# Counting and generating in some classes of lattices

Outline of Ph.D. Thesis

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# Introduction

In this dissertation, our goal is to get a better understanding of the structure of some lattices and some related lattices. We describe slim rectangular lattices by permutations, and we also count these lattices. We search for minimum-sized generating sets of the lattices of quasiorders. Also, we characterize lattices with many congruences. While counting these congruences, we describe the structure of the congruence lattices, too.

This dissertation is based on four of the author's papers. These publications are the following:

1. G. Czédli, T. Dékány, G. Gyenizse and J. Kulin: *The number of slim rectangular lattices*. Algebra Universalis **75/1** (2016), 33–50.

2. G. Czédli and J. Kulin: *A concise approach to small generating sets of lattices of quasiorders and transitive relations*. Acta Sci. Math. (Szeged) **83** (2017), 3–12.

3. J. Kulin: *Quasiorder lattices are five-generated*. Discussiones Mathematicae - General Algebra and Applications **36** (1) (2016), 59–70.

4. C. Mureşan and J. Kulin: *On the largest numbers of congruences of finite lattices*. Order **37** (2020), 445–460.

# The number of slim rectangular lattices

Following the introductory Chapter 1, Chapter 2 is about slim rectangular lattices and is based on [7]. An element of a lattice is *join-irreducible* if it has exactly one lower cover. A finite lattice $L$ is *slim*, if $\mathrm{Ji}\, L$, the set of the join-irreducible elements of $L$, is included in the union of two chains of $L$; see Czédli and Schmidt [10]. Note that, in the semimodular case, this concept was first introduced by Grätzer and Knapp [14] in a different way. Slim lattices are *planar*, that is, they possess planar diagrams.

If $D_1$ and $D_2$ are planar diagrams and $\varphi\colon D_1 \to D_2$ is a bijective map such that $\varphi$ is a lattice isomorphism and it preserves the left-right order of (upper) covers and that of lower covers of each element of $D_1$, then $\varphi$ is called a *similarity map*. Two planar diagrams are *similar* if there exists a similarity map between them. We treat similar diagrams as equal ones. That is, when we count planar diagrams, we always do it up to similarity. By our convention, the lattice properties of a planar lattice diagram $D$ are those of the lattice determined by $D$.

Following Grätzer and Knapp [15], a semimodular diagram $D$ is *rectangular* if its *left boundary chain*, denoted by $\mathrm{C_l}(D)$, has exactly one doubly irreducible element, $\mathrm{lc}(D)$, its *right boundary chain*, $\mathrm{C_r}(D)$, has exactly one doubly irreducible element, $\mathrm{rc}(D)$, and these two elements, called the *corners* of $D$, are complementary, that is, $\mathrm{lc}(D) \wedge \mathrm{rc}(D) = 0$ and $\mathrm{lc}(D) \vee \mathrm{rc}(D) = 1$. Rectangular lattices are those that have rectangular diagrams.

Associated with a slim rectangular diagram $D$, the following three numerical parameters will be of particular interest. As usual, the *length* of $D$ is denoted by $\mathrm{length}\, D$. The *left upper length* and the *right upper length* of $D$, denoted by ${}^{\mathrm{lu}}\mathrm{len}\, D$ and ${}^{\mathrm{ru}}\mathrm{len}\, D$, are the length of the interval $[\mathrm{lc}(D), 1]$ and that of $[\mathrm{rc}(D), 1]$, respectively.

A minimal non-chain region of a planar lattice diagram $D$ is called a *cell*. A four-element cell is a 4-*cell*. A diagram is a 4-*cell diagram* if all of its cells are 4-cells. It was proved in Grätzer and Knapp [14, Lemmas 4 and 5] that $D$ is a slim semimodular diagram iff it is a 4-cell diagram and no two distinct 4-cells have the same bottom. Two prime intervals of a slim semimodular diagram $D$ are *consecutive* if they are opposite sides of a 4-cell. The consecutiveness of two prime intervals in a slim semimodular lattice $L$ does not depend on the planar diagram chosen. Maximal sequences of consecutive prime intervals form a *trajectory*. In other words, a trajectory is a class of the equivalence relation generated by consecutiveness. By Czédli and Schmidt [10, Lemma 2.8], if $T$ is a trajectory of a slim semimodular diagram $D$, then $T$ contains exactly one prime interval of $\mathrm{C_l}(D)$, and the same holds for $\mathrm{C_r}(D)$. Going from left to right, $T$ does not branch out. First $T$ goes up (possibly in zero steps), then it may turn to the lower right, and finally it goes down (possibly, in zero steps). In particular, at most one turn is possible. See Figure 1 for illustration.

