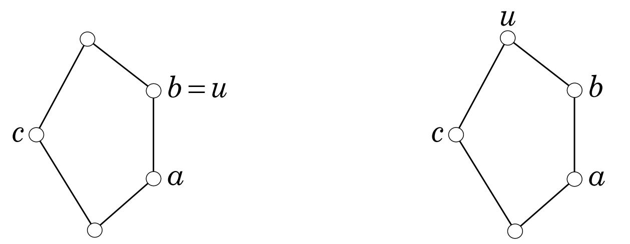


Figure 1.2: The elements a, b and c (with an example u proving $(a, c, b), (b, a, c) \in T$) in case (N_5).

$=$	x	y	z	$x \wedge y$	$x \wedge z$	$y \wedge z$	$x \wedge y \wedge z$
x	–	(a, c, b)	(a, c, b)	(a, c, b)	(b, a, c)	(a, c, b)	(a, c, b)
y	–	–	(a, c, b)	(a, c, b)	(a, c, b)	(a, c, b)	(a, c, b)
z	–	–	–	(a, c, b)	(a, c, b)	(a, c, b)	(a, c, b)
$x \wedge y$	–	–	–	–	(a, c, b)	(b, a, c)	(b, a, c)
$x \wedge z$	–	–	–	–	–	(a, c, b)	(a, c, b)
$y \wedge z$	–	–	–	–	–	–	
$x \wedge y \wedge z$	–	–	–	–	–	–	–

Table 1.3: Counterexamples for case (N_5) showing that these equations do not belong to Eq(T).

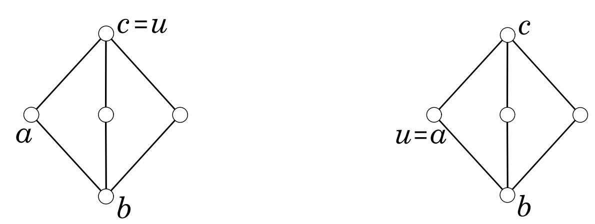


Figure 1.3: The elements a, b and c (with an example u proving $(a, b, c), (a, c, b) \in T$) in case (M_3).

$=$	x	y	z	$x \wedge y$	$x \wedge z$	$y \wedge z$	$x \wedge y \wedge z$
x	–	(a, b, c)	(a, b, c)	(a, b, c)	(a, c, b)	(a, b, c)	(a, b, c)
y	–	–	(a, b, c)	(a, c, b)	(a, b, c)	(a, c, b)	(a, c, b)
z	–	–	–	(a, b, c)	(a, b, c)	(a, b, c)	(a, b, c)
$x \wedge y$	–	–	–	–	(a, b, c)	(a, c, b)	(a, c, b)
$x \wedge z$	–	–	–	–	–	(a, b, c)	(a, b, c)
$y \wedge z$	–	–	–	–	–	–	
$x \wedge y \wedge z$	–	–	–	–	–	–	–

Table 1.4: Counterexamples for case (M_3) showing that these equations do not belong to Eq(T).

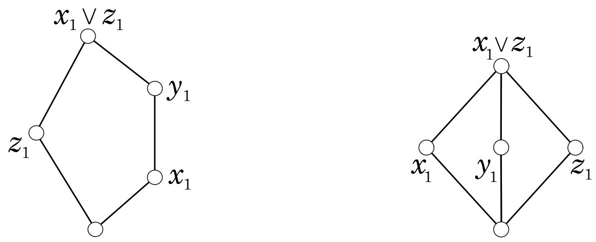


Figure 1.4: (x_1, y_1, z_1) satisfies all equations in Eq(T), but $(x_1, y_1, z_1) \notin T$.

1.4 Some non-polymorphism-homogeneous three-element groupoids: non-extendable partial polymorphisms

A partial operation ϕ	The groupoids G (up to clone equivalence) such that ϕ is a non-extendable partial polymorphism of G
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 1 \end{cases}$	(2), (4), (5), (6), (8), (10), (11), (12), (13), (15), (16), (18), (22), (26), (30), (31), (32), (33), (34), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46), (48), (49), (51), (52), (53), (59), (61), (63), (65), (66), (67), (69), (73), (376), (377), (378), (379), (380), (381), (382), (384), (385), (387), (388), (390), (391), (405), (407), (410), (417), (434), (436), (437), (439), (1014), (1038), (1040), (1066)
$\phi : \begin{cases} 0 \mapsto 2 \\ 2 \mapsto 0 \end{cases}$	(21), (24), (47), (50), (72), (75), (78), (96), (99), (119), (141), (144), (162), (165), (168), (182), (185), (188), (201), (204), (218), (221), (235), (252), (255), (1227), (1233)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$	(60), (147)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{cases}$	(79), (81), (82), (83), (85), (87), (88), (89), (90), (91), (93), (97), (101), (111), (112), (113), (115), (117), (120), (121), (122), (123), (124), (125), (130), (132), (134), (136), (137), (138), (142), (143), (406), (454), (455), (456), (457), (458), (459), (460), (462), (463), (465), (469), (483), (484), (485), (487), (493), (494), (495), (496), (512), (513), (514), (515), (517), (519), (522), (1086), (1107), (1108), (1133)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (1,0) \mapsto 1 \end{cases}$	(80), (100), (102), (239), (241), (1132)
$\phi : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}$	(116), (135), (178), (194), (203), (281), (308), (316), (488), (520), (565), (758), (780), (984), (1793), (1799), (1818), (1962), (2430), (2436), (2539), (2545), (2636)

$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$	(149), (175)
$\phi : \begin{cases} (1,1) \mapsto 2 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$	(176), (198), (306), (320), (563), (756)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (1,1) \mapsto 1 \\ (1,2) \mapsto 2 \\ (2,1) \mapsto 2 \\ (2,2) \mapsto 2 \end{cases}$	(216), (284), (1708), (1837), (2088), (2472), (2558)
$\phi : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 2 \end{cases}$	(219), (222), (2090)
$\phi : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$	(257), (258), (259), (260), (261), (262), (263), (265), (266), (267), (268), (269), (270), (271), (272), (274), (278), (282), (677), (678), (679), (680), (682), (684), (687), (690), (691), (693), (695), (696), (697), (698), (704), (705), (707), (710), (712), (1271), (1277), (1281), (2460), (2461), (2462), (2463), (2464), (2467), (2476), (2478), (2479), (2480), (2483), (2486), (2487), (2493), (2739)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 0 \end{cases}$	(273), (681), (2116)
$\phi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 2 \\ (2,1) \mapsto 2 \\ (2,2) \mapsto 1 \end{cases}$	(283), (287), (353), (356), (359)

$\phi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$	(354)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (2,0) \mapsto 2 \\ (2,2) \mapsto 0 \end{cases}$	(1012), (1084), (1151), (1153), (1176), (1200), (1219), (1221), (1242), (1321), (1433), (1437), (1481), (2102), (2104)
$\phi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$	(2654), (2686), (2698), (2702)

Table 1.5: Partial polymorphisms proving that the corresponding groupoids are not polymorphism-homogeneous.

1.5 Some non-polymorphism-homogeneous three-element groupoids: operation tables

(2)	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

(4)	0	1	2
0	0	0	0
1	0	0	0
2	0	1	0

(5)	0	1	2
0	0	0	0
1	0	0	0
2	0	1	1

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