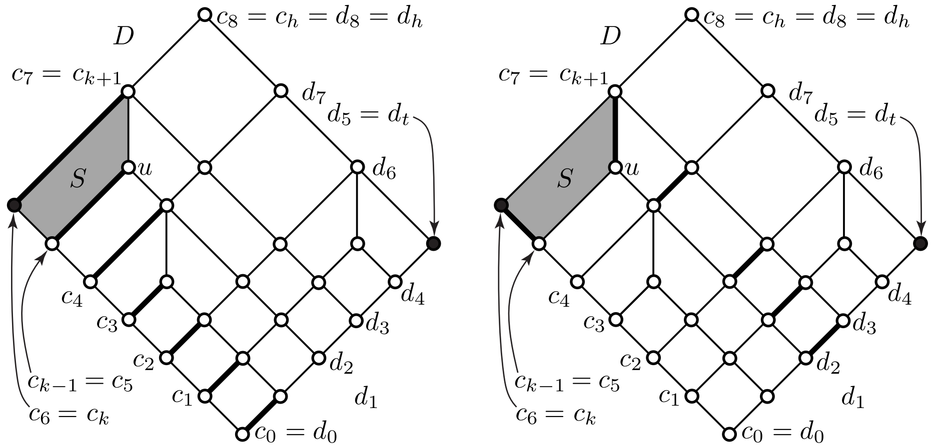


Figure 1: Two trajectories (the bold edges) of a slim rectangular diagram

We denote the set of (the similarity classes of) *slim rectangular diagrams* of length $n$ and that of *slim semimodular diagrams* of length $n$ by the acronyms SRectD($n$) and SSmodD($n$), respectively. Similarly, the set of the isomorphism classes of *slim rectangular lattices* of length $n$, that of *slim semimodular lattices* of length $n$ are denoted by SRectL($n$) and SSmodL($n$).

There are several known tools for examining semimodular lattices; one of them is describing these lattices by permutations. For a slim rectangular diagram $D$ of length $n$, let $\mathrm{C_l}(D) = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $\mathrm{C_r}(D) = \{0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1\}$. Following Czédli and Schmidt [11], the *permutation* $\pi = \pi_D \in S_n$ is defined by the rule $\pi(i) = j$ iff $[c_{i-1}, c_i]$ and $[d_{j-1}, d_j]$ belong to the same trajectory. Czédli and Schmidt proved in [11] that the map SSmodD($n$) $\to S_n$, defined by $D \mapsto \pi_D$, is a bijection.

In Chapter 2, we describe the permutations belonging to slim rectangular lattices.

**Definition 1.** A permutation $\pi \in S_n$ is called *rectangular* if it satisfies the following three properties.

(i) For all $i$ and $j$, if $\pi^{-1}(1) < i < j \leq n$, then $\pi(i) < \pi(j)$.

(ii) For all $i$ and $j$, if $\pi(1) < i < j \leq n$, then $\pi^{-1}(i) < \pi^{-1}(j)$.

(iii) $\pi(n) < \pi(1)$.

**Remark 2.** *If $\pi \in S_n$ is rectangular, then we have*

(iv) $\pi^{-1}(n) < \pi^{-1}(1)$.

*So, $\pi$ is rectangular iff $\pi^{-1}$ is rectangular.*

**Proposition 3.** *A slim, semimodular, planar diagram $D$ of length $n \geq 2$ is rectangular if and only if $\pi = \pi_D \in S_n$ is rectangular. Furthermore, if $D$ is rectangular, then*

$$\pi_D(1) = \operatorname{length} D - {}^{\mathrm{ru}}\mathrm{len}\, D + 1, \quad \pi_D^{-1}(1) = \operatorname{length} D - {}^{\mathrm{lu}}\mathrm{len}\, D + 1.$$

With the help of this description, we give formulas for the numbers of slim rectangular diagrams and slim rectangular lattices.

**Proposition 4.** *For $2 \leq n \in \mathbb{N}$, the number of slim rectangular diagrams of length $n$ is*

$$|\mathrm{SRectD}(n)| = \sum_{\substack{a+b \leq n \\ a,b \in \mathbb{N}}} \binom{n-a-1}{b-1} \binom{n-b-1}{a-1} (n-a-b)!\,.$$

Let $\mathrm{Invl}(k) = \{\pi \in S_k : \pi = \pi^{-1}\}$ denote the set of *involutions* acting on the set $\{1, \ldots, k\}$. For $k \in \mathbb{N}$, the number of involutions in $S_k$ is $|\mathrm{Invl}(k)| = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{k-2j} \cdot (2j-1)!!$.

**Proposition 5.** *For $2 \leq n \in \mathbb{N}$, the number of* (*the isomorphism classes of*) *slim rectangular lattices of length $n$ is*

$$|\mathrm{SRectL}(n)| = \frac{1}{2} \cdot \left( |\mathrm{SRectD}(n)| + \sum_{a=1}^{\lfloor n/2 \rfloor} \binom{n-a-1}{a-1} \cdot |\mathrm{Invl}(n-2a)| \right).$$

Based on the formulas, we are able to give asymptotic results, in which $e \approx 2.71828$.

**Proposition 6.** *The number of (the similarity classes of) slim rectangular diagrams of length $n$ is asymptotically $(n-2)! \cdot e^2$, that is, $|\mathrm{SRectD}(n)| \sim (n-2)! \cdot e^2$.*

This leads to the main result of Chapter 2.

**Theorem 7.** *The number of (the isomorphism classes of) slim rectangular lattices of length $n$ is asymptotically $(n-2)! \cdot e^2/2$, that is,*

$$\lim_{n\to\infty} \frac{|\mathrm{SRectL}(n)|}{(n-2)! \cdot e^2/2} = 1.$$

Based on Propositions 4 and 5, $|\mathrm{SSmodD}(n)|$ and $|\mathrm{SSmodL}(n)|$ can easily be determined for $n \leq 1000$; Table 1 and Table 2 contain some of our results by computer algebra. The numbers in Table 1 are also given in https://oeis.org/A273596 and https://oeis.org/A273988, respectively.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\mathrm{SRectD}(n)\rvert$ | 1 | 3 | 9 | 32 | 139 | 729 | 4 515 | 32 336 |
| $\lvert\mathrm{SRectL}(n)\rvert$ | 1 | 2 | 6 | 19 | 78 | 387 | 2 327 | 16 384 |

| $n$ | 10 | 11 | 12 |
|---|---|---|---|
| $\lvert\mathrm{SRectD}(n)\rvert$ | 263 205 | 2 401 183 | 24 275 037 |
| $\lvert\mathrm{SRectL}(n)\rvert$ | 132 336 | 1 203 145 | 12 146 959 |

Table 1: Computational results for $2 \leq n \leq 12$

| $n$ | 200 | 600 | 1000 |
|---|---|---|---|
| $\lvert\mathrm{SRectD}(n)\rvert$ | $1.4568041 \cdot 10^{371}$ | $2.5975960 \cdot 10^{1403}$ | $2.9732576 \cdot 10^{2562}$ |
| $\lvert\mathrm{SRectL}(n)\rvert$ | $7.2840205 \cdot 10^{370}$ | $1.2987980 \cdot 10^{1403}$ | $1.4866288 \cdot 10^{2562}$ |
| $\frac{\lvert\mathrm{SRectL}(n)\rvert}{(n-2)! \cdot e^2/2}$ | 0.99496227 | 0.99832914 | 0.99899847 |

Table 2: Computational results for $n \in \{200, 600, 1000\}$

# Small generating sets of lattices of quasiorders and transitive relations

In Chapter 3, we aim to determine a minimum-sized generating set of the lattice of quasiorders, also of the lattice of transitive relations. This chapter is based on [8] and [16].

A *quasiorder* is a reflexive and transitive relation. Quasiorders on a set $A$ form a complete lattice $\mathrm{Quo}(A)$. So do the transitive relations on $A$; their complete lattice is denoted by $\mathrm{Tran}(A)$. Similarly, $\mathrm{Equ}(A)$ will stand for the lattice of all equivalences on $A$.

For a subset $X$ of $\mathrm{Equ}(A)$, $\mathrm{Quo}(A)$, or $\mathrm{Tran}(A)$, we say that $X$ generates the complete lattice in question if the only complete sublattice including $X$ is the whole lattice itself. For $k \in \mathbb{N} := \{1, 2, 3, \dots\}$, we say that a complete lattice $L$ is *$k$-generated* if it can be generated by a $k$-element subset $X$. If a complete lattice is generated by a four-element subset $X = \{x_1, x_2, x_3, x_4\}$ such that $x_1 < x_2$ but both $\{x_1, x_3, x_4\}$ and $\{x_2, x_3, x_4\}$ are antichains, then we say that this lattice is *$(1 + 1 + 2)$-generated.*

All sets in Chapter 3 are assumed to be of accessible cardinalities. A cardinal $\kappa$ is *accessible* if it is finite, or it is infinite and for every $\lambda \le \kappa$,

- either $\lambda \le 2^\mu$ for some cardinal $\mu < \lambda$,
- or there is a set $I$ of cardinals such that $\lambda \le \sum_{\mu \in I} \mu$, $|I| < \lambda$, and $\mu < \lambda$ for all $\mu \in I$.

ZFC has a model in which all cardinals are accessible, hence the scope of many of our results includes all sets in an appropriate model of set theory.

It was known by Strietz [18] and [19], Zádori [21], and Czédli [3] that the complete lattice $\mathrm{Equ}(A)$ of all equivalences is four-generated, provided the size $|A|$ of $A$ is an accessible cardinal and $|A| \ge 2$. Also, $\mathrm{Equ}(A)$ cannot be generated by less than four elements if $|A| \ge 4$. We know from Chajda and Czédli [2] and Takách [20] that $\mathrm{Quo}(A)$ is six-generated as a complete lattice, provided that $|A|$ is accessible. Actually, we know from Dolgos [12] for $2 \leq |A| \leq \aleph_0$ that the complete lattice $\mathrm{Quo}(A)$ is five-generated.

We extend Dolgos' result in two ways. The first one is short and states more (about all sets $A$ where $|A|$ is accessible) than the second one, but it is based heavily on Czédli's quite involved and long constructions from [3] and [4]. This justifies the second way: we give an easier, more understandable and self-contained construction for a five-element generating set of $\mathrm{Quo}(A)$ if $|A| \leq 2^{\aleph_0}$, based on Dolgos' work.

**Theorem 8.** *Let $A$ be a set with at least three elements.*

(i) *If $|A|$ is an accessible cardinal, then $\mathrm{Quo}(A)$ is five-generated as a complete lattice.*

(ii) *If $\aleph_0 \leq |A| \leq 2^{\aleph_0}$, then $\mathrm{Quo}(A)$ is five-generated as a complete lattice.*

Following this result, Czédli proved in [5] that the complete lattice $\mathrm{Quo}(A)$ is four-generated for $|A| = \{\aleph_0\} \cup (\mathbb{N} \setminus \{1,4,6,8,10\})$. It is also shown in [5] that the complete lattice $\mathrm{Quo}(A)$ cannot be generated by less than four elements, provided $|A| \geq 3$. Concerning transitive relations, Dolgos [12] has shown that the complete lattice $\mathrm{Tran}(A)$ is eight-generated for $2 \leq |A| \leq \aleph_0$.

So our second goal in Chapter 3 is to show, in a concise but not self-contained way, that $\mathrm{Quo}(A)$ is four-generated if $|A| \neq 4$ and $|A|$ is an accessible cardinal. Furthermore, we prove that $\mathrm{Quo}(A)$ is $(1+1+2)$-generated in many (however not all) cases. We also improve the earlier results on the generating sets of $\mathrm{Tran}(A)$.

**Theorem 9.** *Let $A$ be a non-singleton set. Then the following statements hold.*

- *If $|A| \neq 4$ and $|A|$ is an accessible cardinal, then the complete lattice $\mathrm{Quo}(A)$ is four-generated.*

- *If $|A| \geq 13$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the complete lattice $\mathrm{Quo}(A)$ is $(1+1+2)$-generated.*

- *If $13 \leq |A| \leq \aleph_0$ and either $|A|$ is an odd number, or $|A| \geq 56$ is even, then the lattice $\mathrm{Quo}(A)$ (not a complete one now) contains a $(1+1+2)$-generated sublattice that includes all atoms of $\mathrm{Quo}(A)$.*

**Theorem 10.** *If* $3 \leq |A|$ *and* $|A|$ *is an accessible cardinal, then* $\mathrm{Tran}(A)$ *is six-generated as a complete lattice*

Later, Ahmed and Czédli [1] proved that if $A$ is a finite set such that $|A| \in \{3, 6, 11\}$ or $|A| \geq 13$, then $\mathrm{Quo}(A)$ is $(1+1+2)$-generated. So they extended the scope of the middle part of Theorem 9 by 24 new values of $|A|$. At present, there are seven finite values of $|A|$ such that we do not know whether $\mathrm{Quo}(A)$ is $(1+1+2)$-generated or not.

# On the largest numbers of congruences of finite lattices

Chapter 4 deals with the problem that given a natural number $n$, find the $n$-element finite lattices with the most, second-most, third-most, etc. congruences; also, give the diagram of the lattice of their congruences. This chapter is based on [17].

By Czédli and Mureşan [9], the set of all the congruences of an infinite lattice can be of any size between 2 and the cardinality of the lattice, or it can have the same cardinality as the lattice's subsets. But the situation is quite different for finite lattices. To formulate our results, the following lattice operations and notations are needed.

Let $L$ and $M$ be lattices. If $L$ has a largest element $1^L$ and $M$ has a smallest element $0^M$, then the *glued sum* of $L$ and $M$, denoted by $L \dotplus M$, is obtained from $L$ and $M$ by identifying $1^L$ with $0^M$ and stacking $M$ on top of $L$. If $L$ and $M$ are nontrivial bounded lattices, then the *horizontal sum* of $L$ and $M$, denoted by $L \boxplus M$, is obtained from $L$ and $M$ by identifying their bottom elements $0^L$ and $0^M$, identifying their top elements $1^L$ and $1^M$, and letting every element of $L \setminus \{0^L, 1^L\}$ be incomparable to every element of $M \setminus \{0^M, 1^M\}$ in $L \boxplus M$. For any $n \in \mathbb{N}$, we denote the $n$-element chain by $\mathcal{C}_n$. As usual, $\mathcal{N}_5$ denotes the five-element nonmodular lattice $\mathcal{C}_3 \boxplus \mathcal{C}_4$.

Using these notations, Freese [13] and Czédli [6] determined the largest and second largest numbers of congruences. Namely, if $L$ is a finite lattice with $n$ elements, then $|\mathrm{Con}(L)| \le 2^{n-1}$, also, $|\mathrm{Con}(L)| = 2^{n-1}$ iff $L \cong \mathcal{C}_n$. In other words, a finite lattice can have at most as many congruences as the chain with the same number of elements has. Furthermore, if $|\mathrm{Con}(L)| < 2^{n-1}$, then $|\mathrm{Con}(L)| \le 2^{n-2}$, moreover, $|\mathrm{Con}(L)| = 2^{n-2}$ iff $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-2}$ for some $k \in [1, n-3]$. That means the second largest possible number of congruences is witnessed by a glued sum of two chains with the four-element Boolean algebra. Following the line of Czédli's proof, we obtain the next result about the lattices with the third, fourth and fifth largest possible numbers of congruences. For a better understanding, see Figures 2–4.

**Theorem 11.** *Let $L$ be a finite lattice with $n$ elements.*

(i) *If $|\mathrm{Con}(L)| < 2^{n-2}$, then $n \ge 5$, $|\mathrm{Con}(L)| \le 5 \cdot 2^{n-5} = 2^{n-3} + 2^{n-5}$, and: $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ iff $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$ for some $k \in [1, n-4]$.*

(ii) *If $|\mathrm{Con}(L)| < 5 \cdot 2^{n-5}$, then $|\mathrm{Con}(L)| \le 2^{n-3}$, and: $|\mathrm{Con}(L)| = 2^{n-3}$ iff either $n \ge 6$ and $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$ for some $k \in [1, n-5]$, or $n \ge 7$ and $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-4}$ for some $k, m \in \mathbb{N}$ such that $k+m \le n-5$.*

(iii) *If $|\mathrm{Con}(L)| < 2^{n-3}$, then $|\mathrm{Con}(L)| \le 7 \cdot 2^{n-6} = 2^{n-4} + 2^{n-5} + 2^{n-6}$, and: $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ iff $n \ge 6$ and, for some $k \in [1, n-5]$, $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ or $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$.*

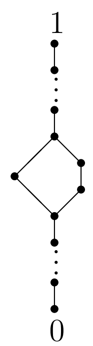


Figure 2: For $L \cong \mathcal{C}_k \dotplus \mathcal{N}_5 \dotplus \mathcal{C}_{n-k-3}$: $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$

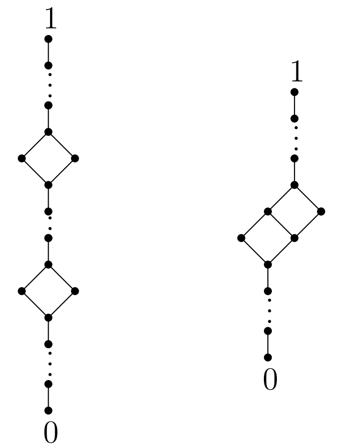

Figure 3: For $L \cong \mathcal{C}_k \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_m \dotplus \mathcal{C}_2^2 \dotplus \mathcal{C}_{n-k-m-4}$ and $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_2 \times \mathcal{C}_3) \dotplus \mathcal{C}_{n-k-4}$: $|\mathrm{Con}(L)| = 2^{n-3}$

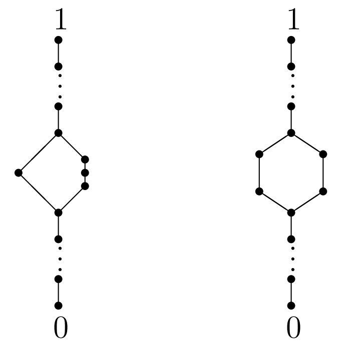

Figure 4: For $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_3 \boxplus \mathcal{C}_5) \dotplus \mathcal{C}_{n-k-4}$ and $L \cong \mathcal{C}_k \dotplus (\mathcal{C}_4 \boxplus \mathcal{C}_4) \dotplus \mathcal{C}_{n-k-4}$: $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$

Combining the earlier theorems with ours, we summarize the results on the lattices of the congruences of a finite lattice with the most, second-most, third-most, etc. congruences.

**Corollary 12.**

(i) $|\mathrm{Con}(L)| = 2^{n-1}$ *iff* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-1}$.

(ii) $|\mathrm{Con}(L)| = 2^{n-2}$ *iff* $n \geq 4$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-2}$.

(iii) $|\mathrm{Con}(L)| = 5 \cdot 2^{n-5}$ *iff* $n \geq 5$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-5} \times (\mathcal{C}_2 \dotplus \mathcal{C}_2^2)$.

(iv) $|\mathrm{Con}(L)| = 2^{n-3}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-3}$.

(v) $|\mathrm{Con}(L)| = 7 \cdot 2^{n-6}$ *iff* $n \geq 6$ *and* $\mathrm{Con}(L) \cong \mathcal{C}_2^{n-6} \times (\mathcal{C}_2^2 \dotplus \mathcal{C}_2^2)$.

# Összefoglaló (Summary in Hungarian)

E disszertációban az a célunk, hogy jobban megértsük bizonyos hálók és bizonyos kísérőhálók szerkezetét.

A bevezető 1. fejezetet követően a 2. fejezet sovány téglalapszerű hálókkal foglalkozik, és a [7] cikkünket dolgozza fel. A *sovány téglalapszerű* hálók speciális síkbarajzolható féligmoduláris hálók. Permutációkkal jellemezzük e hálókat, és a permutációk segítségével megadjuk az adott $n$ hosszúságú sovány téglalapszerű hálók számát. Azt is bebizonyítjuk, hogy a számuk aszimptotikusan $(n-2)! \cdot e^2/2$, ahol $e \approx 2.71828$.

A 3. fejezetben azt vizsgáljuk, hogy legkevesebb hány elemmel generálható a kvázirendezések hálója, valamint a tranzitív relációk hálója. Ez a fejezet a [8] és [16] cikkeinken alapul. Egy reflexív és tranzitív relációt *kvázirendezésnek* nevezünk. Egy $A$ halmaz kvázirendezései, illetve tranzitív relációi teljes hálót alkotnak, melyeket $\mathrm{Quo}(A)$-val, illetve $\mathrm{Tran}(A)$-val jelölünk. Takách [20] cikkében bebizonyította, hogy $\mathrm{Quo}(A)$-t hat elemmel lehet generálni elérhető számosságú $A$ halmazok esetén, Dolgos pedig megmutatta megszámlálható számosságú $A$ halmazokra [12]-ben, hogy $\mathrm{Quo}(A)$ öt elem által is generálható. A disszertáció 3. fejezetében először úgy általánosítjuk a korábbi eredményeket, hogy minden elérhető számosságú $A$ halmazra igazoljuk $\mathrm{Quo}(A)$ ötgeneráltságát. Ezt az eredményünket követően Czédli néhány kivételtől eltekintve majdnem minden megszámlálható $A$ halmazra bebizonyította az [5] cikkében, hogy $\mathrm{Quo}(A)$ négygenerált. Azt is megmutatta [5]-ben, hogy $|A| \geq 3$ esetén $\mathrm{Quo}(A)$ nem generálható négynél kevesebb elemmel. A 3. fejezet második részében általánosítjuk Czédli eredményét, tömör bizonyítást adunk arra, hogy $\mathrm{Quo}(A)$ négygenerált, ha $|A| \neq 4$ és $|A|$ tetszőleges elérhető számosság. Javítunk a $\mathrm{Tran}(A)$ generátorhalmazairól szóló korábbi eredményeken is: Dolgos [12]-ben $\mathrm{Tran}(A)$ nyolcgeneráltságát mutatta meg megszámlálható $A$ halmazok esetén, mi bebizonyítjuk, hogy hat elemmel is lehet generálni a tranzitív relációk hálóját elérhető számosságú alaphalmazok esetén.

A 4. fejezetben azzal a problémával foglalkozunk, hogy adott $n$ természetes szám esetén mely $n$ elemű véges hálóknak van a legtöbb, második legtöbb, harmadik legtöbb, stb. kongruenciája; továbbá azzal, hogy az ilyen hálók kongruenciahálóinak milyen a szerkezete. Ezek az eredmények a [17] cikkünkben jelentek meg. Freese [13]-ban bebizonyította, hogy egy véges hálónak legfeljebb annyi kongruenciája lehet, mint az azonos elemszámú lánc kongruenciáinak a száma. Majd Czédli [6]-ban leírta a lehetséges második legtöbb kongruenciával rendelkező hálókat. A disszertáció 4. fejezetében bemutatjuk az eredményeinket hálók kongruenciáinak harmadik, negyedik és ötödik lehetséges legnagyobb számáról.

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# Társszerzői nyilatkozat

Alulírott nyilatkozom, hogy az alábbi

G. Czédli, T. Dékány, G. Gyenizse and J. Kulin: *The number of slim rectangular lattices*, Algebra Universalis 75/1 (2016) 33-50

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Coauthor's declaration

I, the undersigned, declare that the following paper:

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is a joint work of the authors. In this work, Júlia Kulin's contribution is about 20% (twenty percent). I also declare that I have never used and will never use this paper to obtain a Ph.D. degree.

I agree for Júlia Kulin to include this paper in her Ph.D. thesis.

April 28th, 2024

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